Black holes as generalised Toda molecules

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Abstract

In this note we compare the geodesic formalism for spherically symmetric black hole solutions with the black hole effective potential approach. The geodesic formalism is beneficial for symmetric supergravity theories since the symmetries of the larger target space leads to a complete set of commuting constants of motion that establish the integrability of the geodesic equations of motion, as shown in arXiv:1007.3209. We point out that the integrability lifts straightforwardly to the integrability of the equations of motion with a black hole potential. This construction turns out to be a generalisation of the connection between Toda molecule equations and geodesic motion on symmetric spaces known in the mathematics literature. We describe in some detail how this generalisation of the Toda molecule equations arises.

Contents

1 Introduction

To find solutions in supergravity theories is known to be complicated due to the non-linear structure of the Einstein equations and the complexity of having many fields, especially scalar fields. However, when solutions have enough space-time symmetries there is often only a single coordinate that the fields depend on. For cosmological solutions this is the time coordinate and for spherical stationary black hole solutions this is the radial distance from the singularity. This dependence on a single coordinate implies that the equations of motion reduce to ordinary, but coupled, second-order differential equations of the kind we encounter in Hamiltonian systems. In the specific case of black hole solutions[1] in *massless* supergravity theories there are two known ways to describe these Hamiltonian systems. The first way, originally pioneered in [1] uses the timelike Killing vector to reduce the problem to a supergravity problem in three Euclidean dimensions. In three dimensions all vectors can be dualised to scalars, such that we have gravity coupled to some non-linear sigma model. It can be shown that the Einstein equations become trivial and that the scalar field equations of motion reduce to the equations of motion for a geodesic curve on the sigma model. The inverse of the radial coordinate r turns out to give an affine parametrisation of the geodesic. Hence the effective action can be written as ($\tau = 1/r$)

$$S_a = \int \mathrm{d}\tau \left(\tfrac{1}{2}\, g_{ij} \dot{\phi}^i \dot{\phi}^j \right), \tag{1}$$

where g_{ij} is the sigma model metric in three-dimensional supergravity. This metric is indefinite due to the reduction over a timelike direction [1–4].

The second method for constructing an effective action, which we name S_b, was developed in [5,6] and uncovered the well-known attractor mechanism for some extremal black holes. The construction of S_b works when we assume that the NUT charge vanishes and the stationarity turns into staticity. Then one can integrate out the vector fields in four

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[1]From now on we assume the black hole solutions to be stationary and spherically symmetric.

dimensions in terms of their electric and magnetic charges. The remaining equations of motion can then be derived from the following one-dimensional action

$$S_b = \int \mathrm{d}\tau \left(\dot{U}^2 + \tfrac{1}{2} G_{rs} \dot{\phi}^r \dot{\phi}^s - \mathrm{e}^{2U} V_{BH}(\phi) \right), \tag{2}$$

where U is the so-named black hole warp factor which describes the four-dimensional space-time metric (see below). The scalars ϕ^r are the scalars of the four-dimensional supergravity theory and span a submanifold (together with U) of the sigma model in three dimensions. The function $V_{BH}(\phi)$ is named the black hole effective potential and contains the electric and magnetic charges. We re-derive both effective actions in section 2 of this paper. One can think of the term $\mathrm{e}^{2U} V_{BH}$ as a scalar potential in two dimensions originating from a flux compactification over the two-sphere.

In this paper we specify to the case of "symmetric supergravity theories". By that we mean that the scalars span a symmetric coset space G_D/H_D, where by the subscript D we denote the dimension of the spacetime of the theory. Symmetric supergravity theories that are obtained from dimensionally reducing over time have an indefinite signature of the scalar manifold. This implies that H is not the maximal compact subgroup of G, but some non-compact subgroup.

It is in these circumstances that the geodesic approach seems beneficial since we have a larger symmetry group $G_3 \supset G_4$. This larger symmetry comes with the following concrete benefits:

- The geodesic equations of motion can be recast in a simple form in terms of the symmetric coset matrix $\mathcal{M}$. The solution to these equations is a simple matrix exponential $\mathcal{M}(\tau) = \mathcal{M}(0)\mathrm{e}^{Q\tau}$ [1].

- Solutions can be generated using the large symmetry group G_3 on simple "seed solution", see [1,3,7] (and [8] for a simple pedagogical example). The G_3-orbit structure gives the essential insight for the understanding of regularity and supersymmetry of the various solutions [1,3,9,10][2].

- The geodesic equations of motion also allow a Lax pair form [11–13], of a kind that allows a closed but iterative formula for the coset representative L. Because of the upper-triangular structure of the coset representative L in the Borel gauge this allows to peal of the solutions for the individual scalar fields ϕ^i, something that is more complicated for the matrix $\mathcal{M}$. A different, but equivalent Lax pair formalism can be found in [14].

- The Lax pair formalism has been used to prove the full Liouville integrability of the geodesic equations of motion. This means that for an n-dimensional symmetric space one can find n constants of motion that mutually Poisson commute [15]. This in turn

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[2]See also [4] for a general treatment of dimensionally-reduced black holes in supergravity theories that are not necessarily symmetric.

implies that the system can *always* be integrated in the sense of Hamilton–Jacobi [15]. This has been called the " first-order formalism " or the " fake supergravity " formalism in the supergravity literature, see [16–23] and references therein.

- Finally, we refer to [24, 25] for recent applications of the geodesic approach in understanding properties of black hole solutions, such as the defining harmonic functions, the attractor structure and a canonical way of obtaining non-extremal solutions from extremal ones. We refer to [26, 27] for an interesting connection with quantum information theory[3].

