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Tangency Capacity Notions

Based upon the Profit and Cost Functions:

A Non-Parametric Approach and a General Comparison*

Walter Briec† Kristiaan Kerstens‡ Diego Prior§ Ignace Van de Woestyne¶

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Abstract

This contribution provides a way to define and compute a tangency notion of economic capacity based upon the relation between the various directional distance functions and the profit and cost functions using non-parametric technologies. A new result relating profit and cost function-based tangency capacity notions is established.

Keywords: economic capacity, profit function, cost function, directional distance function, tangency.

JEL: C61, D24.

1 Introduction

Analysing efficiency and productivity using frontier specifications of technology has become a standard empirical tool serving both regulatory and managerial purposes. However, this literature has almost ignored integrating the important notion of capacity utilisation. As a consequence, part of what may be attributed to inefficiency may in fact be due to the short-run fixity of certain inputs, depending on the exact definition of capacity utilisation.

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†University of Perpignan, LAMPS, 52 avenue de Villeneuve, F-66000 Perpignan, France.

‡Corresponding author: CNRS-LEM (UMR 8179), IESEG School of Management, 3 rue de la Digue, F-59000 Lille, France. Tel: +33 (0)320545892, Fax: +33(0)320574855, e-mail: k.kerstens@ieseg.fr.

§Universitat Autònoma de Barcelona, Dpt. d'Economia de l'Empresa, E-08193 Bellaterra, Spain.

¶Hogeschool Universiteit Brussel, Stormstraat 2, B-1000 Brussel, Belgium

Capacity utilisation of fixed inputs is of managerial and policy relevance at various levels of aggregation and in all economic sectors (agriculture, industry, and both private and public services). Traditionally, capacity utilisation is employed as a leading macro-economic indicator to forecast inflationary pressures (e.g., Christiano (1981)). At the industry level, it is important to account for variable capacity utilisation when measuring productivity growth (Luh and Stefanou (1991)). Capacity management has recently become an important issue in fisheries due to declining stocks of this common pool resource. For instance, various capacity measures have been used to evaluate vessel decommissioning schemes (e.g., Walden, Kirkley and Kitts (2003)). Governments worldwide must try to determine sustainable capacity levels and implement a variety of policy measures (e.g., licenses, fishing day restrictions, etc.) to curb overfishing. This has led to the development of short-run industry models based on vessel capacity estimates to plan the industry (e.g., Lindebo (2005)).

At the firm level, fluctuations in capacity utilisation of investments are determined by relative factor prices, demand fluctuations, the lumpiness of certain investments, lead times, strategic issues, etc. A rather well-documented strategic use of excess capacities is entry deterrence in imperfectly competitive markets. However, the relative importance of precautionary and strategic uses of excess capacity vary significantly across industries (e.g., Driver (2000)). The literature on strategic capacity management develops pragmatic models (e.g., drawing upon, amongst others, the aggregate planning, the inventory and supply chain management, and the economics literatures) to determine the size, type and timing of capacity adjustment under uncertainty (see the survey in Van Mieghem (2003)). An example of a recent question of great managerial interest is the impact of IT capital goods on capacity utilisation in complex supply, production and distribution systems coordinated by control systems (Nightingale et al. (2003)).

In brief, the measurement of capacity and its utilisation is important for both managers and policy makers in all sectors of the economy and ideally one would like to have methods of capacity measurement available that are sufficiently general to be applicable to agriculture, manufacturing and services. However, different notions of capacity exist in the economic and managerial literature (see Christiano (1981) or Johansen (1968)). Specifically, it is customary to distinguish between technical (engineering) and economic (mostly defined in terms of cost) capacity concepts. Johansen (1968) pursued a technical approach focusing on a plant capacity notion, defined as the maximal amount that can be produced per unit of time with existing plant and equipment without restrictions on the availability of variable production factors. This definition is clearly in line with the engineering perspective and has been translated into a production frontier framework using output-based efficiency measures by Färe, Grosskopf and Kokkelenberg (1989).

Traditionally, there are three basic ways of defining a cost-based notion of capacity (see, e.g., Nelson (1989)). The purpose of each of these notions is to isolate the short-run excessive or inadequate utilisation of fixed inputs. The first notion of potential output is defined in terms of the output produced at short-run minimum average total cost, given existing plant and factor prices (e.g., Hickman (1964)), and stresses the need to exploit short-run scale economies. The second definition of potential output is defined in terms of the output produced at long-run minimum average total cost (see Cassels (1937), among others). However, it has been little used, probably because it is clearly heavily intertwined with the notion of scale economies. The third definition follows, among others, Klein (1960) and Segerson and Squires (1990) and corresponds to the output at which short- and long-run average total costs curves are tangent. Since this point is also the intersection of short- and long-run expansion paths, this notion has a strong theoretical appeal.

To implement these cost-based notions of capacity utilisation on non-parametric, deterministic frontier technologies, we summarise the possibilities.[1] First, estimation of the short-run minimum average total cost amounts to solving a variable cost function relative to a constant returns to scale technology (see Prior (2003)). Second, long-run minimum average total cost is easily determined by computing a total cost function relative to a constant returns to scale technology (e.g., Hackman (2008) or Ray (2004)). Third, assuming inputs are fixed and cannot be changed, but outputs are adjustable such that installed capacity is utilised ex post at a tangency level we are unaware of any non-parametric method to solve for the tangency capacity notion. Therefore, the first goal of this paper is the development of a tangency capacity notion using non-parametric frontier methodologies which allows for any eventual inefficiencies.[2]

Apart from economic capacity measures defined using the cost function, there also exist capacity notions using other economic objective functions. The case of revenue functions has been treated in Segerson and Squires (1995), while the case of profit functions has been handled in Squires (1987), among others. Furthermore, Coelli, Grifell-Tatjé and Perelman (2002) define

[1]For the options using traditional parametric specifications: see Nelson (1989). For a parametric frontier application, see, e.g., Rodríguez-Álvarez and Lovell (2004).

