Multiple orthogonal polynomials, irrationality and transcendence

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Dedicated to Jerry Lange at the occasion of his 70th birthday.

ABSTRACT. We explain the notion of multiple orthogonal polynomials (poly-orthogonal polynomials), which is closely related to Hermite-Padé rational approximation of a system of Markov functions, and give some explicit examples. As an application we show how multiple orthogonal polynomials can be used to give a constructive proof of irrationality of certain real numbers and also of transcendence of real numbers. Historically Hermite-Padé approximation was introduced by Hermite to prove the transcendence of e .

1. Orthogonal polynomials

The notion of orthogonal polynomials is an old one, going back to the previous century (Chebyshev, Stieltjes). A very good source of information is Szegő's book [Sz] and a more recent exposition can be found in [VA1]. Monic orthogonal polynomials for a positive Borel measure μ on the real line are polynomials P_n ($n = 0, 1, \dots$) of degree n and leading coefficient one such that

$$(1.1) \quad \int P_n(x) x^k d\mu(x) = 0, \quad k = 0, 1, \dots, n-1.$$

These n orthogonality conditions given n linear equations for the n unknown coefficients $a_{k,n}$ ($k = 0, 1, \dots, n-1$) of the monic polynomial $P_n(x) = \sum_{k=0}^n a_{k,n} x^k$ (where $a_{n,n} = 1$). This system of n equations for n unknowns always has a unique solution since the matrix with entries $\int x^{i+j} d\mu(x)$ ($0 \leq i, j \leq n-1$) (this is known as the Gram matrix) is positive definite and hence nonsingular. It is well known that such polynomials satisfy a three-term recurrence relation

$$(1.2) \quad P_{n+1}(x) = (x - b_n)P_n(x) - a_n^2 P_{n-1}(x), \quad n \geq 0,$$

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with $P_0 = 1$ and $P_{-1} = 0$. Often it is more convenient to consider the orthonormal polynomials $p_n(x) = \gamma_n P_n(x)$ for which

$$(1.3) \quad \int p_n(x) p_m(x) d\mu(x) = \delta_{m,n}, \quad m, n \geq 0,$$

so that $\gamma_n = \left(\int P_n^2(x) d\mu(x) \right)^{-1/2} > 0$. The recurrence relation for these orthonormal polynomials is given by

$$(1.4) \quad xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \geq 0$$

where

$$(1.5) \quad a_n = \int xp_{n-1}(x)p_n(x) d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n}, \quad b_n = \int xp_n^2(x) d\mu(x).$$

A very important result in the theory of orthogonal polynomials, known as Favard's theorem, say that a system of polynomials P_n ($n = 0, 1, 2, \dots$) satisfying a three-term recurrence relation of the form (1.2) with $a_n^2 > 0$ for all $n > 0$ is always a system of monic orthogonal polynomials for some positive Borel measure μ on the real line. Hence the orthogonality (1.1) and the recurrence relation (1.2) are two equivalent ways to describe monic orthogonal polynomials.

Starting from the recurrence relation (1.5) we can construct the Jacobi matrix

$$(1.6) \quad J_n = \begin{pmatrix} b_0 & a_1 & & & 0 \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & a_3 & b_3 & \ddots \\ & & & \ddots & \ddots & a_{n-1} \\ 0 & & & & a_{n-1} & b_{n-1} \end{pmatrix}.$$

The recurrence relation (1.5) can then be written as

$$\begin{pmatrix} b_0 & a_1 & & & 0 \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & a_3 & b_3 & \ddots \\ & & & \ddots & \ddots & a_{n-1} \\ 0 & & & & a_{n-1} & b_{n-1} \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \\ p_{n-1}(x) \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \\ p_{n-1}(x) \end{pmatrix} - a_np_n(x) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

from which we conclude that every zero $x_{j,n}$ of p_n is an eigenvalue of J_n with eigenvector $(p_0(x_{j,n}), p_1(x_{j,n}), \dots, p_{n-1}(x_{j,n}))^T$. Since all zeros of orthogonal polynomials are simple, it follows that the eigenvalues of J_n are the same as the zeros of p_n . One can also consider the infinite matrix J_∞ which acts as an operator $J_\infty : \ell_2 \rightarrow \ell_2$ on an appropriate domain. The symmetric matrix J_∞ has selfadjoint extensions to its maximal domain (sometimes several) and the study of such a selfadjoint operator (especially the spectral theory of such operators) gives useful insight in the corresponding orthogonal polynomials.

Let us mention at this point the best known examples: the very classical orthogonal polynomials of Jacobi, Laguerre and Hermite. These three families of orthogonal polynomials satisfy a second order linear differential equation, their derivatives are again orthogonal polynomials of the same family (but with different

parameters) and they can all be obtained from a Rodrigues formula. These properties can be extended to difference operators and q -difference operators and give the classical orthogonal polynomials in the Askey table [KS].

1.1. Jacobi polynomials. The monic Jacobi polynomials $\hat{P}_n^{(\alpha, \beta)}$ satisfy the orthogonality relations

$$(1.7) \quad \int_{-1}^1 \hat{P}_n^{(\alpha, \beta)}(x)(1-x)^\alpha(1+x)^\beta x^k dx = 0, \quad k = 0, 1, \dots, n-1,$$

where $\alpha, \beta > -1$. There are two important differential operators, namely the lowering operator D for which

$$(1.8) \quad D\hat{P}_n^{(\alpha, \beta)}(x) = \left(\hat{P}_n^{(\alpha, \beta)}(x) \right)' = n\hat{P}_{n-1}^{(\alpha+1, \beta+1)}(x),$$

and the raising operator $D_{\alpha, \beta} = (1-x)^{-\alpha}(1+x)^{-\beta}D(1-x)^\alpha(1+x)^\beta$ for which

$$(1.9) \quad (1-x^2)D_{\alpha, \beta}\hat{P}_n^{(\alpha, \beta)}(x) = -(n+\alpha+\beta)\hat{P}_{n+1}^{(\alpha-1, \beta-1)}(x).$$

