Subtraction by Addition in Children with Mathematical Learning Disabilities

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Abstract

In the last decades, strategy variability and flexibility have become major aims in mathematics education. For children with mathematical learning disabilities (MLD), it is unclear whether the same goals can and should be set. Some researchers and policy makers advise to teach MLD children only one solution strategy, others advocate stimulating the flexible use of various strategies, as for typically developing children. To contribute to this debate, we investigated the use of the subtraction by addition strategy to mentally solve two-digit subtractions in children with MLD. We used non-verbal research methods to infer strategy use patterns, and found that MLD children – similar to their typically developing peers – switch between the traditionally taught direct subtraction strategy and subtraction by addition, based on the relative size of the subtrahend. These findings challenge typical special education classroom practices, which only focus on the routine mastery of the direct subtraction strategy.

Key words: mathematical learning disabilities, strategy use, strategy choice, subtraction by addition

Highlights:

- This study focuses on how children with MLD solve multi-digit subtraction problems.
- Children with MLD are able to flexibly solve multi-digit subtraction problems.
- They switch between direct subtraction and subtraction by addition strategies.
- Strategy choices are based on the relative size of the subtrahend.
- Findings in children with MLD are similar to typically developing peers.

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1. Introduction

In the last decades, variety and flexibility in children's strategy use have become major aims of mathematics education (e.g., Freudenthal, 1991; Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Torbeyns, De Smedt, Luwel, & Van Dooren, 2007). To achieve these goals children are stimulated to discover and flexibly use a variety of strategies based on their understanding of number relations and/or the properties of operations. For children with mathematical learning disabilities (MLD), however, the feasibility and suitability of strategy variety and flexibility remains an issue of continued debate in many countries. Some researchers, curriculum developers, and policy makers argue that it is better for these children to develop mastery and confidence in only one way or strategy to solve problems (e.g., Geary, 2003; Milo & Ruijssenaars, 2003; National Mathematics Advisory Panel, 2008). Others claim that the development of strategy variety and flexibility should be educational goals for all students, including those with MLD (e.g., Baroody, 2003; Kilpatrick et al., 2001; Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012; Verschaffel et al., 2007). While this discussion remains to be lively, more scientific evidence is needed.

Mental subtraction is one mathematical subdomain in which strategy variety and flexibility can be stimulated. When solving subtractions such as 81 − 43, the most commonly taught solution strategy[1] is the *direct subtraction* strategy, in which the smaller number (43) is subtracted from the larger number (81) (e.g., 81 − 40 = 41 and 41 − 3 = 40 − 2 = 38). However, for problems with a relatively large subtrahend compared to the difference, such as 81 − 79, *subtraction by addition* appears to be a more clever strategy (e.g., Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). With this strategy, one can solve 81 − 79 very efficiently by determining how much needs to be added to 79 to make 81 (e.g., 79 + 1 = 80, 80 + 1 = 81, so the answer is 1 + 1 = 2). The use of the complementary addition operation on such problems can thus considerably facilitate the calculation process by reducing computational effort and increasing solution efficiency, i.e., fewer and/or smaller calculation steps, which lead faster to a correct answer (e.g., Heinze, Marschick, & Lipowsky, 2009; Verschaffel, Bryant, & Torbeyns, 2012). In contrast, for problems with a relatively small subtrahend compared to the difference, such as 81 − 2, the subtraction by addition strategy does not lead to fewer and/or smaller calculation steps. For these problems the direct subtraction strategy seems to be more efficient.

Previous work on children's and adults' use of subtraction by addition in elementary subtraction indicated that children hardly use the subtraction by addition strategy spontaneously, not even on problems such as 81 − 79 (e.g., Blöte, Van der Burg, & Klein, 2001; De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010; Heinze et al., 2009; Selter, Prediger, Nührenbörger, & Hussmann, 2012; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009). Adults, on the other hand, seem to solve subtractions efficiently and flexibly by means of subtraction by addition (Torbeyns, Ghesquière, & Verschaffel, 2009). These available studies relied on verbal protocol data to infer strategy use. A closer inspection of the speed data in the study by De Smedt et al. (2010) suggested that children sometimes used subtraction by addition even though they reported a direct subtraction strategy. If these children only used direct subtraction, an increase in reaction times should have been observed from items with relatively small subtrahends (81 − 7) over items with medium-sized subtrahends (81 − 43) to items with relatively large subtrahends (81 − 79), since subtracting a larger subtrahend requires more and/or larger calculation steps (Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2010). This reaction time pattern was not found in De Smedt et al. (2010): Problems with a relatively large subtrahend were solved significantly faster than problems with a medium-sized subtrahend, which suggests that the actual use of the subtraction by addition strategy might be larger than revealed by the children’s verbal protocols. In a recent study, Peters, De Smedt, Torbeyns, Ghesquière, and Verschaffel (2012) therefore used two non-verbal methods to infer the use of the subtraction by addition strategy in typically developing children: regression analyses and a format manipulation. They concluded that children, like adults, switched between direct subtraction and subtraction by addition to solve two-digit subtraction problems, based on the relative size of the subtrahend: The children used direct subtraction when the subtrahend was relatively small compared to the difference (as in 83 − 4), and subtraction by addition when the subtrahend was relatively large (as in 83 − 79).