[2]It is important to underline the issue of extrapolating the functions implied in some of these different concepts beyond the data domain. However, this is a problem for parametric and non-parametric estimation methods alike, even when model flexibility is allowed for.

an alternative ray economic capacity measure using non-parametric frontiers that involves short-run profit maximisation whereby the output mix is held constant. While this notion may have some appeal, it does normally not coincide with the tangency capacity notion defined above.[3] Therefore, to cover the most general economic objective function first, this paper defines a tangency economic capacity notion based on the profit function. The case of defining a tangency point of economic capacity for the cost function is completely analogous.[4] The second goal of this contribution is then to establish a relation between profit and cost function based tangency capacity notions, independent of the underlying nature of technology. Such relation is -to the best of our knowledge- new in the production literature.

The next section defines the axioms imposed on technology, the directional distance function as a representation of technology and its dual, the profit function, as well as the basic efficiency decomposition and the ensuing new definition of a tangency economic capacity notion based on the profit function allowing for inefficiencies. A similar analysis is also developed for the case of the cost function. Section 3 develops a computational procedure to obtain the profit and cost function based tangency points using non-parametric, deterministic frontier technologies that impose either constant or variable returns to scale. The fourth section focuses on establishing a new relation between the set of profit and cost tangency points. Then, a sample of Chilean hydro-power plants serves as an empirical illustration. The paper ends with some concluding comments.

2 Technology and Distance Functions: Definitions, Efficiency Decomposition, and Tangency Point

This section introduces the necessary definitions of the production possibility set, the distance functions and the profit functions. To be more specific, first we define the axioms underlying technology. Then, we define both the long- and short-run directional distance functions characterising technology and their corresponding long- and short-run profit functions. Thereafter, we develop the corresponding efficiency decompositions. This allows us to end up with a novel

[3]This proposal can be attractive in case the output mix should remain (more or less) fixed, which may be relevant in certain (mainly industrial) production processes. A more profound analysis as to the relative merits of this new proposal compared to more traditional definitions is being called for.

[4]Of course, the analytical generality of this analysis must be distinguished from the differences in economic interpretation implied by adopting a profit or a cost perspective.

way of characterising profit tangency points in terms of the short- and long-run allocative efficiency components. This generalises existing approaches defining profit capacity that ignore the possibility of inefficiencies. The case for the cost function is developed in a similar way.

2.1 Technology

Production technology transforms inputs $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$ into outputs $y = (y_1, ..., y_m) \in \mathbb{R}^m_+$. The set of all feasible input and output vectors is called the production possibility set T and is defined as follows:

$$T = \left\{(x, y) \in \mathbb{R}^{n+m}_+ : x \text{ can produce } y\right\} . \tag{2.1}$$

It satisfies the following standard assumptions (e.g, Hackman (2008), Shephard (1970)): (T.1) $(0, 0) \in T,\ (0, y) \in T \Rightarrow\ y = 0$, i.e., no free lunch; (T.2) the set $A(x) = \{(u, y) \in T :\ u \leqslant\ x\}$ of dominating observations is bounded $\forall x \in \mathbb{R}^n_+$, i.e., infinite outputs cannot be obtained from a finite input vector; (T.3) T is a closed set; and (T.4) $\forall(x, y) \in T,\ (u, v) \in \mathbb{R}^{n+m}_+$ and $(x, -y) \leqslant (u, -v) \Rightarrow (u, v) \in\ T$, i.e., strong input and output disposability; and (T.5) T is convex.

The estimation of efficiency relative to production frontiers relies on the theory of distance or gauge functions. In economics, Shephard (1970) distance functions are related to the efficiency measures introduced by Farrell (1957). The input distance function is dual to the cost function, while the output distance function is dual to the revenue function (e.g., Shephard (1970)).

2.2 Short-Run Profit Function and Duality

We now discuss the recently introduced directional distance function $D : \mathbb{R}^{n+m}_+ \times \mathbb{R}^{n+m}_+ \to \mathbb{R} \cup \{-\infty, \infty\}$ that involves simultaneous input and output variations:

$$D(x, y; h, k) = \sup_{\delta\in\mathbb{R}} \{\delta :\ (x - \delta h, y + \delta k) \in T\} . \tag{2.2}$$

If there is no δ such that $(x-\delta h, y+\delta k) \in T$ then, by definition, $D(x, y; h, k) = -\infty$. The scalar δ attempts to contract the input vector x and to expand the output vector y in the direction of the vector h respectively k. It is a special case of the shortage function (Luenberger (1995)) and is closely related to the Farrell proportional distance (Briec (1997)), a generalisation of the Debreu-Farrell measure. Input and output distance functions also appear as special cases (see Chambers, Chung and Färe (1998)).[5] We denote the general directional vector $g = (h, k) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$.

This directional distance function is dual to the profit function (Luenberger (1995)) and therefore offers a general framework for economic analysis. This function has proven to be a useful tool in micro-economic theory (for instance, it allows Chavas and Kim (2007) to shed new light on economies of scope from a primal viewpoint).