Both (1.8) and (1.9) can be proved using integration by parts and the orthogonality (1.7). Combining the raising and lowering operators gives the differential equation

$$(1.10) \quad (1-x^2)y''(x) - [\alpha - \beta + (\alpha + \beta + 2)x]y'(x) = -n(n + \alpha + \beta + 1)y(x),$$

where $y(x) = \hat{P}_n^{(\alpha, \beta)}(x)$, and repeated use of the raising operator $D_{\alpha, \beta}$ gives the Rodrigues formula

$$(1.11) \quad (-1)^n \binom{\alpha + \beta + 2n}{n} n! \hat{P}_n^{(\alpha, \beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n} (1-x)^{n+\alpha}(1+x)^{n+\beta}.$$

Using Leibnitz' formula we can obtain the explicit formula

$$(1.12) \quad (-1)^n \binom{\alpha + \beta + 2n}{n} \hat{P}_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (-1)^k (1-x)^{n-k} (1+x)^k.$$

The historical way to normalize the Jacobi polynomials is by setting $P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$ so that $P_n^{(\alpha, \beta)}(x) = 2^{-n} \binom{\alpha+\beta+2n}{n} \hat{P}_n^{(\alpha, \beta)}(x)$. See Szegő [Sz, Chapter IV] for more properties of the Jacobi polynomials.

1.2. Laguerre polynomials. The monic Laguerre polynomials $\hat{L}_n^\alpha$ satisfy the orthogonality conditions

$$(1.13) \quad \int_0^\infty \hat{L}_n^\alpha(x) x^\alpha e^{-x} x^k dx = 0, \quad k = 0, 1, 2, \dots, n-1,$$

where $\alpha > -1$. Differentiation gives again the lowering operation

$$(1.14) \quad D\hat{L}_n^\alpha(x) = n\hat{L}_{n-1}^{\alpha+1}(x),$$

and $D_\alpha = x^{-\alpha}e^x D x^\alpha e^{-x}$ is the raising operator for which

$$(1.15) \quad xD_\alpha \hat{L}_n^\alpha(x) = -\hat{L}_{n+1}^{\alpha-1}(x).$$

Both (1.14) and (1.15) can easily be proved using integration by parts and the orthogonality relation (1.13). Combining the lowering and the raising operator gives the differential equation

$$(1.16) \quad xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x),$$

where $y(x) = \hat{L}_n^\alpha(x)$. Using the raising operator repeatedly gives the Rodrigues formula

$$(1.17) \quad (-1)^n \hat{L}_n^\alpha(x) = x^{-\alpha} e^x \frac{d^n}{dx^n} x^{\alpha+n} e^{-x},$$

from which one can obtain an explicit expression using Leibnitz' formula

$$(1.18) \quad (-1)^n \hat{L}_n^\alpha(x) = n! \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}.$$

The historical normalization for the Laguerre polynomials is to take $L_n^\alpha(0) = \binom{n+\alpha}{n}$ so that $L_n^\alpha(x) = (-1)^n/n! \hat{L}_n^\alpha(x)$. See Szegő [Sz, Chapter V] for more properties of Laguerre polynomials.

1.3. Hermite polynomials. The monic Hermite polynomials $\hat{H}_n$ satisfy the orthogonality relations

$$(1.19) \quad \int_{-\infty}^{\infty} H_n(x) e^{-x^2} x^k dx = 0, \quad k = 0, 1, 2, \dots, n-1.$$

Integration by parts and the orthogonality show that D is a lowering operator

$$(1.20) \quad D\hat{H}_n(x) = \hat{H}'_n(x) = n\hat{H}_{n-1}(x),$$

and we have the raising operator

$$(1.21) \quad e^{x^2} D e^{-x^2} \hat{H}_n(x) = -2\hat{H}_{n+1}(x).$$

Combining the lowering and the raising operator gives the differential equation

$$(1.22) \quad y''(x) - 2xy'(x) = -2ny(x),$$

where $y(x) = \hat{H}_n(x)$. Repeated application of the raising operator gives the Rodrigues formula

$$(1.23) \quad (-1)^n 2^n \hat{H}_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and by using Faa di Bruno's formula for the n th derivative of a composite function we obtain

$$(1.24) \quad \hat{H}_n(x) = 2^{-n} n! \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}.$$

Historically the Hermite polynomials are normalized to have leading coefficient 2^n , hence $H_n(x) = 2^n \hat{H}_n(x)$. For more properties we refer to Szegő [Sz, Chapter 5].

2. Multiple orthogonal polynomials

During the past few years there has been increased interest in multiple orthogonal polynomials, particularly in Eastern Europe. See, e.g., surveys by Aptekarev [A2], de Bruin [dB] and Chapter 4 in the book of Nikishin and Sorokin [NS]. For multiple orthogonal polynomials we will need multi-indices consisting of r positive integers, for which we use the notation $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$, where $r \in \mathbb{N}$. Furthermore we will use the notation $|\vec{n}| = n_1 + n_2 + \dots + n_r$. We distribute the orthogonality conditions over r real intervals $\Delta_1, \Delta_2, \dots, \Delta_r$ with r different measures $\mu_1, \mu_2, \dots, \mu_r$. Two different ways are considered, which give type I and type II multiple orthogonal polynomials.

2.1. Type II multiple orthogonal polynomials. Type II multiple orthogonal polynomials are monic polynomials $P_{\vec{n}}$ of degree $|\vec{n}|$ such that

$$(2.1) \quad \int_{\Delta_1} P_{\vec{n}}(x) x^k d\mu_1(x) = 0, \quad k = 0, 1, \dots, n_1 - 1,$$

$$(2.2) \quad \int_{\Delta_2} P_{\vec{n}}(x) x^k d\mu_2(x) = 0, \quad k = 0, 1, \dots, n_2 - 1,$$

$$(2.3) \quad \begin{array}{c} \vdots \\ \int_{\Delta_r} P_{\vec{n}}(x) x^k d\mu_r(x) = 0, \quad k = 0, 1, \dots, n_r - 1. \end{array}$$

For $r = 1$ we get the ordinary monic orthogonal polynomials, but for $r > 1$ we get multiple orthogonal polynomials (or polyorthogonal polynomials). All together the orthogonality conditions give $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients $a_{k, \vec{n}}$ of the polynomial $P_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k, \vec{n}} x^k$, where $a_{|\vec{n}|, \vec{n}} = 1$. However, the matrix of coefficients of this system can be singular and we need some extra conditions on the r measures $\mu_1, \mu_2, \dots, \mu_r$ in order that the multiple orthogonal polynomial is unique. If the polynomial $P_{\vec{n}}(x)$ is unique, then we say that $\vec{n}$ is a normal index and if all indices are normal then we have a complete system. Our interest is of course in systems of r measures for which all multi-indices are normal.