So far, the use of the subtraction by addition strategy has not been explored in children with MLD, except for the study by Peltenburg et al. (2012). They showed that Dutch special education children (aged 8 to 12, with a mathematics level similar to the end of Grade 2) reported the use of this strategy in more than 50 % of problems with a relatively large subtrahend (e.g., 61 − 59) and significantly less on problems with a medium-sized subtrahend (such as 52 – 36; about 20 %), but they did not include problems with relatively small subtrahends in their problem set. However, these remarkably high percentages may be due to the fact that two thirds of problems were word problems, half of which even reflected an adding-on situation (e.g., “The album has space for 51 cards. 49 are already included. How many more cards can be added?”). This latter type of problems is well known to elicit a lot of subtraction by addition strategies in elementary school children (e.g., Carpenter & Moser, 1982; De Corte & Verschaffel, 1987; Klein & Beishuizen, 1994), even in low-achieving children (Kraemer, 2011). It is thus not surprising that Peltenburg et al. (2012) observed more subtraction by addition strategies on word problems compared to symbolically presented problems (i.e., 70 % on adding-on word problems, 25 % on taking-away word problems, and only 8 % on symbolically presented subtractions). Unfortunately, Peltenburg et al. did not deepen the interaction between number characteristics (i.c., large vs. medium subtrahend) and type of problems (i.c., word problems vs. symbolically presented problems). In this regard, it is important to point out that Peters et al. (2012) observed that typically developing children, when confronted with symbolically presented two-digit subtraction problems, switch between the direct subtraction strategy and the subtraction by addition strategy depending on number characteristics: Direct subtraction was used when the subtrahend was relatively small (as in 83 − 4), subtraction by addition when the subtrahend was relatively large (as in 83 − 79). Against this background, we extended the work by Peltenburg et al. (2012) in children with MLD by investigating the role of the numbers in symbolically presented problems, also including problems with relatively small subtrahends. We verified whether children with MLD showed similar patterns of flexible strategy use as typically developing children.

It also might be that the number of verbal reports of subtraction by addition on the symbolically presented problems in the study of Peltenburg et al. (2012) was an underestimation. As argued by Peters et al. (2012), the subtraction by addition strategy can be executed very fast and quasi-automatic, and it therefore might be that children had difficulties in explaining how they found their answer: They may not have been aware of, or confused by, the steps they performed while calculating and therefore reported a strategy they knew how to explain (e.g., Cooney & Ladd, 1992; Kirk & Ashcraft, 2001). Moreover, children may have hidden the use of the subtraction by addition strategy because they thought it was not valued or allowed to use other strategies than the one(s) taught in the mathematics lessons (e.g., Yackel & Cobb, 1996). These problems might be particularly prominent in children with MLD (see Milo [2003] and Thevenot, Castel, Fanget, & Fayol [2010]). We therefore used two non-verbal methods to answer our research questions: regression analyses in which reaction times were predicted based on different task characteristics, and a method in which speed was contrasted between problems presented in different presentation formats.

2. The Present Study

Extending the data by Peltenburg et al. (2012), we investigated whether children with MLD switch between direct subtraction and subtraction by addition based on number characteristics when solving symbolically presented two-digit subtraction problems. Since verbal self-reports might be less suited to identify the subtraction by addition strategy, especially in children with MLD, two non-verbal methods were used.