It is the aim of this paper to clarify how some of the benefits of the geodesic approach can also be made visible in the black hole effective potential action, especially the issue regarding the integrability. Outside of the black hole context this was investigated by mathematicians in the context of Toda molecule equations. In [28] (see also [29]) it was established that the Toda molecule equations, which are a Hamiltonian system with specific potential, could be viewed as having an underlying geodesic motion on an enlarged target space. Even more, it is conjectured in [28] *that all one-dimensional integrable systems have an underlying geodesic motion on a symmetric space.* We consider it to be very interesting that this is realised in the context of black holes in supergravity theories as the equivalence between the two approaches (geodesic and black hole potential) to construct the effective action. As we demonstrate in detail below, the connection between the two approaches generalises the link between Toda molecules and symmetric spaces in two ways: i) we find generalised Toda molecule equations from truncating the geodesic motion in a more general way then done in [28, 29] and ii) we point out that there exist different choices for the signature of the sigma model without ruining the integrability, which also defines a generalisation of Toda molecule equations.

This paper is organised as follows. In section 2 we rederive the two known effective actions for general theories of gravity coupled to massless scalars and Abelian gauge fields. In section 3 we specify to symmetric supergravity theories and first review the construction of the geodesic motion and its integrability. Then we point out in section 4 how this integrability implies, in a straightforward manner, the Liouville integrability of the action with potential. In section 5 we describe the link with Toda molecule equations and discuss the generalisation that black hole effective actions naturally provide. Finally in section 6 we end with a discussion and a conjecture about integrability of domain wall and cosmological solutions in gauged supergravity.

2 Effective actions

In this section we review the effective actions for spherically symmetric and stationary black hole solutions in theories with multiple Abelian gauge fields B^I, $I = 1, \ldots, n_V$ and

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[3] Much more work on uncovering the structure of supergravity solutions has been carried out in the recent literature using the geodesic approach and we refer the interested reader to the references in the cited papers for further reading.

scalar fields ϕ^r (without mass terms). This has been reviewed in many papers[4] but here we wish to present this again in order to establish our conventions and notation but also because the comparison between the effective actions is our main focus.

The action in four dimensions is given by[5]

$$S_4 = \int \left(\tfrac{1}{2} \star R_4 - \tfrac{1}{2} G_{rs} \star \mathrm{d}\phi^r \wedge \mathrm{d}\phi^s - \tfrac{1}{2} \mu_{IJ} \star G^I \wedge G^J + \tfrac{1}{2} \nu_{IJ} G^I \wedge G^J \right), \qquad (3)$$

where $G^I = \mathrm{d}B^I$, and G_{rs}, μ_{IJ}, ν_{IJ} are symmetric matrices that depend on the scalars ϕ; in particular, G and μ are required to be positive definite. We closely follow the appendix of [15], which itself is based on the original paper by Breitenlohner, Gibbons and Maison [1].

The Ansatz for black hole solutions can be written as

$$\mathrm{d}s_4^2 = -\mathrm{e}^{2U}(\mathrm{d}t + A_{KK})^2 + e^{-2U}\mathrm{d}s_3^2,$$
$$B^I = \tilde{B}^I + Z^I(\mathrm{d}t + A_{KK}), \qquad (4)$$

where $\tilde{B}^I$ and A_{KK} are vectors and U, Z^I are scalar fields in $d = 3 + 0$. When A_{KK} can not be redefined away $(\mathrm{d}A_{KK} \neq 0)$ the black hole has a non-zero NUT charge and is not static anymore but stationary. We have written the Ansatz in the same way as a Kaluza–Klein reduction over time. For a true Kaluza–Klein reduction we would require time to be periodic and this would manifest itself in the Kaluza–Klein tower being discrete instead of continuous. Since we restrict to the zero modes this difference is of no importance.

The Ansatz for the spatial (3-dimensional) part of the metric is

$$\mathrm{d}s_3^2 = \exp[4A(\tau)]\mathrm{d}\tau^2 + \exp[2A(\tau)]\mathrm{d}\Omega_2^2, \qquad (5)$$

where $\mathrm{d}\Omega_2^2$ is the metric on the unit 2-sphere. Consistency with the metric symmetries requires the scalar fields to depend on τ only, $\phi^r = \phi^r(\tau)$.

Let us now reduce the action over the timelike direction

$$S_3 = \int \Big(\tfrac{1}{2} \star R_3 - \star \mathrm{d}U \wedge \mathrm{d}U + \tfrac{1}{4}\mathrm{e}^{4U} \star F_{KK} \wedge F_{KK} - \tfrac{1}{2} G_{rs} \star \mathrm{d}\phi^r \wedge \mathrm{d}\phi^s$$
$$+ \tfrac{1}{2}\mu_{IJ}\mathrm{e}^{-2U} \star \mathrm{d}Z^I \wedge \mathrm{d}Z^J - \tfrac{1}{2}\mathrm{e}^{2U}\mu_{IJ} \star (\tilde{G}^I + Z^I F_{KK}) \wedge (\tilde{G}^J + Z^J F_{KK})$$
$$- \nu_{IJ}(\tilde{G}^I + Z^I F_{KK}) \wedge \mathrm{d}Z^J \Big), \qquad (6)$$

where $\tilde{G}^I = \mathrm{d}\tilde{B}^I$, $F_{KK} = \mathrm{d}A_{KK}$. As usual the vectors A_{KK} and $\tilde{B}^I$ can be dualised to scalars χ and Z_I by adding the following Lagrange multipliers to the action

$$S_3' = S_3 + \chi \mathrm{d}F_{KK} + Z_I \mathrm{d}\tilde{G}^I. \qquad (7)$$

[4]We refer to [30] for a construction of black hole effective actions in a more general setting.

[5]Our discussion trivially extends to any dimension. The existence of electric and magnetic point charges is however specific to four dimensions.