We also need a short-run version of this directional distance function that involves simultaneous proportional variable input and output variations for a given sub-vector of fixed inputs. Therefore, we assume that the input set can be partitioned into two subsets $I_v = \{1, ..., n_v\}$ and $I_f = \{n_v + 1, ..., n\}$. I_v stands for the set of the variable inputs and I_f represents the set of fixed inputs. Obviously, we have $\{1, ..., n\} = I_v \cup I_f$. We adopt a similar decomposition of the output set. Therefore, we assume that $J_v = \{1, ..., m_v\}$ and $J_f = \{m_v + 1, ..., m\}$. J_v stands for the set of the variable outputs and J_f represents the set of fixed outputs and thus we have $\{1, ..., m\} = J_v \cup J_f$. Fixed outputs, that in the short-run prevent adjusting the output mix to its profit maximising level, may occur in case of an exclusive contract to deliver a certain amount of a sub-vector of outputs. Moreover, we assume that the inputs and the outputs are ranged such that each input-output vector is denoted $(x, y) = (x^v, x^f, y^v, y^f)$. Similarly, the short-run direction vector is denoted $g = (h^v, h^f, k^v, k^f)$. The short-run directional distance function is then defined as:

$$SRD(x, y; h^v, 0, k^v, 0) = \sup_{\delta \in \mathbb{R}} \left\{ \delta : \ (x^v - \delta h^v, x^f, y^v + \delta k^v, y^f) \in T \right\} \tag{2.3}$$

$$= D(x^v, x^f, y^v, y^f; h^v, 0, k^v, 0). \tag{2.4}$$

The next element needed for our analysis is the long-run profit function, which can be defined as follows:

$$\Pi(w, p) = \sup_{x,y} \{p.y - w.x : D(x, y; h, k) \geq 0\}. \tag{2.5}$$

Luenberger (1995) and Chambers, Chung and Färe (1998) show duality between the directional distance function (2.2) and the long-run profit function (2.5).

Chambers, Chung and Färe (1998) first define the overall efficiency (OE) index as the quan-

[5] Slightly different generalisations of the Shephard distance functions that are equally related to the profit function have been defined in, e.g., Chavas and Cox (1999) or McFadden (1978). In principle, our analysis could equally be transposed to capacity utilisation measures based on the latter type of distance functions.

tity:

$$OE\left(x,y,p,w\right)=\frac{\Pi\left(w,p\right)-\left(p.y-w.x\right)}{p.k+w.h}. \tag{2.6}$$

Then, they continue by characterising a technical efficiency (TE) index as the quantity:

$$TE(x,y)=D(x,y;g). \tag{2.7}$$

Finally, the allocative efficiency (AE) index is defined as the quantity:

$$AE\left(x,y,p,w\right)=OE\left(x,y,p,w\right)-TE(x,y). \tag{2.8}$$

Obviously, the following additive decomposition identity holds:

$$OE(x,y,w,p)=AE(x,y,w,p)+TE(x,y). \tag{2.9}$$

Notice that all three components are semi-positive, with zero indicating efficiency. This implies that increases in efficiency are reflected in decreasing scores.[6]

Thus, OE is simply the ratio between maximum profit minus observed profits for the observation evaluated over the normalised value of the direction vector $g=(h,k)$ for given input and output prices (w,p). Chambers, Chung and Färe (1998) call this Nerlovian efficiency. Technical efficiency only guarantees reaching a point on the production frontier, not necessarily a point on the frontier maximising the profit function. Allocative efficiency, by contrast, measures the adjustments in input and output mixes along the production frontier needed to achieve the maximum of the profit function given relative prices. Overall efficiency ensures that both these ideals of technical and allocative efficiency are realised simultaneously.

To define short-run or restricted profit functions, it is necessary to distinguish between input prices of variable and fixed inputs $w=(w^v,w^f)$. A similar distinction is needed for output prices: $p=(p^v,p^f)$. The short-run total profit function is then:

$$SR\Pi(w,p,\bar{x}^f,\bar{y}^f)=\sup_{(x^v,\bar{x}^f,y^v,\bar{y}^f)\in T}\left\{p^v.y^v+p^f.\bar{y}^f-w^v.x^v-w^f.\bar{x}^f\right\}, \tag{2.10}$$

where the bar indicates the exogenous parameters. The short-run variable profit function is

[6] It is also possible to define all three components such that they are semi-negative, with zero again indicating efficiency and increasing efficiency scores now reflecting increases in efficiency.

defined as:

$$SRV\Pi(w^v, p^v, \bar{x}^f, \bar{y}^f) = SR\Pi(w, p, \bar{x}^f, \bar{y}^f) - p^f.\bar{y}^f + w^f.\bar{x}^f \tag{2.11}$$

$$= \sup_{x^v, y^v} \{p^v.y^v - w^v.x^v : SRD(x, y; h^v, 0, k^v, 0) \geq 0\}. \tag{2.12}$$

Adapting Blancard et al. (2006) for the case of fixed and variable outputs, it is trivial to define duality between the short-run directional distance function (2.3) and the short-run variable profit function (2.11).

Since the OE definition of Chambers, Chung and Färe (1998) employs a long-run profit function, we first need to define similar components based upon the short-run variable profit function. Short-run overall efficiency ($SROE$) is defined as the quantity:

$$SROE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = \frac{SRV\Pi(w^v, p^v, \bar{x}^f, \bar{y}^f) - (p^v.y^v - w^v.x^v)}{p^v.k^v + w^v.h^v}. \tag{2.13}$$

A short-run technical efficiency ($SRTE$) index can now be defined as the quantity:

$$SRTE(x^v, \bar{x}^f, y^v, \bar{y}^f) = SRD(x^v, \bar{x}^f, y^v, \bar{y}^f; h^v, 0, k^v, 0). \tag{2.14}$$

Obviously, in analogy to expression (2.8), a short-run allocative efficiency ($SRAE$) index bridges the gap between $SROE$ and $SRTE$.