First, we used the reaction times for problems presented in the standard subtraction format to calculate three linear regression models (see Peters et al., 2012; Woods, Resnick, & Groen, 1975). These models represented three different strategy use patterns. The first model, the *DS*-Model, represents the consistent use of the direct subtraction strategy. When children consistently use this strategy, the reaction times should be best predicted by the size of the subtrahend (*S*), because it takes longer to subtract a larger number from the minuend (e.g., 83 – 79 = .) than to subtract a smaller number (e.g., 83 − 4 = .). The second model, the *SBA*-Model, starts from the same idea but represents the consistent use of the subtraction by addition strategy: If children only use subtraction by addition, reaction times should be best predicted by the size of the difference (*D*), because it takes more time to determine how much needs to be added to get at a given number when the difference between both numbers is large ("How much needs to be added to 4 to have 83?") than when it is small ("How much needs to be added to 79 to have 83?"). The third model, the *Switch*-Model, represents switching between both strategies based on the relative magnitude of the subtrahend ($S < D$ vs. $S > D$), and reaction times in this model are best predicted by the minimum of subtrahend and difference (min[*D*, *S*]): For problems with the subtrahend smaller than the difference (e.g., 83 − 4 = . and 84 − 38 = .), problems can be more easily solved by means of the direct subtraction strategy, and therefore reaction times for these problems are expected to increase with the size of the subtrahend. In contrast, problems with the subtrahend larger than the difference (e.g., 83 − 79 = . and 84 − 46 = .) can be more easily solved by means of the subtraction by addition strategy, and therefore reaction times for these problems are expected to increase with the size of the difference. Peters et al. (2012) showed that for typically developing children the *Switch*-Model provided the best fit to their reaction times. For the present study, we wondered whether children with MLD show the same strategy use pattern: Do they also switch between direct subtraction and subtraction by addition based on the magnitude of the subtrahend when solving two-digit subtraction problems (Research Question 1)?

Second, we expected that, besides the magnitude of the subtrahend, the numerical distance between subtrahend and difference would have an influence on strategy selection as well. For problems with a *large* numerical distance between *S* and *D* (such as 83 − 4 or 83 – 79), the computational gain in using one strategy compared to the other is very clear. However, for problems with a *small* numerical distance between *S* and *D* (such as 84 − 38 or 84 − 46) there is no clear computational advantage for one of the two strategies. This interaction was found in the study of Peters et al. (2012) with typically developing children solving symbolically presented problems. For Research Question 2, we thus wondered whether we could replicate these findings in children with MLD. To answer this question, we divided the subtractions into four problem types, based on the combination of the magnitude of *S* ($S < D$ vs. $S > D$) and the numerical distance between *S* and *D* [small vs. large] (see Peters et al., 2012). We then compared reaction times for these four problem types when presented in two different presentation formats: the traditional subtraction format ($M - S = .$) and the (unusual) addition format ($S + . = M$) (see also Campbell, 2008; Peters et al., 2010). Based on Campbell (2008), we expected speed differences between the two presentation formats because the subtraction by addition strategy can be performed more easily when the problem is presented in the addition format, while a time-consuming mental *re*-representation is needed when the same problem is presented in the subtraction format (and vice versa for the direct subtraction strategy on problems in the addition format). So, if children with MLD are switching between direct subtraction and subtraction by addition depending on the combination of the magnitude of S and the numerical distance between S and D, then we should find an interaction between the magnitude of S, the presentation format, and the numerical distance between S and D. For the large-distance problems (such as 83 − 4 or 83 − 79), we expect children to select direct subtraction when $S < D$, and subtraction by addition when $S > D$. This means that large-distance $S < D$ problems (such as 83 − 4 = .) will lead to faster response times in the subtraction format compared to the addition format (4 + . = 83) and, similarly, large-distance $S > D$ problems (such as 83 − 79 = .) will be solved faster in the addition format (79 + . = 83) compared to the same problems in subtraction format. For the small-distance problems (such as 84 − 38 or 84 − 46), we expect no significant magnitude × format interaction, because such a small distance does not yield a clear computational advantage for either direct subtraction or subtraction by addition.

3. Method

3.1 Participants

Participants were 81 children with MLD from the final year of special education for children with specific learning disorders, coming from eight different Flemish special schools of Type 8 (Belgium). In order to get enrolled in these schools for children with learning disorders, children need to have normal general intellectual ability, as evidenced by standardized clinical assessment at school entry. The problems in mathematics experienced by these children are thus not readily explained by lowered general intellectual ability. None of the children was additionally diagnosed with dyslexia. These children all completed a short paper-and-pencil pre-test (see Peters et al. [2012]), to check whether they were able to solve two-digit subtraction problems presented in the (unusual) $S + . = M$ format.

