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Abstract. The history of the axiomatic approach to the ranking of infinite streams starts with Koopmans' (1960) characterization of the discounted utilitarian rule. This rule, however, meets Chichilnisky's axiom of dictatorship of the present and puts future generations offside. Recently, Lauwers (2010a) and Zame (2007) have uncovered the impossibility to combine in a constructible way the requirements of equal treatment, sensitivity, and completeness.

This contribution presents and discusses different axioms proposed to guide the ranking of infinite streams and the criteria they imply. The literature covered in this overview definitely points towards a set of meaningful alternatives to discounted utilitarianism.

JEL Classification Number: D71, D81.

1 Introduction

Global environmental issues—like biodiversity conservation or climate change—are in reality long term issues that are not properly taken into account with traditional models that incorporate the impatience axiom manifested in fixed discount factors and in the use of present discounted utility criteria.

This social impatience conflicts with the utilitarian tradition of moral philosophy where it is recommended to treat different generations equally. As Sidgwick (1907, p414) writes: "the time at which a man exists cannot affect the value of his happiness from a universal point of view; and [...] the interests of posterity must concern a utilitarian as much as those of his contemporaries." Ramsey (1928), in one of the first formal studies on the evaluation of social welfare in an intertemporal framework, strongly endorses this view. Despite this position, he nevertheless introduces a rate of discount in some of the investigations. Simply because undiscounted utilitarianism provides no unique answer in case the maximum total welfare is infinite and this infinite value is achieved by many feasible paths. Later on, the same doubt on the sustainability of the equal treatment principle occurs when Koopmans (1960) characterizes the discounted utilitarian rule on the basis of five appealing axioms. Section 3 recalls this result of Koopmans.

It is only recently (Lauwers 2010a, Zame 2007) that the deep cause of the conflict as experienced by Ramsey and Koopmans was exposed: it is not possible to define in a "*constructive*" way a complete ranking on the set of infinite consumption paths that combines anonymity (an axiom that captures equal treatment) and Pareto (or sensitivity).[1] Since the question of how to evaluate policies that involve the distant future is normative, it should by no means be answered through "*non-constructive*" mathematics such as the Axiom of Choice or Szpilrajn's Lemma.[2] Only constructible and well defined criteria can take part in the discussions. The inevitable implication, then, is that at most two of the three requirements—completeness, anonymity, and Pareto—are compatible. Either one has to drop the requirement of completeness, or one has to weaken the requirement of anonymity and/or Pareto. Anyway, one cannot but accept the above incompatibility and temper the ambition to find a representable ordering (as encapsulated in Koopmans' postulates). Incompleteness of the criterion should be interpreted as a consequence of an unbiased position in the discussion. In order to leave as many options open, the primitive should be a partial ordering. Section 4 starts with the introduction of Diamond's axiom of equal treatment and closes with the statement and the interpretation of the Lauwers-Zame impossibility theorem.

Sections 5 and 6 show two different routes to extend in a constructive way a sequence of finite dimensional orderings towards a partial ordering on the set of infinite streams. If the finite dimensional orderings all satisfy anonymity and Pareto, then the resulting infinite dimensional partial ordering inherits both these properties.

Section 7 drops the Pareto principle, strengthens anonymity, imposes completeness and a restricted form of monotonicity, and discusses rules in the Rawlsian spirit such as the infimum rule and the limit inferior. Also, the rank-discounted utilitarian rule (Zuber and Asheim 2012) is discussed. Although each of these rules violates the weakest form of Pareto, they may be extremely useful in a two step procedure as proposed by Ferejohn and Page (1978, p274):

> *Our result suggests that the search for a fair rate of discount is a vain one. Instead of searching for the right number, i.e. 'the' social rate of discount, we must look to broader principles of social choice to incorporate ideas of intertemporal equity. Once found, these principles might be used as side conditions in a discounting procedure to rule out gross inequities that can arise with discounting, even with a low discount rate*

[1] In his review on intergenerational equity, Asheim (2010, section 3.2) coins this result as the Lauwers-Zame impossibility theorem.

[2] I want to mention already here that the combination of continuity with respect to the sup-topology and representability does not guarantee that the ranking rule is constructible. The other way around, the representation of a non-constructible ordering, is in itself a non-constructible object.

A strongly anonymous welfare function might indeed be used as a first step. As a strongly anonymous welfare function typically has thick levels sets, a second step can further investigate the set of optimal paths obtained in the first.

Sections 8 and 9 return to social welfare functions. At the center of the sustainable discounted utilitarian rule (Asheim and Mitra 2010) is the axiom of Hammond equity for the future, according to which a sustained improvement at the cost of the present generation is considered an improvement only in case this first generation is ex post still the better off. They succeed in modifying the axioms of Koopmans towards a characterization. Finally, Chichilnisky (1996, 1997) introduces the axioms of non-dictatorship of the present and non-dictatorship of the future. She proposes a convex combination of the discounted utilitarian rule and a map that captures the limiting behavior of an infinite stream. The discounted rule prevents the future from dictatorship, the limit-part prevents the present from dictatorship. She coins those social welfare functions as sustainable preferences.

This overview is organized as follows. Section 2 introduces notation and describes the problem. The next sections unfold as described above. As the focus is on the axiomatic approach, we recall and discuss different appealing properties or axioms. William Thomson (2001, p349) motivates as follows:

> *The objective of the axiomatic program is to give as detailed as possible a description of the implications of properties of interest, singly or in combination, and in particular to trace out the boundary that separates combinations of properties that are compatible from combinations of properties that are not.*

Applied to the ranking of infinite streams, one investigates on what ethical conditions various ranking criteria are based and proceeds to evaluate the normative appeal of these conditions. Although not included in this overview, the alternative approach that confronts the criteria with different technological environments and compares the properties of the intergenerational well-being streams that are generated, is undoubtedly a necessary route in the debate.[3]

2 Notation

We consider a model with successive generations, each generation living exactly one period. Time is discrete and starts with period 1. Let $\mathbb{N} = \{1, 2, \ldots, t, \ldots\}$ be the set of natural numbers, $\mathbb{R}$ the set of real numbers, and let Y be a subset of $\mathbb{R}$. Let X be the infinite cartesian product $Y^{\mathbb{N}}$. A sequence $x = (x_1, x_2, \ldots, x_t, \ldots)$ in X is said to be an infinite stream of generational well-being, for each t in $\mathbb{N}$ the real number x_t indicates the average well-being of generation t. The indicator of well-being is assumed to be at least ordinally measurable and level comparable across generations. Also, for each generation, the distribution of resources among the individuals of a same generation is neglected. Furthermore, the population size is assumed to be given and constant over time.

[3] Section 3 provides a simple example to indicate that paths optimal with respect to a sustainable-equitable approach might differ substantially from optima generated by the discounted utilitarian rule.

For each n in $\mathbb{N}$ and for each x in X, we write $x = (x_{-n}, x_{+n})$ with $x_{-n} = (x_1, x_2, \ldots, x_n)$ and $x_{+n} = (x_{n+1}, x_{n+2}, \ldots)$. Note that $x_{-1} = (x_1)$. For each x and y in X, we write

- $x \geq y$ if $x_t \geq y_t$ for each t in $\mathbb{N}$,
- $x > y$ if $x \geq y$ and $x \neq y$,
- $x \gg y$ if $x_t > y_t$ for each t in $\mathbb{N}$,
- $d_s(x, y) = \sup_{t \text{ in } \mathbb{N}} |x_t - y_t|$.

The different inequalities will be used to formulate axioms of monotonicity, the distance function d_s generates the sup-topology and is used to formulate an axiom of continuity.

The object we look for is a partial social welfare ordering, denoted by $\succsim$, on the set X of infinite streams. That is, $\succsim$ is a relation in X, and $x \succsim y$ means that the infinite stream x is at least as good as y. The relation $\succsim$ is assumed to be (i) transitive, for each x, y, and z in X, we have that $x \succsim y$ and $y \succsim z$ implies $x \succsim z$, and (ii) reflexive, for each x in X, we have that $x \succsim x$. The symmetric and the asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$. The partial ordering $\succsim_e$ is said to extend the partial ordering $\succsim$, or $\succsim$ is said to be a subrelation to $\succsim_e$, in case, for each x and y in X, $x \succ y$ implies $x \succ_e y$ and $x \sim y$ implies $x \sim_e y$.

The term social welfare order refers to a partial social welfare ordering $\succsim$ that is complete, i.e. for each x and y in X, we have $x \succsim y$ or $y \succsim x$. The term social welfare function refers to a map $f : X \to \mathbb{R}$ that represents some social welfare order $\succsim$: for each x and y in X, we have $f(x) \geq f(y)$ if and only if $x \succsim y$. Recall that in view of the Lauwers-Zame impossibility result, completeness (or representability) is not a neutral requirement as it already excludes the combination of the axioms of anonymity and Pareto.

3 Discounted utilitarianism

Tjalling Charles Koopmans (1972b)[4] considers a social welfare order $\succsim$ on the set of al bounded consumption streams, i.e. infinite streams for which the supremum and the infimum are both finite,[5] and investigates the next five postulates.

Continuity. The relation $\succsim$ is continuous with respect to the sup-topology, i.e. the topology generated by the distance function d_s.

Sensitivity. There exist infinite streams x and $y = (y_1, x_{+1})$ such that $x \succ y$.