Varying the action S_3' with respect to F_{KK} and $\tilde{G}^I$ gives the equations of motion

$$\mathrm{d}Z_J = -\mathrm{e}^{2U} \star \mu_{IJ}(\tilde{G}^I + Z^I F_{KK}) - \nu_{IJ}\mathrm{d}Z^I\,, \tag{8}$$

$$\mathrm{d}\chi = \tfrac{1}{2}\mathrm{e}^{4U} \star F_{KK} + Z^I \mathrm{d}Z_I\,. \tag{9}$$

Dualisation of the action S_3 is obtained by eliminating F_{KK} and $\tilde{G}^I$ from the action S_3' using (8, 9). If we furthermore define, $2\chi \equiv a + Z^I Z_I$ we find

$$S_3 = \int \Big(\tfrac{1}{2} \star R_3 - \star \mathrm{d}U \wedge \mathrm{d}U - \tfrac{1}{2}G_{rs} \star \mathrm{d}\phi^r \wedge \mathrm{d}\phi^s + \tfrac{1}{2}\mathrm{e}^{-2U} \star \mathrm{d}\mathbf{Z}^T \wedge \mathcal{M}_4 \mathrm{d}\mathbf{Z}$$

$$- \tfrac{1}{4}\mathrm{e}^{-4U} \star (\mathrm{d}a + \mathbf{Z}^T \mathbb{C}\mathrm{d}\mathbf{Z}) \wedge (\mathrm{d}a + \mathbf{Z}^T \mathbb{C}\mathrm{d}\mathbf{Z}) \Big)\,, \tag{10}$$

where

$$\mathbf{Z} \equiv (Z^I, Z_I)\,, \qquad \mathbb{C} = \begin{pmatrix} 0 & -\mathbb{1} \\ +\mathbb{1} & 0 \end{pmatrix}\,, \qquad \mathcal{M}_4 = \begin{pmatrix} \mu + \nu\mu^{-1}\nu & \nu\mu^{-1} \\ \mu^{-1}\nu & \mu^{-1} \end{pmatrix}\,. \tag{11}$$

As explained in many papers (see e.g. [19]) the 3-dimensional Einstein equations for $A(\tau)$ decouple and are trivial to solve. The main task is to solve the equations for the scalars U, ϕ^r and $\mathbf{Z}$. The latter equations can be derived from the one-dimensional geodesic action, denoted S_a,

$$\boxed{S_a = \int \mathrm{d}\tau \left(\dot{U}^2 + \tfrac{1}{2}G_{rs}\dot{\phi}^r\dot{\phi}^s - \tfrac{1}{2}\mathrm{e}^{-2U}\dot{\mathbf{Z}}^T \mathcal{M}_4 \dot{\mathbf{Z}} + \tfrac{1}{4}\mathrm{e}^{-4U}(\dot{a} + \mathbf{Z}^T \mathbb{C}\dot{\mathbf{Z}})^2 \right)\,,} \tag{12}$$

where a dot denotes differentiation with respect to τ.

Let us now consider the NUT charge to be zero. It can be shown that this implies the consistent truncation

$$\dot{a} + \mathbf{Z}^T \mathbb{C}\dot{\mathbf{Z}} = 0\,. \tag{13}$$

Then one observes that the axions $\mathbf{Z}$ appear shift-symmetric in the action such that they can be integrated out in terms of the physical electric and magnetic charges

$$\mathbf{Q} = (m^I, e_I)\,, \tag{14}$$

as follows

$$\mathbf{Q} = \mathrm{e}^{-2U}\mathbb{C}\mathcal{M}_4 \dot{\mathbf{Z}}\,. \tag{15}$$

When we plug this back into the action we have to add an overall minus sign in front of the $\mathbf{Z}$-kinetic term to obtain an effective action that leads to the correct equations of motion. If we furthermore use that $\mathcal{M}_4$ is a symplectic matrix

$$\mathcal{M}_4^{-1} = \mathbb{C}\mathcal{M}_4\mathbb{C}^T\,, \tag{16}$$

we find

$$S_3 = \int \Big(\tfrac{1}{2} \star R_3 - \star \mathrm{d}U \wedge \mathrm{d}U - \tfrac{1}{2}G_{rs} \star \mathrm{d}\phi^r \wedge \mathrm{d}\phi^s - \tfrac{1}{2}\mathrm{e}^{2U}\mathbf{Q}^T \mathcal{M}_4 \mathbf{Q} \Big)\,. \tag{17}$$

The one-dimensional effective action S_b reads

$$\boxed{S_b = \int \mathrm{d}\tau \left(\dot{U}^2 + \tfrac{1}{2}G_{rs}\dot{\phi}^r\dot{\phi}^s + \tfrac{1}{2}\mathrm{e}^{2U}\mathbf{Q}^T \mathcal{M}_4 \mathbf{Q} \right)\,.} \tag{18}$$

3 Symmetric supergravity theories

As we mentioned in the introduction the definition of a symmetric supergravity theory is that the scalars span a symmetric coset space G_4/H_4 and that the coupling to the vectors is such that the dimensionally reduced theory again has a symmetric scalar manifold G_3/H_3 (after dualisation of the vectors). Let us briefly recall some basic facts about the geometry of symmetric spaces.