2.2. Type I multiple orthogonal polynomials. For type I multiple orthogonal polynomials we want to find a vector of polynomials $(A_{\vec{n},1}, A_{\vec{n},2}, \dots, A_{\vec{n},r})$ such that each polynomial $A_{\vec{n},j}$ is of degree $n_j - 1$ and

$$(2.4) \quad \sum_{j=1}^r \int_{\Delta_j} A_{\vec{n},j}(x) x^k d\mu_j(x) = 0, \quad k = 0, 1, 2, \dots, |\vec{n}| - 2.$$

Again for $r = 1$ we find the ordinary orthogonal polynomials, but for $r > 1$ we find multiple orthogonal polynomials. There are $|\vec{n}| - 1$ orthogonality conditions which give $|\vec{n}| - 1$ homogenous linear equations for the $|\vec{n}|$ unknown coefficients of all the polynomials $A_{\vec{n},j}$ ($j = 1, 2, \dots, r$). Hence we can determine the vector $(A_{\vec{n},1}, A_{\vec{n},2}, \dots, A_{\vec{n},r})$ up to a multiplicative factor.

2.3. Operator approach. The three-term recurrence relation (1.2) and the Jacobi matrix (1.6) have an interesting extension for multiple orthogonal polynomials, which was subsequently exploited by Kalyagin [K2] to initiate the spectral

analysis of certain nonsymmetric operators. We construct a special sequence $\vec{s}(n)$ ($n \in \mathbb{N}$) of multi-indices in $\mathbb{N}^r$ by writing $n = kr + j$, with $0 \leq j < r$ and then set

$$(2.5) \quad \vec{s}(n) = (\underbrace{k+1, k+1, \dots, k+1}_{j \text{ times}}, k, k, \dots, k).$$

These indices are nearly diagonal and if all these indices are normal, then we have a weakly complete system. Now consider the monic type II multiple orthogonal polynomials with these indices, i.e., we put $Q_n(x) = P_{\vec{s}(n)}(x)$, then one can use all the orthogonality relations (2.1)–(2.3) to show that they satisfy a recurrence relation of the form

$$(2.6) \quad xQ_n(x) = Q_{n+1}(x) + a_0(n)Q_n(x) + a_1(n)Q_{n-1}(x) + \dots + a_r(n)Q_{n-r}(x),$$

with initial conditions $Q_0 = 1$ and $Q_{-1} = Q_{-2} = \dots = Q_{-r} = 0$. These diagonal polynomials therefore satisfy a linear recurrence relation of order $r+1$, generalizing the second order recurrence relation for ordinary orthogonal polynomials ($r = 1$). We can then construct a banded Hessenberg matrix

$$(2.7) \quad H_n = \begin{pmatrix} a_0(0) & 1 & & & & & 0 \\ a_1(1) & a_0(1) & 1 & & & & \\ \vdots & a_1(2) & a_0(2) & 1 & & & \\ a_r(r) & \vdots & a_1(3) & a_0(3) & 1 & & \\ & a_r(r+1) & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & & & & 1 \\ 0 & & & a_r(n-1) & \dots & a_1(n-1) & a_0(n-1) \end{pmatrix}.$$

Again it turns out that the eigenvalues of H_n coincide with the zeros of the multiple orthogonal polynomial $Q_n = P_{\vec{s}(n)}$. The matrix H_n is not symmetric, thus there is no a priori reason why the eigenvalues should be real. Nevertheless it turns out that in many cases the zeros are indeed real. Hence the coefficients $a_i(n+i)$ ($n \in \mathbb{N}$), with $0 \leq i \leq r$ are not arbitrary and have some hidden structure. An interesting open problem is to find this hidden structure and to find out under which conditions on these coefficients the matrix H_n corresponds to multiple orthogonal polynomials.

3. Hermite-Padé approximation

3.1. Simultaneous Padé approximation. Multiple orthogonal polynomials are closely related to simultaneous Padé approximation, which is nowadays known as Hermite-Padé approximation. Suppose we are given r Markov functions (or Stieltjes functions)

$$\begin{aligned} f_1(z) &= \int_{\Delta_1} \frac{d\mu_1}{z-x}, & z \notin \Delta_1, \\ f_2(z) &= \int_{\Delta_2} \frac{d\mu_2}{z-x}, & z \notin \Delta_2, \\ &\vdots \\ f_r(z) &= \int_{\Delta_r} \frac{d\mu_r}{z-x}, & z \notin \Delta_r, \end{aligned}$$

then our aim is to approximate these r functions simultaneously by rational functions, such that the approximation is of prescribed order near ∞ . Type I Hermite-Padé approximation is to find a vector of polynomials $(A_{\vec{n},1}, A_{\vec{n},2}, \dots, A_{\vec{n},r})$ and a polynomial $B_{\vec{n}}$ such that the degree of $A_{\vec{n},j}$ is less than or equal to $n_j - 1$ and

$$(3.1) \quad A_{\vec{n},1}(z)f_1(z) + \dots + A_{\vec{n},r}(z)f_r(z) - B_{\vec{n}}(z) = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|}}\right), \quad z \rightarrow \infty.$$

Type II Hermite-Padé approximation is to find a polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ and polynomials $Q_{\vec{n},1}, \dots, Q_{\vec{n},r}$ such that

$$(3.2) \quad P_{\vec{n}}(z)f_1(z) - Q_{\vec{n},1}(z) = \mathcal{O}\left(\frac{1}{z^{n_1+1}}\right), \quad z \rightarrow \infty,$$

$$(3.3) \quad P_{\vec{n}}(z)f_2(z) - Q_{\vec{n},2}(z) = \mathcal{O}\left(\frac{1}{z^{n_2+1}}\right), \quad z \rightarrow \infty,$$

$$\vdots$$

$$(3.4) \quad P_{\vec{n}}(z)f_r(z) - Q_{\vec{n},r}(z) = \mathcal{O}\left(\frac{1}{z^{n_r+1}}\right), \quad z \rightarrow \infty.$$