The next axiom appeals to the following rankings generated by $\succsim$ and some fixed reference stream $z = (z_1, z_2, z_3, \ldots, z_t, \ldots)$. The orderings $\succsim_{z+1}$ on Y, $\succsim_{z+2}$ on Y^2, and $\succsim_{z-1}$ on X are defined by

[4]See also Koopmans (1960, 1965, 1972a) and Koopmans et al (1964).

[5]For simplicity, $Y = \mathbb{R}$, x_t is the consumption of generation t, and the set ℓ_∞ of bounded streams takes the role of X.

- for each x and y in Y, we write $x \succsim_{z_{+1}} y$ if $(x, z_{+1}) \succsim (y, z_{+1})$,
- for each (x_1, x_2) and (y_1, y_2) in Y^2,
  we write $(x_1, x_2) \succsim_{z_{+2}} (y_1, y_2)$ if $(x_1, x_2, z_{+2}) \succsim (y_1, y_2, z_{+2})$, and
- for each x and y in X, we write $x_{+1} \succsim_{z_{-1}} y_{+1}$ if $(z_1, x_{+1}) \succsim (z_1, y_{+1})$.

Independence. The three orderings, $\succsim_{z_{+1}}$ on Y, $\succsim_{z_{+2}}$ on Y^2, and $\succsim_{z_{-1}}$ on X, do not depend on the reference stream z.

Stationarity$_0$. There exists an x_1^* in Y such that

$$(x_1^*, x_2, x_3, \ldots, x_t, \ldots) \succsim (x_1^*, y_2, y_3, \ldots, y_t, \ldots)$$

if and only if

$$(x_2, x_3, \ldots, x_t, \ldots) \succsim (y_2, y_3, \ldots, y_t, \ldots).$$

Monotonicity.[6] Let x and y in Y satisfy $x_t \geq y_t$ for each t in $\mathbb{N}$. Then, $x \succsim y$.

We briefly discuss these five axioms. Koopmans (1972a) motivates the continuity axiom: a small change in a prospect cannot drastically change the position of that prospect in the ranking of all other prospects. Continuity in combination with a monotonicity axiom that imposes $x \succ y$ in case $x > y$ and $x \succsim y$ in case $x \geq y$ implies that the ranking $\succsim$ is representable by a real valued function (Diamond 1965, Lauwers 1997a).[7] The usefulness of a representation by a continuous function lies primarily in the availability of stronger mathematical techniques.[8] Sensitivity excludes the ordering $\succsim$ from being trivial in the sense that all infinite streams are equally good. This axiom also prohibits dictatorship of the future (see section 9): the ranking of infinite streams is not solely based upon the limiting behavior of the infinite streams. Independence removes all complementarity between the well-being of different (subsequent) generations, cannot be regarded as a realistic assumption, and should be looked upon as a way to facilitate the investigations (Koopmans 1972b, p83).[9] Independence implies that the particular value x^* in the axiom of stationarity$_0$ can be replaced with any value in Y. In case the axiom of independence is imposed upon the ranking $\succsim$, stationarity$_0$ strengthens to the next axiom.

Stationarity. For each x_1^* in Y we have

$$(x_1^*, x_2, x_3, \ldots, x_t, \ldots) \succsim (x_1^*, y_2, y_3, \ldots, y_t, \ldots)$$

[6] Koopmans considers infinite streams of vectors instead of scalars. The axiom of monotonicity is a one-dimensional version of Koopmans' axiom.

[7] Diamond (1965) follows Gérard Debreu (1954) to prove this result.

[8] This motivation, however, is wrong in case the continuous social welfare function represents a non-constructible order.

[9] See also Fleurbaey and Michel (2003, Section 3.4).

if and only if

$$(x_2, x_3, \ldots, x_t, \ldots) \succsim (y_2, y_3, \ldots, y_t, \ldots).$$

Stationarity compares infinite streams with a common first period value in the same way as the infinite streams that are obtained by deleting these first period values and advancing the timing of all subsequent values by one period. Repeated use of stationarity results in comparing two streams (x_{-t}, x_{+t}) and (x_{-t}, y_{+t}) with the common head x_{-t} in the same way as the infinite tails x_{+t} and y_{+t}. In other words, if the first t generations are not affected the ranking is made as if the present time (date 1) actually was in time t. The passage of time has no effect on preferences. Stationarity also implies that the ranking of two infinite streams is not altered if both streams are postponed by one unit of time and identical values are plugged in at time period 1. Monotonicity demands that if each generation is at least as good off in x than in y, the infinite stream x should be considered at least as good as y.

These five postulates characterize the discounted utilitarian rule.

Theorem 1 (Koopmans 1972b).[10] Let the social welfare ordering $\succsim$ on the set of bounded streams satisfy continuity with respect to the sup-topology, sensitivity, independence, stationarity$_0$, and monotonicity. Then, the ordering $\succsim$ is represented by a continuous function

$$D : (x_1, x_2, \ldots, x_t, \ldots) \longmapsto (1-\alpha) \sum_{t=1}^{\infty} \alpha^{t-1} u(x_t),$$

with u nowhere decreasing and continuous and with α in the open interval $(0, 1)$.[11]

A first step towards this result investigates the representation of $\succsim$ restricted to the subset of infinite streams with a fixed tail, say z_{+t}. Next, the domain of infinite streams with a constant tail is considered. The partial results are then generalized towards the full domain of bounded streams. Discounted utilitarianism satisfies a recursive relation:

$$D(x_1, x_2, \ldots, x_t, \ldots) = (1-\alpha)\, u(x_1) + \alpha\, D(x_2, x_3, \ldots, x_{t+1}, \ldots),$$

for each infinite stream x. The map D attaches the weight $1-\alpha$ to the utility allocated to the present period and the complementary weight α to the aggregated utility of all future periods.

Although (Koopmans 1965, Section 6) holds an ethical preference for neutrality, he provides an argument for the discounted utilitarian rule:

> *We admit to an ethical preference for neutrality as between the welfare of different generations. ... A previous investigation has shown that there does*

[10] Different axiomatizations are obtained by Lauwers (1997c), Bleichrodt et al (2008), and Asheim et al (2012).

[11] The factor $(1-\alpha)$ in the definition of D ensures that $D(x, x, \ldots, x, \ldots) = u(x)$. Hence, the weights with which the $u(x_t)$ are multiplied add up to 1.

not exist a utility function of all consumptions paths, which at the same time exhibits timing neutrality and satisfies other reasonable postulates which all utility functions used sofar have agreed with.

The mathematical conflict between neutrality and the five postulates overrules the ethical preference for neutrality. In his 1975 Nobel Memorial Lecture, Koopmans (1977) returns to the issue of a positive discount factor:

> *Thus, the impatience expressed by a positive discount rate merely denies to uncounted distant generations a permanently higher level of consumption because that would necessitate a substantially smaller present consumption. Perhaps a pity, but not a sin.*

Kenneth Joseph Arrow (1999) considers a world in which an investment by the first generation generates a perpetual stream of benefits. The undiscounted total gain exceeds the finite loss to the first generation, and the optimal path would almost sacrifice the first generation. He concludes "that the strong ethical requirement that all generations be treated alike, itself reasonable, contradicts a very strong intuition that it is not morally acceptable to demand excessively high saving rates of any one generation, or even every generation." This argument, however, assumes a non-decreasing path and leaves open the case where a period of economic growth is followed by a period of economic regression. Then, the 'small sin' might take problematic proportions.

The set of five axioms leads to the 'class' of discounted utilitarian rules. The axioms, however, do not pass any judgment about the value of α and the particular form of u. Let us illustrate the effect of a change in α. The next table (Fleurbaey and Zuber 2012, Table 1) considers α equal to 0.9862 and 0.9737 and shows the minimum return a one dollar investment for the future should have to be considered better than consuming it now.

	$\alpha = 0.9862$	$\alpha = 0.9737$	ratio
time period 50	2.00	3.79	1.89
time period 100	4.02	14.36	3.57
time period 200	16.13	206.11	12.78
time period 1000	1 091 327.24	371 914 916 666.52	340 791.38

The minimal return sufficient to defend a one dollar investment today.

The value $r = (1/\alpha) - 1$ has the interpretation of a discount rate, $\alpha = 0.9862$ (resp. 0.9737) corresponds to $r = 1.4\%$ (resp. 2.7%). The final column has the following alternative interpretation. When the discount rate r jumps from 1.4 to 2.7 percent the minimal return that is sufficient to defend a one dollar investment at year 1 at the benefit of year 200 becomes 12.78 times larger. The huge ratios in the final column show the impact of a change in the discount rate.

We close this section with a simple example illustrating the shortsightedness of the time-discounted utilitarian approach (Lauwers 2012). Consider an economy in which trees are a necessary input to production or consumption. The dynamics of tree reproduction are as follows. If n out of $2n$ subsequent generations cut the forest at a maximal rate, the species become extinct after the $2n$'th generation, in which case there is zero utility at every period from then on. Assume this strategy results in utility streams of the form $u^n = (0.1, 0.1, \ldots, 0.1; 1, 1, \ldots, 1; 0, 0, \ldots)$ with the first (resp. last) 1 at the $n+1$ (resp. $2n$)'th place, in which generations $n+1, \ldots, 2n$ cut at a full capacity and exhaust the forest. When the consumption of the forest is delayed and n becomes larger, the forest slightly expands and more generations can benefit. Alternatively, generations can invest in the forest and only cut at an equilibrium rate which allows the forest to survive. This strategy results in the utility stream $u^\infty = (.25, .25, \ldots, .25, \ldots)$ in which each generation reaches the same utility level. Optimization with respect to a discounted utilitarian rule leads to the elimination of the forest.[12] If the long term future is considered important, the constant stream u^∞ should be ranked strictly above a stream where within a finite horizon the forest is consumed.