Now we provide a definition of a profit tangency point:

Definition 2.1 *Let* $(\bar{w}^v, w^f, \bar{p}^v, p^f) \in \mathbb{R}_+^{n+m}$ *be an input-output price vector, where the bar indicates the exogenous parameters of the problem. Under the assumptions (T.1) to (T.5), an input-output vector* $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ *is a* $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$*-profit tangency point if and only if*

$$\bar{p}^v.y^v + p^f.\bar{y}^f - \bar{w}^v.x^v - w^f.\bar{x}^f = SR\Pi(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f) = \Pi(\bar{w}^v, w^f, \bar{p}^v, p^f).$$

We denote $\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$ the set of all the $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency points for a specific observation. It is clear that if a production vector $(x, y) \in T$ is such that (i) $x^f = \bar{x}^f$ and $y^f = \bar{y}^f$ and (ii) it maximises simultaneously short- and long-run profits, then it is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point.

Proposition 2.2 *Under the assumptions (T.1) to (T.5), if* $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T$ *and*

$$SRAE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = AE(w, p, x, y) = 0,$$

if and only if

$$(\tilde{x}, \tilde{y}) = (\ x - SRD(x, y; g)\,(h^v, 0)\ ,\ y + SRD(x, y; g)\,(k^v, 0)\ )$$

is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$*-profit tangency point.*

Proof: We have $SRAE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = 0$ if and only if $(\tilde{x}, \tilde{y})$ maximises the short-run profit function. Moreover, $AE(w, p, x^v, \bar{x}^f, y^v, \bar{y}^f) = 0$ if and only if $(\tilde{x}, \tilde{y})$ maximises the long-run profit function. This ends the proof. □

Thus, a short-run technically efficient projection point that is both allocatively efficient in the short- and long-run is a profit tangency point. Notice that this definition is novel in that it allows for technical inefficiencies when defining tangency points on the short- and long-run profit function. A possibly technically inefficient point can then be projected onto the technology frontier and eventually satisfy both allocative efficiency conditions simultaneously.

2.3 Short-Run Cost Function and Duality

Notice that the case of the cost function can be rather straightforwardly derived from the profit function: most if not all definitions and results can be obtained by simply setting the second component of the direction vector equal to zero (i.e., $k = 0$). To avoid needless repetition, we briefly introduce the key definitions and results.

The definition of the input directional distance function is derived from the directional distance function (2.2) by taking a direction vector $g = (h, 0)$:

$$D(x, y, h, 0) = \sup_{\delta \in \mathbb{R}} \{\delta :\ (x - \delta h, y) \in T\}\,. \tag{2.15}$$

Then, we define the short-run input directional distance function from the short-run directional distance function (2.3) by taking a direction denoted $g = (h^v, 0, 0, 0)$:

$$SRD(x, y; h, 0) = D(x^v, x^f, y, h^v, 0, 0, 0) \tag{2.16}$$

$$= \sup_{\delta \in \mathbb{R}} \left\{ \delta : (x^v - \delta h^v, x^f, y) \in T \right\}. \tag{2.17}$$

The next element needed for our analysis is the long-run cost function, which can be defined as follows:

$$C(w, y) = \inf_x \{ w.x : D(x, y, h, 0) \geq 0 \}. \tag{2.18}$$

Chambers, Chung and Färe (1996) show duality between the input directional distance function (2.15) and the long-run cost function (2.18).

Following Chambers, Chung and Färe (1998), we first define the input overall efficiency (OE^i) index as the quantity:

$$OE^i(x, y, w) = \frac{w.x - C(w, y)}{w.h} \tag{2.19}$$

Then, they continue by characterising a technical efficiency (TE^i) index as the quantity:

$$TE^i(x, y) = D(x, y; h, 0) \tag{2.20}$$

Finally, the allocative efficiency (AE^i) index is defined as the quantity:

$$AE^i(x, y, w) = OE^i(x, y, w) - TE^i(x, y) \tag{2.21}$$

Overall input efficiency ensures that both these ideals of technical and allocative efficiency are realised simultaneously. Obviously, the following additive decomposition identity holds:

$$OE^i(x, y, w) = AE^i(x, y, w) + TE^i(x, y) \tag{2.22}$$

The short-run total cost function is then:

$$SRC(w, \bar{x}^f, y) = \inf_{x^v} \left\{ w^v.x^v + w^f.\bar{x}^f : (x^v, \bar{x}^f, y) \in T \right\}, \tag{2.23}$$

while the short-run variable cost function is:

$$SRVC(w^v, \bar{x}^f, y) = SRC(w, \bar{x}^f, y) - w^f.\bar{x}^f \tag{2.24}$$

$$= \inf_{x^v} \left\{ w^v.x^v : (x^v, \bar{x}^f, y) \in T \right\}. \tag{2.25}$$

Based upon Blancard et al. (2006), it is straightforward to establish duality between the short-run input directional distance function (2.16) and the short-run variable cost function (2.24).[7]

Short-run input overall efficiency ($SROE^i$) is defined as the quantity:

$$SROE^i(w, x^v, \bar{x}^f, y) = \frac{w^v.x^v - SRC(w, \bar{x}^f, y)}{w^v.h^v} \tag{2.26}$$

A short-run technical efficiency ($SRTE^i$) index can now be defined as the quantity:

$$SRTE^i(x^v, \bar{x}^f, y) = SRD(x, y; h^v, 0, 0). \tag{2.27}$$

Obviously, in analogy to the profit function, a short-run allocative efficiency ($SRAE^i$) index bridges the gap between $SROE^i$ and $SRTE^i$.