We excluded children from further participation based on two criteria. First, based on the results of the pre-test, we excluded 25 children who solved less than 3 out of 18 problems correctly within the 120 seconds time-limit, to avoid that they would become too frustrated in the main test (which contained 64 problems). Second, we excluded two more children because they demonstrated lack of understanding of the $S + . = M$ presentation format on one or more items of the pre-test (by solving these problems as if they were straightforward addition problems, as in giving 87 as the answer for 4 + . = 83).

There were 54 children with MLD remaining for participation in the individual computer test. However, due to absence ($n = 5$) and irregularities ($n = 5$) during the data collection, the final group of participants consisted of 44 children (31 males). Their mean age was 12 years and 5 months ($SD = 6$ months).

3.2 Materials and Procedure

Materials and procedure were exactly the same as in Peters et al. (2012). All 44 participants were asked to mentally solve 64 subtraction problems, which were presented horizontally in the middle of a computer screen. All problems had a two-digit minuend larger than 30 and required borrowing. Half of the problems were presented in the traditional subtraction format (e.g., 83 − 4 = .), the other half were composed by transforming these problems into their corresponding addition format (i.e., 4 + . = 83). The two formats were presented in a mixed order.

All 64 problems could be categorised into four problem types, based on the combination of the magnitude of S ($S < D$ or $S > D$) and the numerical distance between S and D (small or large). For the small-distance problems, S and D differed by less than 10, while for the large-distance problems S and D were differing by at least 10 *and* either S or D was a one-digit number. This resulted in the following categorisation: (a) *large-distance $S < D$ problems*, with subtrahends smaller than 10 (e.g., 83 − 4 = . and 8 + . = 34); (b) *large-distance $S > D$*

problems, with differences smaller than 10 (e.g., 77 − 68 = . and 37 + . = 42); (c) *small-distance S < D problems* (e.g., 92 − 44 = . and 36 + . = 75); and (d) *small-distance S > D problems* (e.g., 32 − 17 = . and 29 + . = 53).

4. Results

Results are presented in two parts. The first part involves a descriptive overview of the accuracy data, whereas in the second part we focus on the reaction times analyses. All analyses were carried out by means of SAS Version 9.3.

4.1 Accuracy

All 44 children solved 64 problems in the computer task. For these analyses, we excluded 37 trials due to incorrect task administration (1 %), resulting in a total of 2779 trials.

As can be expected from children with MLD, accuracy levels were rather low: The mean score was 71 % correct on all problems, ranging from 24 % to 100 %. The descriptive data per problem type shown in Table 1 indicate that large-distance problems were solved better than small-distance problems, and that problems in subtraction format were solved better than problems in addition format for all problem types except large-distance *S* > *D* problems (such as 83 − 79).

Insert Table 1 about here

Most of the 818 incorrect answers represent common errors of children mentally solving subtractions (e.g., Beishuizen, 1993): Errors resulting from the smaller-from-larger bug (e.g., 52 – 48 answered with 16 instead of 4; in 28 % of all incorrect answers), answers that differed only in tens from the correct answer (e.g., 24 + . = 53 answered with 39 instead of 29; in 22 % of all incorrect answers), answers that differed in only 1 unit from the correct answer (e.g., 44 + . = 92 answered with 47 instead of 48; in 8 % of all incorrect answers). Although we excluded children who demonstrated lack of understanding of the *S* + . = *M* format in the pre-test, in 3 % of all incorrect answers the addition operation was used incorrectly (e.g., 3 + . = 31 answered with 34, but also 32 – 17 answered with 49).

4.2 Reaction time analyses

For the reaction time analyses, we excluded 21 more trials due to children correcting their initial answer (1 %). This resulted in a final data set of 2758 trials (i.e., 98 % of all trials). No items were found to be outliers deviating more than 2.5 standard deviations from a participant's cell mean per problem type. We first evaluated the regression models that predicted children's reaction times, and afterwards we compared children's reaction times in the two presentation formats for the four different problem types.

4.2.1 Regression analyses for problems in subtraction format

We first fitted three regression models to the reaction times of all 32 problems presented in the subtraction format (see Peters et al., 2012; Woods et al., 1975). As stated before, these models represented three different strategy use patterns: the use of direct subtraction (*DS*-Model), the use of subtraction by addition (*SBA*-Model), and switching between both strategies based on the magnitude of the subtrahend (*Switch*-Model). Either the size of the subtrahend (*DS*-Model), the size of the difference (*SBA*-Model), or the minimum of subtrahend and difference (*Switch*-Model) was used to predict the reaction times.