Hence type II Hermite-Padé approximation is rational approximation of the functions $f_1, f_2, \dots, f_r$ with a *common denominator*. For $r = 1$ type I and type II Hermite-Padé approximation reduces to ordinary Padé approximation of f_1 . It is well known that the denominator polynomials in Padé approximation to a Markov function (or Stieltjes function) are orthogonal polynomials for the measure μ_1 . For multiple orthogonal polynomials there is a similar relationship: the polynomials $A_{\vec{n},1}, A_{\vec{n},2}, \dots, A_{\vec{n},r}$ in type I Hermite-Padé approximation are precisely the type I multiple orthogonal polynomials for the measures $(\mu_1, \dots, \mu_r)$ and the common denominator $P_{\vec{n}}$ in type II Hermite-Padé approximation is the type II multiple orthogonal polynomial for the measures $(\mu_1, \dots, \mu_r)$. Furthermore, the remaining polynomials $B_{\vec{n}}$ in type I approximation and $Q_{\vec{n},j}$ ($j = 1, 2, \dots, r$) in type II approximation can be obtained from the multiple orthogonal polynomials through the formulas

$$(3.5) \quad B_{\vec{n}}(z) = \sum_{j=1}^r \int_{\Delta_j} \frac{A_{\vec{n},j}(z) - A_{\vec{n},j}(x)}{z - x} d\mu_j(x),$$

and

$$(3.6) \quad Q_{\vec{n},j}(z) = \int_{\Delta_j} \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z - x} d\mu_j(x), \quad j = 1, \dots, r.$$

Combining (3.1) with (3.5) gives a nice formula for the remainder in type I Hermite-Padé approximation in terms of the multiple orthogonal polynomials

$$(3.7) \quad \sum_{j=1}^r A_{\vec{n},j}(z)f_j(z) - B_{\vec{n}}(z) = \sum_{j=1}^r \int_{\Delta_j} \frac{A_{\vec{n},j}(x)}{z - x} d\mu_j(x),$$

and combining (3.2)–(3.4) with (3.6) gives a formula for the remainder in type II Hermite-Padé approximation

$$(3.8) \quad P_{\vec{n}}(z)f_j(z) - Q_{\vec{n},j}(z) = \int_{\Delta_j} \frac{P_{\vec{n}}(x)}{z - x} d\mu_j(x).$$

Hence for Hermite-Padé approximation one only needs to obtain the multiple orthogonal polynomials, from which all other quantities can be deduced.

3.2. Useful systems of functions. In order to get a system for which most multi-indices are normal, one needs to pose additional assumptions on the Markov functions $(f_1, f_2, \dots, f_r)$ or on the measures $(\mu_1, \mu_2, \dots, \mu_r)$. Two kinds of systems turn out to be quite useful.

DEFINITION 3.1 (see [An]). An *Angelesco system* consists of r measures on r pairwise disjoint intervals:

$$(3.9) \quad f_j(z) = \int_{\Delta_j} \frac{d\mu_j(x)}{z-x}, \quad \Delta_i \cap \Delta_j = \emptyset \text{ if } i \neq j.$$

Angelesco systems are always complete systems and are the prototype for studying multiple orthogonal polynomials on r pairwise disjoint intervals. It turns out that the zeros of type II multiple orthogonal polynomials have a very simple location property for Angelesco systems: there will be n_i zeros of $P_{\vec{n}}$ on the interval Δ_i for $i = 1, 2, \dots, r$. The polynomial $P_{\vec{n}}$ can therefore be factored as $P_{\vec{n}}(x) = P_{n_1}(x)P_{n_2}(x) \cdots P_{n_r}(x)$ and each P_{n_i} is an ordinary orthogonal polynomial on Δ_i for the varying measure $\prod_{j \neq i} P_{n_j}(x) d\mu_j(x)$.

For multiple orthogonal polynomials on one interval there is another useful system of functions.

DEFINITION 3.2 (see [N1]). An *AT system* consists of r measures on the same interval Δ such that each measure μ_j is absolutely continuous with respect to $\mu = \mu_1$ and such that the Radon-Nikodym derivatives $u_j = d\mu_j/d\mu$ (with $u_1 = 1$) are such that $\{1, x, \dots, x^{n_1-1}, u_2, xu_2, \dots, x^{n_2-1}u_2, \dots, u_r, xu_r, \dots, x^{n_r-1}u_r\}$ is a Chebyshev system on Δ for every multi-index $(n_1, n_2, \dots, n_r)$.

Recall that a set of n linearly independent functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a Chebyshev system on Δ if every linear combination $\sum_{k=1}^n a_k \phi_k$ has at most $n-1$ zeros on Δ . For an AT system it thus follows that $P_{n_1-1}(x) + P_{n_2-1}(x)u_2(x) + \cdots + P_{n_r-1}(x)u_r(x)$ has at most $|\vec{n}| - 1$ zeros on Δ whenever P_k is a polynomial of degree at most k . A special example of an AT system is a *Nikishin system* [N2] which is defined recursively as follows. Any Markov function

$$f_{1,1}(z) = \int_{\Delta_1} \frac{d\mu_1(x)}{z-x}$$

is a Nikishin system of order one on Δ_1 . A system $(f_{2,1}, f_{2,2})$ is a Nikishin system of order two on Δ_2 if

$$f_{2,1}(z) = \int_{\Delta_2} \frac{d\mu_2(x)}{z-x}, \quad f_{2,2}(z) = \int_{\Delta_2} f_{1,1}(x) \frac{d\mu_2(x)}{z-x},$$

where $f_{1,1}$ is a Nikishin system of order one on Δ_1 and $\Delta_1 \cap \Delta_2 = \emptyset$. In general the system $(f_{r,1}, f_{r,2}, \dots, f_{r,r})$ is a Nikishin system of order r on Δ_r if

$$f_{r,j}(z) = \int_{\Delta_r} f_{r-1,j-1}(x) \frac{d\mu_r(x)}{z-x}, \quad j = 1, 2, \dots, r,$$

where $f_{r-1,0} = 1$ and $(f_{r-1,1}, f_{r-1,2}, \dots, f_{r-1,r-1})$ is a Nikishin system of order $r-1$ on Δ_{r-1} with $\Delta_r \cap \Delta_{r-1} = \emptyset$. Nikishin systems turn out to be the prototype of systems of Markov functions on one interval and allow a rather detailed analysis. See, e.g., Driver and Stahl [DS1] [DS2] for normality of Nikishin systems and

Bustamante and López [BL] and Aptekarev [A3] for asymptotic results. Driver and Stahl [DS1] have shown that the type II multiple orthogonal polynomial $P_{\vec{n}}$ for a Nikishin system of order r on Δ_r has $|\vec{n}|$ simple zeros on this interval Δ_r whenever $n_j \geq \max(n_{j+1}, \dots, n_r) - 1$ for every $j = 1, 2, \dots, r - 1$. In particular this is true for the diagonal indices $\vec{s}(n)$ given in (2.5).