Now we provide a definition of a cost tangency point:

Definition 2.3 *Let $(\bar{w}^v, w^f, \bar{p}) \in \mathbb{R}_+^{n+m}$ be an input-output price vector. Under the assumptions (T.1) to (T.5), an input-output vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point if*

$$\bar{w}^v.x^v + w^f.\bar{x}^f = SRC(\bar{w}^v, \bar{x}^f, y) = C(\bar{w}^v, w^f, y).$$

We denote $\aleph(\bar{w}^v, \bar{p}, \bar{x}^f)$ the set of all the $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency points for a specific observation. It is clear that if a production vector $(x, y) \in T$ is such that $x^f = \bar{x}^f$ and if x minimises the short- and long-run cost, then it is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point.

[7] Under the assumptions (T.1) to (T.5), we have:

a)

$$SRVC(w^v, \bar{x}^f, y) = \inf_{x^v} \left\{ w^v.x^v : SRD(x^v, \bar{x}^f, y; h^v, 0) \geq 0 \right\}$$

b)

$$SRD(x^v, \bar{x}^f, y; h^v, 0) = \inf_{w \geq 0} \left\{ \frac{w^v.x^v - SRVC(w^v, \bar{x}^f, y)}{w^v.h^v} : w^v.h^v \neq 0 \right\}$$

The proof is similar to the proof in Blancard et al. (2006) and is omitted.

Proposition 2.4 *Under the assumptions (T.1) to (T.5), if* $(x^v, \bar{x}^f, y) \in T$ *and*

$$SRAE^i(\bar{w}^v, w^f, x^v, \bar{x}^f, y) = AE^i(\bar{w}^v, w^f, x, y) = 0,$$

if and only if

$$(\check{x}, \check{y}) = (\, x - SRD(x, y; h^v, 0, 0)\,(h^v, 0)\,,\, y\,)$$

is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$*-cost tangency point.*

Proof: Similar to the one for Proposition 2.2 and therefore discarded. □

3 Tangency Points and Non-parametric Technology

3.1 Profit Tangency Points

Now, we focus on non-parametric production models and prove how a tangency point can be computed by finding the solution to a system of linear inequalities.

Let us consider a set of r firms $A = \{(x_1, y_1), ..., (x_r, y_r)\} \in \mathbb{R}_+^{n+m}$. Production technology can be estimated by enveloping observed firms while respecting some basic economic production axioms (see Hackman (2008) or Ray (2004)). First, we focus on the constant and variable returns to scale cases and provide systems of equalities and inequalities for determining solutions for the profit tangency points. This is basically just the dual solution of the mathematical program for computing the directional distance functions. When returns to scale are constant, then the non-parametric technology is the conical hull of the observed data plus the non-negative orthant. When returns to scale are variable, then the non-parametric technology is the convex hull of the observations plus the non-negative Euclidean orthant. The formulation adopted here is of particular importance to provide a system of linear inequalities.

Under constant returns to scale the production set is defined as:

$$T_{CRS} = \left\{ (x, y) : x \geq \sum_{j=1}^{r} \theta_j x_j, y \leq \sum_{j=1}^{r} \theta_j y_j, \theta \geq 0 \right\}. \tag{3.1}$$

Let $C_c(A)$ be the conical hull of A . We have $T_{CRS} = (C_c(A) + K) \cap \mathbb{R}_+^{n+m}$ where $K =$

$\mathbb{R}^n_+ \times \left(-\mathbb{R}^m_+\right)$. Let us calculate the profit function: $\forall (w,p) \in \mathbb{R}^{n+m}_+$, we have:

$$\Pi(w,p) = \begin{cases} +\infty & \text{if} \quad p.y_j - w.x_j > 0 \quad \text{for some} \quad j \\ 0 & \text{if} \quad p.y_j - w.x_j \leq 0 \quad \text{for all} \quad j, \end{cases} \tag{3.2}$$

since profits are either infinite or non-positive in the case of constant returns to scale. This yields a dual program for finding the shadow price solution $(\tilde{w}, \tilde{p})$ that is also the adjusted price function (see Luenberger (1995)):

$$D_{CRS}(x,y;g) = \min_{(w,p)\geq 0} \{\Pi(w,p) - p.y + w.x \; : \; p.k + w.h = 1\} \tag{3.3}$$

$$= \min_{\substack{(w,p)\geq 0 \\ p.k+w.h=1}} \{-p.y + w.x : p.y_j - w.x_j \leq 0 \, , j = 1,...,r\} \, , \tag{3.4}$$

where $(\tilde{w}, \tilde{p})$ is the solution of the program. We deduce the following result:

Proposition 3.1 *Assume that $T = T_{CRS}$. Under the assumptions (T.1) to (T.5), a vector $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point if and only if $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ satisfies the following system of linear inequalities:*

$$(\star) \quad \begin{cases} -\bar{p}^v.y^v - p^f.\bar{y}^f + \bar{w}^v.x^v + w^f.\bar{x}^f & = 0, \\ \bar{p}^v.y^v_j + p^f.\bar{y}^f_j - \bar{w}^v.x^v_j - w^f.\bar{x}^f_j & \leq 0, \quad j = 1,...,r \\ (x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j, & \\ (y^v, \bar{y}^f) \leq \sum_{j=1}^{r} \theta_j y_j, & \\ \theta \geq 0, y^v \geq 0, x^v \geq 0, p^f \geq 0, w^f \geq 0. & \end{cases}$$