In all three regression models, we predicted the reaction times of each problem, averaged across individuals. Results showed that the model in which children switched between direct subtraction and subtraction by addition based on the magnitude of the subtrahend (*Switch*-Model) provided the best fit to the data (Table 2), explaining 47 % of the variance.

Insert Table 2 about here

It might be that aggregating speed over the various participants in these analyses has covered different strategy use patterns at the individual level. Therefore, we additionally predicted the reaction times of each participant individually, using the same three regression models. The frequencies of the best fitting model per individual were: 14 for whom none of the models fitted significantly, 10 for the *DS*-Model, 0 for the *SBA*-Model, and 20 for the *Switch*-model. The model representing a switch between direct subtraction and subtraction by addition based on the magnitude of the subtrahend thus provided most frequently the best fit to the reaction times.

As an answer to Research Question 1, we can thus conclude that children with MLD switch between direct subtraction and subtraction by addition depending on the magnitude of the subtrahend. The *Switch*-Model fits best to the reaction time data averaged across individuals, but also for most individual children for whom a model could be fitted.

4.2.2 Comparison of reaction times between the two presentation formats

To answer Research Question 2, we included the presentation format and numerical distance between *S* and *D* into the analyses. The mean reaction times per problem type and presentation format are depicted in Table 3. We performed a 2 × 2 × 2 repeated measures ANOVA on the reaction time data with magnitude of *S* ($S < D$ vs. $S > D$), numerical distance between *S* and *D* (small vs. large) and presentation format (subtraction vs. addition) as within-subject factors. Tukey-Kramer adjustments were used for post-hoc comparisons.

Insert Table 3 about here

There was no main effect of magnitude, $F(1, 43) = 1.42$, $p = .24$, but there were significant effects of presentation format, $F(1, 43) = 4.60$, $p = .04$, and numerical distance, $F(1, 43) = 126.61$, $p < .01$. Problems were solved faster in the traditional subtraction format (9916 ms) compared to the addition format (10623 ms), and large-distance problems, such as 83 − 4 = . or 79 + . = 83, were solved faster (8235 ms) than small-distance problems, such as 84 − 38 = . or 46 + . = 84 (12304 ms). There was no significant magnitude × distance interaction, $F(1, 172) = 1.05$, $p = .31$, but the format × distance interaction was significant, $F(1, 172) = 7.04$, $p < .01$: There were no significant speed differences between the two formats for the large-distance problems ($p = .99$), but for the small-distance problems items were solved faster in the subtraction (11629 ms) than in the addition format (12980 ms) ($p < .01$). The interaction between magnitude and presentation format was significant as well, $F(1, 172) = 48.64$, $p < .01$: Whereas $S > D$ problems only tended to be solved faster in the addition format compared to the subtraction format ($p = .08$), $S < D$ problems were solved 2402 ms faster in the subtraction format ($p < .01$).

Most importantly, the magnitude × format × distance interaction was significant, $F(1, 172) = 50.64$, $p < .01$ (see Figure 2). Providing an answer to Research Question 2, the post-hoc tests revealed that the magnitude × format interaction was only significant for the large-distance problems, and not for the small-distance problems. The large-distance $S > D$ problems in the addition format (e.g., 79 + . = 83) were solved faster than the same problems in the subtraction format (e.g., 83 − 79 = .), $t(172) = 6.29$, $p < .01$, whereas the large-distance $S < D$ problems were solved faster in the subtraction (e.g., 83 − 4 = .) than in the addition format (e.g., 4 + . = 83), $t(172) = -6.52$, $p < .01$. This result shows that the numerical distance between subtrahend and difference affected strategy selection in children with MLD: They only switch between direct subtraction and subtraction by addition when the numerical distance between subtrahend and difference is large.

Insert Figure 1 about here

5. Discussion

Strategy variety and flexibility have become important goals in mathematics education in the last 20 years. There is, however, still discussion whether these goals should be set also for children with MLD (e.g., Geary, 1993; Kilpatrick et al., 2001). Some scholars claim that it is better to teach these children only one way of solving certain types of problems. Others disagree, and state that also for children with MLD one should aim at the discovery and flexible use of a variety of strategies based on understanding of number relations and/or the properties of operations. In this study, we investigated the use of the subtraction by addition strategy in the domain of mental multi-digit subtraction in children with MLD, a strategy that is rarely explicitly taught, especially in special education. Most often, children only learn the direct subtraction strategy, in which the subtrahend is taken away from the minuend in several smaller steps. However, for problems with a relatively large subtrahend compared to the difference, such as 83 − 79, using the addition operation might be more efficient to determine the difference (Torbeyns, De Smedt, Stassens et al., 2009).