As already mentioned, it is a natural step in the axiomatic approach to confront the rule characterized through a set of axioms with the consequences is generates in specific environments outcome. In case a set of desirable axioms leads to undesirable consequences, there is always the invitation to reconsider the axioms. This iterative process between moral principles and their examination in particular models is supported by, for example, Atkinson (2001). Dasgupta and Heal (1979) state

> *... it is legitimate to revise or criticize ethical norms in the light of their implications.*

As such, the above example motivates the search for alternatives to the discounted utilitarian rule.

4 Finite anonymity

Peter Arthur Diamond (1965) continues the axiomatic approach initiated by Koopmans. He considers infinite utility streams: x_t is the one period utility level associated with consumption in period t and Y is the closed interval $[0, 1]$. In order to investigate whether or not impatience is unavoidable when ranking infinite streams, he introduces the next axiom. Let $\succsim$ be a partial order on the set X.

Finite anonymity. For each $x = (x_1, x_2, \ldots, x_t, \ldots)$ in X and for each t in $\mathbb{N}$, we have

$$x \sim x^t = (x_t, x_2, \ldots, x_{t-1}, x_1, x_{t+1}, \ldots).$$

The infinite stream x^t is obtained from x by switching the coordinates x_1 and x_t. Due to the transitivity of $\succsim$, a finitely anonymous evaluation is indifferent between two infinite streams that are equal up to a finite number switches in the coordinates. A ranking $\succsim$ which

[12]The map $x \mapsto (1-\beta)(x_1 + \beta x_2 + \cdots + \beta^{t-1} x_t + \cdots)$ obtains a maximal value, equal to .3025, in one of the streams of type t^n; while the stream u^∞ obtains a lower value of .25.

treats all generations equally should satisfies this condition. Diamond keeps the axiom of continuity with respect to the sup-topology and imposes the following strengthening of the axiom of sensitivity.

Strong Pareto. For each x and y in X, we have $x \succ y$ as soon $x > y$.

The imposition of strong Pareto requires that the welfare ordering judges a utility stream superior to another as soon at least one period obtains a higher utility while all other periods obtain at least the same utility. This ability to detect an improvement in a single period makes strong Pareto a very demanding axiom in the study of infinite streams. The combination of finite anonymity and strong Pareto generates a partial social welfare ordering: the Suppes-Sen grading principle considers the infinite stream x at least as good as y in case it is possible to obtain from x by means of a finite number of switches a stream x' that Pareto dominates y, i.e. $x' \geq y$. Hence, each partial social welfare ordering that satisfies finite anonymity and strong Pareto extends the Suppes-Sen grading principle. Furthermore, recall that a complete, strongly Paretian, and continuous (with respect to the sup-topology) social welfare relation is representable by a continuous real valued map.

The following theorem, a result that Diamond attributes to Yaari, reveals a fundamental conflict.

Theorem 2 (Diamond 1965)**.** There does not exist a social welfare order $\succsim$ on the set X that satisfies continuity, strong Pareto, and finite anonymity.

This impossibility result has been the starting point of an extensive literature on its robustness. The axiom of continuity with respect to the sup-topology is, in contrast to strong Pareto and finite anonymity, a rather technical condition. Furthermore, as different distance functions generate different notions of continuity, the axiom of continuity is manipulable. For example, with respect to the discrete topology on X, each ordering $\succsim$ becomes continuous and continuity becomes an empty concept. Therefore, axioms of continuity in the infinite dimensional framework are considered as controversial. Svensson (1980) shows the existence of a complete and transitive relation that combines strong Pareto, finite anonymity, and a very weak continuity requirement. Basu and Mitra (2003) insist on representability by a real valued function, drop continuity, and obtain again an impossibility result. Fleurbaey and Michel (2003) consider the following Pareto axiom.

Weak Pareto. For each x and y in X, we have (i) $x \succsim y$ as soon $x \geq y$ and (ii) $x \succ y$ as soon $x \gg y$.

Weak Pareto strengthens monotonicity: a utility stream is considered at least as good as another as soon each period obtains at least the same utility. Furthermore, according to weak Pareto a utility stream is superior to another as soon each period obtains a higher utility. A ranking that satisfies strong Pareto also satisfies weak Pareto. Fleurbaey and Michel (2003) strengthen Diamond's theorem: a social welfare ordering cannot simultaneously satisfy continuity with respect to the sup-topology, finite anonymity, and weak Pareto.

A deep result in this track of research was conjectured by Fleurbaey and Michel (2003) and confirmed by Lauwers (2010a) and Zame (2007). We present the discrete version with $Y = \{0, 1\}$. A discrete version of the Pareto principle is needed.

Intermediate Pareto. For each x and y in X, we have (i) $x \succsim y$ as soon $x \geq y$ and (ii) $x \succ y$ as soon $x_i > y_i$ for infinitely many i in $\mathbb{N}$.

Theorem 3 (Lauwers 2010a)**.** A partial order on the set $X = \{0, 1\}^{\mathbb{N}}$ of infinite utility streams made up of zeros and ones that satisfies intermediate Pareto and finite anonymity either is incomplete, or is a non-constructive object (and hence has no explicit description).

This theorem considers only two levels of utility. Then, intermediate Pareto is the weakest version of Pareto that is non-trivial. In this limited framework a partial order is unable to combine (in a constructive way) completeness, intermediate Pareto, and finite anonymity.

With respect to the change from a continuous towards a discrete setting, one might argue that there exists a smallest unit (or quantum) of utility and that a discrete level set Y is natural. In such a framework, intermediate Pareto seems appropriate. If the set Y of utility levels is equal to $\mathbb{N}$, then the map $X \to \mathbb{R} : x \mapsto \min_t x$ (and the map $x \mapsto \liminf x$ as well) defines a complete, finitely anonymous, and weakly Paretian order on X that violates intermediate Pareto. If, however, the set Y is the unit interval $[0, 1]$, the minimum of an infinite stream is not well defined while the map $x \mapsto \inf_t x$ violates weak Pareto. Indeed, the streams

$$x = (1, 1/2, \ldots, 1/n, \ldots) \quad \text{and} \quad y = (0, 0, \ldots, 0, \ldots)$$

dominate each other ($x \gg y$) while they have the same infimum (and the same limit inferior). This kind of situations do not occur in a discrete setting.[13] Intermediate Pareto and minimum should be seen as the discrete analogue to weak Pareto and infimum.

The above theorem appeals to the concept of constructibility. In order to explain this, consider Brouwer's fixed theorem: each continuous function from a convex compact subset of a Euclidean space to itself has a fixed point. A well known proof of this result is based on algebraic topology and is not constructive: the proof shows the existence of a fixed point but does not specify where it is located. Only later on constructive proofs, algorithms to detect the location of a fixed point, were provided. Theorem 3 is in the same spirit. Svensson (1980) already showed the *existence* of a complete and transitive relation that combines strong Pareto and finite anonymity. Svensson's proof, however, uses the Axiom of Choice, which is within the axiomatic setup of set theory in mathematics a non-constructive axiom. Theorem 3 shows the impossibility to provide a constructive proof of Svensson's result. The use of a non-constructive axiom (in the spirit of the Axiom of Choice) cannot be avoided to obtain Svensson's result. As a consequence, Svensson's existence proof contributes almost nothing to the discussion on how exactly infinite streams should be ordered.

[13] Dubey (2011) and Dubey and Mitra (2011) investigate the role of the set Y of possible utility levels and refine the results of Lauwers (2010a) and Zame (2007).

5 Pareto dominating tails

The framework with an infinite number of periods generates incompatibilities that are easy to reconcile in a finite work. This section explains how to construct, starting from an infinite sequence of finite dimensional partial orderings, a partial ordering on X that combines finite anonymity and strong Pareto. The following idea is at the basis of the construction. If the infinite stream x Pareto dominates y, all generations unanimously (and independently) agree to rank x above y. If, for some T, the infinite tail x_{+T} Pareto dominates y_{+T}, then from $T+1$ onwards each generation agrees to rank x above y. The problem, in this case, reduces to check whether the anonymous aggregative decision of the finite horizon society $\{1, 2, \ldots, T\}$ agrees with the unanimous decision of generations $T+1, T+2, \ldots, T+k, \ldots$.

Before we explain the construction, let us list four well documented ranking relations on the finite dimensional Euclidean space $\mathbb{R}^n$: the Suppes-Sen grading principle $\succsim_n^S$, the utilitarian ordering $\succsim_n^U$, the leximin ordering $\succsim_n^L$, and the generalized Lorenz partial ordering $\succsim_n^G$.[14]

We need some extra notation. For each n-tuple a in $\mathbb{R}^n$ let a^+ be a rearrangement of a that satisfies $a_{[1]} \leq a_{[2]} \leq \cdots \leq a_{[n]}$. Let $\geq_n^L$ denote the lexicographic ordering on the set of non-decreasing n-tuples: $a^+ \geq_n^L b^+$ if $(a_{[1]}, a_{[2]}, \cdots, a_{[k-1]}) = (b_{[1]}, b_{[2]}, \ldots, b_{[k-1]})$ and $a_{[k]} \geq b_{[k]}$ for some $k = 1, 2, \ldots, n$.