Proof: If (x^v, y^v, θ) satisfies $(\star)$, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T_{CRS}$ and moreover it maximises profits. Since $x^f = \bar{x}^f$ and $y^f = \bar{y}^f$, we deduce that $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point. Conversely, we deduce that if $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T^{CRS}$, thus there exists $\theta \geq 0$ such that $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$ and $(y^v, \bar{y}^f) \leq \sum_{j=1}^{r} \theta_j y_j$. Moreover, from the weak version of the separation theorem, it maximises profits, therefore: $-\bar{p}^v.y^v - p^f.\bar{y}^f + \bar{w}^v.x^v + w^f.\bar{x}^f = 0$ and $\bar{p}^v.y^v_j + p^f.\bar{y}^f_j - \bar{w}.x^v_j - w^f.\bar{x}^f_j \leq 0, \; j = 1,...,r$, (x^v, y^v, θ) satisfies system $(\star)$. $\square$

The first equality imposes that profits must be zero at the tangency point (otherwise the invested capacity would be economic obsolete). The next set of r inequalities guarantees that all profits are smaller than or equal to zero (which provides an incentive to exploit the initially build capacity). All other inequalities simply impose the constant returns to scale non-parametric technology.

Thus, assuming competitive conditions, the basic logic is that for all fixed quantities, i.e., fixed inputs and outputs, we look for the corresponding (shadow) prices, i.e., fixed input and output prices, that make the observed configurations of fixed and variable inputs and outputs ex post economically viable. The results are useful to assess whether certain optimal fixed quantities are technically feasible and whether the prices to make them economically viable can be supported by the market. Notice that we suppose a competitive product market, since the firm can sell any output at the optimal price. Equally so, we assume input prices are determined in a competitive setting.[8]

The system of equalities and inequalities ($\star$) can be seen as a so-called constraint satisfaction problem. Several techniques are available for solving these types of problems (see, e.g., Chinneck (2008)), mainly focussing on overconstrained problems. Usually, some penalty function related to the number of satisfied constraints is minimised. However, since system ($\star$) is not overconstrained, a feasible solution can be obtained by applying standard optimisation techniques by adding to the system a constant objective function (see, e.g., Moore, Kearfott and Cloud (2009)). Notice that zero solutions for the optimal fixed input price (p^f) may occur.

In the variable returns to scale case the production technology is:

$$T_{VRS} = \left\{ (x,y) : x \geq \sum_{j=1}^{r} \theta_j x_j, y \leq \sum_{j=1}^{r} \theta_j y_j, \sum_{j=1}^{r} \theta_j = 1, \theta \geq 0 \right\}. \tag{3.5}$$

Let $Co(A)$ be the conical hull of A. We have $T_{VRS} = (Co(A) + K) \cap R_+^{n+m}$ and we have $T_{VRS} + K = Co(A) + K$. The profit function is $\forall\, (w,p) \in R_+^{n+m}$:

$$\Pi(w,p) = \max \left\{ p.y_j - w.x_j \,:\, j = 1, ..., r \right\}. \tag{3.6}$$

[8] While some literature exists on capacity utilisation in non-competitive settings, its development in a framework allowing for inefficiency would necessitate further developments in analysing non-competitive behaviour in a non-parametric framework (see, e.g., Kallio and Kallio (2002) for a start).

Assume that $(x,y) \in T_{VRS} + K$. Since $T_{VRS} + K = Co(A) + K$, we obtain:

$$D_{VRS}(x,y) = \min_{\substack{(w,p)\geq 0 \\ p.k+w.h=1}} \max \{p.(y_j - y) - w.(x_j - x) : j = 1,...,r\}. \tag{3.7}$$

Proposition 3.2 *Assume that $T = T_{VRS}$. Under the assumptions (T.1) to (T.5), a vector $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point if and only if $(x^v, y^v, w^f, p^f, \theta)$ satisfies the following system of linear inequalities:*

$$(\star\star) \quad \begin{cases} \bar{p}^v.y_j^v + p^f.\bar{y}_j^f - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f - (\bar{p}^v.y^v + p^f.\bar{y}^f - \bar{w}^v.x^v - w^f.\bar{x}^f) \leq 0 \quad , j = 1,...,r \\ (x^v, \bar{x}^f) \geq \sum\limits_{j=1}^{r} \theta_j x_j, \\ (y^v, \bar{y}^f) \leq \sum\limits_{j=1}^{r} \theta_j y_j, \\ \sum\limits_{j=1}^{r} \theta_j = 1, \\ \theta \geq 0, y^v \geq 0, x^v \geq 0, p^f \geq 0, w^f \geq 0. \end{cases}$$

Proof: If (x^v, y^v, θ) satisfies $(\star\star)$, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T^{VRS}$. Moreover, since

$$\bar{p}^v.y_j^v + p^f.\bar{y}_j^f - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f \leq \bar{p}^v.y^v + p^f.\bar{y}^f - \bar{w}^v.x^v - w^f.\bar{x}^f \text{ for } j = 1,...,r$$

it maximises the profit. Since $x^f = \bar{x}^f$ and $y^f = \bar{y}^f$, we deduce that $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point. Conversely, if $(x^v, \bar{x}^f, y^v, \bar{y}^f)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-tangency point, then $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T_{VRS}$. Thus, there exists $\theta \geq 0$ such that $\sum_{j=1}^{r} \theta_j = 1$ and $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$. Moreover, from the weak version of the separation theorem, it maximises profits, therefore:

$$\bar{p}^v.y_j^v + p^f.\bar{y}_j^f - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f - \bar{w}^v.x_j^v - w^f.x_j^f + w^v.x^v + w^f.\bar{x}^f \leq 0 \text{ for } j = 1,...,r$$

and (x^v, y^v) satisfies $(\star\star)$. □

The first set of r inequalities guarantees that all observed profits are smaller than the profits at the tangency point. All other inequalities simply impose the variable returns to scale non-parametric technology. Note that the same remarks developed for the constant returns to scale case apply.

