Revealed preference tests of collectively rational consumption behavior: formulations and algorithms*

Talla Nobibon F, Cherchye L, Crama Y, Demuynck T, De Rock B, Spieksma F.

KBI_1317

Revealed preference tests of collectively rational consumption behavior: formulations and algorithms*

Fabrice Talla Nobibon†, Laurens Cherchye‡, Yves Crama§, Thomas Demuynck¶, Bram De Rock‖, Frits C.R. Spieksma**

September 1, 2013

Abstract

This paper focuses on revealed preference tests of the collective model of household consumption. We start by showing that the decision problems corresponding to testing collective rationality are NP-complete. This makes the application of these tests problematic for (increasingly available) large(r) scale data sets. We then present two approaches to overcome this negative result. First, we introduce exact algorithms based on mixed-integer programming (MIP) formulations of the collective rationality tests, which can be usefully applied to medium sized data sets. Next, we propose simulated annealing heuristics, which allow for efficient testing of the collective model in the case of large data sets. We illustrate our methods by a number of computational experiments based on Dutch labor supply data.

Keywords: revealed preference axioms; rationality; mixed-integer programming; global optimization; simulated annealing.

1. Introduction

The word 'economics' stems from the Greek 'oikos nomein', which is literally translated as 'running the household'. Households form the cornerstone of our society and, as a result, household consumption decisions drive a huge part of our economy. It is by now well established that the unitary model, which assumes that households behave as single decision makers, is not adequate to describe the behavior of households with multiple members. Therefore, as a more useful alternative, Chiappori (1988, 1992) and Apps and Rees (1988) suggested the collective model of household consumption. This model explicitly recognizes that households consist of multiple individuals (or decision makers) with their own (rational) preferences. It assumes that these individuals engage into intrahousehold bargaining processes that obtain Pareto efficient intrahousehold allocations.

*The project leading to these results has been partially funded by the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy Office, Grant P7/36.

†Senior Operations Research Analyst at Federal Express (FedEx) Belgium. Fabrice Talla Nobibon gratefully acknowledges the Fund for Scientific Research - Flanders (FWO-Vlaanderen) for his postdoctoral fellowship.

‡Center for Economic Studies, Campus Kortrijk, Faculty of Economics and Business, KU Leuven, Belgium. Laurens Cherchye gratefully acknowledges the Research Fund KULeuven for his grant STR1/08/004.

§QuantOM, HEC-Management School, University of Liège, Belgium

¶Center for Economic Studies, Campus Kortrijk, Faculty of Economics and Business, KU Leuven, Belgium. Thomas Demuynck gratefully acknowledges the Fund for Scientific Research - Flanders (FWO-Vlaanderen) for his postdoctoral fellowship.

‖ECARES and ECORE, Université Libre de Bruxelles, Belgium. Bram De Rock gratefully acknowledges the European Research Council (ERC) for his Starting Grant.

**ORSTAT, Faculty of Economics and Business, KU Leuven, Belgium. E-mail: frits.spieksma@kuleuven.be

It is well-documented that this collective model provides a better empirical fit of multi-individual household behavior than the unitary model. See, for example, Fortin and Lacroix (1997); Browning and Chiappori (1998); Cherchye and Vermeulen (2008); Cherchye, De Rock, Sabbe, and Vermeulen (2008) and Cherchye, De Rock, and Vermeulen (2009).

To verify the empirical adequacy of a particular consumption model, it is important to come up with reliable tests that can be applied to household consumption data. These tests check whether the observed household behavior is consistent with the model predictions; such consistent behavior is then commonly referred to as "rational" in terms of the model subject to testing. Preferably, rationality tests avoid all types of ad-hoc choices. From this perspective, the recently developed revealed preference tests of the collective model are particularly appealing (see Cherchye, De Rock, and Vermeulen (2007, 2010, 2011a)). These revealed preference tests of the collective model build on early work of Samuelson (1938, 1948); Afriat (1967); Diewert (1973) and Varian (1982), who focused on revealed preference tests of the unitary model. Rather than describing a model's testable restrictions on observed household behavior in terms of derivatives of certain unobservable functions (e.g., symmetry of the cross-derivatives of the consumer's cost function), the revealed preference approach defines testable implications in terms of finite systems of inequalities that only involve the household consumption choices that are actually observed (and summarized in terms of prices and quantities). As such, the approach effectively minimizes the risk of specification error. It avoids that the collective model is rejected simply because of a wrongly specified functional structure (rather than a bad empirical fit of the model per se). Interestingly, revealed preference tests not only allow for deciding which model is the most appropriate one to describe the observed household decisions. They can also be used to subsequently recover preferences of individual household members, and to predict household decisions in new situations. Given the very nature of revealed preference tests, such recovery and counterfactual analyses will not be contaminated by ad-hoc functional choices.

The starting result of the current paper is that the revealed preference tests of the collective model are NP-complete. Importantly, this contrasts sharply with existing results for the unitary model, for which polynomial time testing algorithms do exist (see, for example, Varian (1982); Chung-Piaw and Vohra (2003) and Talla Nobibon, Smeulders, and Spieksma (2012b)). This result fits into the emerging literature that utilizes insights from computational complexity theory in the study of economic-theoretical decision problems. We mention here Gilboa and Zemel (1989); Chu and Halpern (2001); Cechlarova and Hajdukova (2002); Fang, Zhu, Cai, and Deng (2002); Woeginger (2003); Baron, Durieu, Haller, and Solal (2004); Baron, Durieu, Haller, Savani, and Solal (2008); Brandt and Fisher (2008); Conitzer and Sandholm (2008); Kalyanaraman and Umans (2008); Procaccia and Rosenschein (2008); Galambos (2009); Hudry (2009); Brandt, Fisher, Harrenstein, and Mair (2010); Deb (2011); Apesteguia and Ballester (2010); Talla Nobibon and Spieksma (2010); Cherchye, Demuynck, and De Rock (2011b) and Demuynck (2013). Essentially, this trend signals that testing the collective model tests can become very difficult in the case of large data sets. Given this first, negative finding, we subsequently propose two practical approaches to apply revealed preference tests of the collective model to large(r) data sets. First, we develop exact algorithms based on a mixed-integer programming (MIP) formulation of the revealed preference tests. Cherchye, De Rock, and Vermeulen (2011a) already introduced similar MIP formulations, and showed that these formulations are well-suited for meaningfully testing the collective model in practice. As we will discuss below, our new MIP algorithms are - from a computational viewpoint - substantially more attractive than the algorithms originally proposed in Cherchye, De Rock, and Vermeulen (2011a). Specifically, we show that these exact MIP algorithms can be very useful for medium sized data sets (i.e., datasets consisting of up to 100-150 observations). However, given their mixed integer programming nature, they become less useful for analyzing large scale data sets. Therefore, as a second approach, we propose simulated annealing heuristics that solve a global optimization formulation of the collective rationality tests.

Our main results are particularly relevant when testing the collective consumption model on large(r) scale data sets. Importantly, such large data sets are increasingly available, and the use of revealed preference tests to analyze these data has recently been advocated in the literature. A first example concerns large scale scanner data sets, which contain information on (many) household-level purchases collected at checkout scanners in supermarkets. Echenique, Lee, and Shum (2010, 2011) convincingly argue that scanner data are obvious candidates for using revealed preference insights to better understand household behavior. A second prime example concerns cross-sectional data gathered for a large number of households (e.g. through national budget surveys). Again, the use of revealed preference tests to such data allows for meaningfully assessing the empirical validity of the collective model (see, for example, Cherchye and Vermeulen (2008)). In this study, we will illustrate this last application for a large set of Dutch labor supply data. By using computational experiments, we will demonstrate the practical usefulness of both our newly proposed MIP methods and our specially tailored simulating annealing heuristic.

The rest of our paper unfolds as follows. Section 2 sets the stage by presenting the revealed preference tests that we focus on. Section 3 establishes our NP-completeness result. Section 4 introduces our MIP formulation. Section 5 shows the global optimization formulation, which will then form the basis for the simulated annealing heuristics that we present in Section 6. Section 7 contains our computational experiments. Section 8, finally, concludes.

2. Setting the stage

We consider households with two decision makers acting in an economy with m commodities or goods. We assume that these goods can only be consumed privately without externalities, as opposed to public goods. (However, the algorithms proposed in this paper can be modified to test rationality of collective households consuming public goods.) *Private* consumption of a good means that its consumption by one household member affects the supply available for the other member; as an example, drinking water can only be consumed privately. Consumption *externalities* refer to the fact that one member gets utility from another member's consumption; for an illustration, consider a wife enjoying her husband's nice clothes. *Public* consumption of a good means that consumption of that good by one member does not affect the supply available for another member, and no one can be excluded from consuming it; as an example, a movie watched by both members of the household is a public good.

We suppose that we have observed n household consumption quantity bundles $q_t := (q_{t,1}, \ldots, q_{t,m}) \in \mathbb{R}^m_+$ (non-negative) with corresponding positive prices $p_t := (p_{t,1}, \ldots, p_{t,m}) \in \mathbb{R}^m_{++}$ $(t = 1, \ldots, n)$. The component $q_{t,i}$ (respectively $p_{t,i}$), for $i = 1, \ldots, m$, corresponds to the quantity of good i bought by the household (respectively, the unit price of good i) at the time t of observation. Note that the scalar product $p'q$ represents the total price of bundle $q \in \mathbb{R}^m_+$ at the prices $p \in \mathbb{R}^m_{++}$. We denote the set of observations by $S := \{(p_t, q_t) : t \in N\}$, where $N := \{1, \ldots, n\}$ and we refer to S as the *data set*. For ease of exposition, throughout this paper, we use $t \in N$ to refer to the observation (p_t, q_t).

For a given observation $t \in N$, a *feasible personalized quantity* vector $\left(x_t^1, x_t^2\right)$ is one of the infinitely many feasible split ups of the observed quantity vector q_t into a pair of non-negative vectors x_t^1, $x_t^2 \in \mathbb{R}^m_+$ such that $q_t = x_t^1 + x_t^2$. The true split up of q_t, which represents the quantity of each good effectively consumed by each member in the household, is unobserved. Given the personalized quantity vectors, we define the *personalized consumption data sets* by $S_1 := \left\{\left(p_t, x_t^1\right) : t \in N\right\}$ for the first member and $S_2 := \left\{\left(p_t, x_t^2\right) : t \in N\right\}$ for the second member of the household. The following terminology is well-established, see Varian (1982, 2006).

Definition 1. *Member $\ell \in \{1,2\}$ reveals that he or she* directly prefers *bundle x_s^ℓ over bundle x_t^ℓ if and only if $p_s' x_s^\ell \geq p_s' x_t^\ell$.*

In words, member ℓ reveals that he or she directly prefers bundle x_s^ℓ over bundle x_t^ℓ if ℓ has chosen bundle x_s^ℓ while bundle x_t^ℓ was affordable and could have been chosen. For ease of exposition, when this happens, we simply say that member ℓ directly prefers s over t. Considering the transitive closure of the direct preference relation leads to the next definition.

Definition 2. *Member $\ell \in \{1,2\}$* prefers *s over t if and only if there exists a sequence $s_1, s_2, \ldots, s_k \in N$, with $s = s_1$ and $t = s_k$, such that ℓ directly prefers s_i over s_{i+1}, for $i = 1, 2, \ldots, k-1$.*

These notions are used in the definition of the following well-known axioms of revealed preference, for households with a single decision maker (see, e.g., Varian (1982)).

Definition 3 (GARP). *A personalized consumption data set S_ℓ ($\ell = 1,2$) satisfies the* Generalized Axiom of Revealed Preference (GARP) *if and only if, for all observations s and t, when member ℓ prefers s over t, then $p_t' x_t^\ell \leq p_t' x_s^\ell$.*

Definition 4 (SARP). *A personalized consumption data set S_ℓ ($\ell = 1,2$) satisfies the* Strong Axiom of Revealed Preference (SARP) *if and only if, for all observations s and t, when $x_s^\ell \neq x_t^\ell$ and member ℓ prefers s over t, then $p_t' x_t^\ell < p_t' x_s^\ell$.*

Observe that if S_ℓ satisfies SARP then it also satisfies GARP. One may consider that member ℓ has a "rational" consumption behavior if S_ℓ satisfies one of these axioms. Testing whether S_ℓ satisfies GARP or SARP can be done in time $O(n^2)$, using graph theory algorithms (Varian, 1982; Talla Nobibon, Smeulders, and Spieksma, 2012b).

The previous axioms can be extended to collective households, as formalized by the following definitions, where the prefix 2 indicates that we consider households with two decision makers (Cherchye and Vermeulen, 2008; Sabbe, 2010; Cherchye, De Rock, and Vermeulen, 2011a).

Definition 5 (2-GARP). *A data set S is consistent with the collective consumption model* 2-GARP *if and only if there exist feasible personalized quantities $\left(x_t^1, x_t^2\right)$, with $x_t^1 + x_t^2 = q_t$, such that, for each member ℓ, the personalized consumption data set S_ℓ satisfies* GARP *($\ell = 1,2$).*

Definition 6 (2-SARP). *A data set S is consistent with the collective consumption model* 2-SARP *if and only if there exist feasible personalized quantities $\left(x_t^1, x_t^2\right)$, with $x_t^1 + x_t^2 = q_t$, such that, for each member ℓ, the personalized consumption data set S_ℓ satisfies* SARP *($\ell = 1,2$).*

For ease of exposition, when S is consistent with the collective consumption model 2-GARP (respectively 2-SARP), we simply say that S satisfies 2-GARP (respectively 2-SARP). Notice that if S satisfies 2-SARP then it also satisfies 2-GARP. Cherchye, De Rock, and Vermeulen (2011a) establish that 2-GARP provides a necessary and sufficient condition for collective rationality, in the following sense: 2-GARP implies that the consumption decisions (taken jointly by both members) result in Pareto efficient intra-household allocations, whereas any consumption data set S that does not satisfy 2-GARP can be directly interpreted as a Pareto inefficiency (implying collectively irrational behavior).

In practice, the collective consumption models described above may lead to weak tests, because they rely on the assumption that we observe only *aggregate* household consumption. As a consequence, most consumption data sets S satisfy 2-SARP, and hence 2-GARP, and these models may fail to recognize irrational collective behavior. To strengthen the models, existing *assignable information* is often taken into account explicitly (Bourguignon, Browning, and Chiappori, 2009;

Chiappori, 1988; Cherchye and Vermeulen, 2008; Sabbe, 2010; Cherchye, De Rock, and Vermeulen, 2011a). We say that a good is *assignable* if, for at least one member in the household, one has observed a positive quantity of that good consumed by that member, for at least one time t (a common example of assignable good is clothing (Bourguignon, Browning, and Chiappori, 2009)). For member $\ell \in \{1, 2\}$ and observation $t \in N$, we denote by q_t^ℓ a known vector of assignable quantities, where we assume that $q_t^1 \geq 0$, $q_t^2 \geq 0$, and $q_t^1 + q_t^2 \leq q_t$ for all $t \in N$. In the presence of assignable information, the data set is now represented as $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$. In the rest of this paper, we rely on this representation of S; when there is no assignable information, we set $q_t^1 = q_t^2 := \mathbf{0}$ for all $t \in N$, where $\mathbf{0}$ is the zero vector.

The axioms for 2-GARP and 2-SARP immediately extend to the setting with assignable information, provided that we adopt the following definition.

Definition 7. *The pair* $(x_t^1, x_t^2) \in \mathbb{R}_+^{2m}$ *is a pair of* feasible personalized quantities *for the data set* $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$ *if and only if* $q_t^1 \leq x_t^1$, $q_t^2 \leq x_t^2$, *and* $q_t = x_t^1 + x_t^2$ *for all* $t \in N$.

In this paper, we investigate the following computational questions: for a given consumption data set S of a two-member household, (i) does S satisfy 2-GARP, (ii) does S satisfy 2-SARP? If we are given S_1 and S_2, then the problems reduce to checking whether the personalized quantity vectors are feasible, and whether S_ℓ satisfies GARP, or SARP, for $\ell = 1, 2$. These two queries can be answered in polynomial time. However, the global problems are considerably harder. The contributions of this paper include:

(1) A proof that testing 2-GARP, as well as 2-SARP is NP-complete, even when there is no assignable information.

(2) New MIP formulations of the problems.

(3) Global optimization formulations and simulated annealing heuristics for their solution.

(4) Extensive computational experiments on real-world data.

3. Problem statement

We first give a formal definition of the problems in Section 3.1, and we review existing literature on integer programming models for testing rationality in Section 3.2.

3.1 Problem description

The problems that we study are formally defined as the following *decision problems*.

Problem 2-GARP
Instance: A data set $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$, where $N := \{1, \ldots, n\}$.
Question: Do there exist x_t^1, $x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$ and $q_t^1 \leq x_t^1$, $q_t^2 \leq x_t^2$ for each observation $t \in N$ such that the sets $S_\ell := \{(p_t, x_t^\ell) : t \in N\}$ satisfy GARP for $\ell = 1, 2$?

Problem 2-SARP
Instance: A data set $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$, where $N := \{1, \ldots, n\}$.
Question: Do there exist x_t^1, $x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$ and $q_t^1 \leq x_t^1$, $q_t^2 \leq x_t^2$ for each observation $t \in N$ such that the sets $S_\ell := \{(p_t, x_t^\ell) : t \in N\}$ satisfy SARP for $\ell = 1, 2$?

Thus, an instance of 2-GARP (respectively 2-SARP) is entirely defined by the consumption data set S. Finding out whether an instance satisfies 2-GARP, or 2-SARP is difficult:

Theorem 1. *Problem* 2-GARP *and problem* 2-SARP *are* NP*-complete, even when there is no assignable information.*

Proof: See Appendix A. □

Theorem 1 implies that any method of which the aim is to solve Problem 2-GARP or 2-SARP is likely to experience sharply increasing running times when the number of observations grows.

If there is no assignable information ($q_t^1 = q_t^2 := \mathbf{0}$, for all t) then any data set $S := \{(p_t, q_t) : t \in N\}$ that satisfies GARP (respectively SARP) also satisfies 2-GARP (respectively, 2-SARP). Furthermore, any instance with at most two observations is a YES instance of 2-SARP (and hence of 2-GARP), as evidenced by the personalized quantity vectors $x_1^1 := q_1$, $x_2^1 := \mathbf{0}$, $x_1^2 := \mathbf{0}$ and $x_2^2 := q_2$. However, when assignable information is available, then instances with NO answer can be built that consist of only two observations, for both 2-GARP and 2-SARP. As mentioned earlier, checking whether S satisfies GARP or SARP can be performed in time $O(n^2)$ (Varian, 1982; Talla Nobibon, Smeulders, and Spieksma, 2012b).