For each a and b in $\mathbb{R}^n$ we have

$$
\begin{array}{lll}
a \succsim_n^S b & \text{if} & a^+ \geq b^+, \\
a \succsim_n^U b & \text{if} & a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n, \\
a \succsim_n^L b & \text{if} & a^+ \geq_n^L b^+, \\
a \succsim_n^G b & \text{if} & a_{[1]} + a_{[2]} + \cdots + a_{[k]} \geq b_{[1]} + b_{[2]} + \cdots + b_{[k]}, \ k = 1, 2, \ldots, n,
\end{array}
$$

All four ranking rules combine (finite) anonymity and strong Pareto. As a matter of fact, the Suppes-Sen grading principle $\succsim_n^S$ is a subrelation to each partial ordering on $\mathbb{R}^n$ that satisfies anonymity and strong Pareto. Next, the rankings $\succsim_n^U$ and $\succsim_n^L$ are both complete. The utilitarian rule orders vectors according to the sum of the utilities. The Suppes-Sen grading principle, the leximin rule, and the generalized Lorenz criterion make decisions after rewriting the n-tuples in increasing order. The leximin rule judges the n-tuple with the highest lowest utility level as being better; if these lowest levels are the same for the two n-tuples, then the ranking is based on the second lowest utilities; and so forth. The generalized Lorenz criterion is incomplete, e.g. it is unable to compare the vectors $(0, 3)$ and $(1, 1)$. All these finite dimensional relations are easy to extend towards the framework of infinite streams. We will explain how this can be done by means of an arbitrary sequence of orderings.

[14]We refer to Suppes (1966), Sen (1971), Hammond (1976), d'Aspremont and Gevers (1977), and Shorrocks (1983). Bossert and Weymark (2004) provide an excellent overview.

Consider a sequence $\succsim_1, \succsim_2, \ldots, \succsim_n, \ldots$ of (partial) rankings the subscript of which reflects the dimension or length of the vectors it compares, i.e. for each n, the relation $\succsim_n$ is defined on $\mathbb{R}^n$. A first method to construct a relation $\succsim_\infty$ on the set X is as follows:

$$x \succsim_\infty y \quad \text{if there exists a } T \text{ in } \mathbb{N} \text{ such that } x_{-T} \succsim_T y_{-T} \text{ and } x_{+T} \geq y_{+T}.$$

Two infinite streams x and y for which the infinite tail x_{+T} dominates y_{+T} are comparable if, according to $\succsim_T$, the head x_{-T} is not worse than y_{-T}. In order to decide whether the infinite stream x is at least as good as y it is necessary that (i) from some generation $T+1$ onwards each individual generation t prefers x to y on the basis that the level x_t is at least as good as y_t and (ii) the finite horizon society $\{1, 2, \ldots, T\}$ considers x_{-T} at least as good as y_{-T} on the basis of the relation $\succsim_T$. In other words, a finite society judges on the basis of the finite social welfare relation and all future generations unanimously concur this judgement. If the relation $\succsim_\infty$ is able to rank two infinite streams, then Pareto dominance applies to their tails. The relation $\succsim_\infty$ is a partial ranking. The following transfer of properties from $\succsim_t$ towards $\succsim_\infty$ is obvious: in case each $\succsim_t$ satisfies anonymity or strong Pareto, then the infinite version $\succsim_\infty$ also meets the axiom.

To illustrate the relations $\succsim_\infty^U$, $\succsim_\infty^L$, $\succsim_\infty^S$, and $\succsim_\infty^G$ we consider the streams

$$x = (\underbrace{.3, .3, \ldots, .3}_{3001 \text{ times}}, .5, .5, \ldots, .5, \ldots),$$

$$y = (\underbrace{.2, .2, \ldots, .2}_{2000 \text{ times}}, .5, .5, \ldots, .5, \ldots),$$

$$z = (\underbrace{.1, .1, \ldots, .1}_{1000 \text{ times}}, \underbrace{.3, .3, \ldots, .3}_{1000 \text{ times}}, .5, .5, \ldots, .5, \ldots).$$

The infinite horizon utilitarian rule $\succsim_\infty^U$ considers y and z equally good, and strictly better than x. The infinite horizon leximin rule $\succsim_\infty^L$ ranks x strictly above y, and y strictly above z. The rule $\succsim_\infty^G$ ranks y strictly above z and is unable to compare x and y, and x and z. The infinite Suppes-Sen grading principle $\succsim_\infty^S$ is a subrelation to each ranking criterion that satisfies finite anonymity and strong Pareto. The Suppes-Sen grading principle is unable to compare any pair of the streams x, y, and z. Characterizations were obtained by means of the following axioms.

Partial translation scale invariance (Basu and Mitra 2007)**.** For each x and y in X and for each α in $\mathbb{R}^{\mathbb{N}}$, if $x \succsim y$, $x_{+T} = y_{+T}$, and $x + \alpha$ and $y + \alpha$ belong to X, then $x + \alpha \succsim y + \alpha$.

The axiom requires that preferences are invariant to changes in the origins of the utility indices used in the various periods and should be interpreted as an infinite version of unit interpersonal comparability (Sen 1977, d'Aspremont and Gevers 1977).

Hammond equity (Hammond 1976, Asheim and Tungodden 2004)**.** For each x and y in X, for each i and j in $\mathbb{N}$, if $x_t = y_t$ for each t different from i and j, and $y_j > x_j > x_i > y_i$, then $x \succsim y$.

Start from an infinite stream y. Bring a better-off generation j and a worse-off generation i closer to each other. The resulting infinite stream x is not worse than the original stream y.

Strict transfer principle (Hara et al 2008, Bossert et al 2007)**.** For each x and y in X, for each i and j in $\mathbb{N}$, if $x_t = y_t$ for each t different from i and j, $y_j > x_j \geq x_i > y_i$, and $x_i + x_j = y_i + y_j$, then $x \succsim y$.

Start from an infinite stream y. Execute a Pigou-Dalton transfer, i.e. a transfer of a positive amount from a better-off to a worse-off generation so that the relative ranking of the two agents does not change. The resulting infinite stream x is not worse than the original stream y.

The next theorem characterizes the four partial orderings.

Theorem 4. A partial ordering $\succsim$ on the set X satisfies strong Pareto and finite anonymity if and only if $\succsim^S_\infty$ is a subrelation to $\succsim$. A partial ordering $\succsim$ on the set X satisfies strong Pareto, finite anonymity, and partial translation scale invariance (resp. Hammond equity, or the strict transfer principle) if and only if $\succsim^U_\infty$ (resp. $\succsim^L_\infty$, or $\succsim^G_\infty$) is a subrelation to $\succsim$.

For the characterization of the infinite horizon Suppes-Sen grading principle $\succsim^S_\infty$ we refer to Banerjee (2006b) and Mitra and Basu (2007). Basu and Mitra (2007) introduce and characterize the infinite horizon utilitarian rule $\succsim^U_\infty$. The transfer-sensitive infinite horizon rule $\succsim^G_\infty$ and the infinite horizon leximin rule $\succsim^L_\infty$ are introduced and characterized by Bossert et al (2007).

6 Decisive sets of horizons

Again we start from an infinite sequence $\succsim_1, \succsim_2, \ldots, \succsim_n, \ldots$ of finite dimensional criteria. Let x and y be two infinite streams. For each t let the finite horizon society $\{1, 2, \ldots, t\}$ decide, on the basis of $\succsim_t$, whether or not x_{-t} should be considered at least as good as y_{-t}. Consider the set

$$N(x, y) \; = \; \{t \text{ in } \mathbb{N} \,|\, x_{-t} \; \succsim_t \; y_{-t}\}.$$

The set $N(x, y)$ collects all finite time horizons t for which the truncated vector x_{-t} is at least as good as y_{-t}. In this section, the problem of ranking x not below y is reduced to the question of whether or not the set $N(x, y)$ is large enough. In case $N(x, y)$ is equal to $\mathbb{N}$ up to a finite set, one can argue that the infinite stream x should not be ranked below y. Indeed, for each t larger than some T, the aggregative judgement (on the basis of $\succsim_t$) of the finite horizon society $\{1, 2, \ldots, t\}$ considers x not worse than y.

Let $\mathcal{F}$ denote the collection of all subsets of $\mathbb{N}$ that are equal to $\mathbb{N}$ up to a finite set. Then, we can define the following relation on the set X of infinite streams:

$$x \; \succsim_{\mathcal{F}} \; y \qquad \text{if} \qquad N(x, y) \; \in \; \mathcal{F}.$$

An element of the collection $\mathcal{F}$ is a subset of $\mathbb{N}$ and can be interpreted as a decisive set. If the set $N(x, y)$ belongs to $\mathcal{F}$, then $N(x, y)$ is decisive and the stream x is judged to be at least as good as y. If, in addition, also $N(y, x)$ belongs to $\mathcal{F}$, then x and y are considered equally good. If, however, $N(y, x)$ does not belong to $\mathcal{F}$, then x is strictly preferred to y.[15]

Let us list the relevant properties of $\mathcal{F}$. The empty set does not belong to $\mathcal{F}$, the empty set is never decisive. The collection $\mathcal{F}$ is closed for intersection, i.e. for each A and B in $\mathcal{F}$, the intersection $A \cap B$ also belongs to $\mathcal{F}$. As a consequence, the relation $\succsim_{\mathcal{F}}$ is transitive. Furthermore, the collection $\mathcal{F}$ is closed for supersets, i.e. if A belongs to $\mathcal{F}$, then a superset $B \supset A$ also belongs to $\mathcal{F}$. A superset of a decisive coalition is in its turn decisive. These three properties turn the collection $\mathcal{F}$ into a filter.