Consider a general group G and its associated Lie algebra $\mathbb{G}$. The symmetric space property is defined through an involution θ that respects the Lie bracket. This induces a natural grading of Lie algebra elements of $\mathbb{G}$ into "even" and "odd"

$$\text{even}: \theta(T) = +T\,, \qquad \text{odd}: \theta(T) = -T\,. \tag{19}$$

Because θ respects the Lie bracket one can show that the even generators form a subalgebra $\mathbb{H}$. The real group associated to the Lie algebra $\mathbb{H}$ is accordingly denoted H. The vector subspace of odd generators is denoted $\mathbb{K}$, such that we can write

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K}\,. \tag{20}$$

This is called the Cartan decomposition of the algebra and θ is accordingly called the Cartan involution. The properties of the involution imply, besides $\mathbb{H}$ being a subalgebra, that

$$[\mathbb{K}, \mathbb{K}] \subset \mathbb{H}\,, \qquad [\mathbb{K}, \mathbb{H}] \subset \mathbb{K}\,. \tag{21}$$

By letting θ act through the exponent we can naturally define the Cartan involution θ on the level of the group G. This also allows us to define the coset space G/H as spanned by the elements $x(g)$, with $g \in G$,

$$x(g) = g\theta(g)^{-1} \tag{22}$$

such that $\theta(x) = x^{-1}$. For some coordinate system y on G/H we can parametrize the coset elements as elements of G, in some representation, $L(y) \in G$, where we assume that the isometry group G acts from the right: $L \to gL$ and the local isotropy group from the left $L \to Lh$ [6]. This way the combination (22)

$$\mathcal{M} = L\theta(L^{-1}) \tag{23}$$

is a proper coset element that is invariant under H and transforms under G as

$$\mathcal{M} \to g\mathcal{M}\theta(g^{-1})\,. \tag{24}$$

We name $\mathcal{M}$ the "symmetric" coset matrix and L a coset representative.

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[6]There is some difference with the notation we used in previous papers. Often one uses the symbol $\mathbb{L}$ for the coset representative and the symbol L for the Lax operator. In this paper we choose L for the coset representative and $\mathcal{V}$ for the Lax operator.

There exists a natural choice of coordinates on symmetric spaces called horospherical coordinates. In the supergravity literature this is known under the name "Borel gauge". It is a popular coordinate frame for many reasons, one of them being that these coordinates are in a simple 1-1 correspondence with the 10-dimensional supergravity degrees of freedom in a canonical basis [31]. This is called a gauge because a coordinate frame can be seen as a choice for the local isotropy "compensator" that keeps the coset representative in a fixed basis $L = \exp(\phi^i T_i)$. The Borel gauge is then the gauge in which the T_i are the elements of the Borel subalgebra, which for the maximally non-compact real form of a simple complex algebra is the algebra formed by the positive step operators E_α and the Cartan generators H_i:

$$ L = \exp(\sum_\alpha \chi^\alpha E_\alpha) \exp(\tfrac{1}{2} \sum_a \Phi^a H_a) \,. \tag{25} $$

In this equation, we have denoted the positive roots by α and the simple roots by a. The simple roots a have been used to pick a basis of the Cartan subalgebra, given by the generators $H_a = 2a \cdot H/a^2$. The scalars Φ^a are referred to as "dilatons"[7]. and the χ^α as "axions". In the Riemannian case, where H is the maximal compact subgroup, the consistency of this gauge is a consequence of the Iwasawa decomposition, which implies that the horospherical coordinate system covers the whole manifold. In the non-Riemannian case, where H is non-compact, the Iwasawa decomposition doesn't hold everywhere, but it is still a good local description. Especially in the case of black holes the patch covered by the horospherical coordinates is the whole physical patch (see [11, 15] for a discussion on this).

The metric on the symmetric space is induced from the Cartan–Killing metric of the Lie algebra $\mathbb{G}$ restricted to the coset part $\mathbb{K}$. This defines the metric on the tangent space at the origin and using the isometries this defines the metric everywhere. This proceeds as follows. The left-invariant one-form $L^{-1}\mathrm{d}L$ is $\mathbb{G}$-valued and can therefore be split according to the Cartan decomposition

$$ L^{-1}\mathrm{d}L = \mathcal{V} + \mathcal{W} \,, \qquad \text{where} \qquad \mathcal{V} \in \mathbb{K} \,, \qquad \mathcal{W} \in \mathbb{H} \,. \tag{26} $$

In particular this means that

$$ \mathcal{V} = \tfrac{1}{2}\Big(L^{-1}\mathrm{d}L - \theta(L^{-1}\mathrm{d}L)\Big) \,, \qquad \mathcal{W} = \tfrac{1}{2}\Big(L^{-1}\mathrm{d}L + \theta(L^{-1}\mathrm{d}L)\Big) \,. \tag{27} $$

The metric on G/H is then defined as

$$ g_{ij}(\phi)\, \mathrm{d}\phi^i \otimes \mathrm{d}\phi^j = \alpha \mathrm{Tr}(\mathcal{V} \otimes \mathcal{V}) \,, \tag{28} $$

where α is a non-zero positive constant that can be chosen at will and determines the length-scale, or curvature, of the space. For a given supergravity theory, however, this number cannot be chosen at will and is fixed by supersymmetry.

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[7]This differs slightly from the supergravity literature where the dilatons are conjugate to the H_i that are mutually othogonal $Tr[H_i H_j] \sim \delta_{ij}$. Therefore the dilatons in our context are non-orthogonal linear combinations of the dilatons in the usual supergravity context.

The simplest way of understanding the geodesic equations of motion

$$\ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^j\dot{\phi}^k = 0\,, \tag{29}$$

is obtained by first rewriting the coset metric in terms of the matrix one-form $\mathcal{M}^{-1}\mathrm{d}\mathcal{M}$

$$\mathcal{M}^{-1}\mathrm{d}\mathcal{M} = 2\theta(L)\mathcal{V}\theta(L^{-1})\,. \tag{30}$$

We then find

$$\mathrm{Tr}(\mathcal{V}\otimes\mathcal{V}) = -\tfrac{1}{4}\mathrm{Tr}(\mathrm{d}\mathcal{M}\otimes\mathrm{d}\mathcal{M}^{-1})\,. \tag{31}$$

The variation of the latter form of the action gives the following equations of motion

$$\ddot{\mathcal{M}} - \dot{\mathcal{M}}\mathcal{M}\dot{\mathcal{M}} = 0 \qquad \Leftrightarrow \qquad \boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathcal{M}^{-1}\dot{\mathcal{M}}\right) = 0\,.} \tag{32}$$

This equation can be solved trivially

$$\mathcal{M}(\tau) = \mathcal{M}(0)\mathrm{e}^{Q\tau}\,, \tag{33}$$

where the matrix Q and $\mathcal{M}(0)$ are constrained by the condition

$$\theta(\mathcal{M}(\tau)) = \mathcal{M}^{-1}(\tau)\,, \tag{34}$$

which gives the right number of integration constants (namely $2n$ for an n-dimensional symmetric space).