The traditional literature on capacity utilisation ignores the possibility of technical and other inefficiencies. The challenge of formulating a decomposition of a suitable overall efficiency measure based on the tangency profit notion has to the best of our knowledge never been dealt with in the literature. Denoting by $\Pi^{\tan} = \Pi(\bar{w}^v, w^f, \bar{p}^v, p^f)$ the profit level corresponding to a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point (where w^f and p^f are as in Proposition 3.1 or 3.2), one can define the following overall efficiency measure:

$$OE^{\tan}\left(x, y, w, p\right) = \frac{\Pi^{\tan} - (p.y - w.x)}{p.k + w.h}. \tag{3.8}$$

which takes the normalised difference between tangency profits and observed profits for the the observation under evaluation. Clearly, if we consider a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point, then $OE^{\tan}\left(x, y, p, w\right) = 0$. In general, this new overall efficiency measure differs from $OE\left(x, y, p, w\right)$:

$$OE^{\tan}\left(x, y, w, p\right) \gtreqless OE\left(x, y, w, p\right). \tag{3.9}$$

This new measure can be decomposed by relating it to the traditional overall efficiency measure. To transform this unsigned relation into an equality, we introduce a new capacity efficiency measure as the quantity:

$$\begin{aligned} CE^{\tan}\left(x, y, w, p\right) &= OE^{\tan}\left(x, y, w, p\right) - OE\left(x, y, w, p\right) && (3.10)\\ &= \frac{\Pi^{\tan} - (p.y - w.x)}{p.k + w.h} - \frac{\Pi\left(w, p\right) - (p.y - w.x)}{p.k + w.h} && (3.11)\\ &= \frac{\Pi^{\tan} - \Pi\left(w, p\right)}{p.k + w.h}. && (3.12)\end{aligned}$$

Notice that, in contrast to all other efficiency components defined so far that are non-negative, this capacity efficiency measure may be smaller, equal or larger than zero. When $CE^{\tan}\left(x, y, w, p\right) = 0$, then capacity is used at optimal level. When $CE^{\tan}\left(x, y, w, p\right) > 0$ (< 0), then the existing capacity is underutilised (overutilised). Thus, the firm is producing too little (much) output relative to the tangency point compatible with existing fixed inputs.

Obviously, this leads to the following additive decomposition identity:

$$OE^{\tan}\left(x, y, w, p\right) = OE\left(x, y, w, p\right) + CE^{\tan}(x, y, w, p), \tag{3.13}$$

where the overall efficiency measure ($OE\,(x,y,p,w)$) can be further decomposed into a technical ($TE\,(x,y)$) and an allocative ($AE\,(x,y,p,w)$) efficiency index (as defined in subsection 2.2). Notice that the above decomposition could equally well be defined relative to the short-run overall efficiency measure defined in subsection 2.2 and its corresponding technical and allocative efficiency components. This implies defining a short-run capacity efficiency measure as a normalised difference between tangency profits and the short-run profit function.

Shadow prices of fixed inputs (p^f) resulting from system ($\star$) or ($\star\star$) can also be compared to their observed prices ($\bar{p}^f$): when the shadow price is larger (smaller) than the observed price, then the fixed input could be expanded (reduced). This way of putting things is very similar to the traditional parametric approach to determine tangency points for an economic value function (see, e.g., Segerson and Squires (1990)), except that here no differentiability is assumed. It is well-known in the literature that multiple fixed inputs can lead to ambiguous situations whereby the equality between profits at the $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point and observed profits is maintained despite the fact that shadow and actual prices of fixed inputs diverge for each fixed input due to offsetting effects (see Berndt and Fuss (1986)).

3.2 Cost Tangency Points

Reasoning in an analogous way, to determine a cost tangency point requires solving a somewhat similar system of equations. However, the following condition is not completely similar to the condition concerning the profit function, because the actual output price need not be known. While the profit function seeks an optimal input and output mix for given input and output prices, a cost function just looks for optimal inputs for given outputs and input prices. Hence, output prices are no longer parameters and the system ($\star\star\star$) becomes non-linear (see below).

Proposition 3.3 *Assume that $T = T_{CRS}$. Under the assumptions (T.1) to (T.5):*
a) If there is some $p \geq 0$ such that (x^v, y, p, w^f, θ) satisfies the following system of inequalities:

$$(\star\star\star)\quad \begin{cases} -p.y + \bar{w}^v.x^v + w^f.\bar{x}^f & = \quad 0, \\ p.y_j - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f & \leq \quad 0,\ j = 1, ..., r \\ (x^v, \bar{x}^f) \geq \sum\limits_{j=1}^{r} \theta_j x_j, & \\ y \leq \sum\limits_{j=1}^{r} \theta_j y_j, & \\ \theta \geq 0, y \geq 0, x^v \geq 0, p \geq 0, w^f \geq 0, & \end{cases}$$

then the vector $(x^v, \bar{x}^f, y)$ *is a* $(\bar{w}^v, \bar{p}, \bar{x}^f)$*-cost tangency point.*

b) If the vector $(x^v, \bar{x}^f, y)$ *is a* $(\bar{w}^v, \bar{p}, \bar{x}^f)$*-cost tangency point, then there is some* $p \in \mathbb{R}^m_+$ *such that* (x^v, y, p, w^f, θ) *is a solution of* $(\star\star\star)$.