To illustrate the approach, start from the sequence $\succsim_n^U$ of utilitarian orderings and consider the infinite streams

$$u = (.2, 0, .1, 0, .1, 0, .1, 0, \ldots) \qquad \text{and} \qquad v = (0, .1, 0, .1, 0, .1, 0, .1, \ldots).$$

The odd-indexed generations prefer u to v, while the even-indexed generations prefer v to u.[16] Hence, the tails do not dominate each other and the almost-unanimity-approach, discussed in the previous section and denoted by $\succsim_\infty^U$, is unable to rank these streams. On the other hand, the set $N(u, v)$ coincides with $\mathbb{N}$ and $N(v, u)$ is empty, each finite horizon society has an aggregative strict preference for u. We obtain

$$N(u, v) \in \mathcal{F} \text{ and } N(v, u) \notin \mathcal{F}, \qquad \text{hence} \qquad u \succ_{\mathcal{F}}^U v.$$

Whatever the horizon t, the finite horizon society $\{1, 2, \ldots, t\}$ when equipped with the utilitarian rule $\succsim_t^U$ considers—as a group—the stream u better than v.

The relation $\succ_{\mathcal{F}}^U$ is known as the utilitarian overtaking criterion (Atsumi 1965, von Weizsäcker 1965). Fleurbaey and Michel (2003) study different versions and extensions of this overtaking criterion and introduce the method of decisive horizons. We refer to Brock (1970), Asheim and Tungodden (2004), Basu and Mitra (2007), and Asheim and Banerjee (2010) for the axiomatizations of the utilitarian and the leximin overtaking rule and their catching-up versions. Starting from the axiomatizations in Theorem 4, an additional consistency demand is sufficient.

Let us return to the sequence $\succsim_1, \succsim_2, \ldots, \succsim_n, \ldots$ and the filter $\mathcal{F}$. It is obvious that the relation $\succsim_{\mathcal{F}}$ is not complete, even in case each $\succsim_n$ is complete. In order to reduce the incompleteness of $\succsim_{\mathcal{F}}$ it is sufficient to enlarge the collection $\mathcal{F}$ of decisive sets. The more decisive sets, the more pairs of streams can be ranked.

A strengthening of the finite anonymity axiom can be used to extend the partial order $\succsim_{\mathcal{F}}$. Let us introduce some further notation. A permutation is a bijective map $\pi : \mathbb{N} \to \mathbb{N}$. For each infinite stream x, the composition $x \circ \pi$ is a map from $\mathbb{N}$ to Y and can be written as

$$x \circ \pi \ = \ (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(t)}, \ldots).$$

---

[15] Note the similarity with the decisive sets in Arrow's impossibility theorem. See also Fleurbaey and Michel (2003).

[16] Basu and Mitra (2007) discuss this example.

A permutation is said to be finite if for some T in $\mathbb{N}$ we have $\pi(t) = t$ for each $t \geq T$. A permutation is said to be fixed step if there exists a natural number k, such that $\pi(\{1, 2, \ldots, kn\}) = \{1, 2, \ldots, kn\}$ for each n in $\mathbb{N}$. The permutation σ that switches the numbers $2j - 1$ and $2j$ for each j is a fixed step permutation with $k = 2$. Applied to the stream $y = (0, .1, 0, .1, 0, .1, 0, .1, \ldots)$ we obtain

$$y \circ \sigma \;=\; (.1, 0, .1, 0, .1, 0, .1, 0, \ldots).$$

Note that a finite permutation is fixed step. We now formulate an anonymity axiom that is stronger than finite anonymity.

Fixed step anonymity (Lauwers 1997b)**.** Let x be an infinite stream and let π be a fixed step permutation. Then, $x \circ \pi$ and x are equally good.

The imposition of fixed step anonymity forces indifference between the utility streams

$$v = (0, .1, 0, .1, 0, .1, 0, .1, \ldots) \qquad \text{and} \qquad z = v \circ \sigma = (.1, 0, .1, 0, .1, 0, .1, 0, \ldots).$$

A preference for z above v indeed reveals some form of impatience. Fixed step anonymity is compatible with strong Pareto (Lauwers 1997b). Moreover, the combination of fixed step anonymity and Pareto dominance ranks rank $u = (.2, 0, .1, 0, .1, 0, .1, 0, \ldots)$ above v.

In contrast to finite anonymity, however, the imposition of fixed step anonymity conflicts with the combination of strong Pareto and stationarity. Indeed, a fixed step anonymous rule considers the streams

$$a = (1, 0, 1, 0, \ldots, 1, 0, \ldots),\; b = (0, 1, 1, 0, \ldots, 1, 0, \ldots),\; \text{and } c = (0, 1, 0, 1, \ldots, 0, 1, \ldots)$$

equally valuable. Stream b is equal to a up to the switch in the first two coordinates, and stream c is obtained from a after switching the odd and even coordinates. Stationarity implies indifference between b_{+1} and c_{+1}. Since c_{+1} coincides with a, we have indifference between c_{+1} and c. Because of strong Pareto, the stream b_{+1} is strictly preferred to c. Thus, the incompatibility is established.[17]

Let $\mathcal{G}$ be the collection of sets that are up to a finite set equal to $k\mathbb{N} = \{k, 2k, \ldots, nk, \ldots\}$ for some k. The collection $\mathcal{G}$ is a filter. If the finite dimensional relations $\succsim_n$ are anonymous, then the relation $\succsim_{\mathcal{G}}$ is fixed step anonymous and extends the relation $\succsim_F$. Lauwers (1997b), Fleurbaey and Michel (2003), Kamaga and Kojima (2009, 2010), Asheim and Banerjee (2010), and Sakai (2010) discuss and axiomatize fixed step anonymous criteria.

In order to extend the partial ordering $\succsim_{\mathcal{G}}$ to a complete ordering, the relations $\succsim_n$ need to be complete and the collection $\mathcal{U}$ of decisive sets should meet the following requirement:

$$\text{for each } A \subset \mathbb{N}, \text{either } A \in \mathcal{U} \text{ or } \mathbb{N} - A \in \mathcal{U},$$

the either-or being exclusive. Indeed, if the collection $\mathcal{U}$ satisfies this demand, then for an arbitrary pair x and y, either the set $N(x, y)$, or $N(y, x)$, or both belong to $\mathcal{U}$ and the infinite streams x and y are comparable. The following definition and lemma summarizes.

Definition (ultrafilter)**.** A collection $\mathcal{U}$ of subsets of $\mathbb{N}$ is said to be an ultrafilter, if

[17] Demichelis et al (2010) study axioms of anonymity in combination with strong Pareto and stationarity.

- the empty set $\varnothing$ does not belong to $\mathcal{U}$,
- for each A and B in $\mathcal{U}$, the intersection $A \cap B$ belongs to $\mathcal{U}$,
- for each $A \subset \mathbb{N}$ either $A \in \mathcal{U}$ or its complement $\mathbb{N} - A \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ that includes $\mathcal{F}$ is said to be free and satisfies $\cap_{\mathcal{U}} A = \varnothing$. A finite subset of $\mathbb{N}$ does not belong to a free ultrafilter. The existence of a free ultrafilter follows from the Axiom of Choice (formulated as Zorn's Lemma). Free ultrafilters are non-constructible objects. As a consequence, the next lemma (Fleurbaey and Michel 2003, Lauwers 2010a) provides a way to obtain non-constructive existence results. In contrast, the filter $\mathcal{G}$ and the relation is $\succsim_{\mathcal{G}}$ are well defined.

Lemma. Let $\succsim_t$ be a relation on Y^t for each t in $\mathbb{N}$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Define the relation $\succsim_{\mathcal{U}}$ on $X = Y^{\mathbb{N}}$ by

$$x \succsim_{\mathcal{U}} y \qquad \text{if} \qquad N(x, y) \in \mathcal{U}.$$

If each relation $\succsim_t$ satisfies the axiom of transitivity, completeness, finite anonymity, strong Pareto, Hammond equity, or the strict transfer principle, then the relation $\succsim_{\mathcal{U}}$ satisfies the same axioms.

While the relation $\succsim_{\mathcal{U}}$ is not relevant from a practical point of view, the subrelation $\succsim_{\mathcal{G}}$ will be incomplete, but is still defined in a constructive way. On the other hand, the above lemma cannot be used to conclude that a certain set of axioms necessarily leads to a ranking rule that is non-constructible. The proof of Theorem 3, for example, appeals to the concept of non-Ramsey sets and shows how the existence of an ordering that combines intermediate Pareto and finite anonymity implies the existence of such a non-constructible object.[18]

7 Strong anonymity

The principles of finite anonymity and fixed step anonymity are concepts of procedural equity: the action of a permutation upon an infinite stream does not change the distribution of the levels in the infinite stream. In contrast equity principles as Hammond's equity and the Pigou-Dalton transfer principle are called consequentialist equity concepts: they judge the effect of a change in the distribution in the levels. This section discusses the strongest form of procedural equity, labeled strong anonymity.