As explained in [15] one can, in principal, extract explicit expressions for the various coordinates $\phi^i(\tau)$ from $\mathcal{M}$ but this is a laborious and unpractical task. Furthermore we would like to have some insight on the integrability of the geodesic equations on the level of the coordinates (scalar fields). For this purpose one introduces the Lax pair form of the equations of motion. This form is obtained by differentiation of equation (30), which after some algebra gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\mathcal{M}^{-1}\dot{\mathcal{M}}] = 2\theta(L)\left(\dot{V} + [\mathcal{W},\mathcal{V}]\right)\theta(L^{-1})\,. \tag{35}$$

Hence we find the Lax pair form of the geodesic equations of motion

$$\boxed{\dot{\mathcal{V}} = [\mathcal{V},\mathcal{W}]\,.} \tag{36}$$

From equation (30) we find that the constant matrix Q is related to the Lax matrix $\mathcal{V}$ as follows

$$Q = 2\theta(L)\mathcal{V}\theta(L^{-1})\,. \tag{37}$$

4 Integrability

Recently the Liouville integrability of the geodesic equations of motion (29) was proven [15], based on earlier work by Fré and Sorin [12]. Liouville integrability is the statement that for a Hamiltonian system of n degrees of freedom we can find n constants of motion f_i that mutually Poisson commute

$$\{f_i, f_j\} = 0 \,, \tag{38}$$

where the Poisson bracket and the generalised momenta are defined in the usual way

$$\{f, g\} = \sum_i \frac{\partial f}{\partial \phi^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial \phi^i} \frac{\partial f}{\partial p_i} \,, \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} \,. \tag{39}$$

For the case at hand the generalised momenta are simply $p_i = g_{ij}(\phi)\dot{\phi}^j$.

Liouville integrability is equivalent to the fact that the Hamilton-Jacobi problem has a complete solution. This means that we can find a globally defined, and generically multi-valued, generating function $W(\phi_i, f_i)$ that describes the following canonical transformation

$$(\phi^i, p_i) \quad \xrightarrow{W} \quad (\phi^i, f_i) \,, \tag{40}$$

such that the new generalised momenta are the conserved quantities. It then follows from the theory of generating functions that we have the following *first-order* equations

$$\dot{\phi}^i = g^{ij}\partial_i W \,, \tag{41}$$

where W can effectively be seen as a function of the ϕ^i since the f are constants. It was observed and emphasized in [20] that this is what underlies the first-order, or fake supergravity, formalism for black holes (see [32] for earlier remarks for the case of domain walls and cosmologies).

Consider a vanishing NUT charge such that the $\mathbf{Z}$ become cyclic, which implies that their generalised momenta, denoted $\mathbf{P}$, are constants of motion. Then the function $W(\phi^i, f_i)$ can be written as [20]

$$W(\phi^i) = W_4(\phi^r, U) + \mathbf{Z} \cdot \mathbf{P} \,. \tag{42}$$

Where the function $W_4(\phi^r, U)$ does not depend on the $\mathbf{Z}$ and describes the generating function for the system with black hole effective potential. Hence the cyclic property of the $\mathbf{Z}$ coordinates guarantees that the integrability of the geodesic motion projects to the integrability of the black hole effective potential system (18). As we explain below, projected integral motions are generically *not* integrable and it is a non-trivial result when they are.

The integrability of (18) can also be understood from the point of view of Poisson commuting constants of motion. For that we use some properties of the constants of motion for the geodesics that were constructed in [12, 15]. For our purposes it suffices to mention that the constants are divided into three classes, which we denote as

- $\mathcal{H}_m(\mathcal{V})$: the polynomial Lax constants of motion.

- $\mathcal{H}_r(\mathcal{V})$: the rational Lax constants of motion.

- $\mathcal{H}_A(Q)$: the Q-constants of motion .

The constants $\mathcal{H}_m(\mathcal{V}), \mathcal{H}_r(\mathcal{V})$ are functions of the components of the Lax operator *along the Borel algebra*. The constants $\mathcal{H}_A(Q)$ are obtained by replacing the Lax components along the Borel algebra by the components of Q along the Borel algebra for those $\mathcal{H}(\mathcal{V})$ that are not Casimirs [15]. As the names suggest the polynomial constants are polynomial functions of $\mathcal{V}$. The polynomial Lax constants are combinations of the $\text{Tr}(\mathcal{V}^N)$ functions, which were known to Poisson commute [33]. One of those, namely $\text{Tr}\mathcal{V}^2$ corresponds to the Hamiltonian.

A crucial property of the constants of motion is that *the number of Q-constants $\mathcal{H}_A(Q)$ is equal to the number of $\mathbf{Z}-$ axions that are integrated out.* In fact, when the Taub-NUT charge vanishes one can replace the Q-constants with the electric and magnetic charges $\mathbf{Q}$ since they are the same in number and also mutually commute amongst themselves and with the constants built from the Lax operator. It is now easy to show that the remaining constants $\mathcal{H}(\mathcal{V})$ are commuting constants of motion for the smaller system. To prove this it is sufficient to observe that the Lax operator $\mathcal{V}$ only contains the axions related to the vectors in a total derivative i.e. as $\dot{\mathbf{Z}}$. Hence,

$$\mathcal{H}(\mathcal{V}) = \mathcal{H}(\mathcal{V})[U, \phi^r; p_U, p_r, \mathbf{P}] \,. \tag{43}$$

This can be seen as a function of the phase space variables of the truncated system $(U, \phi^r; p_U, p_r)$ since the $\mathbf{P}$ are constants. Now it is straightforward to check that the Poisson bracket relations on the larger phase space imply that the Poisson bracket on the smaller phase space also vanish.