Recently Gonchar, Rakhmanov and Sorokin [GRS] have considered a mixture of Angelesco and Nikishin systems and have analyzed the asymptotic behavior of the corresponding Hermite-Padé approximants. The asymptotic behavior of the Hermite-Padé approximants for Angelesco and Nikishin systems was obtained in [GR] and [S5]. One of the important results is the asymptotic distribution of the zeros of the polynomials. For ordinary orthogonal polynomials the distribution of zeros is usually obtained by investigating an extremal problem for logarithmic potentials. For multiple orthogonal polynomials one needs to study an extremal problem in logarithmic potential theory for vector potentials.

4. Examples of multiple orthogonal polynomials

The very classical orthogonal polynomials of Jacobi, Laguerre, and Hermite have several possible extensions as multiple orthogonal polynomials. We will mention some of them. See also Aptekarev, Marcellán and Rocha [AMR] for some Angelesco type multiple orthogonal polynomials.

4.1. Jacobi-Angelesco polynomials. Kalyagin [K1] [KR] studied multiple Jacobi polynomials on two intervals $[a, 0]$ and $[0, 1]$, with $a < 0$. The measures are $d\mu_1(x) = |x|^\gamma(x-a)^\alpha(1-x)^\beta dx$ on $[a, 0]$ and $d\mu_2(x) = x^\gamma(x-a)^\alpha(1-x)^\beta dx$ on $[0, 1]$, with $\alpha, \beta, \gamma > -1$. These intervals have only one point in common and this point will not influence the nature of the multiple orthogonal polynomials so that we can still consider this as an Angelesco system. There is a raising operator for the monic diagonal type II multiple orthogonal polynomials, namely

$$(4.1) \quad \left(x^\gamma(x-a)^\alpha(1-x)^\beta P_{n,n}^{(\alpha,\beta,\gamma)}(x) \right)' \\ = -(\alpha + \beta + \gamma + 2n)x^{\gamma-1}(x-a)^{\alpha-1}(1-x)^{\beta-1} P_{n+1,n+1}^{(\alpha-1,\beta-1,\gamma-1)}(x),$$

and by using this formula repeatedly we obtain the Rodrigues formula

$$(4.2) \quad (-1)^n n! \binom{\alpha + \beta + \gamma + 3n}{n} P_{n,n}^{(\alpha,\beta,\gamma)}(x) \\ = x^{-\gamma}(x-a)^{-\alpha}(1-x)^{-\beta} \frac{d^n}{dx^n} (x^{\gamma+n}(x-a)^{\alpha+n}(1-x)^{\beta+n}).$$

These polynomials satisfy a third order differential equation

$$(4.3) \quad x(x-1)(x-a)y''' + (6x^2 - 4(a+1)x + 2a)y'' \\ + (n-1)(n+2)(-3x+a+1)y' - 2n(n^2+3n+2)y = 0,$$

where $y(x) = P_{n,n}^{(\alpha,\beta,\gamma)}(x)$. Observe that the coefficients of y''' and y'' are polynomials independent of n but that the coefficient of y' depends on n , so that Sturm-Liouville theory of linear differential equations can not be used here. If we write $P_{2n} = P_{n,n}^{(\alpha,\beta,\gamma)}$ and $P_{2n+1} = P_{n+1,n}^{(\alpha,\beta,\gamma)}$, then we have a third order recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) + d_n P_{n-2}(x),$$

for which

$$\begin{aligned}\lim_{n \rightarrow \infty} b_{2n} &= \frac{a+1}{9} + \frac{2}{3}x_1, & \lim_{n \rightarrow \infty} b_{2n+1} &= \frac{a+1}{9} + \frac{2}{3}x_2, \\ \lim_{n \rightarrow \infty} c_n &= \frac{4}{81}(a^2 - a + 1), \\ \lim_{n \rightarrow \infty} d_{2n} &= -\frac{4}{27}x_1(x_1 - a)(x_1 - 1), & \lim_{n \rightarrow \infty} d_{2n+1} &= -\frac{4}{27}x_2(x_2 - a)(x_2 - 1)\end{aligned}$$

where x_1 and x_2 are the roots of the quadratic equation $h'(x) = 0$, with $h(x) = x(x-a)(x-1)$.

4.2. Jacobi-Pineiro polynomials. Pineiro [P] considered multiple Jacobi polynomials on $[0, 1]$ with Jacobi weights

$$(4.4) \quad d\mu_j(x) = x^{\alpha_j}(1-x)^{\alpha_0} dx, \quad j = 1, 2, \dots, r,$$

where $\alpha_k > -1$ for $k = 0, 1, \dots, r$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$ and $i, j \neq 0$. Originally Pineiro only considered the case $\alpha_0 = 0$ but this more general case can easily be obtained without too much extra effort (see, e.g., [NS, p. 162]). If we set $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ then we have r raising operators for the monic type II multiple orthogonal polynomials:

$$(4.5) \quad \left(x^{\alpha_j}(1-x)^{\alpha_0} P_{\vec{n}}^{\alpha_0, \vec{\alpha}}(x) \right)' \\ = -(|\vec{n}| + \alpha_0 + \alpha_j) x^{\alpha_j-1} (1-x)^{\alpha_0-1} P_{\vec{n} + \vec{e}_j}^{\alpha_0-1, \vec{\alpha} - \vec{e}_j}(x), \quad j = 1, \dots, r,$$

where $\vec{e}_j$ is the j th standard unit vector $(0, \dots, 0, 1, 0, \dots, 0)$. Combining all these raising operators gives the Rodrigues formula

$$(4.6) \quad (-1)^{|\vec{n}|} \prod_{j=1}^r (|\vec{n}| + \alpha_0 + \alpha_j + 1)_{n_j} P_{\vec{n}}^{\alpha_0, \vec{\alpha}}(x) \\ = (1-x)^{-\alpha_0} \prod_{j=1}^r \left[x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j + \alpha_j} \right] (1-x)^{\alpha_0 + |\vec{n}|},$$

where the order in the product of r differential operators can be changed. For $r = 2$ we put $P_{2n} = P_{n,n}^{(\alpha_0, \alpha_1, \alpha_2)}$ and $P_{2n+1} = P_{n+1,n}^{(\alpha_0, \alpha_1, \alpha_2)}$ and we have the third order recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) + d_n P_{n-2}(x),$$

for which the recurrence coefficients satisfy

$$\begin{aligned}\lim_{n \rightarrow \infty} b_n &= 3 \frac{4}{27}, \\ \lim_{n \rightarrow \infty} c_n &= 3 \left(\frac{4}{27} \right)^2, \\ \lim_{n \rightarrow \infty} d_n &= \left(\frac{4}{27} \right)^3.\end{aligned}$$