[18] A non-Ramsey set is a subset $\mathcal{N}$ of the collection $\mathbb{N}_\infty$ of all infinite subsets of $\mathbb{N}$ such that for each element J in $\mathcal{N}$ the collection of infinite subsets of J intersects both $\mathcal{N}$ and its complement $\mathbb{N}_\infty - \mathcal{N}$. The technique developed in Lauwers (2010a) to define non-Ramsey sets has been used by Dubey and Mitra (2013) to show that a complete ranking that combines strong Pareto and Hammond equity (or the strict transfer principle) is non-constructible. See also Dubey (2011), Dubey and Mitra (2011, 2012), and Banerjee and Dubey (2013).

Strong anonymity. For each infinite stream x in X and each permutation $\pi : \mathbb{N} \to \mathbb{N}$, we have $x \sim x \circ \pi$.

This axiom conflicts with weak Pareto (Fleurbaey and Michel 2003). We present two examples. The first one comes from Fleurbeay and Michel (2003):

$$\begin{array}{llllllllllll} t & (\ldots, & 9, & 7, & 5, & 3, & 1, & 2, & 4, & 6, & 8, & \ldots), \\ x = & (\ldots, & \frac{1}{9}, & \frac{1}{7}, & \frac{1}{5}, & \frac{1}{3}, & 1, & 2\text{-}\frac{1}{2}, & 2\text{-}\frac{1}{4}, & 2\text{-}\frac{1}{6}, & 2\text{-}\frac{1}{8}, & \ldots), \\ y = & (\ldots, & \frac{1}{7}, & \frac{1}{5}, & \frac{1}{3}, & 1, & 2\text{-}\frac{1}{2}, & 2\text{-}\frac{1}{4}, & 2\text{-}\frac{1}{6}, & 2\text{-}\frac{1}{8}, & 2\text{-}\frac{1}{10}, & \ldots). \end{array}$$

The first line lists the moments in time. The even indexed moments are written in increasing order, the odd indexed moments are written in decreasing order (or reading from right to left, in increasing order). The second line presents the infinite stream x. The even indexed values increase an have a limit equal to 2. The odd indexed values decrease (as time moves forward) and have a limit equal to 0. The third line lists the very same values in the same order as in the previous line. Each value, however, is shifted one place to the left. The result is that the stream y is just a permutation of the stream x that strongly dominates x, i.e. for each t we have $y_t > x_t$. Conclude that strong anonymity conflicts with weak Pareto.

The second example considers a discrete setting. The next two streams shelter an infinite number of zeroes and an infinite number of ones:

$$z_{99} = (\underbrace{1, 1, \ldots, 1}_{99 \text{ times}}, 0; \ldots; \underbrace{1, 1, \ldots, 1}_{99 \text{ times}}, 0; \ldots) \sim z_1 = (1, \underbrace{0, 0, \ldots, 0}_{99 \text{ times}}; \ldots; 1, \underbrace{0, 0, \ldots, 0}_{99 \text{ times}}; \ldots).$$

There is a bijective map (permutation) that transforms z_1 into z_{99}. Hence, according to strong anonymity, the infinite stream z_{99} with 99 percent of the generations at level 1 is equally good as the stream z_1 with 1 percent at level 1. In the discrete setting, with $Y = \{0, 1\}$ strong anonymity conflicts with intermediate Pareto (Van Liedekerke and Lauwers 1997).

Note the difference between the two examples. In the first example, the distribution of the different levels remains untouched after being permuted (because all the different levels occur only a finite number of times over the infinite number of different periods). In contrast, in the second example the volume—and hence the distribution—of zeros changes from 1 percent to 99 percent when permuting z_{99} into z_1.

The most familiar rules that satisfies strong anonymity are the Rawlsian infimum-rule and the limit inferior-rule (Rawls 1999). Both these rules satisfy strong anonymity, monotonicity (if $x \geq y$, then $x \succsim y$), continuity with respect to the sup-topology, and Hammond equity. Characterizations are obtained by Lauwers (1997c) and Chambers (2009).[19]

We close this section with the rank-discounted utilitarian rule. This rule, introduced by Zuber and Asheim (2012), also satisfies strong anonymity. First, we discuss two sets, X^+ and $\bar{X}$, of infinite utility streams for which the axioms of strong Pareto and strong

[19]Doyen and Martinet (2012) apply the maximin rule in a general dynamic economic model.

anonymity are not in conflict. The domain X^+ collects the nowhere decreasing streams, i.e. streams x with $x_t \leq x_{t+1}$ for each t in $\mathbb{N}$. For each stream x in X^+, it is impossible to permute x into y such that y strongly Pareto dominates x. The set $\bar{X}$ is defined as the set of infinite streams that can be rewritten as a nowhere decreasing stream. The set $\bar{X}$ is closed under all permutations and within the set $\bar{X}$ the axioms of strong anonymity and strong Pareto are compatible. In addition, each infinite stream in $\bar{X}$ has a well defined lowest level, second lowest level, and so forth. Hence, for each infinite stream x in $\bar{X}$ a corresponding non-decreasing stream $x_{[]} = (x_{[1]}, x_{[2]}, \ldots, x_{[t]}, \ldots)$ is defined. Furthermore, the limit $\ell(x)$ is well defined for each x in $\bar{X}$. Indeed, the increasing stream $x_{[]}$ associated to x has a unique point of accumulation. The infinite stream x in $\bar{X}$ either remains below $\ell(x)$ (for each t, $x_t < \ell(x)$), or reaches $\ell(x)$ after a finite number of time moments (for each t big enough, $x_t = \ell(x)$).

Zuber and Asheim (2012) define the rank-discounted utilitarian rule

$$R : \bar{X} \longrightarrow \mathbb{R} : x \longmapsto R(x) = (1-\beta) \sum \beta^{r-1} u(x_{[r]}),$$

with $0 < \beta < 1$ and u a continuous and increasing map. The factor $(1-\beta)$ normalizes the total sum of the weights to 1. In contrast to the time-discounted utilitarian rule, the weight $(1-\beta)\beta^{r-1}$ corresponds to the rank a particular value $x_t = x_{[r]}$ obtains after rewriting x in increasing order. The highest weight $(1-\beta)$ is attached to the moment t for which x_t is the lowest level, the second highest weight is attached to the moment with the second lowest level, and so forth. The lower the valuex_t, the higher the weight attached to generation t.

The axiomatization of Koopmans' rule can be used to characterize the rank-discounted utilitarian rule R when restricted to the domain X^+ of increasing streams. Roughly, the axioms of Koopmans are not imposed upon the whole collection X of infinite streams. Restricted axioms are imposed to order the set X^+. Strong anonymity then extends the rule R to the set $\bar{X}$. When applied to the set $\bar{X}$, the criterion R attaches weights to a generation on the basis of the rank this generation obtains. Koopmans' rule uses discounting according to the position in time, in contrast, the welfare function R uses discounting according to the rank after rewriting the stream in increasing order.

Finally, Zuber and Asheim (2012) introduce an extended rank-discounted utilitarian social welfare function on the set X of all infinite streams:

$$R : X \longrightarrow \mathbb{R} : x \longmapsto R(x) \;=\; u(\ell(x)) + (1-\beta) \sum_{r=1}^{|L(x)|} \beta^{r-1} \Big(u(x_{[r]}) - u(\ell(x)) \Big),$$

with $0 < \beta < 1$, $\ell(x)$ the limit inferior of x, $L(x) \subset \mathbb{N}$ the set of indices t for which $x_t < \ell(x)$, and u a continuous and increasing real valued function. The length of the discounted sum in $R(x)$ either is finite (if $|L(x)| < +\infty$) or infinite (if $|L(x)| = +\infty$). In words, take an infinite stream x, let $\ell(x) = \liminf(x)$ be the limit inferior of x, let $L(x)$ collect all generations t for which $x_t < \ell(x)$, and apply the rank-discounted utilitarian rule upon the stream x restricted to $L(x)$ which is, if necessary, supplemented with infinitely many values $\ell(x)$.

For example, the welfare function R attaches value zero to the above infinite streams z_1 and z_{99}. The imposition of strong anonymity (on the set X) implies a cost: the welfare function R does not satisfy weak Pareto. The infinite stream $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$ is equally good as the zero stream. In conclusion, when applied to the set X of all infinite streams, the criterion R refines the 'Rawlsian' limit inferior as it only pays attention to those generations that obtain a level below or equal to this limit inferior.[20]

8 Sustainable discounted utilitarianism

The infinite horizon utilitarian rule $\succsim_\infty^U$ imposes

$$x = (x_1, z+\varepsilon, z+\varepsilon, \ldots, z+\varepsilon, \ldots) \succ y = (y_1, z, z, \ldots, z, \ldots),$$

for each x_1, y_1, z, and $\varepsilon > 0$. Whatever the sacrifice $y_1 - x_1$ of the first generation, the infinitely many ε's bridge the gap and overtake the infinite stream y. The axiom of Hammond equity for the future modifies the above ranking as follows.

Hammond equity for the future (Asheim and Tungodden 2004)**.** For each x_1, y_1, z, and $\varepsilon > 0$, we have

$$x = (x_1, z+\varepsilon, z+\varepsilon, \ldots, z+\varepsilon, \ldots) \succ y = (y_1, z, z, \ldots, z, \ldots),$$

as soon $y_1 > x_1 > z + \varepsilon$.