5   Toda molecule equations

The procedure of integrating out the $\mathbf{Z}$-axions (and the a axion) generalizes the connection between geodesics on symmetric spaces and Toda molecules [28, 29]. The connection between Toda molecules and geodesics is established by integrating out specific directions in order to obtain a non-geodesic motion on the truncated space. Let us here apply this specifically to the case of coset spaces relevant to describing black holes in $D = 4$ and stick to the case of maximally split cosets. These results also extend to the case of non-maximally split cosets and sigma models not specific to $D = 4$ black holes.

Consider the sigma model in $D = 4$, denoted G_4/H_4, where H_4 is the maximal compact subgroup of G_4. Dimensional reduction then gives a coset G_3/H_3. $\mathbb{G}_3$ contains the two disjoint subalgebras, A_1 and $\mathbb{G}_4$. The A_1 subalgebra is built from the extra Cartan generator H_{β_0} that is generated from the reduction from four to three dimensions and β_0 denotes the corresponding simple root. The adjoint of $\mathbb{G}_3$ decomposes as follows under $A_1 \times \mathbb{G}_4$

$$\mathbb{G}_3 \to (1, \mathbb{G}_4) \oplus (3, 1) \oplus (2, \mathbf{R}) \,, \tag{44}$$

where $\mathbf{R}$ is some subspace whose representation we do not specify any further.

We can divide the positive roots α in three classes, depending on their behavior under A_1, which can be read off from the weight the generator has w.r.t. H_{β_0}

1. E_{β_0} has weight two under H_{β_0}.

2. the positive roots E_γ, that constitute the representation $\mathbf{R}$, have weight one under H_{β_0}.

3. the positive roots in $\mathbb{G}_4$, denoted E_β, are orthogonal to β_0 (and thus have no weight under H_{β_0}).

Because of the grading structure, one can show via the Jacobi identity, that the positive roots γ that build the representation $\mathbf{R}$ obey

$$[E_{\gamma_1}, E_{\gamma_2}] = \mathbb{C}_{\gamma_1 \gamma_2} E_{\beta_0} \,. \tag{45}$$

In the context of dimensional reductions to three dimensions one should think of the roots and the corresponding axions χ as follows. The χ^{β_0} corresponds to the Hodge-dual of the KK vector that arises in going from four to three dimensions. The χ^γ are the axions $\mathbf{Z}$ that come from the vector potentials in three dimensions and the χ^β are the axionic scalars already present in four dimensions. Their grading structure can be read off from the power of e^{-2U} that is in front of their kinetic terms (12), since the scalar U is the dilaton that multiplies H_{β_0} in the coset representative.

This grading structure allows us to define the integrations/truncations we are looking for: 1) The truncation of χ^{β_0}, 2) The truncation of $\chi^{\beta_0}, \chi^\gamma$ and 3) The truncation of all axions χ^α.

The direction related to the root β_0 can be integrated out, as it appears shift symmetric. This means that

$$\mathrm{e}^{-4U}(\dot{a} + \mathbf{Z}^T \mathbb{C} \dot{\mathbf{Z}}) = n \,, \tag{46}$$

where n is a constant proportional to the Taub-NUT charge. After this integration the effective action is given by

$$S = \int d\tau \left(\dot{U}^2 + \tfrac{1}{2} G_{rs} \dot{\phi}^r \dot{\phi}^s + \tfrac{1}{2} e^{2U} \mathbf{Q}^T \mathcal{M} \mathbf{Q} + \tfrac{1}{4} e^{4U} n^2 \right) , \tag{47}$$

where, in this case, the physical electric and magnetic charges take the form

$$\mathbf{Q} = \mathbb{C}\mathbf{P} - \frac{1}{2}\mathbf{Z}n \,. \tag{48}$$

where the momentum $\mathbf{P}$ contains $\dot{\mathbf{Z}}$ and is defined as the momentum conjugate to $\mathbf{Z}$. This effective action can be seen as an effective action of the variables $U, \phi^r, \mathbf{Z}$ or of the smaller set of variables where the $\mathbf{Z}$ are integrated out because the $\mathbf{Q}$ in (47) are constants of motion. In case we include the variables $\mathbf{Z}$, integrability is again inherited from the geodesic as follows. The Taub-Nut charge n coincides with one of the Hamiltonians of the

geodesic motion [15], and furthermore the remaining Hamiltonians do not depend on a and only depend on $\dot{a}$ via the TN charge n. Therefore the remaining Hamiltonians are the required Hamiltonians necessary for Liouville integrability. We expect the integrability of the geodesic to also project down to the integrability of this effective action regarded as a function of U and ϕ^r only. However, this has not yet been shown because some of the Hamiltonians depend explicitly on $\mathbf{Z}$. This is currently under investigation.

From the effective action (47) we notice that only when $n = 0$ the $\mathbf{Z}$ axions appear shift symmetric. This is because the Heisenberg algebra (45) becomes Abelian. The shift symmetry implies that we can integrate out the χ^γ for any value of their velocities. For the black hole case this leads to the action with the black hole effective potential (18). As we explained in the previous section the Liouville integrability of the geodesic motion carries over to this truncated system.

Finally, one can integrate out all the axions χ^α but this is not possible for generic velocities. It is explained in [29] when this is possible and we recall this in what follows since this is how the link with conventional Toda molecules appears. We also immediately extend the analysis by allowing an isotropy group H which is not necessarily compact, as for the case at hand. We consider a Cartan involution that is defined as follows

$$\theta(H_i) = -H_i\,, \qquad \theta(E_\alpha) = -\epsilon_\alpha E_{-\alpha}\,, \tag{49}$$

where ϵ_α is a specific sign. For the case of dimensional reductions from four to three we have the conventional choice, $\epsilon_\alpha = +1$, when we reduce over a spacelike direction. This leads to a Riemannian coset for which H is the maximal compact subgroup of G. For dimensional reductions over time we have $\epsilon_\alpha = (-1)^{\alpha \cdot \beta}$, such that for the roots γ we have the unconventional sign. We will however keep the sign ϵ_α arbitrary, since the following argument works for more general Cartan involutions.