4.3. Multiple Laguerre polynomials. In a similar way, Sorokin [S2] [S6] considered multiple Laguerre polynomials on $[0, \infty)$ with weights

$$(4.7) \quad d\mu_j(x) = x^{\alpha_j} e^{-x} dx, \quad j = 1, 2, \dots, r,$$

with $\alpha_j > -1$ for $j = 1, 2, \dots, r$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$. These polynomials are a limiting case of the Jacobi-Pineiro polynomials:

$$(4.8) \quad \lim_{\alpha_0 \rightarrow \infty} \alpha_0^{|\vec{n}|} P_{\vec{n}}^{\alpha_0, \vec{\alpha}}(x/\alpha_0) = L_{\vec{n}}^{\vec{\alpha}}(x),$$

in a similar way as the Laguerre polynomials are a limiting case of the Jacobi polynomials. There are again r raising operators

$$(4.9) \quad (x^{\alpha_j} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x))' = -x^{\alpha_j-1} e^{-x} L_{\vec{n}+\vec{e}_j}^{\vec{\alpha}-\vec{e}_j}(x), \quad j = 1, 2, \dots, r,$$

and combining these raising operators gives the Rodrigues formula

$$(4.10) \quad (-1)^{|\vec{n}|} L_{\vec{n}}^{\vec{\alpha}}(x) = e^x \prod_{j=1}^r \left[x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right] e^{-x}.$$

For $r = 2$ we put $P_{2n} = L_{n,n}^{(\alpha_1, \alpha_2)}$ and $P_{2n+1} = L_{n+1,n}^{(\alpha_1, \alpha_2)}$ to find the third order recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) + d_n P_{n-2}(x),$$

with

$$\begin{aligned} b_{2n} &= 3n + \alpha_1 + 1 \\ b_{2n+1} &= 3n + \alpha_2 + 2 \\ c_{2n} &= n(3n + \alpha_1 + \alpha_2) \\ c_{2n+1} &= (n+1)(3n + \alpha_1 + \alpha_2) - \alpha_2 + 1 \\ d_{2n} &= n(n + \alpha_1)(n + \alpha_1 - \alpha_2) \\ d_{2n+1} &= n(n + \alpha_2)(n + \alpha_2 - \alpha_1). \end{aligned}$$

Observe that

$$b_n = 3\frac{n}{2} + \mathcal{O}(1), \quad c_n = 3\left(\frac{n}{2}\right)^2 + \mathcal{O}(n), \quad d_n = \left(\frac{n}{2}\right)^3 + \mathcal{O}(n^2).$$

Another way to obtain multiple Laguerre polynomials on $[0, \infty)$ is to use the weights

$$(4.11) \quad d\mu_j(x) = x^{\alpha} e^{-c_j x} dx, \quad j = 1, 2, \dots, r,$$

with $\alpha > -1$ and $c_1, c_2, \dots, c_r$ distinct positive real numbers (see [NS, p. 160] where $\alpha = 0$). The raising operators are

$$(4.12) \quad \left(x^{\alpha} e^{-c_j x} P_{\vec{n}}^{\vec{c}, \alpha}(x) \right)' = -c_j x^{\alpha-1} e^{-c_j x} P_{\vec{n}+\vec{e}_j}^{\vec{c}, \alpha-1}(x), \quad j = 1, 2, \dots, r,$$

and the corresponding Rodrigues formula is

$$(4.13) \quad (-1)^{|\vec{n}|} \prod_{j=1}^r c_j^{n_j} P_{\vec{n}}^{\vec{c}, \alpha}(x) = x^{-\alpha} \prod_{j=1}^r \left[e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right] x^{|\vec{n}|+\alpha},$$

and again the product of the differential operators can be taken in any order.

4.4. Jacobi-Laguerre polynomials. Sorokin [S1] considered an Angelesco case containing a Jacobi and a Laguerre part. Consider the measures $d\mu_1(x) = (1+x)^\alpha |x|^\beta e^{-x} dx$ on the interval $[-1, 0]$ and $d\mu_2(x) = (1+x)^\alpha x^\beta e^{-x} dx$ on $[0, \infty)$, with $\alpha, \beta > -1$, then the raising operator for the diagonal type II multiple orthogonal polynomial is

$$(4.14) \quad \left((1+x)^\alpha x^\beta e^{-x} P_{n,n}^{(\alpha,\beta)}(x) \right)' = -(1+x)^{\alpha-1} x^{\beta-1} e^{-x} P_{n+1,n+1}^{(\alpha-1,\beta-1)}(x),$$

and applying this rule repeatedly gives the Rodrigues formula

$$(4.15) \quad (-1)^n P_{n,n}^{(\alpha,\beta)}(x) = (1+x)^{-\alpha} x^{-\beta} e^x \frac{d^n}{dx^n} (1+x)^{n+\alpha} x^{n+\beta} e^{-x}.$$