Similar to Hammond's equity axiom, bringing the levels y_1 and z closer to each other (towards x_1 and $z+\varepsilon$) results in a better stream. In contrast to Hammond's axiom, where only two generations are involved, now each generation is involved. The transfer from the better-off first generation leads to a sustained increase in the level of all subsequent generations while the first generation remains the better of.[21] The early generations are not condemned to starvation in order to maximize the welfare for later generations.

The imposition of Hammond equity of the future comes at the cost of weakening strong Pareto and the axiom of independence. For example, an improvement for the first generation is not taken into account in case this first generation is better off than the future generations. Asheim et al (2010, 2012) introduce and characterize the following class of sustainable discounted utilitarian rules:

$$S : X \longrightarrow \mathbb{R} : x \longmapsto \begin{cases} (1-\beta)U(x_1) + \beta S(x_{+1}) & \text{if } U(x_1) \leq S(x_{+1}), \\ S(x_{+1}) & \text{if } U(x_1) > S(x_{+1}), \end{cases}$$

with β in the open interval $(0, 1)$ and $U(x) = S(x, x, \ldots, x, \ldots)$. Observe the similarity with the time-discounted utilitarian rule,

$$D(x) = (1-\alpha)\, u(x_1) + \alpha\, D(x_{+1}).$$

[20] Asheim and Zuber (2013) study the behavior of the rank-discounted utilitarian rule as β goes to zero and show the convergence of R towards a strongly anonymous leximin relation.

[21] Also Banerjee (2006a), Asheim et al (2007), Alcantud and García-Sanz (2010), Dubey and Mitra (2013) consider Hammond equity for the future.

Both social welfare functions are defined in a recursive way. The sustainable discounted utilitarian rule gives zero weight to those generations that are better off then their future generations. As a consequence the sustainable discounted utilitarian rule violates weak Pareto: again, the infinite stream $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$ and the zero stream are considered equally good. On the other hand, when restricted to the domain of non-decreasing streams, the rule coincides with both the time-discounted and the rank-discounted utilitarian rule.

The sustainable discounted utilitarian rule satisfies continuity with respect to the sup-topology, monotonicity, stationarity, and the following weakening of the independence axiom.

Separable future. For each x and y in X and for each t in $\mathbb{N}$,

$$\text{if } x \succsim (x_{-t}, y_{+t}), \text{ then } (y_{-t}, x_{+t}) \succsim y.$$

In words, if two infinite streams have the same head up to time T, then the ranking of these streams does not depend upon this common head. Asheim (2010) provides the following example to motivate the rejection of the other part of Koopmans' independence axiom. Consider the streams

$$a = (0, .75, 1, 1, \ldots, 1, \ldots) \text{ and } b = (.25, .25, 1, 1, \ldots, 1, \ldots).$$

Assume that a is preferred to b. Then, it is not obvious to rank the modifications

$$a^* = (0, .75, .25, .25, .25, \ldots, .25, \ldots) \text{ and } b^* = (.25, .25, .25, .25, \ldots, .25, \ldots)$$

in the same way. The common tails might influence our look upon the conflict between the couple $(0, .75)$ and $(.25, .25)$. The value .75 is the second worst-off in a, while it is the best-off in a^*. The sustainable discounted utilitarian rule S treats the same value but at different ranks in a different way.

9 Chichilnisky's sustainable preference

Graciela Chichilnisky (1996, 1997) introduces the axioms of non-dictatorship and characterizes a whole class of social welfare orderings that satisfies these axioms in combination with completeness, continuity, and representability. We recall these axioms.

Dictatorship of the present. For each x, y, v, and w in X, if $x \succ y$, then there exists a T in $\mathbb{N}$ such that

$$(x_{-(T+k)}, v_{+(T+k)}) \succ (y_{-(T+k)}, w_{+(T+k)}) \text{ for each } k \text{ in } \mathbb{N}.$$

A rule that satisfies this axiom ranks two infinite streams on the basis of their heads (i.e. the truncated streams). Time-discounted utilitarianism is the prime example that satisfies this axiom. The time-discounted rule just puts the (very) long run offside.

Dictatorship of the future. For each x, y, v, and w in X, if $x \succ y$, then there exists a T in $\mathbb{N}$ such that

$$(v_{-(T+k)}, x_{+(T+k)}) \succ (w_{-(T+k)}, x_{+(T+k)}) \text{ for each } k \text{ in } \mathbb{N}.$$

A rule that satisfies this axiom ranks two infinite streams on the basis of their tails. The map $x \mapsto \liminf x$ is an example. This map looks for the infimum of the set of accumulation points, and is not sensitive for changes in the head of the infinite stream. Furthermore, observe the conflict between the axioms of strong Pareto and of dictatorship of the future.

The axioms of non-dictatorship impose that the axioms of dictatorship do not hold.[22] Many rules meet the axioms of non-dictatorship. The Pareto dominating tail rules $\succsim^S_\infty$, $\succsim^U_\infty$, $\succsim^L_\infty$, and $\succsim^U_\infty$; the fixed step anonymous $\mathcal{G}$-rules $\succsim^S_\mathcal{G}$, $\succsim^U_\mathcal{G}$, $\succsim^L_\mathcal{G}$, and $\succsim^U_\mathcal{G}$ all satisfy non-dictatorship of the future (as they all satisfy strong Pareto) and non-dictatorship of the present (the criteria are sensitive for shifts in the tails). None of these criteria, however, is complete.

Theorem 5 (Chichilnisky 1996)**.** Let the ordering $\succsim$ on the set of bounded streams satisfy continuity with respect to the sup-topology, independence, non-dictatorship of the present, and non-dictatorship of the future. Then, the ordering $\succsim$ is represented by a continuous function

$$C : x \longmapsto \sum_{t=1}^{\infty} \lambda_t \, x_t \; + \; \varphi(x),$$

where the real numbers λ_t are all positive and add up to a finite number, and φ a purely finitely additive map.

Chichilnisky uses the term sustainable preferences for the social welfare functions characterized by the previous theorem. The sustainable preference C decomposes into two parts. The first part, $x \mapsto \sum \lambda_t \, x_t$, is countably additive, satisfies strong Pareto, and captures the short run. The second part, $x \mapsto \varphi(x)$, is purely finitely additive and captures the long run. This decomposition follows from the representation of a finitely additive measure on the set $\mathbb{N}$, i.e. a map that (i) assigns to each subset of $\mathbb{N}$ a nonnegative number, and (ii) assigns to the union of two disjoint sets the sum of their numbers.[23]

A purely finitely additive map is typically obtained by means of Hahn-Banach's theorem or by means of a free ultrafilter (e.g. it selects the unique accumulation point the converging subsequence of which is indexed by a set that belongs to the free ultrafilter). Such a purely finitely additive map, although it is continuous with respect to the sup-topology and finitely additive, is a non-constructible object (Lauwers 2009, 2010b). Obviously, the representability of a non-constructible relation does not change the non-constructible

[22] Ayong Le Kama et al (2014) study social welfare functions and introduce two axioms related to non-dictatorship: never-decisiveness of the future and never-decisiveness of the present. Depending upon the domain, a possibility or an impossibility result is obtained.

[23] We refer to Yosida and Hewitt (1952), Rao (1958), and Peressini (1967).

nature of the relation. Here, the map φ provides us an example of a continuous and additive representation of a non-constructible relation. Representability by means of a continuous, monotonic, and additive map does not imply constructibility.[24]

There are, however, at least two ways to circumvent this problem. One can restrict the domain to, for example, those infinite utility streams which exhibit a well defined—without recurse to non-constructive mathematics—and finite limiting behavior (Chichilnisky 2009). Alternatively, one can replace the map φ with, for example, the map $\liminf$ which looks for the infimum of the set of accumulation points.[25]

Finally, we mention the similarities and differences between the results of Chichilnisky and Koopmans. Both social welfare functions satisfy continuity with respect to the sup-topology, strong Pareto, and independence. Stationarity is violated by Chichilnisky's criterion but satisfied by discounted utilitarianism. Non-dictatorship of the present is satisfied by Chichilnisky's criterion but violated by the discounted utilitarian rule.

References

Alcantud JCR, MD García-Sanz (2010) Paretian evaluation of infinite utility streams: an egalitarian criterion. Economics Letters 106, 209-211.

Arrow KJ (1999) Discounting, morality, and gaming. In: Discounting and intergenerational equity. Portney JP, Weyant JP (eds). Washington DC, Resources for the future.

Asheim GB (2010) Intergenerational Equity. In: Book Series, Annual Review of Economics. Arrow KJ, Breshanan TF (eds). Volume 2, 197-222.

Asheim GB, K Banerjee (2010) Fixed-step anonymous overtaking and catching-up. International Journal of Economic Theory 6, 149-165.

Asheim GB, T Mitra (2010) Sustainability and discounted utilitarianism in models of economic growth. Mathematical Social Sciences 59, 148-169.

Asheim GB, T Mitra, B Tungodden (2007) A new equity condition for infinite utility streams and the possibility of being Paretian. In: Intergenerational equity and sustainability. Roemer J, Suzumura K (eds) London: Palgrave.

Asheim GB, T Mitra, B Tungodden (2012) Sustainable recursive social welfare functions. Economic Theory 49, 267-292.

Asheim GB, B Tungodden (2004) Resolving distributional conflicts between generations. Economic Theory 24, 221-230.

[24] Dubey and Mitra (2013) show the existence of a non-constructible relation on X that satisfies the Pigou-Dalton transfer principle while its representation has been established by Sakamoto (2012).