3.2 Integer Programming approaches towards testing rationality

Over the last decades, several researchers have investigated different approaches to model and to test the rationality of collective households consumption (Bourguignon, Browning, and Chiappori, 2009; Cherchye and Vermeulen, 2008; Browning and Chiappori, 1998; Chiappori, 1988, 1992; Cherchye, De Rock, and Vermeulen, 2010, 2007, 2012, 2011a; Cherchye, De Rock, Sabbe, and Vermeulen, 2008). Some of these approaches extend unitary households revealed preference axioms to collective households (Cherchye and Vermeulen, 2008; Cherchye, De Rock, and Vermeulen, 2010; Cherchye, De Rock, Sabbe, and Vermeulen, 2008; Cherchye, De Rock, and Vermeulen, 2007, 2012, 2011a), and this is also the setting considered in our paper. The objective of the present section is not to provide an exhaustive list of papers that describe revealed preference axioms for collective households, but rather to focus on computational methods for testing collective consumption models.

Cherchye, De Rock, Sabbe, and Vermeulen (2008) rely on an integer programming formulation for testing a variant of the collective model, called *Collective Axiom of Revealed Preference*; testing this axiom was later proven to be NP-complete by Talla Nobibon and Spieksma (2010). Heuristics based on a graph-coloring approach are proposed for the same problem by Talla Nobibon, Cherchye, De Rock, Sabbe, and Spieksma (2011). Deb (2011) proves that testing the "situation-dependent dictatorship" version of the collective consumption model is NP-complete, and he proposes a heuristic based on graph coloring for solving the problem.

Cherchye, De Rock, and Vermeulen (2011a) propose an exact testing procedure based on a MIP formulation for 2-GARP. Their formulation uses the binary decision variables y_{st}^ℓ defined for $\ell = 1, 2$, and $s \neq t \in N$, with the interpretation that $y_{st}^\ell = 1$ if member ℓ prefers s over t. They also introduce continuous (vectors of) decision variables x_t^ℓ for $\ell = 1, 2$, and $t \in N$ corresponding with the consumption of member ℓ in observation t. The constraints of their formulation are given by:

$$\begin{aligned}
x_t^1 + x_t^2 &= q_t, && t \in N, && && (1)\\
x_t^\ell &\geq q_t^\ell, && t \in N; && \ell = 1, 2, && (2)\\
y_{su}^\ell + y_{ut}^\ell - y_{st}^\ell &\leq 1, && s \neq t \neq u \neq s \in N; && \ell = 1, 2, && (3)\\
p_s'\big(x_s^\ell - x_t^\ell\big) - M_s y_{st}^\ell &< 0, && s \neq t \in N; && \ell = 1, 2, && (4)\\
p_t'\big(x_t^\ell - x_s^\ell\big) + p_t' q_t y_{st}^\ell &\leq p_t' q_t, && s \neq t \in N; && \ell = 1, 2, && (5)\\
y_{st}^\ell &\in \{0, 1\}, && s \neq t \in N; && \ell = 1, 2. && (6)
\end{aligned}$$

We refer to the formulation (1)–(6) as F1:2-GARP. The first set of constraints (1)–(2) model the split of the observed quantity vectors into feasible personalized quantity vectors. The constraints (3) enforce the transitivity of the preference relations modeled by the y^ℓ_{st}-variables. The constraints (4) ensure that y^ℓ_{st} takes value 1 when $p'_s x^\ell_s \geq p'_s x^\ell_t$; in these constraints, M_s is any constant such that $M_s > p'_s q_s$. The constraints (5) ensure that if $y^\ell_{st} = 1$, then $p'_t x^\ell_t \leq p'_t x^\ell_s$, as required by the definition of GARP. A related formulation for 2-SARP is given in Appendix B.

4. New mixed-integer programming formulations

The formulation F1:2-GARP given above involves $\Theta(mn)$ continuous variables, $\Theta(n^2)$ binary variables, and $\Theta(n^3)$ constraints (where the Θ-notation indicates the order of growth, up to a constant factor). For medium-sized data sets, this results in very large formulations which are hard to solve, as evidenced by our computational experiments; see Section 7. Therefore, we proceed in Section 4.2 and Section 4.3 with a description of new, more compact MIP formulations for 2-GARP and for 2-SARP. In order to achieve this goal, however, we first need to introduce some interpretations of GARP and SARP in terms of graph theory, which will prove useful in the remainder of the paper.

4.1 Graph interpretation of GARP and SARP

All graphs considered in this paper are *finite, directed graphs* of the form $G = (V, A)$, where V (also denoted $V(G)$) is the vertex-set of G and A is its arc-set. A graph G is *strongly connected* if, for every pair of distinct vertices i and $j \in V(G)$, there is a directed path from i to j and a directed path from j to i in G. The subgraph of G *induced by* $W \subseteq V$ is the graph $G(W) = (W, U)$ where U consists of all arcs whose endpoints are both in W. A *strongly connected component* (SCC) of G is an induced subgraph $G(W)$ of G which is strongly connected and such that W is maximal with respect to this property. For every directed graph G, the vertex sets of the strongly connected components of G define a (unique) partition of $V(G)$. We refer to Golumbic (2004) for additional terminology and concepts.

Given the personalized consumption data set $S_\ell := \{(p_t, x^\ell_t) : t \in N\}$ of member ℓ, we build the directed graph $G^\ell = (N, A^\ell)$ as follows: Each vertex corresponds to an observation in S_ℓ. There is an arc $s \to t \in A^\ell$ if and only if $s \neq t$ and $p'_s x^\ell_s \geq p'_s x^\ell_t$, that is, if $s \neq t$ and member ℓ directly prefers s over t. We refer to G^ℓ as the *directed graph associated* with S_ℓ, or as the *preference graph* of member ℓ. Let A^ℓ_{scc} be the set of all the arcs contained in strongly connected components of G^ℓ, so that $G^\ell_{\text{scc}} = (N, A^\ell_{\text{scc}})$ is simply the union of the strongly connected components of G^ℓ. The following characterizations are direct consequences of Definition 3 and Definition 4, respectively.

Proposition 1. *The data set $S_\ell := \{(p_t, x^\ell_t) : t \in N\}$ satisfies* GARP *if and only if every arc $s \to t \in A^\ell_{scc}$ satisfies $p'_s x^\ell_s = p'_s x^\ell_t$.*

Proposition 2. *The data set $S_\ell := \{(p_t, x^\ell_t) : t \in N\}$ satisfies* SARP *if and only if every arc $s \to t \in A^\ell_{scc}$ satisfies $x^\ell_s = x^\ell_t$.*

4.2 New MIP formulation for 2-GARP

The formulation F1:2-GARP decomposes into two parts, with equations (1)–(2) expressing feasibility of the personalized quantities, and equations (3)–(6) expressing the validity of GARP for each of the personalized consumption data sets S_1 and S_2. In order to obtain a new formulation for 2-GARP, we focus on the second part and we state yet another alternative characterization of GARP.

Proposition 3. *The set $S_\ell := \{(p_t, x_t^\ell) :\ t \in N\}$ satisfies* GARP *if and only if there exist $U_t^\ell \in \mathbb{R}$ for all $t \in N$ such that (i) if $p_s' x_s^\ell \geq p_s' x_t^\ell$ then $U_s^\ell \geq U_t^\ell$, and (ii) if $p_s' x_s^\ell > p_s' x_t^\ell$ then $U_s^\ell > U_t^\ell$.*

Proposition 3 is a variant of many similar results where U_t^ℓ can be viewed as the utility level of bundle x_t^ℓ, and the utility levels are consistent with the preference relation of member ℓ; see Varian (2006, 1982). For the sake of completeness and in order to prepare the ground for our next results, we provide a self-contained proof of this characterization.

Proof: Suppose that there exist $U_t^\ell \in \mathbb{R}$, for all $t \in N$, satisfying (i) and (ii), and suppose that member ℓ prefers s over t. Thus, according to Definition 2, there exists a sequence $s, u, v, \ldots, z, t$ of observations such that $p_s' x_s^\ell \geq p_s' x_u^\ell$, $p_u' x_u^\ell \geq p_u' x_v^\ell$, $\ldots$, $p_z' x_z^\ell \geq p_z' x_t^\ell$. In view of (i), $U_s^\ell \geq U_u^\ell \geq U_v^\ell \geq \cdots \geq U_z^\ell \geq U_t^\ell$. By (ii), this implies in turn that $p_t' x_t^\ell \leq p_t' x_s^\ell$, as required by Definition 3 of GARP.

Conversely, suppose now that S_ℓ satisfies GARP. Consider the directed graph $G^\ell = (V, A^\ell)$ associated with S_ℓ, and let G_1^ℓ, G_2^ℓ, $\ldots$, $G_{\alpha_\ell}^\ell$ be the strongly connected components of G^ℓ ($\alpha_\ell \leq n$). We can assume that the strongly connected components are indexed from 1 to α_ℓ in *reverse topological order,* that is, in such a way that every arc of G^ℓ goes from a component G_i^ℓ to a component G_j^ℓ with $i \geq j$ (see, e.g., Ahuja, Magnanti, and Orlin (1993)). Now, for every vertex $t \in G^\ell$, define $U_t^\ell := \frac{i_t}{n}$, where i_t^ℓ is the index of the strongly connected component that contains t. Then, property (i) is satisfied by construction. On the other hand, if $p_s' x_s^\ell > p_s' x_t^\ell$, then $s \to t \in A^\ell$, but Proposition 1 implies that $s \to t \notin A_{\text{scc}}^\ell$. Hence, arc $s \to t$ has its endpoints in two distinct components, and we conclude that $U_s^\ell > U_t^\ell$. □

The utility levels U_t^ℓ can explicitly be modeled as variables, thereby leading to the following MIP formulation for 2-GARP, denoted by F2:2-GARP.

$$
\text{(F2:2-GARP)} \qquad
\begin{array}{rcllr}
x_t^1 + x_t^2 & = & q_t, & t \in N, & (7)\\
x_t^\ell & \geq & q_t^\ell, & t \in N; \quad \ell = 1,2, & (8)\\
U_s^\ell - U_t^\ell & < & y_{st}^\ell, & s \neq t \in N; \quad \ell = 1,2, & (9)\\
y_{st}^\ell - 1 & \leq & U_s^\ell - U_t^\ell, & s \neq t \in N; \quad \ell = 1,2, & (10)\\
p_s'(x_s^\ell - x_t^\ell) - M_s y_{st}^\ell & < & 0, & s \neq t \in N; \quad \ell = 1,2, & (11)\\
p_t'(x_t^\ell - x_s^\ell) + p_t' q_t y_{st}^\ell & \leq & p_t' q_t, & s \neq t \in N; \quad \ell = 1,2, & (12)\\
y_{st}^\ell & \in & \{0,1\}, & s \neq t \in N; \quad \ell = 1,2, & (13)
\end{array}
$$

where M_s is a strict upper-bound on $p_s' q_s$. The reader should note that in this formulation, contrary to F1:2-GARP, the y_{st}^ℓ-variables do not model the (revealed) preferences of member ℓ, but rather reflect the order imposed by the utility levels: indeed, constraints (9)–(10) imply that, in every feasible solution, $y_{st}^\ell = 1$ if and only if $U_s^\ell \geq U_t^\ell$.

We have the following result.

Proposition 4. *The consumption data set $S := \{(p_t, q_t; q_t^1, q_t^2) :\ t \in N\}$ satisfies* 2-GARP *if and only if the domain defined by* (7)–(13) *is nonempty.*

Proof: Suppose that $S := \{(p_t, q_t; q_t^1, q_t^2) :\ t \in N\}$ satisfies 2-GARP. Then there exist x_t^1, $x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$ and $q_t^1 \leq x_t^1$, $q_t^2 \leq x_t^2$ for each observation $t \in N$ such that the consumption sets $S_\ell := \{(p_t, x_t^\ell) :\ t \in N\}$ satisfy GARP, for $\ell = 1, 2$. Fix these feasible personalized quantities. For $\ell = 1, 2$, consider the graph $G^\ell = (V, A^\ell)$ associated with S_ℓ and its strongly connected components G_1^ℓ, G_2^ℓ, $\ldots$, $G_{\alpha_\ell}^\ell$ indexed in reverse topological order, as in the proof of Proposition 3. Define the point $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$, where the components of $\mathbf{X}$ (respectively, $\mathbf{Y}$ and $\mathbf{U}$) are x_t^ℓ (respectively, y_{st}^ℓ and U_t^ℓ) and are specified as follows for all $s \neq t \in N$ and $\ell \in \{1, 2\}$:

- x_t^ℓ is the quantity consumed by member ℓ in observation t;
- if $i_s^\ell \geq i_t^\ell$ then $y_{st}^\ell := 1$; otherwise $y_{st}^\ell := 0$;
- $U_t^\ell := \frac{i_t^\ell}{n}$.

We now argue that the point $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ belongs to the domain defined by (7)–(13). Indeed:

Constraints (7)–(8) **and** (13)**:** These are satisfied by assumption.

Constraints (9)–(10)**:** For $s \neq t \in N$ and $\ell \in \{1,2\}$, if $i_s^\ell \geq i_t^\ell$ then $y_{st}^\ell = 1$ and $U_s^\ell \geq U_t^\ell$; if $i_s^\ell < i_t^\ell$ then $y_{st}^\ell = 0$ and $U_s^\ell < U_t^\ell$. In both cases, (9)–(10) hold since $U_t^\ell \in [\frac{1}{n}, 1]$ implies $\left|U_t^\ell - U_s^\ell\right| < 1$.

Constraints (11)**:** For $s \neq t \in N$ and $\ell \in \{1,2\}$, if $p_s' x_s^\ell < p_s' x_t^\ell$ then (11) is trivially satisfied. Otherwise, if $p_s' x_s^\ell \geq p_s' x_t^\ell$ then $s \to t \in A^\ell$ by construction, hence $i_s^\ell \geq i_t^\ell$ and $y_{st}^\ell = 1$. Therefore (11) holds.

Constraints (12)**:** For $s \neq t \in N$ and $\ell \in \{1,2\}$, if $p_t' x_s^\ell \geq p_t' x_t^\ell$ then (12) trivially holds. Otherwise, if $p_t' x_s^\ell < p_t' x_t^\ell$ then $i_s^\ell \neq i_t^\ell$ (by Proposition 1) and $y_{st}^\ell = 0$. Therefore (12) holds, and all constraints are satisfied.

Conversely, suppose now that $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ is in the domain defined by (7)–(13). For $\ell = 1, 2$, let $S_\ell := \left\{\left(p_t, x_t^\ell\right) : t \in N\right\}$, where x_t^ℓ are the components of $\mathbf{X}$. We now rely on Proposition 3 to prove that the personalized consumption sets $S_\ell := \left\{\left(p_t, x_t^\ell\right) : t \in N\right\}$ $(\ell = 1, 2)$ satisfy GARP.

If $p_s' x_s^\ell \geq p_s' x_t^\ell$ then constraint (11) implies that $y_{st}^\ell = 1$ and hence, in view of constraint (10), $U_s^\ell \geq U_t^\ell$. On the other hand, if $p_s' x_s^\ell > p_s' x_t^\ell$ then $y_{ts}^\ell = 0$ by constraint (12), and constraint (9) implies $U_s^\ell > U_t^\ell$. Thus, properties (i) and (ii) hold in Proposition 3, and S satisfies 2-GARP. □

Note that, as compared to F1:2-GARP, the formulation F2:2-GARP involves $2n$ additional continuous variables, but only $\Theta(n^2)$ constraints. For large values of n, this is a very significant reduction in size. One might be tempted to compare the linear programming relaxations of the two formulations F1:2-GARP and F2:2-GARP. There are, however, two issues that prevent such a comparison to be meaningful: (i) the y-variables in the two formulations have different interpretations (as mentioned above), and, perhaps more importantly, (ii) since Problem 2-GARP is a decision problem, both formulations lack an objective function.

4.3 New MIP formulation for 2-SARP

We now extend the previous results by providing a new MIP formulation for 2-SARP. This formulation is based on Proposition 5, which is the counterpart of Proposition 3 for SARP.

Proposition 5. *The set $S_\ell := \left\{\left(p_t, x_t^\ell\right) : t \in N\right\}$ satisfies* SARP *if and only if there exist $U_t^\ell \in \mathbb{R}$ for $t \in N$ such that (i) $x_s^\ell = x_t^\ell$ if and only if $U_t^\ell = U_s^\ell$, and (ii) if $x_s^\ell \neq x_t^\ell$ and $p_s' x_s^\ell \geq p_s' x_t^\ell$ then $U_s^\ell > U_t^\ell$.*

Proof: Suppose that there exist $U_t^\ell \in \mathbb{R}$, for all $t \in N$, satisfying (i) and (ii), and suppose that member ℓ prefers s over t, with $x_s^\ell \neq x_t^\ell$. Thus, there exists a sequence $s, u, v, \ldots, z, t$ of observations such that $p_s' x_s^\ell \geq p_s' x_u^\ell$, $p_u' x_u^\ell \geq p_u' x_v^\ell$, $\ldots$, $p_z' x_z^\ell \geq p_z' x_t^\ell$. Therefore, $U_s^\ell \geq U_u^\ell \geq U_v^\ell \geq \cdots \geq U_z^\ell \geq U_t^\ell$. By (i), this implies that $U_t^\ell < U_s^\ell$, and again by (ii), we obtain $p_t' x_t^\ell < p_t' x_s^\ell$, as required by Definition 4 of SARP.

Conversely, if S_ℓ satisfies SARP, construct U^ℓ_t as in the proof of Proposition 3. Note that Property (ii) is satisfied as before. For (i), note that if $U^\ell_t = U^\ell_s$, then t and s are in the same strongly connected component of G^ℓ, and hence $x^\ell_s = x^\ell_t$ by Proposition 2. On the other hand, if $x^\ell_s = x^\ell_t$, then (trivially) x^ℓ_s is preferred over x^ℓ_t, and x^ℓ_t is preferred over x^ℓ_s by member ℓ, so that s and t are in the same component of G^ℓ, and $U^\ell_t = U^\ell_s$. □

Proposition 5 leads to the following MIP formulation for 2-SARP, denoted by F2:2-SARP.

$$
\text{(F2:2-SARP)} \quad
\begin{array}{rcll}
x^1_t + x^2_t & = & q_t, & t \in N, \qquad (14)\\
x^\ell_t & \geq & q^\ell_t, & t \in N; \quad \ell = 1,2. \qquad (15)\\
U^\ell_s - U^\ell_t & < & y^\ell_{st}, & s \neq t \in N; \quad \ell = 1,2, \qquad (16)\\
y^\ell_{st} - 1 & \leq & U^\ell_s - U^\ell_t, & s \neq t \in N; \quad \ell = 1,2, \qquad (17)\\
p'_s\big(x^\ell_s - x^\ell_t\big) - M_s y^\ell_{st} & < & 0, & s \neq t \in N; \quad \ell = 1,2, \qquad (18)\\
\dfrac{1}{q_{s,i}}\left(x^\ell_{s,i} - x^\ell_{t,i}\right) & \leq & 2 - y^\ell_{st} - y^\ell_{ts}, & s \neq t \in N; \quad i = 1,\ldots,m; \quad \ell = 1,2, \qquad (19)\\
y^\ell_{st} & \in & \{0,1\}, & s \neq t \in N; \quad \ell = 1,2, \qquad (20)
\end{array}
$$

where M_s is a strict upper-bound on $p'_s q_s$.