Previous research by Peters et al. (2012) has shown that typically developing children use both the direct subtraction and the subtraction by addition strategy when solving symbolically presented two-digit subtraction problems, and that they switch between these strategies in a flexible way: They use direct subtraction when the subtrahend is relatively small compared to the difference (such as 83 − 4), and subtraction by addition when the subtrahend is relatively large (such as 83 – 79). In children with MLD, the use of the subtraction by addition strategy has hardly been investigated. Peltenburg et al. (2012) showed that MLD children reported this strategy very often on word problems, but hardly on symbolically presented subtractions. Unfortunately, these authors did not investigate whether number characteristics might have influenced strategy choice on this latter type of problems.

We therefore decided to investigate whether children with MLD use the subtraction by addition strategy on symbolically presented subtraction problems, taking into account the influence of number characteristics. As in Peters et al. (2010, 2012), we used non-verbal research methods to infer strategy use, since the subtraction by addition strategy might be very hard to explain, especially by children with MLD who never got any instruction in this mental calculation strategy (see also Cooney & Ladd, 1992; Kirk & Ashcraft, 2001; Peters et al., 2010, 2012; Yackel & Cobb, 1996). First, we conducted regression analyses in which reaction times were predicted based on different task characteristics, and second, we contrasted speed between problems presented in different formats. Both methods converged to the conclusion that, like their typically developing peers (Peters et al., 2012) and adults (Peters et al., 2010), children with MLD solve symbolically presented two-digit subtraction problems by switching between direct subtraction and subtraction by addition, depending on the combination of the magnitude of *S* and the numerical distance between *S* and *D*.

We first fitted the mean reaction times of problems presented in the traditional subtraction format to three regression models, representing three different strategy use patterns (based on Woods et al., 1975). The minimum of subtrahend and difference showed to be the best predictor, which suggests that children with MLD switched between the direct subtraction and the subtraction by addition strategy based on the magnitude of the subtrahend: Direct subtraction when the subtrahend was smaller than the difference, and subtraction by addition when the subtrahend was larger than the difference (answering Research Question 1).

Secondly, we tested whether the numerical distance between subtrahend and difference also had an influence on strategy choice (Research Question 2). We therefore compared reaction times on problems presented in the standard subtraction format with its corresponding addition format (based on Campbell [2008] and Peters et al. [2010, 2012]), and predicted to find significant differences between these two formats for $S < D$ and $S > D$ large-distance problems, but not for the two types of small-distance problems. Our prediction was confirmed: When the numerical distance between the subtrahend and the difference was large, $S < D$ problems (such as 83 − 4) were solved faster in the subtraction than in the addition format, while $S > D$ problems (such as 83 – 79) were solved faster in the addition than in the subtraction format. When the numerical distance was small, there were no format effects. This three-way interaction suggests that children switched between the two strategies when solving two-digit subtractions, but only when the numerical distance between subtrahend and difference is large.

As stated above, these results are comparable to similar research in typically developing children (Peters et al., 2012). Although these latter children were younger (from fourth- to sixth-grade, with mean ages from 9 years 9 months to 11 years 9 months) and overall faster and more accurate, the same reaction time patterns were found when using the same non-verbal methods. Although the subtraction by addition strategy was taught explicitly for solving symbolically presented two-digit subtractions in one of the eight participating special education schools, the overall effect was not caused by these MLD children only. This thus suggests that children with MLD can do more than often is expected from them (see also Peltenburg, 2012), and that their mathematics instruction does not has to be restricted to focussing on the routine mastery of the direct subtraction strategy.

These results are thus an important addition to the discussion about the feasibility and suitability of strategy variety and flexibility in children with MLD, in the sense that they show that we might have to re-consider the typical mathematics instruction about mental calculation strategies for children with MLD. Intervention studies in which different instructional settings are tested should therefore be an important next step in this research domain. In this respect, we have high hopes for a setting in which the current focus on the direct subtraction strategy is supplemented by the teaching of subtraction by addition as an alternative strategy (for similar suggestions in other mathematical domains, see Baker, Gersten, & Lee, 2002; Kroesbergen, 2002; Swanson & Hoskyn, 1998), and by explicit attention to its conceptual underpinnings (i.e., the inverse relation between addition and subtraction) and when to apply this particular strategy, followed by plenty of opportunities to practice and flexibly apply both strategies, not only for children with MLD but also for typically developing children.