4.5. Multiple Macdonald polynomials. For $r = 1$ the very classical orthogonal polynomials are those of Jacobi, Laguerre and Hermite. For multiple orthogonal polynomials there seem to be more classical families, which is at least suggested by the following example [VAY]. Consider the weight function

$$(4.16) \quad d\mu_1(x) = x^\alpha \rho_\nu(x) dx, \quad d\mu_2(x) = x^\alpha \rho_{\nu+1}(x) dx \quad x \in [0, \infty),$$

where $\alpha > -1$ and $\rho_\nu(x) = x^{\nu/2} K_\nu(2\sqrt{x})$ ($\nu \geq 0$), with K_ν the Macdonald function (modified Bessel function of the second kind). This AT system has a lot of properties that very classical polynomials usually have. The differential operator D is a lowering operator for the type II multiple orthogonal polynomials near the diagonal

$$(4.17) \quad DP_{n,n}^\alpha(x) = 2n P_{n,n-1}^{\alpha+1}(x), \quad DP_{n,n-1}^\alpha(x) = (2n-1) P_{n-1,n-1}^{\alpha+1}(x),$$

and $D_\alpha = x^{-\alpha} D x^\alpha$ is a raising operator for the type I multiple orthogonal polynomials: putting $Q_{m,n}^\alpha = A_{(m,n),1}^\alpha \rho_\nu + A_{(m,n),2}^\alpha \rho_{\nu+1}$ we have

$$(4.18) \quad x D_\alpha Q_{n,n}^\alpha(x) = Q_{n+1,n}^{\alpha-1}(x), \quad x D_\alpha Q_{n,n-1}^\alpha(x) = Q_{n,n}^{\alpha-1}(x).$$

Using the raising operators repeatedly gives a Rodrigues formula for the type I multiple orthogonal polynomials

$$(4.19) \quad x^\alpha Q_{n,n-1}^\alpha(x) = \frac{d^{2n}}{x^{2n}} x^{2n+\alpha} \rho_\nu(x), \quad x^\alpha Q_{n,n}^\alpha(x) = \frac{d^{2n+1}}{x^{2n+1}} x^{2n+1+\alpha} \rho_\nu(x).$$

Putting $P_{2n} = P_{n,n}^\alpha$ and $P_{2n+1} = P_{n+1,n}^\alpha$ gives the recurrence relation

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) + d_n P_{n-2}(x),$$

with

$$\begin{aligned} b_n &= (n+\alpha+1)(3n+\alpha+2\nu) - (\alpha+1)(\nu-1), \\ c_n &= n(n+\alpha)(n+\alpha+\nu)(3n+2\alpha+\nu), \\ d_n &= n(n-1)(n+\alpha-1)(n+\alpha)(n+\alpha+\nu-1)(n+\alpha+\nu) \end{aligned}$$

so that

$$b_n = 3n^2 + \mathcal{O}(n), \quad c_n = 3n^4 + \mathcal{O}(n^3), \quad d_n = n^6 + \mathcal{O}(n^5).$$

5. Irrationality

Multiple orthogonal polynomials turn out to be quite useful in proving irrationality of certain real numbers. The key to such irrationality proofs is the following lemma:

LEMMA 5.1. *Let x be a real number. Suppose there exists integers p_n and q_n ($n \in \mathbb{N}$) such that*

1. $q_n x - p_n \neq 0$ for all $n \in \mathbb{N}$,
2. $\lim_{n \rightarrow \infty} (q_n x - p_n) = 0$.

Then x is irrational.

PROOF. Suppose x is rational so that $x = p/q$ with integers p and q . Then

$$q_n x - p_n = \frac{q_n p - p_n q}{q}.$$

By assumption (1) we know that $q_n p - p_n q$ is an integer different from zero, hence $|q_n p - p_n q| \geq 1$. But then

$$|q_n x - p_n| \geq \frac{1}{q},$$

and as $n \rightarrow \infty$ we find that this is in contradiction with assumption (2). Hence x cannot be rational. $\square$

This lemma shows that if one can find good rational approximants p_n/q_n such that $|x - p_n/q_n| = o(1/q_n)$, then x has to be irrational. Such irrational numbers can therefore be approximated very well by rational numbers. Constructing rational approximants to x can be done using Padé approximation or Hermite-Padé approximation, especially when x is the value of a Markov function $f(z)$ with z far enough away from the support of the measure. This is the case for some values of the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1,$$

in particular we have for all positive integers

$$\zeta(j+1) = \frac{(-1)^j}{j!} \int_0^1 \frac{\log^j x}{1-x} dx.$$

We can use that information and multiple orthogonal polynomials for the measures $d\mu_j(x) = (-1)^j \log^j(x) dx$ on $[0, 1]$ for $j = 0, 1, 2$ to prove the irrationality of $\zeta(3)$. When Apéry [A $\mathbf{p}$] first announced the proof of irrationality of $\zeta(3)$, the proof was considered as rather mysterious. Later Beukers [B $\mathbf{1}$] [B $\mathbf{2}$] showed how Apéry's approximants can be obtained using Hermite-Padé approximation.

THEOREM 5.2 (Apéry). *$\zeta(3)$ is irrational.*

PROOF. In order to obtain good rational approximants to $\zeta(3)$ we will use Hermite-Padé approximation to the three Markov functions

$$f_1(z) = \int_0^1 \frac{1}{z-x} dx, \quad f_2(z) = - \int_0^1 \frac{\log x}{z-x} dx, \quad f_3(z) = \frac{1}{2} \int_0^1 \frac{\log^2 x}{z-x} dx,$$

and use that $f_3(1) = \zeta(3)$. We want to find a vector of polynomials (A_n, B_n) , where A_n and B_n are polynomials of degree n , and polynomials C_n and D_n such that

$$(5.1) \quad A_n(1) = 0,$$

$$(5.2) \quad A_n(z)f_1(z) + B_n(z)f_2(z) - C_n(z) = \mathcal{O}(1/z^{n+1}),$$

$$(5.3) \quad A_n(z)f_2(z) + 2B_n(z)f_3(z) - D_n(z) = \mathcal{O}(1/z^{n+1}).$$

Observe that (5.2) is a type I approximation problem for the system (f_1, f_2) and (5.3) is a type I approximation problem for (f_2, f_3) , whereas the combination (5.2)–(5.3) is a vector type II approximation problem with common vector denominator (A_n, B_n) . The solution of this problem in Hermite-Padé approximation is given by the generalized polynomial $A_n(x) - B_n(x) \log x$ for which the following orthogonality relations hold:

$$(5.4) \quad \int_0^1 (A_n(x) - B_n(x) \log x) x^k dx = 0, \quad k = 0, 1, \dots, n-1$$

$$(5.5) \quad \int_0^1 (A_n(x) - B_n(x) \log x) \log(x) x^k dx = 0, \quad k = 0, 1, \dots, n-1,$$

together with the extra condition that $A_n(1) = 0$. Let P_n be the Legendre polynomials of degree n on $[0, 1]$,

$$(5.6) \quad P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} x^n (1-x)^n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k x^k,$$