Proposition 6. *The consumption data set $S := \big\{\big(p_t, q_t; q^1_t, q^2_t\big) : \ t \in N\big\}$ satisfies* 2-SARP *if and only if the domain defined by* (14)–(20) *is nonempty.*

Proof: The proof is a slight modification of that of Proposition 4. If S satisfies 2-SARP, then define $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ as in the proof of Proposition 4. Since SARP implies GARP, we already know that constraints (14)–(18) and (20) are satisfied. For (19), note that if $x^\ell_s \neq x^\ell_t$ then necessarily $i^\ell_s \neq i^\ell_t$ by Proposition 2. Hence the construction of $\mathbf{Y}$ entails $y^\ell_{st} + y^\ell_{ts} \leq 1$, so that (19) is satisfied.

Conversely, if $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ is a solution of (14)–(20), then again, $\mathbf{U}$ satisfies condition (ii) of Proposition 5. For condition (i), assume that $U^\ell_s = U^\ell_t$. Then, due to constraints (16), $y^\ell_{st} = y^\ell_{ts} = 1$, and constraints (19) (applied to the pairs (s,t) and (t,s)) imply $x^\ell_s = x^\ell_t$. The converse implication follows from constraints (17) and (18). □

The formulation F2:2-SARP involves the same variables as F2:2-GARP, and $\Theta(mn^2)$ constraints. In practical applications m is typically much smaller than n (see Section 7), and therefore the new formulation is more compact than the formulation given in Appendix B.

4.4 Additional comments about the MIP formulations

For computational purposes, the formulations F2:2-GARP and F2:2-SARP can be improved in a number of ways. First, it immediately follows from the proofs of Proposition 4 and Proposition 6 that all variables U^ℓ_s can be restricted to lie between 0 and 1, for all $s \neq t \in N$ and $\ell = 1,2$. Moreover, a closer examination of these proofs reveals that the constraints (9) and (16), which involve strict inequalities, can be replaced by the stronger constraints:

$$U^\ell_s - U^\ell_t \ \leq \ y^\ell_{st} - \frac{1}{n}, \qquad s \neq t \in N; \quad \ell = 1,2. \qquad (21)$$

In the sequel, whenever we refer to the MIP formulations F2:2-GARP and F2:2-SARP, we implicitly assume that $U^\ell_s \in [0,1]$ and that (9) and (16) have been replaced by (21).

After these modifications, both MIP formulations still involve the strict inequalities (11) (or (18)), which are not desirable from the numerical point of view. However, if the domain of F2:2-GARP is nonempty, then there exists a small positive ε such that the domain remains nonempty when we replace the constraints (11) by:

$$p'_s\big(x^\ell_s - x^\ell_t\big) - M_s y^\ell_{st} \quad \le \quad -\varepsilon, \qquad\qquad s \neq t \in N; \quad \ell = 1, 2. \tag{22}$$

Indeed, suppose that $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ satisfies all constraints (7)–(13). Fixing the values of y^ℓ_{st} (for all $s \neq t \in N$ and $\ell = 1, 2$) in the constraints yields a feasible system of linear inequalities. From linear programming theory (Chvátal, 1983), we know that there exists a small $\varepsilon > 0$ such that this system of inequalities remains feasible if we replace the right-hand side of (11) by $-\varepsilon$. Since the number of possible assignments of binary values to $\mathbf{Y}$ is finite, it follows that there is a smallest value of ε which can be used without affecting the feasibility of F2:2-GARP. The same reasoning applies to the formulation F2:2-SARP. Note that M_s should be at least $p'_s q_s + \epsilon$ for the previous reasoning to be valid.

Finally, we end this section by describing a set of *cuts* (that is, valid inequalities) that can be used to strengthen the formulations F2:2-GARP and F2:2-SARP. Note that, in the proof of Proposition 4, at least one of y^ℓ_{st} and y^ℓ_{ts} takes value 1, for all $s \neq t \in N$. This condition actually holds for every feasible solution, as can be deduced by taking the sum of the constraints (9) respectively associated with the pairs (s, t) and (t, s). As a consequence, the following inequalities are valid and can safely be added to each of the formulations F2:2-GARP and F2:2-SARP without affecting their feasibility:

$$y^\ell_{st} + y^\ell_{ts} \ge 1, \qquad\qquad s < t \in N; \quad \ell = 1, 2. \tag{23}$$

The effect of these cuts on the solvability of the MIP models is investigated in Section 7.

5. Global optimization formulations

The MIP formulations presented in previous sections turn out to be hard to solve for large instances of 2-SARP and 2-GARP. Therefore, we have also developed heuristic algorithms which rely on global optimization formulations to be presented in Section 5.1. In Section 5.2, we describe a graph-based representation of solutions and we discuss some of their properties.

5.1 Global optimization formulations for 2-GARP and 2-SARP

In the sequel, for ease of exposition, we often use the variables $x_s := x^1_s$, and we implicitly set $x^2_s := q_s - x^1_s$ for all $s \in N$. Consider the box $\mathbf{D} := \big\{(x_1, x_2, \ldots, x_n) : q^1_s \le x_s \le q_s - q^2_s, \, \forall s \in N\big\}$ corresponding to the set of feasible personalized quantities, and denote by $\mathbf{X} := (x_1, x_2, \ldots, x_n) \in \mathbf{D}$ an arbitrary point in this box. Each such point $\mathbf{X}$ can be associated with personalized data sets $S_\ell := \big\{\big(p_t, x^\ell_t\big) : \; t \in N\big\}$ and with their associated preference graphs $G^\ell = \big(N, A^\ell\big)$, for $\ell = 1, 2$. Let $\widehat{A}^\ell_{\text{scc}} \subseteq A^\ell_{\text{scc}}$ be the subset of A^ℓ_{scc} containing all arcs $s \to t \in A^\ell_{\text{scc}}$ such that $p'_s x^\ell_s \neq p'_s x^\ell_t$. Similarly, let $\widetilde{A}^\ell_{\text{scc}} \subseteq A^\ell_{\text{scc}}$ contain all arcs $s \to t \in A^\ell_{\text{scc}}$ with $x^\ell_s \neq x^\ell_t$. Note that we have the inclusion $\widehat{A}^\ell_{\text{scc}} \subseteq \widetilde{A}^\ell_{\text{scc}}$. Proposition 1 and Proposition 2 are equivalently reformulated as follows.

Proposition 7. *The data set $S_\ell := \big\{\big(p_t, x^\ell_t\big) : \; t \in N\big\}$ satisfies* GARP *if and only if $\widehat{A}^\ell_{scc}$ is empty.*

Proposition 8. *The data set $S_\ell := \big\{\big(p_t, x^\ell_t\big) : \; t \in N\big\}$ satisfies* SARP *if and only if $\widetilde{A}^\ell_{scc}$ is empty.*

Let us now introduce the following integer-valued functions on the domain $\mathbf{D}$:

$$f_{\text{G}}\left(\mathbf{X}\right) = f_{\text{G}}\left(x_1, x_2, \ldots, x_n\right) := \left|\widehat{A}^1_{\text{scc}}\right| + \left|\widehat{A}^2_{\text{scc}}\right|, \tag{24}$$

$$f_{\mathrm{S}}(\mathbf{X}) = f_{\mathrm{S}}(x_1, x_2, \ldots, x_n) := \left|\widetilde{A}^1_{\mathrm{scc}}\right| + \left|\widetilde{A}^2_{\mathrm{scc}}\right|, \tag{25}$$

and consider the unconstrained optimization problems

$$\begin{array}{lll} (\text{G:2-}\textsc{garp}) & \min & f_{\mathrm{G}}(\mathbf{X}) \\ & \text{s.t.} & \mathbf{X} \in \mathbf{D}, \end{array}$$

and

$$\begin{array}{lll} (\text{G:2-}\textsc{sarp}) & \min & f_{\mathrm{S}}(\mathbf{X}) \\ & \text{s.t.} & \mathbf{X} \in \mathbf{D}. \end{array}$$

Proposition 9. *The consumption data set $S := \left\{\left(p_t, q_t; q^1_t, q^2_t\right) : \ t \in N\right\}$ is a feasible instance of* 2-GARP *(respectively,* 2-SARP*) if and only if the optimal value of problem G:*2-GARP *(respectively, G:*2-SARP*) is equal to* 0*.*

Proof: This is immediate in view of Proposition 7 and Proposition 8. □

For each $\mathbf{X} \in \mathbf{D}$, the functions $f_{\mathrm{S}}(\mathbf{X})$ and $f_{\mathrm{G}}(\mathbf{X})$ can be evaluated in polynomial time. Note, however, that they are neither continuous nor convex, so that G:2-GARP and G:2-SARP must be viewed as *global optimization problems*.

When comparing the formulation G:2-GARP with F2:2-GARP, we observe that the constraints (7)–(8) expressing feasibility of the personalized quantities are enforced by $\mathbf{X} \in \mathbf{D}$, whereas the constraints (9)–(13) expressing the validity of GARP for each of the personalized consumption data sets S_1 and S_2 are transferred to the objective function of G:2-GARP. Similar observations apply to G:2-SARP and F2:2-SARP. Many other objective functions could be substituted for $f_{\mathrm{S}}(\mathbf{X})$ and for $f_{\mathrm{G}}(\mathbf{X})$ in G:2-GARP and in G:2-SARP. The functions (24) and (25) have been chosen because they are well-suited for the algorithmic developments described in subsequent sections.

5.2 Graphs associated with a solution

We have seen that every feasible solution $\mathbf{X} \in \mathbf{D}$ can be associated with a unique *pair of directed graphs* (G^1, G^2). But the converse does not hold: Clearly, certain pairs of directed graphs on N are not associated with any point in $\mathbf{D}$. Moreover, two points $\mathbf{X}$ and $\mathbf{X}'$ in $\mathbf{D}$ may give rise to the same pair of graphs (G^1, G^2). When this is the case, we say that $\mathbf{X}$ and $\mathbf{X}'$ are *equivalent*: indeed, then, Propositions 7-8 imply that knowledge of (G^1, G^2) suffices to determine whether the personalized data sets associated with each of $\mathbf{X}$, $\mathbf{X}'$ satisfy GARP or SARP, and we actually have $f_{\mathrm{G}}(\mathbf{X}) = f_{\mathrm{G}}(\mathbf{X}')$ and $f_{\mathrm{S}}(\mathbf{X}) = f_{\mathrm{S}}(\mathbf{X}')$.

Thus, the pair of graphs (G^1, G^2) associated with $\mathbf{X} \in \mathbf{D}$ can be viewed as providing an incomplete, but sufficient, representation of $\mathbf{X}$. The main advantage of this representation is its *combinatorial* structure: namely, the number of pairs (G^1, G^2) is finite, as opposed to the number of points in $\mathbf{D}$. Therefore, the representation based on graph pairs better lends itself to the application of local search approaches.

In order to understand what solutions are associated with a given pair of graphs, let us consider the following decision problem:

Problem PC(G^1, G^2)
Instance: A data set $S = \left\{\left(p_t, q_t; q^1_t, q^2_t\right) : \ t \in N\right\}$ and two directed graphs $G^1 = (N, A^1)$ and $G^2 = (N, A^2)$.

Question: Does there exist a point $\mathbf{X} \in \mathbf{D}$ such that the pair of directed graphs associated with $\mathbf{X}$ is exactly (G^1, G^2)?

We argue now that problem $\text{PC}(G^1, G^2)$ can be solved in polynomial time. For an (unknown) point $\mathbf{X} \in \mathbf{D}$ and for each pair $s, t \in N$, $s \neq t$, of distinct observations, we are going to set up inequalities of the following type:

$$lhs[s,t] \leq \quad p'_s x_s - p'_s x_t \quad \leq rhs[s,t]. \tag{26}$$

The values of the left-hand side $lhs[s,t]$ and of the right-hand side $rhs[s,t]$ in (26) are set according to Table 1, where ε is a very small and positive number, and NA means *non applicable*.

Table 1: Values of $lhs[s,t]$ and $rhs[s,t]$ when G^1 and G^2 are given.

four cases	$lhs[s,t]$	$rhs[s,t]$
$s \to t \in A^1$ and $s \to t \in A^2$	0	$p'_s q_s - p'_s q_t$
$s \to t \in A^1$ and $s \to t \notin A^2$	$\max\{0,\, p'_s q_s - p'_s q_t + \varepsilon\}$	NA
$s \to t \notin A^1$ and $s \to t \in A^2$	NA	$\min\{-\varepsilon,\, p'_s q_s - p'_s q_t\}$
$s \to t \notin A^1$ and $s \to t \notin A^2$	$p'_s q_s - p'_s q_t + \varepsilon$	$-\varepsilon$

We denote by $\mathcal{P}(G^1, G^2)$ the system of $n(n-1)$ constraints of type (26) constructed in this way, together with all the constraints enforcing assignable restrictions: $q_s^1 \leq x_s \leq q_s - q_s^2$, for all $s \in N$. The system $\mathcal{P}(G^1, G^2)$ involves $\Theta(mn)$ continuous variables and $\Theta(n^2)$ constraints. We have the following straightforward result.

Proposition 10. *For ε sufficiently small, the solutions to the system $\mathcal{P}(G^1, G^2)$ are exactly the points $\mathbf{X} \in \mathbf{D}$ such that the pair of graphs (G^1, G^2) is associated with $\mathbf{X}$. In particular, the system $\mathcal{P}(G^1, G^2)$ has a solution if and only if the problem $\text{PC}(G^1, G^2)$ has a Yes answer, and these conditions can be tested in polynomial time.*

Proof: For all $\mathbf{X} \in \mathbf{D}$, the inequalities (26) simply express the conditions under which an arc is either present or absent in each of the graphs G^1, G^2. Since the system $\mathcal{P}(G^1, G^2)$ is a linear program, we infer that problem $\text{PC}(G^1, G^2)$ can be decided in polynomial time (Khachiyan, 1979; Karmarkar, 1984). □

The above analysis requires only minor adaptations to work when G^1 and G^2 represent partial preferences. More precisely:

Proposition 11. *If E^1, F^1, E^2, F^2 are subsets of $N \times N$ such that $E^1 \cap F^1 = E^2 \cap F^2 = \emptyset$, one can test in polynomial time whether there exists a point $\mathbf{X} \in \mathbf{D}$ such that the graphs $G^1 = (N, A^1)$ and $G^2 = (N, A^2)$ associated with $\mathbf{X}$ satisfy $E^\ell \subseteq A^\ell$ and $A^\ell \cap F^\ell = \emptyset$ for $\ell = 1, 2$.*

Proof: In the statement, E^ℓ stands for the set of direct preferences that must be present in G^ℓ, whereas F^ℓ stands for the set of preferences that must be absent from G^ℓ, for each $\ell = 1, 2$. The conclusion follows, since the conditions on $\mathbf{X}$ can be tested by imposing the inequalities (26) for all pairs $(s,t) \in E^1 \cup F^1 \cup E^2 \cup F^2$, and by adapting the rows of Table 1 accordingly. □

6. Simulated annealing algorithm

We present a simulated annealing (SA) algorithm for solving G:2-GARP and G:2-SARP. Similar algorithms have been successfully applied to global optimization problems where the objective function is neither continuous nor convex; see, e.g., Dekkers and Aarts (1991); Goffe, Ferrier, and Rogers (1994); Locatelli (2000).

The SA algorithm implemented in this paper follows the scheme described by Dekkers and Aarts (1991), and requires the following parameters: A nonnegative real number c, called *control parameter*, which is used to determine whether a solution with an objective value worse than that of the current solution is accepted as the next current solution or not. A positive integer L representing the length of each *Markov chain* (sequence of trials), and the function $r[0,1)$ that generates uniformly a random value in $[0,1)$. Finally, *stop* is a boolean function used to halt the algorithm. The pseudocode of the SA algorithm is depicted in **Algorithm** 1, where $f(\mathbf{X})$ can be replaced either by $f_{\text{G}}(\mathbf{X})$ or by $f_{\text{S}}(\mathbf{X})$. Concretely, the algorithm starts from an initial solution $\mathbf{X}$ and a given value of c. The function *stop* is initialized to "false". While *stop* remains false, the algorithm generates at each iteration a sequence of L trial solutions. Each solution can be set as the new current solution, depending on whether its objective value is smaller than that of the current solution or, if not, based on the value of the control parameter c. This procedure is repeated until either a solution with objective value zero is found, or the function *stop* takes the value "true". Upon termination, the algorithm outputs either 'Yes' or 'Undecided'. Observe that, since SA is not an exact optimization algorithm, it may sometimes return 'Undecided' when running on 'Yes' instances. This case will be examined in Section 7. We next elaborate on each step of **Algorithm** 1.

Algorithm 1 *Simulated Annealing*

initialize $\mathbf{X} := \mathbf{X_0}$, $c := c_0$, and $stop :=$ false
while $stop =$ false **do**
  for $i := 1$ to L **do**
    generate $\mathbf{X}'$ either as a neighbor of $\mathbf{X}$ or as a random solution
    if $f(\mathbf{X}') = 0$ **then** $\mathbf{X} := \mathbf{X}'$, and go to 9
    else if $f(\mathbf{X}') < f(\mathbf{X})$ **then** $\mathbf{X} := \mathbf{X}'$
    else if $\exp\left(\dfrac{f(\mathbf{X}) - f(\mathbf{X}')}{c}\right) > r[0,1)$ **then** $\mathbf{X} := \mathbf{X}'$
  decrease the value of c and update *stop*
if $f(\mathbf{X}) = 0$ **then** return "Yes"
else return "Undecided"

6.1 Initialization

We first try to produce an initial solution by successively considering the following procedures **P1, P2, P3**. Each of these procedures proceeds by building a pair of directed graphs (G^1, G^2) that represent either the full preferences or the partial preferences of the two household members.

P1: Let $\pi : N \to \{1, 2, \ldots, n\}$ be an arbitrary (random) permutation of the set of observations. We build (G^1, G^2) as follows: for every pair $s \neq t \in N$, we set $s \to t \in A^1$ if and only if $\pi(s) < \pi(t)$, and $s \to t \in A^2$ otherwise. The resulting graphs G^1 and G^2 are tentatively taken to represent the complete preference graphs of members 1 and 2. Note that they are acyclic. Therefore, if $\mathcal{P}(G^1, G^2)$ is nonempty then the SA algorithm stops and returns Yes: the instance satisfies 2-SARP.