We will now focus on the limitations of the present study. A first limitation deals with the way in which we selected the large-distance problems. As in the work of Peters et al. in adults (2010) and typically achieving children (2012), we only choose problems that involved a single-digit number either as subtrahend (as in 83 − 4) or as difference (as in 83 − 79). Based on a rational task analysis, we assumed that if children with MLD did not use subtraction by addition on extreme problems such as 83 − 79 = 4, they certainly would not use it for problems with a less extreme difference between subtrahend and difference, such as 84 − 68 = 16. However, this specific problem limits the generalizability of our findings to subtraction in the number domain 20-100 with extremely large subtrahends or extremely small differences. It remains to be determined whether similar flexible strategy choices between direct subtraction and subtraction by addition will occur when children with MLD, but also adults and typically achieving children, are confronted with large-distance problems including only two-digit subtrahends and differences, such as 84 − 16 and 84 − 68.

Secondly, the two non-verbal methods we applied to investigate children's strategic behaviour did not allow us to identify strategies at an item level. However, reliable data on that item level are necessary, both for scientific and practical reasons. Verbal reports might seem very useful to solve this issue, but – as stated in the introduction – using only verbal reports for investigating strategy use, and particularly strategies such as subtraction by addition, is questionable. Confronting verbal strategy reports with a combination of several non-verbal research methods – such as reaction time data (as we showed in the present study), eye-movements, and neuroscientific data – might help in further understanding what really happens when people solve a problem.

Thirdly, the design of our study does not allow us to draw conclusions about the differences and/or similarities in the development of the subtraction by addition strategy between typically developing children and children with MLD. In this respect, future research with a chronological-age/ability-level-match design can be very interesting (e.g., Brankaer, Ghesquière, & De Smedt, 2011; Torbeyns, Verschaffel, & Ghesquière, 2004). With such a design, it would be possible to investigate whether children with MLD are (only) slower in developing the subtraction by addition strategy (showing a delay; i.e., when children with MLD only differ from their chronological age matched peers, but not from younger ability matched peers), or whether children with MLD demonstrate a different developmental trajectory in their strategy development (i.e., when children with MLD differ from both their chronological age matched peers and younger ability matched peers).

Finally, we did not include any cognitive factors that might explain individual differences in strategy use in general, and in the development of the subtraction by addition strategy in particular. For example, magnitude comparison skills might play an important role, because the flexible use of the subtraction by addition strategy requires a comparison process of the numbers in the problem. Arguably, such a process relies on a fast and (quasi-)automatic estimation process rather than a precise and deliberate calculation. Against the background of large individual differences in magnitude comparison skill between children of different levels of mathematical ability (e.g., De Smedt, Verschaffel, & Ghesquière, 2009; Gilmore, McCarthy, & Spelke, 2007; Holloway & Ansari, 2009; Vanbinst, Ghesquière, & De Smedt, 2011), the individual differences in the regression models for the problems presented in subtraction format (i.e., 20 children fitting best to the *Switch*-model and 10 to the *DS*-Model) might be explained by such individual differences in magnitude comparison skill, but more research is needed to investigate this issue.

Related to the above, individual differences in inhibition and shifting skills might also explain the individual differences in strategy use. These skills are needed to switch between direct subtraction and subtraction by addition on problems presented in the subtraction format: Children have to be able to inhibit the taking-away interpretation of subtraction, which is represented by the minus sign (e.g., Van den Heuvel-Panhuizen & Treffers, 2009). Additionally, they should see the advantage of shifting from direct subtraction to subtraction by addition (or vice versa) based on the number characteristics in the problem. Since previous research has observed individual differences in inhibition and shifting in children of varying mathematical ability (e.g., Bull & Scerif, 2001; van der Sluis, de Jong, & van der Leij, 2004), such differences might also play a role in individual differences in strategy use. Further research is needed to shed further light on this issue.

6. Conclusion

The finding that children with MLD switch between direct subtraction and subtraction by addition based on the relative size of the subtrahend, similar to typically developing peers and adults, is of great relevance for the theory and practice of special mathematics education. It challenges the typical current classroom practice in special education, which only focuses on the routine mastery of the direct subtraction strategy and of mental calculation strategies in general.

7. Footnotes

[1] In the present study, we categorise the variety of subtraction strategies based on the main operation that is used, i.e., either subtraction or addition. Different categorisations are used by other researchers (e.g., Beishuizen, 1993; Blöte et al., 2001; Buys, 2001; Peltenburg et al., 2012), such as focusing on the manipulation of the numbers during problem solving, which leads to a classification into jump, split and varying strategies.