then the Mellin convolution

$$(5.7) \quad F_n(x) = \int_x^1 P_n(x/t) P_n(t) \frac{dt}{t}$$

is of the form $A_n(x) - B_n(x) \log x$ with $A_n(1) = 0$. Taking the Mellin transform gives

$$\int_0^1 F_n(x) x^s dx = \left(\int_0^1 P_n(x) x^s dx \right)^2,$$

and for $s = 0, 1, \dots, n-1$ this gives the orthogonality conditions in (5.4). Taking derivatives gives

$$\int_0^1 F_n(x) x^s \log x dx = 2 \int_0^1 P_n(x) x^s dx \int_0^1 P_n(x) x^s \log x dx,$$

and for $s = 0, 1, \dots, n-1$ this gives the orthogonality conditions (5.5). The remainder in (5.3) is

$$- \int_0^1 \frac{A_n(x) - B_n(x) \log x}{z - x} \log x dx,$$

and hence, choosing $z = 1$ in (5.3) gives

$$2B_n(1)\zeta(3) - D_n(1) = - \int_0^1 \frac{A_n(x) - B_n(x) \log x}{1 - x} \log x dx.$$

Using the representation (5.7), the Rodrigues formula for P_n and

$$-\frac{\log x}{1-x} = \int_0^1 \frac{dz}{1 - (1-x)z}$$

gives

$$\begin{aligned} & - \int_0^1 \frac{A_n(x) - B_n(x) \log x}{1-x} \log x \, dx \\ & = \int_0^1 \int_0^1 \int_0^1 \frac{x^n y^n z^n (1-x)^n (1-y)^n (1-z)^n}{(1-(1-xy)z)^{n+1}} \, dx \, dy \, dz > 0, \end{aligned}$$

so that $2B_n(1)\zeta(3) - D_n(1) > 0$. A simple estimation of $xyz(1-x)(1-y)(1-z)/(1-(1-xy)z)$ on the unit cube gives the upper bound

$$2B_n(1)\zeta(3) - D_n(1) \leq 2\zeta(3) (\sqrt{2} - 1)^{4n}.$$

Explicit calculation gives

$$B_n(1) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

and

$$\begin{aligned} D_n(1) &= - \int_0^1 \frac{A_n(1) - A_n(x)}{1-x} \log(x) \, dx + \int_0^1 \frac{B_n(1) - B_n(x)}{1-x} \log^2 x \, dx \\ &= \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{j=0}^{k-1} \frac{2}{(j+1)^3} \\ &\quad + \sum_{k=0}^n \sum_{\substack{j=0 \\ j \neq k}}^n \binom{n}{k} \binom{n}{j} \binom{n+k}{k} \binom{n+j}{j} (-1)^{k+j} \frac{H_k^{(2)} - H_j^{(2)}}{j-k}, \end{aligned}$$

where $H_k^{(2)} = \sum_{j=1}^k 1/j^2$. Clearly $B_n(1)$ is a positive integer, but $D_n(1)$ contains rational terms. However, if we multiply $D_n(1)$ by a common multiple of the numbers $2^3, 3^3, \dots, n^3$ then we would get an integer. Taking $d_n = \text{lcm}(1, 2, 3, \dots, n)$ (least common multiple) then

$$0 < 2d_n^3 B_n(1)\zeta(3) - d_n^3 D_n(1) \leq d_n^3 \zeta(3) (\sqrt{2} - 1)^{4n}.$$

Taking the n th root and using the asymptotic property

$$\lim_{n \rightarrow \infty} d_n^{1/n} = e,$$

then gives, with $p_n = 2d_n^3 B_n(1) \in \mathbb{N}$ and $q_n = d_n^3 D_n(1) \in \mathbb{Z}$

$$\limsup_{n \rightarrow \infty} (p_n \zeta(3) - q_n)^{1/n} \leq e^3 (\sqrt{2} - 1)^4 < 1,$$

hence Lemma 5.1 implies that $\zeta(3)$ is irrational. $\square$

6. Transcendence

Historically, Hermite introduced the notion of Hermite-Padé approximation to prove the transcendence of e . The basis lemma for proving transcendence is a generalization of Lemma 5.1, which goes as follows:

LEMMA 6.1. *Let x be a real number. Suppose that for every positive integer $m \in \mathbb{N}$ and for all integers $a_0, a_1, \dots, a_m \in \mathbb{Z}$ we can find integers $p_{0,n}, p_{1,n}, \dots, p_{m,n}$ ($n \in \mathbb{N}$) such that*

1. $\sum_{k=0}^m a_k p_{k,n} \neq 0$ for all $n \in \mathbb{N}$,

2. $\lim_{n \rightarrow \infty} (p_{0,n} x^k - p_{k,n}) = 0$ for $k = 1, 2, \dots, m$.

Then x is transcendental.

PROOF. Suppose that x is algebraic, then there exists a positive integer m and integers $a_0, a_1, \dots, a_m$ such that $\sum_{k=0}^m a_k x^k = 0$. By adding and subtracting we find

$$\sum_{k=0}^m a_k (p_{0,n} x^k - p_{k,n}) + \sum_{k=0}^m a_k p_{k,n} = 0.$$

We know by assumption (1) that the integer $\sum_{k=0}^m a_k p_{k,n}$ is different from zero, hence in absolute value it is at least 1. As a consequence

$$\left| \sum_{k=0}^m a_k (p_{0,n} x^k - p_{k,n}) \right| \geq 1.$$

Taking $n \rightarrow \infty$, then assumption (2) gives a contradiction. Hence x has to be transcendental. $\square$

Proving transcendence can therefore be done constructively by producing good simultaneous approximants to consecutive powers of x . Type II Hermite-Padé approximants can then be used in case $x^j = f_j(z)$ for a complete system of Markov functions $(f_1, f_2, \dots, f_m)$. Hermite first showed that $(e^{c_1 z}, e^{c_2 z}, \dots, e^{c_r z})$ is a complete system for Hermite-Padé approximation near zero (rather than near ∞ , which we described in this paper) whenever all c_i are distinct complex numbers. He then used the type II Hermite-Padé approximants for the system $(e^z, e^{2z}, \dots, e^{mz})$ to produce simultaneous rational approximants satisfying the conditions of Lemma 6.1. See [NS, pp. 130–131] or [VA2] for details of this proof.

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