After having tested a number of permutations π, if we have not been able to conclude that the data set satisfies 2-SARP, then we continue with procedure **P2** hereunder, where $G = (N, A)$ is the directed graph associated with S.

P2: For every pair $s \neq t \in N$, if $s < t$ and $s \to t \in A$, then we impose $s \to t \in A^1$ and $s \to t \notin A^2$. Otherwise, if $s > t$ and $s \to t \in A$, then we impose $s \to t \notin A^1$ and $s \to t \in A^2$. Note that we do not formulate any condition when $s \to t \notin A$, so that the preference graphs are only partially determined by these conditions. Proposition 11 can be used to find a point $\mathbf{X_0} \in \mathbf{D}$ such that the graphs associated with $\mathbf{X_0}$ satisfy the imposed conditions, if there is one. Any such point provides an initial solution for SA.

When **P2** fails to provide an initial solution, we turn to the following procedure.

P3: For every pair $s \neq t \in N$, if $s > t$ and $s \to t \in A$, then we impose $s \to t \notin A^1$ and $s \to t \in A^2$. Once again, any point $\mathbf{X_0}$ satisfying these conditions can be used to initialize SA.

Finally, when all three procedures **P1**, **P2**, **P3** fail, we simply choose $\mathbf{X_0} := random()$, where $random()$ is a function that randomly generates a point in $\mathbf{D}$. In our numerical experiments, this function is implemented as follows: for each observation s and for each good i, we set $x_{s,i} := \alpha q^1_{s,i} + (1-\alpha)\big(q_{s,i} - q^2_{s,i}\big)$, where α is a random value in $[0,1]$.

This two-step procedure was adopted after some preliminary experiments involving a comparison with a purely random initial solution, or with a variant of procedures **P2**, **P3** where the arc-sets A^1 and A^2 were extended to their transitive closures. These alternative implementations, however, did not produce better results than the one described.

We adopt the procedure proposed by Dekkers and Aarts (1991) to initialize the control parameter c: namely, given $\mathbf{X_0}$, the initial value $c = c_0$ is chosen large enough so that almost all randomly generated transitions in $\mathbf{D}$ are accepted by SA.

6.2 Neighborhood of a feasible solution and generation of new solutions

In order to describe the *neighborhood* $\mathcal{N}(\mathbf{X})$ of a feasible solution $\mathbf{X} \in \mathbf{D}$, let (G^1, G^2) be the pair of preference graphs associated with $\mathbf{X}$ as in Section 5.1. We want the neighbors $\mathbf{X'} \in \mathcal{N}(\mathbf{X})$ to be associated with a pair of preference graphs that is different from (G^1, G^2), since otherwise $\mathbf{X}$ and $\mathbf{X'}$ could be viewed as equivalent, as explained in Section 5.2. Accordingly, we define $\mathcal{N}(\mathbf{X})$ as follows. Let A^1_{scc} and A^2_{scc} denote the arc-sets of the strongly connected components of G^1 and G^2, respectively. For each arc $s \to t \in A^1_{scc} \cup A^2_{scc}$, we define a subset $\mathcal{N}_{st}(\mathbf{X})$ of the neighborhood of $\mathbf{X}$; namely, $\mathcal{N}_{st}(\mathbf{X})$ contains all solutions $\mathbf{X'} \in \mathbf{D}$ such that the associated pair of directed graphs (H^1, H^2), $H^1 = (N, E^1)$, $H^2 = (N, E^2)$, satisfies the conditions:

- if $s \to t \in A^1$, then $A^1 \setminus \{s \to t\} \subseteq E^1$, $s \to t \notin E^1$, and $A^2 \subseteq E^2$;
- else, $A^1 \subseteq E^1$, $A^2 \setminus \{s \to t\} \subseteq E^2$, and $s \to t \notin E^2$.

Finally, the neighborhood of $\mathbf{X}$ is $\mathcal{N}(\mathbf{X}) := \bigcup_{s \to t \in A^1_{scc} \cup A^2_{scc}} \mathcal{N}_{st}(\mathbf{X})$.

The rationale behind this definition is to try to remove arcs $s \to t$ from $A^1_{scc} \cup A^2_{scc}$, so as to decrease the value of the objective function (24) or (25). The condition of the form $s \to t \notin E^\ell$, where $\ell \in \{1, 2\}$, ensures that $(H^1, H^2) \neq (G^1, G^2)$. (More complex neighborhoods could be designed by considering simultaneously several arcs in $A^1_{scc} \cup A^2_{scc}$, but this option has not been investigated.)

Note that $\mathcal{N}(\mathbf{X})$ does not contain $\mathbf{X}$ nor any solution equivalent to $\mathbf{X}$. Therefore, there is no guarantee that $\mathcal{N}(\mathbf{X})$ is nonempty. Furthermore, $\mathcal{N}(\mathbf{X})$ is described implicitly because its definition relies on the preference graphs, rather than points in $\mathbf{D}$.

Given a point $\mathbf{X} \in \mathbf{D}$ and an arc $s \to t \in A^1_{scc} \cup A^2_{scc}$, we can rely again on Proposition 11 to find a point $\mathbf{X}'_{st}$ in $\mathcal{N}_{st}(\mathbf{X})$ or to prove that $\mathcal{N}_{st}(\mathbf{X})$ is empty. In our implementation, when we need to find a point in $\mathcal{N}(\mathbf{X})$, we run this procedure over successive choices of $s \to t \in A^1_{scc} \cup A^2_{scc}$, as prescribed by the local search procedure $LS(\mathbf{X})$ (see **Algorithm** 2). $LS(\mathbf{X})$ searches within the neighborhood $\mathcal{N}(\mathbf{X})$ for a solution with a better value than $\mathbf{X}$. If it cannot find such an improving solution, then the algorithm returns the best neighbor found when $\mathcal{N}(\mathbf{X})$ is nonempty, and it returns a random solution otherwise.

Algorithm 2 $LS(\mathbf{X})$

```
1: for all s → t ∈ A^1_scc ∪ A^2_scc do
2:     if N_st(X) ≠ ∅ then find X'_st ∈ N_st(X)
3:         if f(X'_st) < f(X) then set X' := X'_st and go to 6
4: if N(X) = ∅ then generate X' := random() randomly and go to 6
5: else let X'_st be the best solution found in N(X) and set X' := X'_st
6: return X'
```

Finally, Step 3 of the simulated algorithm SA can be implemented as follows: given a solution $\mathbf{X}$, we generate a new solution $\mathbf{X}'$ using either the function $random()$ or the local search procedure. The choice between these two approaches is governed by the following rule:

$$\mathbf{X}' \;:=\; \begin{cases} LS(\mathbf{X}) & \text{with probability } 1-\gamma, \\ random() & \text{otherwise,} \end{cases} \tag{27}$$

where γ is a fixed number in the interval $[0, 1)$. Here again, we see that the new solution $\mathbf{X}'$ does not need to improve the current one and can even be random. This allows SA to diversify the search and to avoid the traps of local optima.

6.3 Control parameter and stopping criterion

After each iteration of the SA algorithm (characterized by the generation of a Markov chain consisting of L trial solutions), the new value for the control parameter, say c', is computed based on the current value c using the following expression (Dekkers and Aarts, 1991):

$$c' \;:=\; c\left(1 + \frac{c\,\log(1+\delta)}{3\,\sigma(c)}\right), \tag{28}$$

where $\sigma(c)$ is the standard deviation of the objective value of the L solutions in the Markov chain at c, and δ is a given parameter.

We stop the SA algorithm when either we have found a solution with objective value 0 (and return "Yes"), or the time limit is reached, or the value of c is smaller than a predefined threshold (and return "Undecided").

6.4 Preprocessing

In this section, we present a few conditions which, when satisfied, allow us to directly conclude that a given instance of 2-GARP or 2-SARP is either a Yes-instance or a No-instance. Since 2-SARP is stronger than 2-GARP, we focus on 2-SARP when identifying Yes-instances and on 2-GARP for No-instances. We separately consider the cases with and without assignable information.

Let first S be a consumption data set without assignable information (i.e., $q_t^1 = q_t^2 := \mathbf{0}$, for all $t \in N$) and let $G = \big(N, A\big)$ be the directed graph associated with S: each observation in S defines a vertex, and there is an arc $s \to t \in A$ if and only if $p_s' q_s \geq p_s' q_t$.

Lemma 1. *If the vertices of G can be partitioned into two subsets such that each subset induces an acyclic subgraph, then S satisfies* 2-SARP*, and hence* 2-GARP*.*

Proof: Suppose that N can be partitioned into two subsets N_1 and N_2 such that each subset induces an acyclic subgraph. We define personalized quantity vectors as follows. For all $s \in N_1$, we set $x_s^1 := q_s$ and $x_s^2 := \mathbf{0}$, and for all $s \in N_2$, we set $x_s^1 := \mathbf{0}$ and $x_s^2 := q_s$. It is not difficult to see that the resulting personalized consumption data sets S_ℓ, for $\ell = 1, 2$, satisfy SARP, and hence GARP. □

The setting exploited in Lemma 1 is known in economics as *situation-dependent dictatorship* because, for each observation, the complete bundle is consumed by one member of the household (Cherchye, De Rock, and Vermeulen, 2007; Deb, 2011). Note that testing the conditions stated in Lemma 1 requires testing whether graph G can be partitioned into two acyclic subgraphs, and the latter problem is known to be NP-complete (Talla Nobibon, Hurkens, Leus, and Spieksma, 2012a). We next present two properties that can be checked in polynomial time.

Let us define the vectors $\tilde{q}_t^1 := q_t - q_t^2$ and $\tilde{q}_t^2 := q_t - q_t^1$, for $t \in N$. Consider the directed graphs $\tilde{H}^1 = \big(N, \tilde{A}_1\big)$ and $\tilde{H}^2 = \big(N, \tilde{A}_2\big)$ where there is an arc $s \to t \in \tilde{A}_1$ (respectively, $s \to t \in \tilde{A}_2$) if and only if $p_s' \tilde{q}_s^1 \geq p_s' q_t^1$ (respectively, $p_s' \tilde{q}_s^2 \geq p_s' q_t^2$).

Lemma 2. *If $\tilde{H}^1$ and $\tilde{H}^2$ are acyclic, then S satisfies* 2-SARP*, and hence* 2-GARP*.*

Proof: Note that for any pair of feasible personalized consumption sets $S_\ell := \{\big(p_t, x_t^\ell\big) : \; t \in N\}$, $\ell = 1, 2$, member ℓ consumes at least q_t^ℓ and at most $\tilde{q}_t^\ell$ in each observation t: $q_t^\ell \leq x_t^\ell \leq \tilde{q}_t^\ell$. Therefore, the directed graph $G^\ell = \big(V, A^\ell\big)$ associated with S_ℓ is a subgraph of $\tilde{H}^\ell$, and when $\tilde{H}^\ell$ is acyclic, its subgraph G^ℓ is also acyclic. This implies that S_ℓ $(\ell = 1, 2)$ satisfies SARP, and hence GARP. □

The next result identifies a condition that leads to a NO answer. We consider the directed graphs $\overline{H}^1 = \big(N, \overline{A}_1\big)$ and $\overline{H}^2 = \big(N, \overline{A}_2\big)$ where there is an arc $s \to t \in \overline{A}_1$ (respectively, $s \to t \in \overline{A}_2$) if and only if $p_s' q_s^1 > p_s' \tilde{q}_t^1$ (respectively, $p_s' q_s^2 > p_s' \tilde{q}_t^2$).

Lemma 3. *If either $\overline{H}^1$ or $\overline{H}^2$ contains a cycle, then S does not satisfy* 2-GARP*, and hence S does not satisfy* 2-SARP*.*

Proof: The proof follows a similar reasoning as that of Lemma 2. □

All conditions described by Lemmas 1–3 can be verified in a preprocessing step before the core of the SA algorithm is executed, as explained in the next section.

7. Computational experiments

We present the real-world data used for our experiments in Section 7.1, we describe some issues related to the implementation of the algorithms in Section 7.2, and we subsequently discuss computational results in Section 7.3.

7.1 Data

The algorithms described in this paper are used for testing the rationality of the behavior of Dutch households. More precisely, we use the data sample presented in Cherchye and Vermeulen (2008). This sample is based on eleven waves of the DNB Household Survey (formerly known as the CentER Savings Survey), drawn from 1995 to 2005. Each household consists of a couple, and each member of each couple is employed, is aged between 25 and 55, and has no child. The goods are aggregated into three groups: the first group contains goods that are shared by the two members of the household, the second one contains leisure female goods, and the third one contains leisure male goods. Natural assignable information supposes that goods in the second group are consumed exclusively by one member, and goods in the third group are consumed exclusively by the other member. For ease of exposition, in the rest of this section we simply assume that we have three goods. We refer to Cherchye and Vermeulen (2008) for additional details about the data.

The complete consumption data set contains more than 500 observations. From this large data set, we create smaller ones as follows. For each value of n ($n \geq 10$), we generate 20 instances, where each instance contains n randomly-drawn distinct observations. We organize all the instances into four sets. The first one, denoted Data I, contains all the data sets generated for different values of n, where assignable information is ignored. This corresponds with the setting where each member in the household can consume any good. The second set of instances, Data II, contains the same instances as Data I, but with assignable information specifying that the second good is consumed exclusively by one member, whereas the third good is consumed exclusively by the other member.

We create two additional collections of data sets by adding proportional assignable information $q^{\ell}_{t,1}$ on non-assignable good(s) for the instances in Data I and Data II. We consider seven different pairs of proportions $(q^1_{t,1}, q^2_{t,1})$: (20, 70), (25, 70), (40, 40), (40, 45), (45, 40), (70, 25), and (70, 20). In the first setting, for instance, the assumption is that member 1 consumes at least 20% of the first good ($q^1_{t,1} = 0.20$), whereas member 2 consumes at least 70% of the first good ($q^2_{t,1} = 0.70$). The third (respectively, fourth) set Data III (respectively, Data IV) contains instances from Data I (respectively, Data II) with at most $n = 100$ observations. Each instance in Data I (respectively, Data II) gives rise to seven instances in Data III (respectively, Data IV), corresponding with the seven pairs listed above. All the instances described in this section can be found at `http://users.aims.ac.za/~tal/programmingcodes.html`

7.2 Implementation issues

In this section, we provide some details related to the implementation of the different solution approaches. The MIP formulations F1:2-GARP, F1:2-SARP, F2:2-GARP and F2:2-SARP are solved using CPLEX 12.4. More precisely, each formulation is turned into an optimization problem by adding to it one of four alternative objective functions, namely: (**obj1**) Maximize the consumption of member 1 (sum of the variables x^1_t); (**obj2**) Maximize the "number of preferences" of member 1 (sum of the variables y^1_{st}); (**obj3**) Maximize the total consumption difference between the two members (sum of the differences $x^1_t - x^2_t$); and (**obj4**) Minimize a constant. We halt the solver after the first feasible solution is found, and a time limit of 30 minutes is imposed on the computation time. In these MIP formulations, we set $\varepsilon := 10^{-6}$, which is the tolerance value for CPLEX, and we use $M_s := p_s q_s + 1$ in constraints (22). For F2:2-GARP and F2:2-SARP, we also consider alternative implementations that include the cuts describe by equation (23).

For the SA algorithm, a preprocessing phase consists in checking the conditions of Lemma 1–3. We use the backtracking algorithm proposed by Talla Nobibon, Hurkens, Leus, and Spieksma (2012a) to investigate whether the vertices of a directed graph can be partitioned into two acyclic subgraphs (with a time limit of 2 minutes), and we use the topological ordering algorithm (Ahuja, Magnanti, and Orlin, 1993) to check whether a directed graph is acyclic. The value of the parameters of the SA algorithm are chosen as proposed (and used) by Dekkers and Aarts (1991): $\delta := 0.1$, $L := 10$ and $\gamma := 0.75$. The boolean indicator *stop* takes the value "true" when either $c < 10^{-4}$ or when the running time of the algorithm exceeds a limit of 30 minutes. Furthermore, *stop* is also set to "true" after 20 iterations of SA without improvement of the objective function. This value was chosen after some preliminary experiments.

The initial solution is obtained as explained in Section 6.1 by considering ten distinct permutations in procedure **P1**. We start with the identity permutation and we subsequently consider nine other randomly generated permutations. Given the preference graphs G^1 and G^2, we use CPLEX 12.4 to check whether $\mathcal{P}(G^1, G^2)$ defines a nonempty domain.

Finally, the local search procedure $LS(\mathbf{X})$ (**Algorithm** 2) investigates the neighborhoods $\mathcal{N}_{st}(\mathbf{X})$ successively, starting with the arc $s \to t$ that has the smallest value $m_{st} = \max\{p'_s x^1_s - p'_s x^1_t,\, p'_s x^2_s - p'_s x^2_t\}$. It then explores the arc with the second smallest value of m_{st}, and so on. The rationale behind this choice is that the elimination of an arc $s \to t$ with the smallest value m_{st} is unlikely to create a new cycle.

7.3 Computational results

All the algorithms presented in this paper have been coded in C using Microsoft Visual Studio C++ 2010. The experiments were run on a Dell Optiplex 760 personal computer with a Pentium R processor, 3.16 GHz clock speed and 3.21 GB RAM, equipped with Windows XP. All CPU times are expressed in *seconds*, and the plots of the average CPU times are smooth Bézier approximations (Farin, 2006) of the real plots. We compare different implementations of each MIP formulation in Section 7.3.1, and we compare the formulations among themselves in Section 7.3.2. The best MIP formulation is compared to the SA algorithm in Section 7.3.3. Finally, we further investigate the efficiency of the SA algorithm in Section 7.3.4.

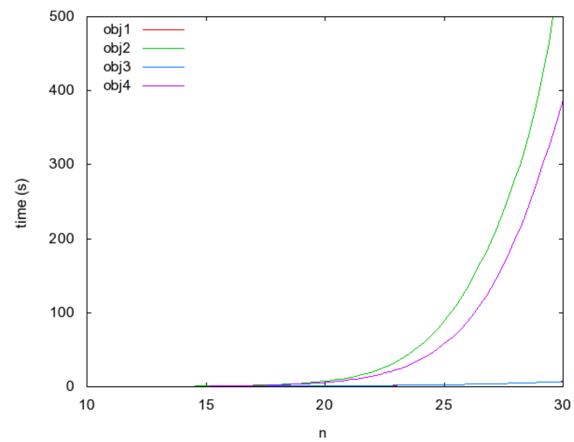
7.3.1 Comparison of different implementations of the MIP formulations

We use instances from Data I with at most 30 observations to compare four different implementations of each MIP formulation. These four implementations correspond with the four different objective functions presented in the previous section.