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9. References

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Table 1.

Mean percentage correct (and standard deviations) per format and problem type

	S < *D*		*S* > *D*	
	Large-distance (e.g., 83 − 4 = .)	Small-distance (e.g., 84 − 38 = .)	Small-distance (e.g., 84 − 38 = .)	Large-Distance (e.g., 83 − 79 = .)
Subtraction Format	93 % (13 %)	66 % (33 %)	65 % (34 %)	70 % (31 %)
Addition Format	81 % (24 %)	53 % (35 %)	52 % (34 %)	86 % (26 %)

Table 2.

Model specifications of the linear regression models for problems presented in subtraction format

Model	Model Specifications			R^2	Effect Estimates	
	DF	*F*-value	*p*-value		Intercept	Parameter
DS-Model	1, 30	12.82	.0012	0.2995		
SBA-Model	1, 30	3.21	.0834	0.0966		
Switch-Model	1, 30	26.37	<.0001	0.4678	7895	115

Table 3.

Mean reaction times (and standard deviations) in ms per format and problem type

	S < *D*		*S* > *D*	
	Large-distance (e.g., 83 − 4 = .)	Small-distance (e.g., 84 − 38 = .)	Small-distance (e.g., 84 − 38 = .)	Large-Distance (e.g., 83 − 79 = .)
Subtraction Format	6471 (2741)	11377 (5239)	11881 (5059)	9937 (4640)
Addition Format	9958 (4814)	12694 (5680)	13266 (5632)	6573 (3342)

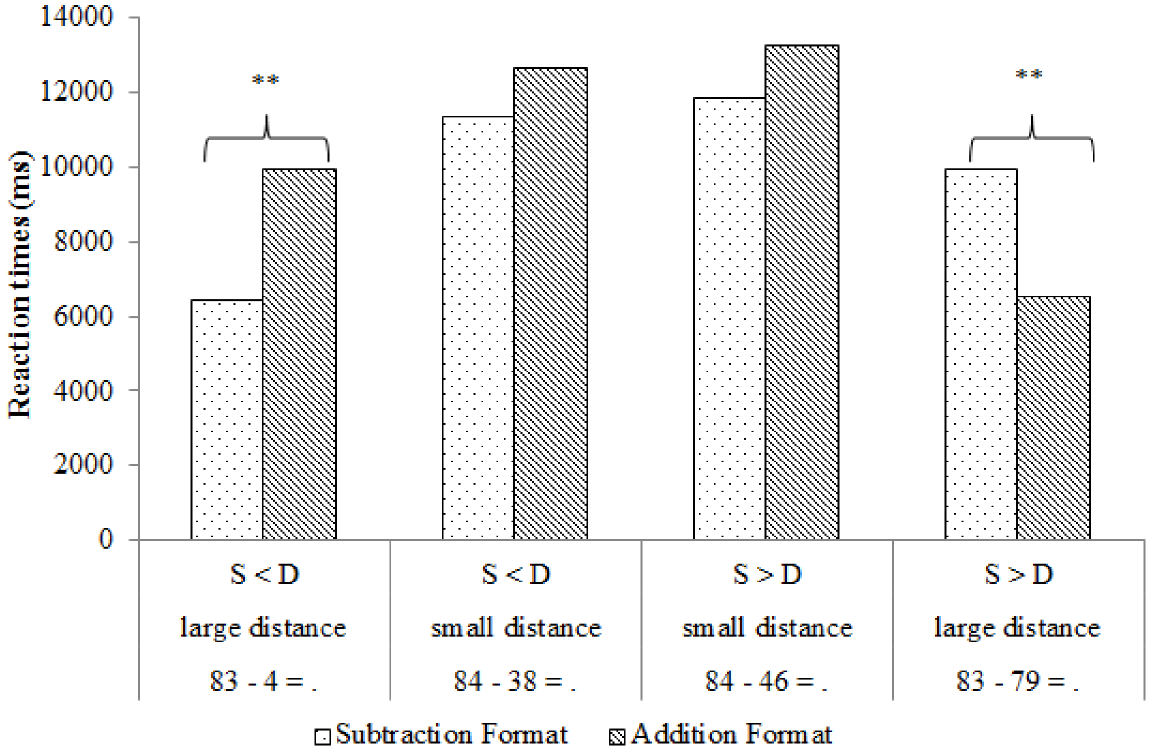

Figure 1. Graph showing the three-way interaction between magnitude, numerical distance, and format.** $p < .01$