For the purpose of integrating out we have to consider the (pull-back of the) one-forms $L^{-1}\mathrm{d}L$ and $\mathcal{M}^{-1}\mathrm{d}\mathcal{M}$, built from the coset representative in the Borel gauge. We find [8]

$$\Omega = L^{-1}\dot{L} = \tfrac{1}{2}\sum_a \dot{\Phi}^a H_a + \sum_\alpha V_\alpha \mathrm{e}^{-\frac{1}{2}\sum_a K_{\alpha a}\Phi^a} E_\alpha\,, \tag{50}$$

where V_α are functions of the χ and $\dot{\chi}$ defined via

$$\sum_\alpha V_\alpha E_\alpha = \mathrm{e}^{-\sum_\alpha \chi^\alpha E_\alpha}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\mathrm{e}^{\sum_\alpha \chi^\alpha E_\alpha}\,, \tag{51}$$

and K is defined as

$$K_{\alpha a} = 2\frac{\alpha \cdot a}{a^2}\,. \tag{52}$$

The expression for $\mathcal{M}^{-1}\dot{\mathcal{M}}$ is more involved and we write it as

$$\mathcal{M}^{-1}\dot{\mathcal{M}} = \mathrm{e}^{-\sum_\alpha \epsilon_\alpha \chi^\alpha E_{-\alpha}}\left(\sum_\alpha V_\alpha \mathrm{e}^{-\sum_a K_{\alpha a}\Phi^a} E_\alpha + \sum_a \dot{\Phi}^a H_a + \sum_\alpha T_\alpha E_{-\alpha}\right)\mathrm{e}^{\sum_\alpha \epsilon_\alpha \chi^\alpha E_{-\alpha}}\,, \tag{53}$$

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[8]One should use the Hadamard lemma : $\mathrm{e}^T S \mathrm{e}^{-T} = \mathrm{e}^{adjT} S$.

where the T_α are functions of the axions and their derivatives, defined as follows

$$\sum_\alpha T_\alpha E_{-\alpha} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\mathrm{e}^{-\sum_\alpha \epsilon_\alpha \chi^\alpha E_{-\alpha}}\right) \mathrm{e}^{\sum_\alpha \epsilon_\alpha \chi^\alpha E_{-\alpha}}. \tag{54}$$

If we use the standard Cartan-Weyl commutation relations it is easy to see that the coefficient in front of the highest positive root E_α-term in (53) is simply $V_\alpha \mathrm{e}^{-\sum_a K_{\alpha a}\Phi^a}$. Since $\mathcal{M}^{-1}\dot{\mathcal{M}}$ is constant, so is that term. If we therefore take it to vanish initially, it vanishes at all times. Clearly we can subsequently repeat this for the new highest root and so on such that we end up with the simple roots only. Then one can similarly infer that the coefficients in front of the E_a are simply $V_a \mathrm{e}^{-\sum_b K_{ab}\Phi^b}$. Hence they are constant and we denote these constants by C_a. This can be seen as a way to integrate out the dependence on the axions and their derivatives since the only place where they appeared, after all the truncations made sofar, is in V_a. But we can write V_a in terms of the dilatons

$$V_a = C_a \mathrm{e}^{\sum_b K_{ab}\Phi^b}. \tag{55}$$

If we plug this in the definition of $\mathcal{V}$ and $\mathcal{W}$, we find (after the truncations we made)

$$\mathcal{V} = \tfrac{1}{2}\sum_a \dot{\Phi}^a H_a + \tfrac{1}{2}\sum_a C_a \mathrm{e}^{\frac{1}{2}\sum_b K_{ab}\Phi^b}(E_a + \epsilon_a E_{-a}), \tag{56}$$

$$\mathcal{W} = \tfrac{1}{2}\sum_a C_a \mathrm{e}^{\frac{1}{2}\sum_b K_{ab}\Phi^b}(E_a - \epsilon_a E_{-a}). \tag{57}$$

The Lax equation (36) then leads to

$$\boxed{\ddot{\phi}^a = -\epsilon_a C_a^2 \mathrm{e}^{\frac{1}{2}\sum_b K_{ab}\Phi^b}.} \tag{58}$$

Either by choosing initial conditions or by shifting the Φ^a we can take $C_a^2 = 1$ and this then defines the Toda molecule equations, when $\epsilon_a = 1$, i.e., when the isotropy group is compact. When not all the ϵ_a are equal to 1 we find a generalization[9] of the Toda molecule equations which still inherits the integrability of the underlying geodesic motion, as can easily be shown using the same arguments of [29, 33]. These arguments are based on the existence of an operator $\mathbb{P}$ that obeys a set of relations, called the fundamental Poisson relations. Essentially, the operator $\mathbb{P}$ allows one to rewrite the Poisson bracket in terms of a Lie algebra commutator. The existence of the operator $\mathbb{P}$ makes it manifest that the quantities $\mathrm{Tr}\mathcal{V}^N$ are in involution

$$\{\mathrm{Tr}\mathcal{V}^N, \mathrm{Tr}\mathcal{V}^M\} = 0. \tag{59}$$

As the number of independent quantities $\mathrm{Tr}\mathcal{V}^N$ is given by the rank of the Lie algebra, this establishes Liouville integrability of the Toda molecule equations. The operator $\mathbb{P}$ constructed in [29, 33] has a uniform structure, that depends on the root system of the underlying algebra $\mathbb{G}_3$. We have checked explicitly that the same operator $\mathbb{P}$ also establishes the Liouville integrability of the ϵ-generalised Toda molecule equations, as suggested in [29].