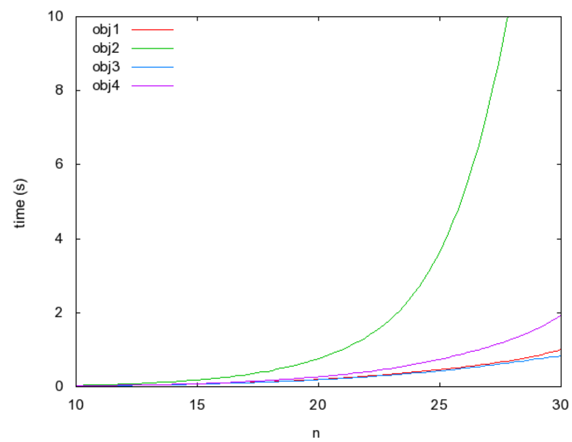
In Figure 1 we plot the CPU time of the two MIP formulations for 2-GARP. Figure 1(a) shows the average CPU time of F1:2-GARP for each of the four objectives. The first objective (**obj1**) and the third objective (**obj3**) yield the smallest average CPU time, while the remaining two objectives (**obj2** and **obj4**) lead to considerably higher average CPU time. Figure 1(b) shows the average CPU time of F2:2-GARP for the four objectives. The objective function **obj3** leads again to smaller average CPU time than the other ones, with **obj1** a close second. Figure 1(c) displays the average CPU time of F2:2-GARP with the cuts (23), and we observe that the objective functions **obj1** and **obj3** also dominate the other objectives. Finally, Figure 1(d) plots the best average CPU time of F2:2-GARP for **obj3**, with and without the cuts (23). Clearly, adding the cuts (23) is beneficial for the formulation F2:2-GARP. Therefore, in the rest of the section, whenever we refer to the formulation F2:2-GARP, we implicitly mean that we include the cuts (23) and that we use the objective function **obj3**. The comparison of the MIP formulations for 2-SARP yields the same results: using the objective function **obj3** and the cuts (23) leads to the best implementation among those investigated.

7.3.2 Comparison of the MIP formulations

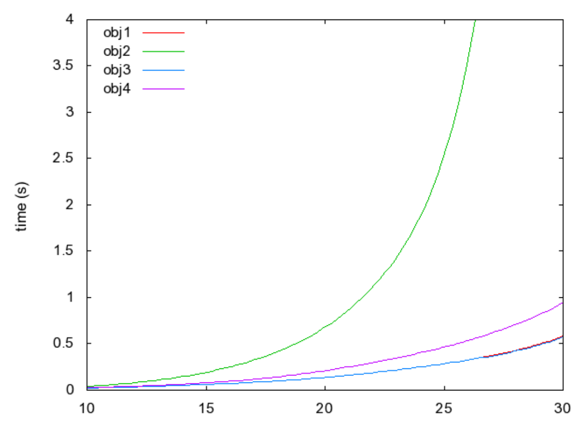
We now turn to a more extensive evaluation of the alternative formulations F1 and F2 for each of the problems 2-GARP and 2-SARP. In Figure 2, we compare the best implementations of F1:2-GARP and F2:2-GARP on instances from Data I and Data II. We perform a similar comparison for F1:2-SARP and F2:2-SARP in Figure 3. With respect both to the average CPU time and to the number of instances solved to optimality within the time limit (30 minutes), we find that F2:2-GARP dominates F1:2-GARP, and that F2:2-SARP dominates F1:2-SARP. On average, F2:2-GARP (respectively, F2:2-SARP) solves instances containing two to three times more observations than the instances that can be solved using F1:2-GARP (respectively, F1:2-SARP). When looking at the average CPU time, we observe that instances of 2-SARP tend to be more difficult than instances of 2-GARP with the same size. This may be due to the constraints (40)–(41) in formulation F1:2-SARP and to the constraints (19) in formulation F2:2-SARP, which significantly increase the size of these MIP formulations.



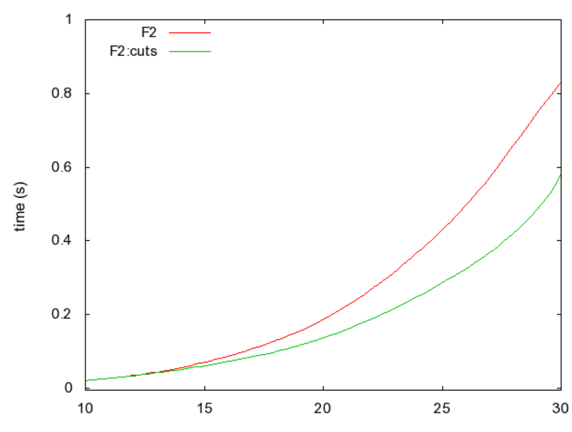
(a) Different objectives for F1:2-GARP



(b) Different objectives for F2:2-GARP



(c) Different objectives for F2:2-GARP with cuts (23)



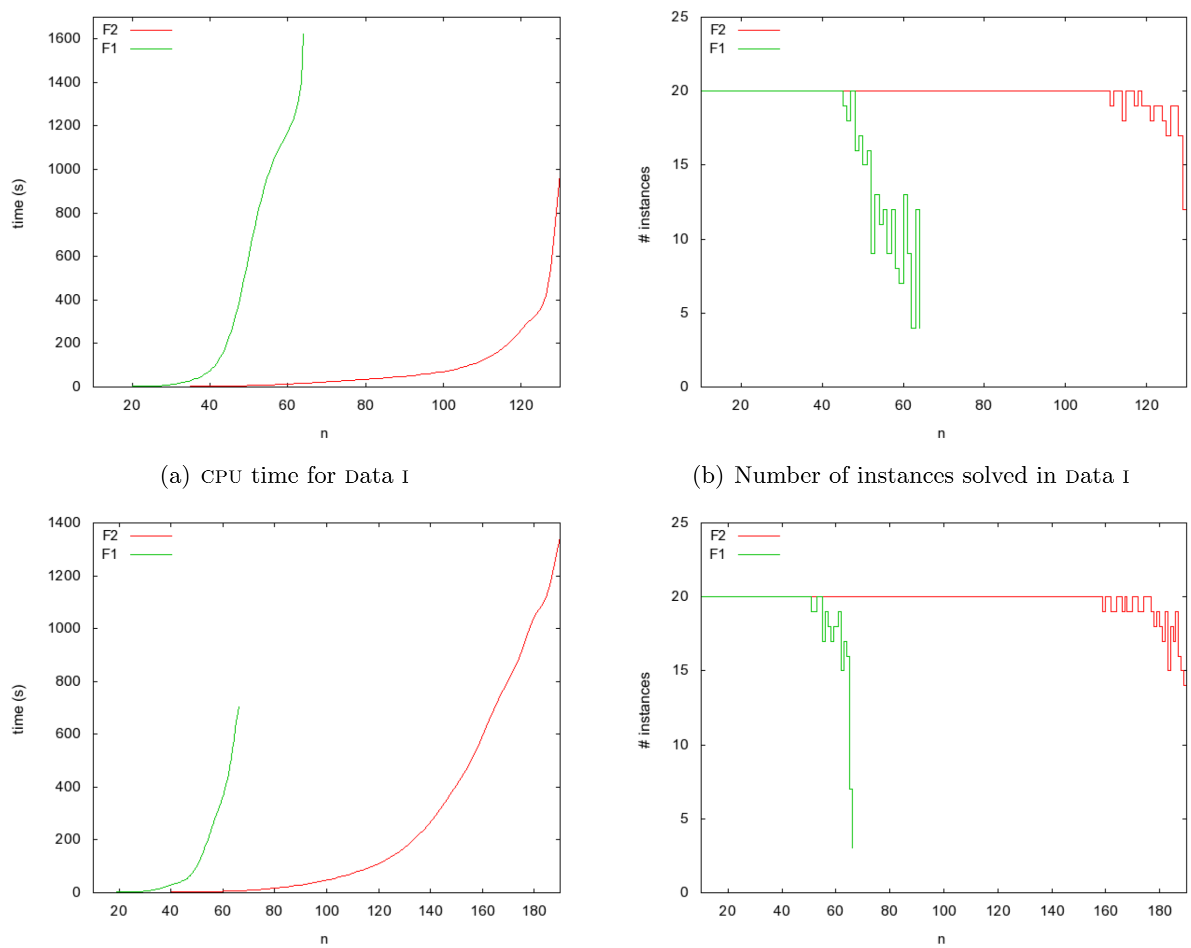
(d) Comparison of F2:2-GARP with and without cuts (23)

Figure 1: Different implementations of the MIP formulations for 2-GARP.

7.3.3 Comparison of the MIP formulations with the SA algorithm

Figure 4 shows the average CPU time and the number of instances solved within the time limit of 30 minutes by the SA algorithm and the best MIP formulations, when applied to instances from Data I and Data II. For 2-GARP, the SA algorithm solves instances with size on average twice the size of instances that can be solved using F2:2-GARP, irrespective of whether the instances are drawn from Data I or Data II. On the other hand, for instances of 2-SARP from Data II the SA algorithm solves instances with size on average twice the size of instances that can be solved using F2:2-SARP. For instances in Data I, that factor increases up to four. Furthermore, for these instances the difference between the average CPU time required by the SA algorithm for 2-GARP and 2-SARP is very small; this is not the case for the MIP formulations.

Overall, the SA algorithm is effective for solving large instances of 2-GARP and 2-SARP, with or without assignable information. We mention that by ignoring the time limit when running the SA algorithm we have found that the initial real-world consumption data set (with more than 500 observations) satisfies 2-SARP, and hence 2-GARP. The good results achieved by the SA algorithm



(a) CPU time for Data I

(b) Number of instances solved in Data I

(c) CPU time for Data II

(d) Number of instances solved in Data II

Figure 2: Comparison of the MIP formulations F1 and F2 for 2-GARP

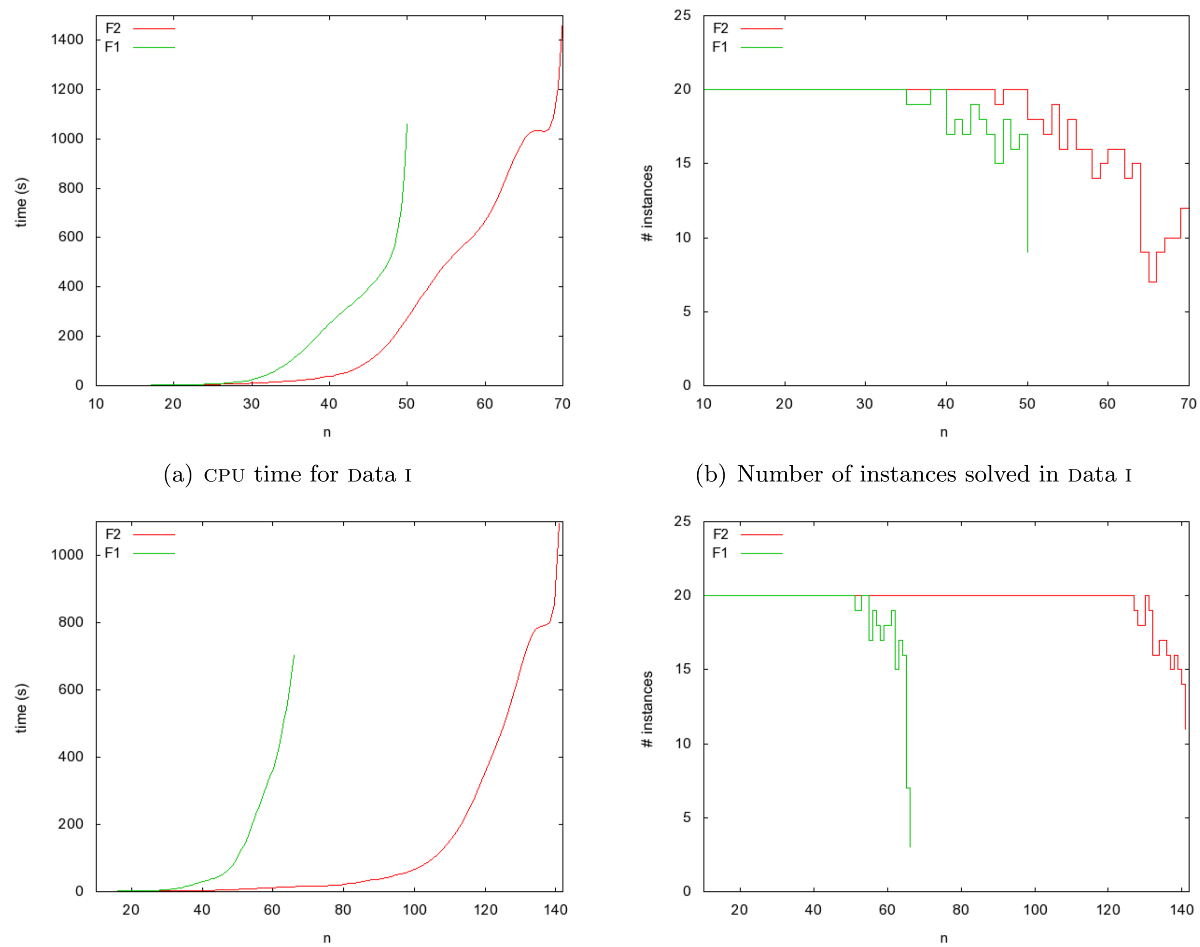
.

should not conceal the fact that it is a heuristic and that there may exist consumption data that are consistent with 2-GARP or 2-GARP for which the SA algorithm does not find a feasible solution (with objective function value of zero), even though such a solution exists. This phenomenon is evaluated in the next section.

7.3.4 Effectiveness of the SA algorithm

In this section, we first evaluate the effectiveness of the SA algorithm for solving 2-GARP instances from Data III and Data IV. The assumption underlying these experiments is that the consideration of proportional assignable information will: (1) increase the difficulty of the instances and (2) turn some "Yes" instances from Data I and Data II into "No" instances. We use a time limit of five minutes when applying the SA algorithm.

Figure 5 displays the average CPU time and the number of Yes instances found by the MIP formulation F2:2-GARP and the SA algorithm. From Figure 5(b), we observe that the SA algorithm identifies almost all the Yes instances in Data III, except five. The plot focuses on instances with



(a) CPU time for Data I

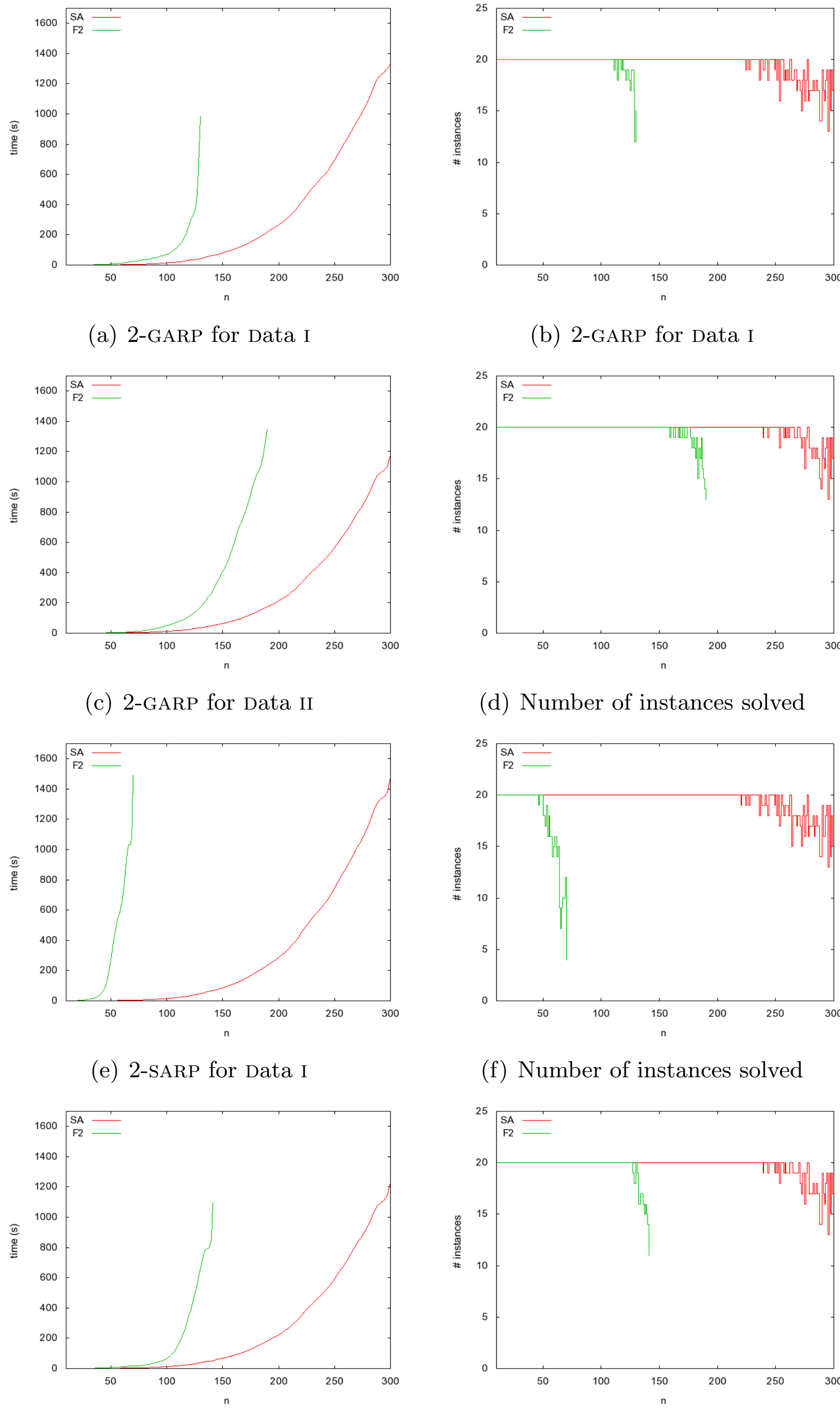
(b) Number of instances solved in Data I

(c) CPU time for Data II

(d) Number of instances solved in Data II

Figure 3: Comparison of the MIP formulations F1 and F2 for 2-SARP

.



(a) 2-GARP for Data I

(b) 2-GARP for Data I

(c) 2-GARP for Data II

(d) Number of instances solved

(e) 2-SARP for Data I

(f) Number of instances solved

(g) 2-SARP for Data II

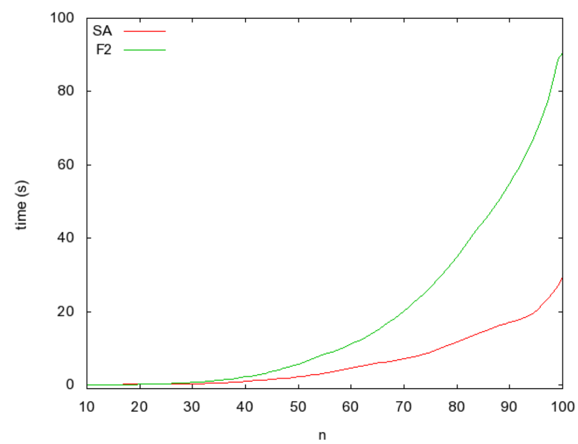
(h) Number of instances solved

Figure 4: Comparison of the best MIP formulations and the SA algorithm.