[25] The map $\liminf$ violates additivity: let $x = (1, 0, 1, 0, \ldots)$ and $y = (0, 1, 0, 1, \ldots)$, then $\liminf(x) = \liminf(y) = 0$ while $\liminf(x + y) = 1$. The map $\liminf$, however, still fits in the Chichilnisky approach.

Asheim GB, S Zuber (2013) A complete and strongly anonymous leximin relation on infinite streams. Social Choice and Welfare 41, 819-834.

Atkinson AB (2001) The strange disappearance of welfare economics. Kyklos 54, 193-206.

Atsumi H (1965) Neoclassical growth and the efficient program of capital accumulation. Review of Economic Studies 32, 127-136.

Ayong Le Kama A, T Ha-Huy, C Le Van, K Schubert (2014) A never-decisive and anonymous criterion for optimal growth models. Economic Theory 55, 281-306.

Banerjee K (2006a) On the equity-efficiency trade off in aggregating infinite utility streams. Economic Letters 93, 63-67.

Banerjee K (2006b) On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams. Social Choice and Welfare 27, 327-339.

Banerjee K, Dubey RS (2013) Do all constructive strongly monotonic inter-temporal orders exhibit impatience. Working paper: Florida Atlantic University, Montclair State University, USA.

Basu K, T Mitra (2003) Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian. Econometrica 71, 1557-1563.

Basu K, T Mitra (2007) Utilitarianism for infinite utility streams: a new welfare criterion and its axiomatic characterization. Journal of Economic Theory 133, 350-373.

Bleichrodt H, KIM Rohde, PP Wakker (2008) Koopmans' constant discounting for intertemporal choice: a simplification and a generalization. Journal of Mathematical Psychology 52, 341-347.

Bossert W, Y Sprumont, K Suzumura (2007) Ordering infinite utility streams. Journal of Economic Theory 135, 579-589.

Bossert W, JA Weymark (2004) Utility in social choice. In: Handbook of Utility Theory, Volume 2. Barbera S, Hammond PJ, Seidl CH (eds) Amsterdam: Kluwer Academic Publishers.

Brock WA (1970) An axiomatic basis for the Ramsey-Weizsäcker overtaking criterion. Econometrica 38, 927-929.

Chambers CP (2009) Intergenerational equity: sup, inf, lim sup, and lim inf. Social Choice and Welfare 32, 243-252.

Chichilnisky G (1996) An axiomatic approach to sustainable development. Social Choice and Welfare 13, 231-257.

Chichilnisky G (1997) What is sustainable development? Land Economics 73, 467-491.

Chichilnisky G (2009) Avoiding extinction: equal treatment of the present and the future. Economics, The Open-Access, Open-Assessment E-Journal 3, Article number 200932. http://www.economics-ejournal.org/economics/journalarticles/2009-32

Dasgupta PS, GM Heal (1979) Economic theory and exhaustible resources. Cambridge UK: Cambridge University Press.

d'Aspremont C, L Gevers (1977) Equity and the informational basis of collective choice. Review of Economic Studies 46, 199-210.

Debreu G (1954) Representation of a preference ordering by a numerical function. Chapter XI in Decision Processes. Thrall RM, Coombs CH, Davis RL (eds) New York.

Demichelis S, T Mitra, G Sorger (2010) Intergenerational equity and stationarity. Working Paper No 1003: Department of Economics, University of Vienna.

Diamond PA (1965) The evaluation of infinite utility streams. Econometrica 33, 170-177.

Doyen L, V Martinet (2012) Maximin, viability and sustainability. Journal of Economic Dynamics and Control 36, 1414-1430.

Dubey RS (2011) Fleurbaey-Michel conjecture on equitable weak Paretian social welfare order. Journal of Mathematical Economics 47, 434-439.

Dubey RS, T Mitra (2011) On equitable social welfare functions satisfying the weak Pareto axiom: a complete characterization. International Journal of Economic Theory 7, 231-250.

Dubey RS, T Mitra (2012) On monotone social welfare orders satisfying the strong equity axiom: construction and representation. Working paper: Montclair State University, Cornell University Ithaca, USA.

Dubey RS, T Mitra (2013) On representation and construction of equitable social welfare orders on infinite utility streams. Working paper: Montclair State University, Cornell University Ithaca, USA.

Ferejohn J, T Page (1978) On the foundations of intertemporal choice. American Journal of Agricultural Economics 60, 269-275.

Fleurbaey M, P Michel (2003) Intertemporal equity and the extension of the Ramsey criterion. Journal of Mathematical Economics 39, 777-802.

Fleurbaey M, S Zuber (2012) Climate policies deserve a negative discount rate, Fondation Maison des sciences de l'homme, WP-2012-19.

Hammond PJ (1976) Equity, Arrow's conditions, and Rawls' difference principle. Econometrica 44, 793-804.

Hara C, T Shinotsuka, K Suzumura, Y Xu (2008) Continuity and egalitarianism in the evaluation of infinite utility streams. Social Choice and Welfare 31, 179-191.

Kamaga K, T Kojima (2009) Q-anonymous social welfare relations on infinite utility streams. Social Choice and Welfare 33, 405-413.

Kamaga K, T Kojima (2010) On the leximin and utilitarian overtaking criteria with extended anonymity. Social Choice and Welfare 35, 377-392.

Koopmans TC (1960) Stationary ordinal utility and impatience. Econometrica 28, 287-309.

Koopmans TC (1965) On the concept of optimal economic growth. Pontificiae Academiae Scientiarum Scripta Varia 28, 225-300.

Koopmans TC (1972a) Representation of preference orderings with independent components of consumption. In: Decision and organization. McGuire CB, Radner R (eds) Amsterdam: North-Holland.

Koopmans TC (1972b) Representation of preference orderings over time. In: Decision and organization. McGuire CB, Radner R (eds) Amsterdam: North-Holland.

Koopmans TC (1977) Concepts of optimality and their uses. American Economic Review 67(3), 261-274.

Koopmans TC, Diamond PA, Williamson RE (1964) Stationary utility and time perspective. Econometrica 32, 82-100.

Lauwers L (1997a) Continuity and equity with infinite horizons. Social Choice and Welfare 14, 345-356.

Lauwers L (1997b) Infinite utility, insisting on strong monotonicity. Australasian Journal of Philosophy 75, 222-233.

Lauwers L (1997c) Rawlsian equity and generalized utilitarianism with an infinite population. Economic Theory 9, 143-150.

Lauwers L (2009) A report on Graciela Chichilnisky: Avoiding extinction equal treatment of the present and the future. Economics, The Open-Access, Open-Assessment E-Journal.
http://www.economics-ejournal.org/economics/discussionpapers/2009-8#comments

Lauwers L (2010a) Ordering infinite utility streams comes at the cost of a non-Ramsey set. Journal of Mathematical Economics 46, 32-37.

Lauwers L (2010b) Purely finitely additive measures are non-constructible objects. KU Leuven, Center for Economic Studies, DPS 1010.

Lauwers L (2012) Intergenerational equity, efficiency, and constructibility. Economic Theory 49, 227-242.

Mitra T, Basu K (2007) On the existence of Paretian social welfare relations for infinite utility streams with extended anonymity. In: Intergenerational equity and sustainability. Roemer J, Suzumura K (eds) London: Palgrave.

Peressini A (1967) Ordered topological vector spaces. New York: Harper and Row.

Ramsey FP (1928) A mathematical theory of savings. The Economic Journal 38, 543-559.

Rao RR (1958) A note on finitely additive measures. The Indian Journal of Statistics 19, 27-28.

Rawls J (1999) A theory of justice, revised edition. The Belknap Press of the Harvard University Press, Cambridge.

Sakai T (2010) Intergenerational equity and an explicit construction of welfare criteria. Social Choice and Welfare 35, 393-414.

Sakamoto N (2012) Impossibilities of Paretian social welfare functions for infinite utility streams with distributive equity. Hitotsubashi Journal of Economics 53, 60-64.

Sen AK (1971) Collective choice and social welfare. Edinburgh: Oliver and Boyd.

Sen AK (1977) On weights and measures: informational constraints in social welfare analysis. Econometrica 45, 1539-1572.

Shorrocks AF (1983) Ranking income disctributions. Economica 50, 3-17.

Sidgwick H (1907) The methods of ethics. London: Macmillan.

Suppes P (1966) Some formal models of grading principles. Synthese 6, 284-306.

Svensson L-G (1980) Equity among generations. Econometrica 48, 1251-1256.

Szpilrajn E (1930) Sur l'extension du l'ordre partiel. Fundamenta Mathematicae 16, 386-389.

Thomson W (2001) On the axiomatic method and its recent applications to game theory and resource allocation, Social Choice and Welfare 18(2), 327-386.

Van Liedekerke L, L Lauwers (1997) Sacrificing the patrol: utilitarianism, future generations and infinity. Economics and Philosophy 13, 159-174.

von Weizsäcker CC (1965) Existence of optimal programs of accumulation for an infinite time horizon. Review of Economic Studies 32, 85-104.

Yosida K, E Hewitt (1952) Finitely additive measures. Transactions of the American Mathematical Society 72, 46-66.

Zame WR (2007) Can intergenerational equity be operationalised. Theoretical Economics 2, 187-202.

Zuber S, GB Asheim (2012) Justifying social discounting: The rank-discounted utilitarian approach. Journal of Economic Theory 147, 1572-1601.