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[9]A similar generalisation appears in [34], where also the relation with geodesic curves was used.

An example

Consider the Einstein–Maxwell–dilaton action

$$S = \int \sqrt{|g|}\left(\tfrac{1}{2}\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{\sqrt{6}\phi}F^2\right). \tag{60}$$

The action (60) is just the circle reduction of pure gravity in $d = 5$. The reduction over time leads to the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$-coset. The black hole solutions of this theory have first been considered in [35]. After truncation of the Taub-NUT direction the SL(3)-sigma model becomes

$$S_a = \int \mathrm{d}\tau \left\{\dot{U}^2 + \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}e^{-2U+\sqrt{6}\phi}(\dot{Z}^1)^2 - \tfrac{1}{2}e^{-2U-\sqrt{6}\phi}(\dot{Z}^2)^2\right\}, \tag{61}$$

If we subsequently integrate out the $\mathbf{Z}$ axions we find the action

$$S_a = \int \mathrm{d}\tau \left\{\dot{U}^2 + \tfrac{1}{2}\dot{\phi}^2 + e^{2U}V(\mathbf{Q},\phi)\right\}, \tag{62}$$

where the positive definite potential $V(\mathbf{Q},\phi)$ has the following form

$$V = \frac{1}{2}\mathbf{Q}^T\mathcal{M}_4\mathbf{Q} = \frac{1}{2}\mathbf{Q}^T\begin{pmatrix} e^{\sqrt{6}\phi} & 0 \\ 0 & e^{-\sqrt{6}\phi} \end{pmatrix}\mathbf{Q}, \qquad \mathbf{Q} = (m, e), \tag{63}$$

with

$$\mathbf{P} = \left(-e^{-2U+\sqrt{6}\phi}(\dot{Z}^1), -e^{-2U-\sqrt{6}\phi}(\dot{Z}^2)\right), \qquad \mathbf{Q} = \mathbb{C}\mathbf{P}. \tag{64}$$

Let us now show how the generalized Toda molecule equation (58) reproduces the equations of motion following from (62), namely,

$$\ddot{U} = e^{2U}V, \qquad \ddot{\phi} = e^{2U}\frac{\partial V}{\partial\phi}. \tag{65}$$

The only thing we need is the Cartan matrix for SL(3)

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{66}$$

Thus, the Toda molecule equation (58) can be written out to yield

$$\ddot{\Phi}^1 = C_1^2 e^{\frac{1}{2}(2\Phi^1 - \Phi^2)}, \tag{67}$$

$$\ddot{\Phi}^2 = C_2^2 e^{\frac{1}{2}(-\Phi^1 + 2\Phi^2)}. \tag{68}$$

One can go from equations (67)-(68) to equations (65) by simply performing the following field redefinitions

$$\Phi_1 = 4\left(U + \frac{1}{\sqrt{6}}\phi\right), \tag{69}$$

$$\Phi_2 = 4\left(U - \frac{1}{\sqrt{6}}\phi\right), \tag{70}$$

together with the identification $C_1 = 2m$, $C_2 = 2e$.

Few other examples exist for which it was known that the black hole equations of motion reduce to a Toda system, see e.g. [36, 37].

6 Discussion

The fact that solutions to the system

$$\mathcal{L} = \tfrac{1}{2} g_{rs} \dot{\phi}^r \dot{\phi}^s - V(\phi^r) \tag{71}$$

can sometimes be seen as a subset of solutions to a free system of *more* degrees of freedom[10]

$$\mathcal{L} = \tfrac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^i \tag{72}$$

finds a concrete realisation for black hole equations of motion in supergravity. Note that (72) is not unique since there can exist multiple extensions of the configuration space such that a truncation of geodesics on the larger space coincide with the equations of motion coming from (71). The reformulation of (71) as (72) is useful when (72) exhibits more symmetries. This is the case for black hole solutions in symmetric supergravity theories and the reformulation (72) establishes the full integrability of (72) and (71).

The equivalence between (72) and (71) is more general than black holes in supergravity, see for instance [38] in the context of cosmological solutions. In fact it applies to general "brane" solutions[11]. This can easily be understood due to the fact that there are two ways to describe the effective action for brane solutions [39]. Either one reduces the brane over its flat worldvolume and then the effective action is a geodesic motion [3] or one reduces the brane over that part of the transversal space that does not include the "radial direction", such as a sphere for spherical brane solutions. This maps the brane solution to a domain wall solution of a gauged supergravity in a lower dimension. The latter effective action is of the kind (71). As explained in [39] this is equally valid for time-dependent brane solutions, such as S-branes, which get mapped to cosmological solutions in the lower-dimensional theory, which are again described by an action of the form (71). This gives a higher-dimensional interpretation to the observations made in [38].

The above considerations lead us to conjecture that domain wall (and cosmology) effective actions in gauged maximal supergravity might be integrable because of a hidden geodesic motion on a larger target space, maybe in some Kac-Moody sigma model related to the E_{10} or E_{11} algebra. This picture comes about when one assumes that all domain walls (and cosmologies) in maximal supergravity have some kind of brane origin in 10 or 11 dimensions. Given the fact that the standard branes (and S-branes) in 10 and 11 dimensions can be found as solutions to the E_{10} or E_{11}-sigma model [40,41] it makes perfect sense to conjecture the hidden geodesic origin of all domain walls (or cosmologies) of maximal gauged supergravity.

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[10]Since the configuration space of the geodesic has more dimensions this is different from the Maupertuis–Jacobi principle.

[11]In this language we regard a FRW cosmology as a S-brane solution with one transversal direction being time. Just like a stationary domain wall where the transversal direction is spacelike.

Acknowledgements

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