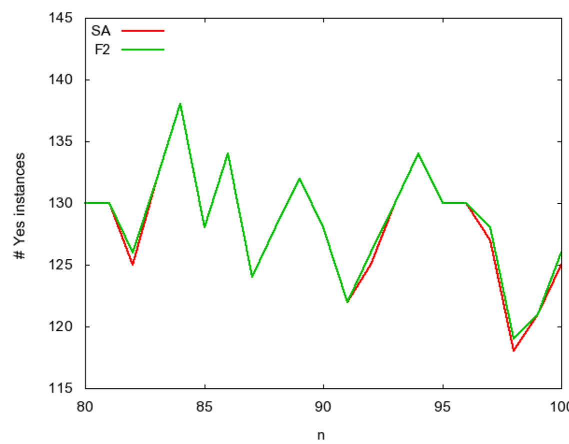
more than 80 observations; for smaller values of n, the SA algorithm and F2:2-GARP find exactly the same number of Yes instances. The average CPU time of the SA algorithm is clearly smaller than that of F2:2-GARP as shown in Figure 5(a). For instances from Data IV, the SA algorithm fails to identify 28 Yes instances out of a total of 10,724 Yes instances. (Note that for each value of n, there are 140 instances. For two instance sizes, $n = 77$ and $n = 98$, the SA algorithm fails to identify two Yes instances. For the remaining value of n, the SA algorithm either finds the same number of Yes instances as the MIP formulation F2:2-GARP or only finds one less than the MIP formulation.) Figure 5(c) indicates that the average CPU time of the SA algorithm increases significantly, and even exceeds that of the MIP formulation F2:2-GARP for large values of n. This is mainly due to the increasing number of No instances present in Data IV, some of which force the SA algorithm to run until the time limit. Overall, we conclude from Figure 5 that the SA algorithm is efficient for identifying Yes instances of 2-GARP.

Similar results are reported for 2-SARP in Figure 6. In this case, F2:2-SARP could not solve most of the larger instances (say, with $n > 50$) in Data III within the time limit of 5 minutes. As a result, many Yes instances were not correctly identified by this approach, which explains the observations in Figures 6(a)-6(b).

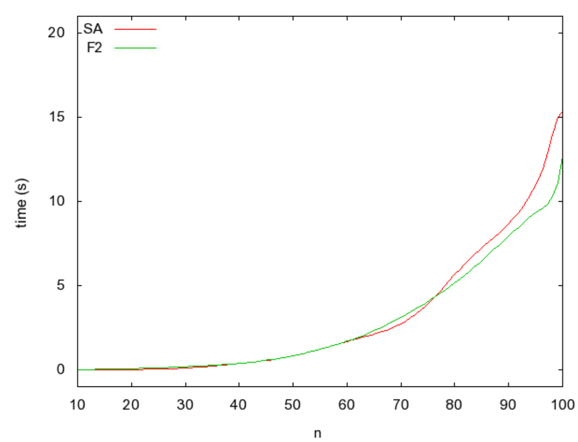
To summarize, the SA algorithm can solve much larger instances than the MIP formulations, and seldom outputs 'Undecided' when 'Yes' is the correct answer.



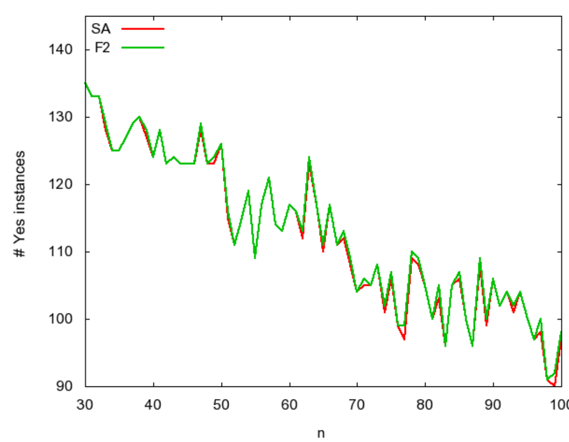
(a) CPU time for Data III



(b) Number of instances solved in Data III



(c) CPU time for Data IV



(d) Number of instances solved in Data IV

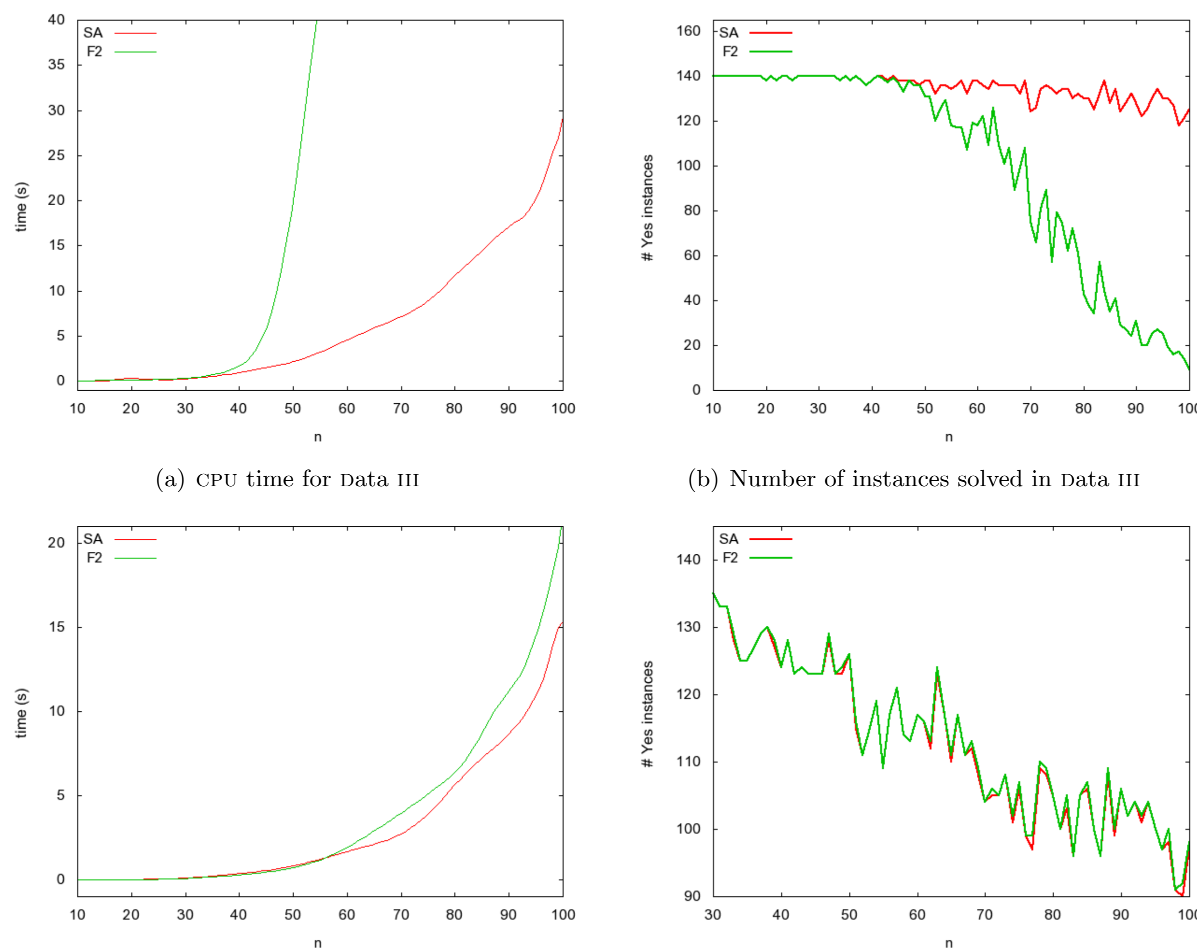
Figure 5: Comparison of the number of Yes instances of 2-GARP found by F2 and by SA.

8. Conclusion

We consider the problem of testing whether household behavior involving multiple decision makers is consistent with the Generalized Axiom of Revealed Preference (GARP), and the Strong Axiom of Revealed Preference (SARP). More concretely, we consider the collective consumption models called 2-GARP and 2-SARP, which are extensions of GARP and SARP to households with two decision makers. The models allow for so-called *assignable information*, meaning that information concerning consumption of individual members of the household can be taken into account.

We prove that it is NP-complete to test these two models, even if there is no assignable information. Next, based on a particular characterization of GARP relying upon utility levels, we propose new MIP formulations for these models. The resulting formulations have significantly less constraints than the previous models from literature. Also, we propose an intricate simulated annealing (SA) heuristic, which is based on a representation of a continuous solution space as a solution space with a combinatorial structure.

Finally, to test the formulations and the SA heuristic, we have conducted extensive computational experiments. These experiments reveal the following basic insights. Firstly, the exact



(a) CPU time for Data III

(b) Number of instances solved in Data III

(c) CPU time for Data IV

(d) Number of instances solved in Data IV

Figure 6: Comparison of the number of Yes instances of 2-SARP found by F2 and by SA.

algorithms based on our new MIP formulations are effective for solving medium-size instances of both 2-GARP and 2-SARP. More specifically, all instances with up to 120 observations in case of 2-GARP, and 60 observations in case of 2-SARP, are solved within ten minutes. These results are a huge improvement compared to the results that correspond to previous models from literature. Secondly, the SA heuristic is able to solve even larger instances of our problems: instances of up to 300 observations are solved within half an hour. Moreover, it happens only very rarely that the SA-heuristic does not identify a Yes-instance as such.

Arguably, testing rationality is only a first step in the analysis of household consumption data. Next steps involve recovery and counterfactual analysis, which also raises challenging issues from a computational point of view. We anticipate that the methods and models proposed here can also play a supporting role in this other context.

Appendix

A. Proof of Theorem 1

We argue in the introduction that 2-GARP and 2-SARP belong to the class NP. To complete the proof of Theorem 1, we show that 2-GARP and 2-SARP are at least as hard as a known NP-complete problem. Below, we prove this result for 2-GARP and we show how it can be modified to hold for 2-SARP. We proceed in two steps: First, we build a polynomial-time reduction from an intermediary problem (that we call 2-BUDGET) to 2-GARP. This reduction implies that the problem 2-GARP is at least as hard as 2-BUDGET. Subsequently, we build a reduction from the problem of testing the Collective Axiom of Revealed Preference (CARP) to 2-BUDGET, which shows that 2-BUDGET is at least as hard as CARP. Because the problem CARP is NP-complete (Talla Nobibon and Spieksma, 2010) and 2-BUDGET belongs to the class NP, we infer that 2-BUDGET is also NP-complete and hence that 2-GARP is NP-complete.

For ease of exposition, because we do not use assignable information in this proof we simplify the notation by ignoring the vectors containing assignable information. The decision problems 2-BUDGET and CARP are defined next.

Problem 2-BUDGET:
Instance: A set $A := \{a_{st} : \; s, t \in M\}$ of $|M|^2$ positive numbers.
Question: Do there exist non-negative numbers a^1_{st} and a^2_{st} for all $s, t \in M$ such that:

1. we have $a^1_{st} + a^2_{st} = a_{st}$ for all $s, t \in M$;
2. if there exist a sequence $s, u, v, \ldots, w, t \in M$ and $\ell \in \{1, 2\}$ satisfying $a^\ell_{ss} \geq a^\ell_{su}$, $a^\ell_{uu} \geq a^\ell_{uv}$, $\ldots$, $a^\ell_{ww} \geq a^\ell_{wt}$ then $a^\ell_{tt} \leq a^\ell_{ts}$.

The problem 2-BUDGET belongs to the class NP. Indeed, suppose that we are given a^1_{st} and a^2_{st} for all $s, t \in M$; we can easily check that there are all non-negative and satisfied 1.; that is: $a^1_{st} + a^2_{st} = a_{st}$ for all $s, t \in M$. To verify 2., we proceed as follows. For each ℓ, we build a directed graph $G^\ell = \left(M, A^\ell\right)$ where each node corresponds with a unique $i \in M$ and there is an arc from node s to node t ($s \to t \in A^\ell$) if and only if $a^\ell_{ss} \geq a^\ell_{st}$. Furthermore, we compute the subgraph $G^\ell_{\text{scc}} = \left(M, A^\ell_{\text{scc}}\right)$ of G^ℓ consisting of the union of all the strongly connected components of G^ℓ. The condition 2. is satisfied if and only if every arc $s \to t \in A^\ell_{\text{scc}}$ satisfies $a^\ell_{ss} = a^\ell_{st}$. Because all these steps are completed in polynomial time, we conclude that 2-BUDGET belongs to the class NP.

Problem CARP:

Instance: A consumption data set $S := \{(p_t, q_t) : t \in \mathbb{T}\}$, where $\mathbb{T} := \{1, \ldots, T\}$.
Question: Do there exist two binary relations H_0^1 and H_0^2 (and hence their transitive closures H^1 and H^2) that satisfy for all $s, t, t_1, t_2 \in \mathbb{T}$:

Rule 1: if $p'_s q_s \geq p'_s q_t$ then $s\, H_0^1\, t$ or $s\, H_0^2\, t$;

Rule 2: if $p'_s q_s \geq p'_s q_t$ and $t\, H^{\ell_1}\, s$ then $s\, H_0^{\ell_2}\, t$, with $\ell_1 \neq \ell_2$ for $\ell_1, \ell_2 \in \{1, 2\}$;

Rule 3: if $p'_s q_s \geq p'_s (q_{t_1} + q_{t_2})$ and $t_1\, H^{\ell_1}\, s$ then $s\, H_0^{\ell_2}\, t_2$, with $\ell_1 \neq \ell_2$ for $\ell_1, \ell_2 \in \{1, 2\}$;

Rule 4: if $p'_s q_s > p'_s q_t$ then either $\neg\left(t\, H^1\, s\right)$ or $\neg\left(t\, H^2\, s\right)$;

Rule 5: if $p'_s q_s > p'_s (q_{t_1} + q_{t_2})$ then $\neg\left(t_1\, H^1\, s\right)$ or $\neg\left(t_2\, H^2\, s\right)$;

where $\neg\left(s\, H^\ell\, t\right)$ means that s is not in relation with t with respect to H^ℓ. Talla Nobibon and Spieksma (2010) prove that CARP is an NP-complete problem. Furthermore, their proof remains valid even if for all $s \neq t \neq u \neq s \in \mathbb{T}$, we have $p'_s q_s \neq p'_s q_t$ and $p'_s q_s \neq p'_s (q_t + q_u)$.

Reduction from 2-BUDGET to 2-GARP

Given an instance $A := \{a_{st} : s, t \in M\}$ of 2-BUDGET, we build an instance $S := \{(p_t, q_t) : t \in N\}$ of 2-GARP with $n = |N| := |M|$ observations over $m := |M|$ goods. For observation t, the price of good t is 1 whereas that of the remaining goods is ε, where ε is a very small positive number. Therefore $p_t := (\varepsilon, \ldots, \varepsilon, 1, \varepsilon, \ldots, \varepsilon)$, where 1 is the t^{th} component of that vector. The quantity vector q_t is the solution to the system of equations:

$$\mathbf{P} q_t \;=\; A_t, \tag{29}$$

where $\mathbf{P}$ is the $n \times n$ matrix of stacked (row) price vectors, i.e.

$$\mathbf{P} := \begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon & \ldots & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & \varepsilon & \ldots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \ldots & \varepsilon & \varepsilon \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \ldots & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \ldots & \varepsilon & \varepsilon & 1 \end{pmatrix}$$

and $A'_t := (a_{1t}, a_{2t}, \ldots, a_{m,t})$, for $t = 1, \ldots, n := |M|$. The quantity vector q_t is well defined if and only if equation (29) has a solution whose components are non-negative. We use the following result to prove that this is indeed the case.

Theorem 2 (Walters (1969)). *Let $\bar{B}$ be a non-negative invertible matrix with a positive diagonal $D > 0$ and let $\bar{y} > 0$ be a positive vector. Let $B := D^{-1}\bar{B}$ and $y := D^{-1}\bar{y}$. If $0 < 2y - By$ then $2y - By \leq x \leq y$, where $x := B^{-1}y$.*

We show that the hypotheses of Theorem 2 hold. By taking the matrix $\bar{B} = \mathbf{P}$, it has a positive diagonal and it is invertible (for an appropriate value of ε). Each component of the vector $\bar{y}' := (a_{1t}, a_{2t}, \ldots, a_{m,t})$ is positive. Because $D = I$ we have $D^{-1} = I$, which implies that $y = \bar{y}$ and $B = \bar{B}$. Therefore, $2y - By = 2y - \mathbf{P}y$ and for ε small enough, we have $2y - \mathbf{P}y > 0$. Theorem 2 implies that $q_t := x = B^{-1}y = \mathbf{P}^{-1}y$ is a solution of (29) with positive (and hence non-negative) components. This completes the description of the instance S of 2-GARP, which is built from $A := \{a_{st} : s,t \in M\}$. Note that this construction is completed in polynomial time.

We now argue that S satisfies 2-GARP if and only if $A := \{a_{st} : s,t \in M\}$ is a YES instance of 2-BUDGET. On the one hand, suppose that S satisfies 2-GARP. There exist $x_t^1,\, x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$, for each $t \in N$, such that for $\ell = 1,2$, the set $S_\ell := \left\{\left(p_t, x_t^\ell\right) :\ t \in N\right\}$ satisfies GARP. We define $a_{ts}^1 := p_t' x_s^1$ and $a_{ts}^2 := p_t' x_s^2$, for all $s,t \in M$. Note that because $x_t^1, x_t^2 \in \mathbb{R}_+^m$ for all $t \in N$, we infer that each a_{ts}^ℓ (for $\ell = 1,2$ and $s,t \in M$) is non-negative. We now prove that a_{ts}^ℓ ($\ell = 1,2$ and $s,t \in M$) satisfy the conditions defining 2-BUDGET.

1. For $s,t \in M$ we have $a_{ts}^1 + a_{ts}^2 = p_t' x_s^1 + p_t' x_s^2 = p_t' \left(x_s^1 + x_s^2\right) = p_t' q_s = a_{ts}$.

2. Let us suppose that there exist a sequence $s,u,v,\ldots,w,t \in M$ and $\ell \in \{1,2\}$ such that $a_{ss}^\ell \geq a_{su}^\ell$, $a_{uu}^\ell \geq a_{uv}^\ell$, $\ldots$, $a_{ww}^\ell \geq a_{wt}^\ell$. These inequalities imply that member ℓ prefers s over t because $a_{ij}^\ell := p_i' x_j^\ell$. If we would have $a_{tt}^\ell > a_{ts}^\ell$ then that will mean that $p_t' x_t^\ell > p_t' x_s^\ell$. Together with the previous observation, we will have that member ℓ prefers s over t and $p_t' x_t^\ell > p_t' x_s^\ell$. This will contradict the fact that $S_\ell := \left\{\left(p_t, x_t^\ell\right) :\ t \in N\right\}$ satisfies GARP. Therefore, $a_{tt}^\ell \leq a_{ts}^\ell$ and hence $A := \{a_{st} : s,t \in M\}$ is a YES instance of 2-BUDGET.

On the other hand, we suppose that $A := \{a_{st} : s,t \in M\}$ is a YES instance of 2-BUDGET and we want to show that S satisfies 2-GARP. Let a_{st}^1, a_{st}^2 for $s,t \in M$ be a solution to the instance of 2-BUDGET. We define x_t^1 and x_t^2 to be solutions to the following system of equations:

$$\mathbf{P} x_t^1 \;=\; A_t^1, \quad t = 1,\ldots,m, \tag{30a}$$

and

$$\mathbf{P} x_t^2 \;=\; A_t^2, \quad t = 1,\ldots,m, \tag{30b}$$

where the transpose of A_t^ℓ is given by $\left(a_{1t}^\ell, a_{2t}^\ell, \ldots, a_{m,t}^\ell\right)$ for $\ell = 1,2$. There exist solutions to (30a) and (30b) because $\mathbf{P}$ is invertible. Furthermore, from the proof of Theorem 2 we infer that these solutions are non-negative, which implies that x_t^1 and x_t^2 are well-defined.

We now prove that each set $S_\ell := \left\{\left(p_t, x_t^\ell\right) :\ t \in N\right\}$ (for $\ell = 1,2$) satisfies GARP. If that was not the case then there will exist two distinct observations s and t such that a member ℓ prefers s over t and $p_t' x_t^\ell > p_t' x_s^\ell$. This would mean that there exists a sequence $s,u,v,\ldots,w,t \in M$ such that $p_s' x_s^\ell \geq p_s' x_u^\ell$, $p_u' x_u^\ell \geq p_u' x_v^\ell$, $\ldots$, $p_w' x_w^\ell \geq p_w' x_t^\ell$ and $p_t' x_t^\ell > p_t' x_s^\ell$; in other words, $a_{ss}^\ell \geq a_{su}^\ell$, $a_{uu}^\ell \geq a_{uv}^\ell, \ldots, a_{ww}^\ell \geq a_{wt}^\ell$ and $a_{tt}^\ell > a_{ts}^\ell$. This will contradict the fact that we have a YES instance of 2-BUDGET. Therefore, each $S_\ell := \left\{\left(p_t, x_t^\ell\right) :\ t \in N\right\}$ ($\ell = 1,2$) satisfies GARP and hence S satisfies 2-GARP. This completes the first part of the proof: 2-GARP is at least at hard as 2-BUDGET.

Modifications for 2-SARP

For 2-SARP, we adapt the definition of 2-BUDGET by replacing condition 2. by the following:

2'. if there exist a sequence $s,u,v,\ldots,w,t \in M$ and $\ell \in \{1,2\}$ satisfying $a_{ss}^\ell \geq a_{su}^\ell$, $a_{uu}^\ell \geq a_{uv}^\ell$, $\ldots$, $a_{ww}^\ell \geq a_{wt}^\ell$ then $a_{tt}^\ell < a_{ts}^\ell$.

It is not difficult to see that the reduction described in the previous section is still valid when we replace 2-GARP by 2-SARP.

Reduction from CARP to 2-BUDGET

We consider an instance $S := \{(p_t, q_t) : t \in \mathbb{T}\}$ (where $\mathbb{T} := \{1, \ldots, T\}$) of CARP satisfying $p'_s q_s \neq p'_s q_t$ and $p'_s q_s \neq p'_s (q_t + q_u)$, for all $s \neq t \neq u \neq s \in \mathbb{T}$. These restrictions have no impact on the complexity of CARP; see Talla Nobibon and Spieksma (2010). We build an instance $A := \{a_{ij} : i, j \in M\}$ of 2-BUDGET as follows. The set M is defined by:

- for each $t \in \mathbb{T}$, there are three elements t^0, t^1, and t^2 in M;
- for $s, t \in \mathbb{T}$ with $s < t$, there are two elements $o(s;t)$ and $o(t;s)$ in M;
- for $s, t, u \in \mathbb{T}$ with $s < t < u$, there are three elements $o(s;t,u)$, $o(t;s,u)$, and $o(u;s,t)$ in M.

Hereinafter, $o(s;t,u)$ and $o(s;u,t)$ refer to the same element in M. In total, the set M contains $3T + T(T-1) + T(T-1)(T-2)/2 \equiv O(T^3)$ elements, which is polynomial in T. We now specify the value of a_{ij}, for all $i, j \in M$.

1. For t^0, t^1, and t^2 corresponding with the same observation $t \in \mathbb{T}$, we have:
   $a_{t^0,t^0} := 3, \quad a_{t^0,t^1} := 2, \quad a_{t^0,t^2} := 2,$
   $a_{t^1,t^0} := 1, \quad a_{t^1,t^1} := 3, \quad a_{t^1,t^2} := 1,$
   $a_{t^2,t^0} := 1, \quad a_{t^2,t^1} := 1, \quad a_{t^2,t^2} := 3.$
2. For all $s \neq t \in \mathbb{T}$, we have:
   $a_{s^0,o(s;t)} := 0.5$; furthermore, if $p'_s q_s > p'_s q_t$ then $a_{o(s;t),o(s;t)} := 2$ and $a_{o(s;t),t^0} := 1$.
3. For all $s \neq t \neq u \neq s \in \mathbb{T}$, we have:
   $a_{s^0,o(s;t,u)} := 0.5$; and
   if $p'_s q_s > p'_s(q_t + q_u)$ then $a_{o(s;t,u),o(s;t,u)} := 3$, $a_{o(s;t,u),t^0} := 1$, and $a_{o(s;t,u),u^0} := 1$.
4. For all the values that are not yet specified, we set $a_{ij} := 1$ if $i = j$ and $a_{ij} := 10$ otherwise.

This completes the construction of the instance $A := \{a_{st} : s, t \in M\}$ of 2-BUDGET. Note that this construction is achieved in polynomial time. We now make the following observation.

Claim 1. *If the above set A is a* Yes *instance of* 2-BUDGET *then any feasible solution a^1_{ij} and a^2_{ij} for all $i, j \in M$ is such that for all $s \neq t \neq u \neq s \in \mathbb{T}$ and $\ell \in \{1, 2\}$, we have $a^\ell_{s^0,s^0} > a^\ell_{s^0,o(s;t)}$ and $a^\ell_{s^0,s^0} > a^\ell_{s^0,o(s;t,u)}$.*

Proof: From the values of a_{ij} for all $i, j \in M$, we observe that for all $s \in \mathbb{T}$ we have $a_{s^1,s^1} > a_{s^1,s^2} + a_{s^1,s^0}$; this means that any split a^1_{ij} and a^2_{ij} of a_{ij} satisfying $a^1_{ij} + a^2_{ij} = a_{ij}$, for all $i, j \in M$, will satisfy $a^1_{s^1,s^1} + a^2_{s^1,s^1} > \left(a^1_{s^1,s^2} + a^1_{s^1,s^0}\right) + \left(a^2_{s^1,s^2} + a^2_{s^1,s^0}\right)$. This implies that:

$$a^1_{s^1,s^1} > a^1_{s^1,s^2} \text{ and } a^1_{s^1,s^1} > a^1_{s^1,s^0} \tag{31a}$$

or

$$a^2_{s^1,s^1} > a^2_{s^1,s^2} \text{ and } a^2_{s^1,s^1} > a^2_{s^1,s^0}. \tag{31b}$$

Similarly, from the inequality $a_{s^2,s^2} > a_{s^2,s^1} + a_{s^2,s^0}$ we conclude that

$$a^1_{s^2,s^2} > a^1_{s^2,s^1} \text{ and } a^1_{s^2,s^2} > a^1_{s^2,s^0} \tag{32a}$$

or

$$a^2_{s^2,s^2} > a^2_{s^2,s^1} \text{ and } a^2_{s^2,s^2} > a^2_{s^2,s^0}. \tag{32b}$$

By applying the same reasoning to $a_{s^0,s^0} > a_{s^0,o(s;\gamma)} + a_{s^0,s^1}$, where γ is either a single observation or a pair of two distinct observations, we arrive at the conclusion that:

$$a^1_{s^0,s^0} > a^1_{s^0,o(s;\gamma)} \text{ and } a^1_{s^0,s^0} > a^1_{s^0,s^1} \tag{33a}$$

or

$$a^2_{s^0,s^0} > a^2_{s^0,o(s;\gamma)} \text{ and } a^2_{s^0,s^0} > a^2_{s^0,s^1}. \tag{33b}$$

Finally, the inequality $a_{s^0,s^0} > a_{s^0,o(s;\gamma)} + a_{s^0,s^2}$ implies that

$$a^1_{s^0,s^0} > a^1_{s^0,o(s;\gamma)} \text{ and } a^1_{s^0,s^0} > a^1_{s^0,s^2} \tag{34a}$$

or

$$a^2_{s^0,s^0} > a^2_{s^0,o(s;\gamma)} \text{ and } a^2_{s^0,s^0} > a^2_{s^0,s^2}. \tag{34b}$$

On the one hand, suppose that (31a) holds; that is $a^1_{s^1,s^1} > a^1_{s^1,s^2}$ and $a^1_{s^1,s^1} > a^1_{s^1,s^0}$. Observe that if (32a) holds; that is: $a^1_{s^2,s^2} > a^1_{s^2,s^1}$ and $a^1_{s^2,s^2} > a^1_{s^2,s^0}$, then we have $a^1_{s^1,s^1} > a^1_{s^1,s^2}$ and $a^1_{s^2,s^2} > a^1_{s^2,s^1}$, which contradicts the fact that A is a Yes instance of 2-BUDGET. Therefore, (32a) does not hold. A similar reasoning allows to conclude that (33a) does not hold. Therefore, both (32b) and (33b) are satisfied. However, (32b) implies that (34b) does not hold (using the same reasoning as before). We conclude that if (31a) holds then both (33b) and (34a) hold, and Claim 1 is true.

On the other hand, if (31b) holds then neither (32b) nor (33b) hold (using the same reasoning as before), which imply that both (32a) and (33a) hold. Because (32a) holds, (34a) does not hold. We then conclude that both (33a) and (34b) hold; this implies that Claim 1 holds. □

We now argue that $A := \{a_{ij} : i,j \in M\}$ is a Yes instance of the problem 2-BUDGET if and only if $S := \{(p_t, q_t) : t \in \mathbb{T}\}$ is a Yes instance of CARP. On the one hand, we suppose that A is a Yes instance of 2-BUDGET. There exist a^1_{ij} and a^2_{ij} for all $i,j \in M$ that satisfy the conditions defining 2-BUDGET. We build the hypothetical relations H^1_0 and H^2_0 as follows: for all $s \neq t \in \mathbb{T}$, $s\,H^1_0\,t$ (respectively $s\,H^2_0\,t$) if and only if there exists a sequence $u, v, \ldots, w \in M$ such that the inequalities $a^1_{s^0,s^0} \geq a^1_{s^0,u}$, $a^1_{u,u} \geq a^1_{u,v}$, $\ldots$, $a^1_{w,w} \geq a^1_{w,t^0}$ (respectively $a^2_{s^0,s^0} \geq a^2_{s^0,u}$, $a^2_{u,u} \geq a^2_{u,v}$, $\ldots$, $a^2_{w,w} \geq a^2_{w,t^0}$) hold. Furthermore, we set $s\,H^1\,s$ and $s\,H^2\,s$ for all $s \in \mathbb{T}$. The binary relations H^1_0 and H^2_0 are well-defined because all the possible pairs of observations in $\mathbb{T}$ are considered. We now verify that the rules defining CARP hold for H^1_0 and H^2_0.

Rule 1: Let $s \neq t \in \mathbb{T}$ such that $p'_s q_s > p'_s q_t$. This implies that $a_{o(s;t),o(s;t)} > a_{o(s;t),t^0}$ and therefore, either $a^1_{o(s;t),o(s;t)} > a^1_{o(s;t),t^0}$ or $a^2_{o(s;t),o(s;t)} > a^2_{o(s;t),t^0}$. Claim 1 implies that we have either $a^1_{s^0,s^0} > a^1_{s^0,o(s;t)}$ and $a^1_{o(s;t),o(s;t)} > a^1_{o(s;t),t^0}$ or $a^2_{s^0,s^0} > a^2_{s^0,o(s;t)}$ and $a^2_{o(s;t),o(s;t)} > a^2_{o(s;t),t^0}$; which means that either $s\,H^1_0\,t$ or $s\,H^2_0\,t$.

Rule 2: We suppose that there are $s \neq t \in \mathbb{T}$ satisfying $p'_s q_s > p'_s q_t$ and $t\,H^1\,s$. The latter inequality implies that $a_{o(s;t),o(s;t)} > a_{o(s;t),t^0}$. Using the definition of H^1_0 we infer from $t\,H^1\,s$ that there exists a sequence $u, v, \ldots, w \in M$ such that $a^1_{t^0,t^0} \geq a^1_{t^0,u}$, $a^1_{u,u} \geq a^1_{u,v}$, $\ldots$, $a^1_{w,w} \geq a^1_{w,s^0}$. Furthermore, Claim 1 implies that $a^1_{s^0,s^0} > a^1_{s^0,o(s;t)}$. We infer that $a^1_{o(s;t),o(s;t)} < a^1_{o(s;t),t^0}$; otherwise condition 2. of 2-BUDGET will be violated. Therefore, $a_{o(s;t),o(s;t)} > a_{o(s;t),t^0}$ implies that $a^2_{o(s;t),o(s;t)} > a^2_{o(s;t),t^0}$, and using Claim 1 we conclude that $s\,H^2_0\,t$.

Rule 3**:** We now consider $s \neq t \neq u \neq s \in \mathbb{T}$ such that $p'_s q_s > p'_s(q_t + q_u)$ and $t\, H^1\, s$. The latter inequality implies that either $a^1_{o(s;t,u),o(s;t,u)} > a^1_{o(s;t,u),t^0} + a^1_{o(s;t,u),u^0}$ or $a^2_{o(s;t,u),o(s;t,u)} > a^2_{o(s;t,u),t^0} + a^2_{o(s;t,u),u^0}$. The relation $t\, H^1\, s$ implies that there exists a sequence $v, \ldots, w \in M$ such that $a^1_{t^0,t^0} \geq a^1_{t^0,v}$, $\ldots$, $a^1_{w,w} \geq a^1_{w,s^0}$ whereas Claim 1 implies that $a^1_{s^0,s^0} > a^1_{s^0,o(s;t,u)}$. We infer that it is not possible to have $a^1_{o(s;t,u),o(s;t,u)} > a^1_{o(s;t,u),t^0} + a^1_{o(s;t,u),u^0}$ because condition 2. of 2-BUDGET will be violated, otherwise. As a consequence, we have $a^2_{o(s;t,u),o(s;t,u)} > a^2_{o(s;t,u),t^0} + a^2_{o(s;t,u),u^0}$. This implies that $a^2_{o(s;t,u),o(s;t,u)} > a^2_{o(s;t,u),u^0}$, and using Claim 1 we conclude that $s\, H^2_0\, u$.

Rule 4**:** We suppose that there exist $s \neq t \in \mathbb{T}$ satisfying $p'_s q_s > p'_s q_t$, $t\, H^1\, s$ and $t\, H^2\, s$. Because $t\, H^1\, s$ there exists a sequence $u, \ldots, v \in M$ such that $a^1_{t^0,t^0} \geq a^1_{t^0,u}$, $\ldots$, $a^1_{v,v} \geq a^1_{v,s^0}$; and Claim 1 implies that $a^1_{s^0,s^0} \geq a^1_{s^0,o(s;t)}$. We infer that $a^1_{o(s;t),o(s;t)} < a^1_{o(s;t),t^0}$; otherwise condition 2. of 2-BUDGET will be violated. By applying a similar reasoning using the fact that $t\, H^2\, s$, we arrive at the conclusion that $a^2_{o(s;t),o(s;t)} < a^2_{o(s;t),t^0}$. Together with the previous inequality, we obtain that $a_{o(s;t),o(s;t)} < a_{o(s;t),t^0}$; which does not comply with our instance because $p'_s q_s > p'_s q_t$ implies that $a_{o(s;t),o(s;t)} = 2$ and $a_{o(s;t),t^0} = 1$.

Rule 5**:** Let us suppose that there exist $s \neq t \neq u \neq s \in \mathbb{T}$ such that $p'_s q_s > p'_s(q_t + q_u)$, $t\, H^1\, s$ and $u\, H^2\, s$. Because $t\, H^1\, s$ there exists a sequence $v, \ldots, w \in M$ such that $a^1_{t^0,t^0} \geq a^1_{t^0,v}$, $\ldots$, $a^1_{w,w} \geq a^1_{w,s^0}$ whereas $u\, H^2\, s$ implies that there exists a sequence $x, \ldots, y \in M$ such that $a^2_{u^0,u^0} \geq a^2_{u^0,x}$, $\ldots$, $a^2_{y,y} \geq a^2_{y,s^0}$. These inequalities together with Claim 1 imply that $a^1_{o(s;t,u),o(s;t,u)} < a^1_{o(s;t,u),t^0}$ and $a^2_{o(s;t,u),o(s;t,u)} < a^2_{o(s;t,u),u^0}$. This contradicts the fact that $p'_s q_s > p'_s(q_t + q_u)$, $t\, H^1\, s$.

This completes the verification that H^1_0 and H^2_0 satisfy the rules defining CARP.

On the other hand, suppose that S is a YES instance of CARP. There exist two binary relations H^1_0 and H^2_0 satisfying **Rule** 1–5. We are going to specify the value of a^1_{ij} and a^2_{ij} for all $i, j \in M$ and show that they satisfy the conditions defining 2-BUDGET.

- For t^0, t^1, and t^2 in M corresponding with the same observation $t \in \mathbb{T}$, we have:

$$
\begin{array}{lll}
a^1_{t^0,t^0} := 1.5, & a^1_{t^0,t^1} := 1.9, & a^1_{t^0,t^2} := 0.1, \\
a^2_{t^0,t^0} := 1.5, & a^2_{t^0,t^1} := 0.1, & a^2_{t^0,t^2} := 1.9, \\
a^1_{t^1,t^1} := 2.9, & a^1_{t^1,t^2} := 0.5, & a^1_{t^1,t^0} := 0.5, \\
a^2_{t^1,t^1} := 0.1, & a^2_{t^1,t^2} := 0.5, & a^2_{t^1,t^0} := 0.5, \\
a^1_{t^2,t^2} := 0.1, & a^1_{t^2,t^1} := 0.5, & a^1_{t^2,t^0} := 0.5, \\
a^2_{t^2,t^2} := 2.9, & a^2_{t^2,t^1} := 0.5, & a^2_{t^2,t^0} := 0.5.
\end{array}
$$

- For $s \neq t \neq v \neq s \in \mathbb{T}$, we fix $a^1_{t^0,o(t;v,s)} := 0.25$, $a^2_{t^0,o(t;v,s)} := 0.25$, $a^1_{t^0;o(t;v)} := 0.25$ and $a^2_{t^0;o(t;v)} := 0.25$.

- For $s \neq t \in \mathbb{T}$ we have:

if $p'_t q_t > p'_t q_s$ **then**
  $a^1_{o(t;s),s^0} := 0.5$ and $a^2_{o(t;s),s^0} := 0.5$
  if $t\, H^1\, s$ **then**
    if $t\, H^2\, s$ **then** $a^1_{o(t;s),o(t;s)} := 1$, $a^2_{o(t;s),o(t;s)} := 1$
    else $a^1_{o(t;s),o(t;s)} := 1.9$, $a^2_{o(t;s),o(t;s)} := 0.1$

else
if $t\,H^2\,s$ **then** $a^1_{o(t;s),o(t;s)} := 0.1$, $a^2_{o(t;s),o(t;s)} := 1.9$
else $a^1_{o(t;s),o(t;s)} = a^2_{o(t;s),o(t;s)} := \frac{1}{2} a_{o(t;s),o(t;s)} = 1$
else $a^1_{o(t;s),o(t;s)} = a^2_{o(t;s),o(t;s)} := 0.5$ and $a^1_{o(t;s),s^0} = a^2_{o(t;s),s^0} := 5$

- For $s \neq t \neq v \neq s \in \mathbb{T}$ we have:

if $p_t' q_t > p_t'(q_s + q_v)$ **then**
if $t\,H^1\,s$ **then**
if $t\,H^1\,v$ **then**
if $t\,H^2\,s$ **then**
if $t\,H^2\,v$ **then**
$a^1_{o(t;s,v),o(t;s,v)} := 1.5$, $a^2_{o(t;s,v),o(t;s,v)} := 1.5$, $a^1_{o(t;s,v),s^0} := 0.5$, $a^2_{o(t;s,v),s^0} := 0.5$, $a^1_{o(t;s,v),v^0} := 0.5$, and $a^2_{o(t;s,v),v^0} := 0.5$
else
$a^1_{o(t;s,v),o(t;s,v)} := 2.2$, $a^2_{o(t;s,v),o(t;s,v)} := 0.8$, $a^1_{o(t;s,v),s^0} := 0.5$, $a^2_{o(t;s,v),s^0} := 0.5$, $a^1_{o(t;s,v),v^0} := 0.1$, and $a^2_{o(t;s,v),v^0} := 0.9$
else
if $t\,H^2\,v$ **then**
$a^1_{o(t;s,v),o(t;s,v)} := 2.2$, $a^2_{o(t;s,v),o(t;s,v)} := 0.8$, $a^1_{o(t;s,v),v^0} := 0.5$, $a^2_{o(t;s,v),v^0} := 0.5$, $a^1_{o(t;s,v),s^0} := 0.1$, and $a^2_{o(t;s,v),s^0} := 0.9$
else
$a^1_{o(t;s,v),o(t;s,v)} := 2.2$, $a^2_{o(t;s,v),o(t;s,v)} := 0.8$, $a^1_{o(t;s,v),s^0} := 0.1$, $a^2_{o(t;s,v),s^0} := 0.9$, $a^1_{o(t;s,v),v^0} := 0.1$, and $a^2_{o(t;s,v),v^0} := 0.9$
else
if $t\,H^2\,s$ **then**
if $t\,H^2\,v$ **then**
$a^1_{o(t;s,v),o(t;s,v)} := 0.8$, $a^2_{o(t;s,v),o(t;s,v)} := 2.2$, $a^1_{o(t;s,v),s^0} := 0.5$, $a^2_{o(t;s,v),s^0} := 0.5$, $a^1_{o(t;s,v),v^0} := 0.9$, and $a^2_{o(t;s,v),v^0} := 0.1$
else
$a^1_{o(t;s,v),o(t;s,v)} := 1.5$, $a^2_{o(t;s,v),o(t;s,v)} := 1.5$, $a^1_{o(t;s,v),s^0} := 0.5$, $a^2_{o(t;s,v),s^0} := 0.5$, $a^1_{o(t;s,v),v^0} := 0.5$, and $a^2_{o(t;s,v),v^0} := 0.5$
else
if $t\,H^2\,v$ **then**
$a^1_{o(t;s,v),o(t;s,v)} := 1.5$, $a^2_{o(t;s,v),o(t;s,v)} := 1.5$, $a^1_{o(t;s,v),s^0} := 0.1$, $a^2_{o(t;s,v),s^0} := 0.9$, $a^1_{o(t;s,v),v^0} := 0.9$, and $a^2_{o(t;s,v),v^0} := 0.1$
else
$a^1_{o(t;s,v),o(t;s,v)} := 1.5$, $a^2_{o(t;s,v),o(t;s,v)} := 1.5$, $a^1_{o(t;s,v),s^0} := 0.1$, $a^2_{o(t;s,v),s^0} := 0.9$, $a^1_{o(t;s,v),v^0} := 0.5$, and $a^2_{o(t;s,v),v^0} := 0.5$
else {use the same reasoning as above while inverting the role of H^1 and H^2}
else $a^1_{o(t;s,v),o(t;s,v)} := 1.5$, $a^2_{o(t;s,v),o(t;s,v)} := 1.5$, $a^1_{o(t;s,v),s^0} := 0.5$, $a^2_{o(t;s,v),s^0} := 0.5$, $a^1_{o(t;s,v),v^0} := 0.5$, and $a^2_{o(t;s,v),v^0} := 0.5$

- For all the remaining values that are not yet specified, we fix $a^1_{t,v} = a^2_{t,v} := \frac{1}{2} a_{t,v}$.

Claim 2. *Consider the values of a^1_{ij} and a^2_{ij} for all $i, j \in M$ defined above. For a pair of distinct observations $s \neq t \in \mathbb{T}$ if there exist $\ell \in \{1, 2\}$ and a sequence $u_1, \ldots, u_n \in M$ such that $u_1 = s^0$, $u_n = t^0$ and for all $i = 1, \ldots, n-1$ we have $a^\ell_{u_i,u_i} \geq a^\ell_{u_i,u_{i+1}}$ then $s\,H^\ell\,t$.*

Proof: Let us consider $s \neq t \in \mathbb{T}$ and a sequence $u_1, \ldots, u_n \in M$ such that $u_1 = s^0$, $u_n = t^0$ and for all $i = 1, \ldots, n-1$ we have $a^1_{u_i,u_i} \geq a^1_{u_i,u_{i+1}}$. Because $u_1 = s^0$, u_2 is either $o(s;v)$, $o(s;v,w)$ or s^2 for some $v, w \in \mathbb{T}$. However, $a^1_{s^2,s^2} < a^1_{s^2,v}$ for all $v \in M$ implies that $u_2 \neq s^2$. If $u_2 = o(s;v)$ then $u_3 = t^0$ and $s\,H^1\,t$ whereas if $u_2 = o(s;v,w)$ then u_3 is either v^0 or w^0 and $s\,H^1\,v$ or $s\,H^1\,w$. By repeating this argument and using the transitivity of H^1, we end up with $s\,H^1\,t$. This completes the proof of Claim 2. □

These values of a^1_{ij} and a^2_{ij} for all $i, j \in M$ satisfy the conditions defining 2-BUDGET because if that was not the case then there will be a cycle either in H^1 or in H^2. The existence of such cycle would contradict the assumption that we have a YES instance of CARP.

This completes the proof that we have a YES instance of 2-BUDGET. Therefore, the problem 2-BUDGET is at least as hard as CARP; and because the latter problem is NP-complete we infer that 2-BUDGET is also NP-complete. Note that the above reduction from CARP to 2-BUDGET remains valid even if we consider the modified version of 2-BUDGET corresponding with 2-SARP. In summary, we have shown that 2-GARP and 2-SARP are at least as hard as 2-BUDGET, which is also at least as hard as CARP. Therefore, 2-GARP and 2-SARP are at least as hard as CARP. Because CARP is an NP-complete problem, we conclude that 2-GARP and 2-SARP are also NP-complete.

B. An MIP formulation for 2-SARP

As in Section 3.2, we use the binary decision variables y^ℓ_{st} and the continuous (vectors of) decision variables x^ℓ_t, for $\ell = 1, 2$ and $s \neq t \in N$, with the interpretations provided in Section 3.2. In addition, there are continuous variables $z^\ell_{st} \in [0,1]$ for $\ell = 1, 2$, and $s < t \in N$, which take the value $z^\ell_{st} = 0$ if and only if $x^\ell_s = x^\ell_t$. The constraints are the following:

$$
\begin{array}{rcllr}
x^1_t + x^2_t & = & q_t, & t \in N, & (35)\\
x^\ell_t & \geq & q^\ell_t, & t \in N; \quad \ell = 1,2, & (36)\\
p'_s\big(x^\ell_s - x^\ell_t\big) - M y^\ell_{st} & < & 0, & s \neq t \in N; \quad \ell = 1,2, & (37)\\
y^\ell_{su} + y^\ell_{ut} - y^\ell_{st} & \leq & 1, & s \neq t \neq u \neq s \in N; \ \ell = 1,2, & (38)\\
y^\ell_{st} + y^\ell_{ts} + z^\ell_{st} & \leq & 2, & s < t \in N; \quad \ell = 1,2, & (39)\\
Q z^\ell_{st} + x^\ell_{t,i} - x^\ell_{s,i} & \geq & 0, & s < t \in N; \quad \ell = 1,2; \quad i = 1,\ldots,m, & (40)\\
Q z^\ell_{st} + x^\ell_{s,i} - x^\ell_{t,i} & \geq & 0, & s < t \in N; \quad \ell = 1,2; \quad i = 1,\ldots,m. & (41)
\end{array}
$$

The constraints (35)–(38) have the same meaning as in F1:2-GARP. The constraints (39) entail that if $x^\ell_s \neq x^\ell_t$ then member ℓ cannot prefer both s over t and t over s. The sets of constraints (40)–(41) ensure that if $x^\ell_s \neq x^\ell_t$ then $z^\ell_{st} > 0$; in these constraints, the constant $Q := \max\{q_{s,i} \ : \ i = 1, \ldots, m; s = 1, \ldots, n\}$. We denote the formulation (35)–(41) as F1:2-SARP.

References

Afriat, S. N., 1967. The construction of utility functions from expenditure data. International Economic Review 8, 67–77.

Ahuja, R. K., Magnanti, T. L., Orlin, J. B., 1993. Network Flows: Theory, Algorithms, and Applications. Prentice Hall.

Apesteguia, J., Ballester, M., 2010. The computational complexity of rationalizing behavior. Journal of Mathematical Economics 46, 356–363.

Apps, P. F., Rees, R., 1988. Taxation and the household. Journal of Public Economics 35, 355–369.

Baron, R., Durieu, J., Haller, H., Savani, R., Solal, P., 2008. Good neighbors are hard to find: Computational complexity of network formation. Review of Economic Design 12, 1–19.

Baron, R., Durieu, J., Haller, H., Solal, P., 2004. Finding a Nash equilibrium in spatial games is an NP-complete problem. Economic Theory 23, 445–454.

Bourguignon, F., Browning, M., Chiappori, P., 2009. Efficient intra-household allocations and distribution factors: Implications and identification. Review of Economic Studies 76, 503–528.

Brandt, F., Fisher, F., 2008. Computing the minimal covering set. Mathematical Social Sciences 56, 254–268.

Brandt, F., Fisher, F., Harrenstein, P., Mair, M., 2010. A computational analysis of the tournament equilibrium set. Social Choice and Welfare 34, 597–609.

Browning, M., Chiappori, P., 1998. Efficient intra-household allocations: A general characterization and empirical tests. Econometrica 66, 1241–1278.

Cechlarova, K., Hajdukova, J., 2002. Computational complexity of stable partitions with B-preferences. International Journal of Game Theory 31, 353–364.

Cherchye, L., De Rock, B., Sabbe, J., Vermeulen, F., 2008. Nonparametric tests of collectively rational consumption behavior: An integer programming procedure. Journal of Econometrics 147, 258–265.

Cherchye, L., De Rock, B., Vermeulen, F., 2007. The collective model of household consumption: A nonparametric characterization. Econometrica 75, 553–574.

Cherchye, L., De Rock, B., Vermeulen, F., 2009. Opening the black box of intra-household decision-making. Journal of Political Economy 117, 1074–1104.

Cherchye, L., De Rock, B., Vermeulen, F., 2010. An Afriat theorem for the collective model of household consumption. Journal of Economic Theory 145, 1142–1163.

Cherchye, L., De Rock, B., Vermeulen, F., 2011a. The revealed preference approach to collective consumption behavior: Testing and sharing rule recovery. Review of Economic Studies 78, 176–198.

Cherchye, L., De Rock, B., Vermeulen, F., 2012. Collective household consumption behavior: Revealed preference analysis. Foundations and Trends in Economics 4, 225–312.

Cherchye, L., Demuynck, T., De Rock, B., 2011b. Testable implications of general equilibrium models: an integer programming approach. Journal of Mathematical Economics 47, 564–575.

Cherchye, L., Vermeulen, F., 2008. Nonparametric analysis of household labor supply: Goodness of fit and power of the unitary and the collective model. The Review of Economics and Statistics 90, 267–274.

Chiappori, P. A., 1988. Rational household labor supply. Econometrica 56, 63–89.

Chiappori, P. A., 1992. Collective labor supply and welfare. Journal of Political Economy 100, 437–467.

Chu, F., Halpern, J., 2001. On the NP-completeness of finding an optimal strategy in games with common payoffs. International Journal of Game Theory 30, 99–106.

Chung-Piaw, T., Vohra, R., 2003. Afriat's theorem and negative cycles. Mimeo, Northwestern University.

Chvátal, V., 1983. Linear Programming. W.H. Freeman and Company.

Conitzer, V., Sandholm, T., 2008. New complexity results about Nash equilibria. Games and Economic Behavior 63, 621–641.

Deb, R., 2011. An efficient nonparametric test of the collective household model. Mimeo, University of Toronto, Canada.

Dekkers, A., Aarts, E., 1991. Global optimization and simulated annealing. Mathematical Programming 50, 367–393.

Demuynck, T., 2013. The computational complexity of rationalizing Pareto optimal choice behavior. to appear in Social Choice and Welfare (DOI 10.1007/s00355-013-0735-1).

Diewert, W. E., 1973. Afriat and revealed preference theory. The Review of Economic Studies 40, 419–425.

Echenique, F., Lee, S., Shum, M., 2010. Revealed preference tests using supermarket data: the money pump. Social science working paper 1328, California Institute of Technology, USA.

Echenique, F., Lee, S., Shum, M., 2011. The money pump as a measure of revealed preference violations. Journal of Political Economy 119, 1201–1223.

Fang, Q., Zhu, S., Cai, M., Deng, X., 2002. On computational complexity of membership test in flow games and linear production games. International Journal of Game Theory 31, 39–45.

Farin, G., 2006. Class A Bézier curves. Computer Aided Geometric Design 23, 573–581.

Fortin, B., Lacroix, G., 1997. A test of the unitary and collective models of household labour supply. Economic Journal 107, 933–955.

Galambos, A., 2009. The complexity of Nash rationalizability. Tech. rep., Lawrence University.

Gilboa, I., Zemel, E., 1989. Nash and correlated equilibria: Some complexity considerations. Games and Economic Behavior 1, 80–93.

Goffe, W. L., Ferrier, G. D., Rogers, J., 1994. Global optimization of statistical functions with simulated annealing. Journal of Econometrics 60, 65–99.

Golumbic, M. C., 2004. Algorithmic Graph Theory and Perfect Graphs, 2nd Edition. Vol. 57 of Annals of Discrete Mathematics. North Holland.

Hudry, O., 2009. A survey on the complexity of tournament solutions. Mathematical Social Sciences 57, 292–303.

Kalyanaraman, S., Umans, C., 2008. The complexity of rationlizing matchings. In: Proceedings of ISAAC2008, Lecture Notes in Computer Science 5369, 171–182.

Karmarkar, N., 1984. A new polynomial-time algorithm for linear programming. Combinatorica 4, 373–395.

Khachiyan, L. G., 1979. A polynomial algorithm in linear programming. Doklady Akademii Nauk SSSR 244, 1093–1096, translated in *Soviet Mathematics Doklady* 20: 191–194 (1979).

Locatelli, M., 2000. Simulated annealing algorithms for continuous global optimization: Convergence conditions. Journal of Optimization Theory and Applications 104, 121–133.

Procaccia, A. D., Rosenschein, J. S., 2008. On the complexity of achieving proportional representation. Social Choice and Welfare 30, 353–362.

Sabbe, J., 2010. Revealed preference analysis of consumption and production behavior: Applications of the collective model. Ph.D. thesis, KU Leuven.

Samuelson, P. A., 1938. A note on the pure theory of consumer's behaviour. Economica 5, 61–71.

Samuelson, P. A., 1948. Consumption theory in terms of revealed preference. Economica 15, 243–253.

Talla Nobibon, F., Cherchye, L., De Rock, B., Sabbe, J., Spieksma, F. C. R., 2011. Heuristics for deciding collectively rational consumption behavior. Computational Economics 38, 173–204.

Talla Nobibon, F., Hurkens, C. A. J., Leus, R., Spieksma, F. C. R., 2012a. Coloring graphs using two colors while avoiding monochromatic cycles. INFORMS Journal on Computing 24, 485–499.

Talla Nobibon, F., Smeulders, B., Spieksma, F., 2012b. An $O(n^2)$ algorithm for testing the generalized axiom of revealed preference. Working paper KBI 1204, KU Leuven, Belgium.

Talla Nobibon, F., Spieksma, F. C. R., 2010. On the complexity of testing the collective axiom of revealed preference. Mathematical Social Sciences 60, 123–136.

Varian, H. R., 1982. The nonparametric approach to demand analysis. Econometrica 50, 945–974.

Varian, H. R., 2006. Revealed preference. In: Szenberg, M., Ramrattan, L., Gottesman, A. (Eds.), Samuelsonian economics and the twenty-first century. Oxford University Press, USA, pp. 99–115.

Walters, J. A., 1969. Nonnegative matrix equations having positive solutions. Mathematics of Computation 23, 827–827.

Woeginger, G. J., 2003. Banks winners in tournaments are difficult to recognize. Social Choice and Welfare 20, 523–528.

FACULTY OF ECONOMICS AND BUSINESS
Naamsestraat 69 bus 3500
3000 LEUVEN, BELGIË
tel. + 32 16 32 66 12
fax + 32 16 32 67 91
info@econ.kuleuven.be
www.econ.kuleuven.be