Proof: a) If (x^v, y, p, θ) satisfies $(\star\star\star)$, then $(x^v, \bar{x}^f, y) \in T_{CRS}$ and moreover it maximises profits. Thus, $(x^v, \bar{x}^f)$ minimises costs. Since $x^f = \bar{x}^f$ we deduce that $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point. b) Conversely, if $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then $(x^v, \bar{x}^f, y) \in T_{CRS}$. Thus, there exists $\theta \geq 0$ such that $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$ and $y \leq \sum_{j=1}^{r} \theta_j y_j$. Moreover, from the weak version of the separation theorem, there exists $p \in \mathbb{R}^m_+$ such that $(x^v, \bar{x}^f, y)$ maximises profits. Therefore: $-p.y + w^v.x^v + w^f.\bar{x}^f = 0$ and $p.y_j - w.x_j \leq 0,\ j = 1, ..., r$. Thus, (x^v, y, p, θ) satisfies $(\star\star\star)$. □

The first equality imposes that profits must be zero at the cost tangency point. The next set of r inequalities guarantees that all profits are smaller than or equal to zero. All other inequalities simply impose the constant returns to scale non-parametric technology. Notice that the above system is non-linear since the first equation looks for output prices and outputs and in addition variable inputs that make given fixed inputs generate a zero profit, whereby the non-linearity resides in the first term. Thus, the system looks for output shadow prices in addition to shadow prices for fixed inputs that render the observed fixed inputs economically viable for a certain level of outputs and variable inputs.

But, if one fixes an output price vector, then we return to the linear case developed for profit maximisation under a similar returns to scale assumption in the main contribution. However, by fixing the output prices, some cost tangency points corresponding to other output price vector are omitted. Therefore, it is clear that a profit-tangency point is a cost tangency point, but the converse is not true.

A symmetrical approach applies under a variable returns to scale assumption.

Proposition 3.4 *Assume that* $T = T_{VRS}$*. Under the assumptions (T.1) to (T.5):*

a) If there is some $p \geq 0$ such that (x^v, y, p, w^f, θ) satisfies the following system of inequalities:

$$(\star\star\star\star) \quad \begin{cases} p.y_j - \bar{w}^v.x_j^v - w^f.\bar{x}_j^f - (p.y - \bar{w}^v.x^v - w^f.\bar{x}^f) \leq 0, \; j = 1, ..., r \\ (x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j, \\ y \leq \sum_{j=1}^{r} \theta_j y_j, \\ \sum_{j=1}^{r} \theta_j = 1, \\ \theta \geq 0, y \geq 0, x^v \geq 0, p \geq 0, w^f \geq 0, \end{cases}$$

then the vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point.
b) If the vector $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then there is some $p \in \mathbb{R}_+^m$ such that (x^v, y, p, w^f, θ) is a solution of $(\star\star\star\star)$.

Proof: a) If (x^v, y, p, θ) satisfies $(\star\star\star\star)$, then $(x^v, \bar{x}^f, y) \in T_{VRS}$ and moreover since

$$p.y_j - w.x_j \leq p.y - w^v.x^v - w^f.\bar{x}^f \text{ for } j = 1, ..., r$$

it maximises profits. Thus, $(x^v, \bar{x}^f)$ minimises costs. Since $x^f = \bar{x}^f$ we deduce that $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point. b) Conversely, if $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then $(x^v, \bar{x}^f, y) \in T_{VRS}$. Thus, there exists $\theta \geq 0$ such that $\sum_{j=1}^{r} \theta_j = 1$, $(x^v, \bar{x}^f) \geq \sum_{j=1}^{r} \theta_j x_j$ and $y \leq \sum_{j=1}^{r} \theta_j y_j$. Moreover, from the weak version of the separation theorem, there exists $p \in \mathbb{R}_+^m$ such that it maximises profits. Therefore:

$$p.y_j - w.x_j \leq p.y - w^v.x^v - w^f.\bar{x}^f; \; j = 1, ..., r$$

and (x^v, y, p, θ) satisfies $(\star\star\star\star)$.$\square$

The first set of r inequalities guarantees that all profits at optimal output prices are smaller than the profits at the tangency point. All other inequalities simply impose the variable returns to scale non-parametric technology.

The same approach developed for integrating the profit tangency notion in a suitable efficiency decomposition can equally well be applied in a cost context. Denoting by $C^{\tan}$ the cost level corresponding to a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, one can define the following overall efficiency measure:

$$OE^{i,\tan}(x,y,w) = \frac{w.x - C^{\tan}}{w.h}. \tag{3.14}$$

which takes the normalised difference between observed costs and tangency costs for the the observation under evaluation. This overall efficiency measure differs from $OE^{i}(x,y,w)$:

$$OE^{i,\tan}(x,y,w) \gtreqless OE^{i}(x,y,w). \tag{3.15}$$

This new measure can be decomposed by relating it to the traditional overall efficiency measure. To transform this unsigned relation into an equality, we introduce a new capacity efficiency measure as the quantity:

$$\begin{aligned} CE^{i,\tan}(x,y,w) &= OE^{i,\tan}(x,y,w) - OE^{i}(x,y,w) && (3.16)\\ &= \frac{w.x - C^{\tan}}{w.h} - \frac{w.x - C(w,y)}{w.h} && (3.17)\\ &= \frac{C(w,y) - C^{\tan}}{w.h}. && (3.18) \end{aligned}$$

Notice that this capacity efficiency measure may be smaller, equal or larger than zero. When $CE^{i,\tan}(x,y,w) = 0$, then capacity is optimal. When $CE^{i,\tan}(x,y,w) > 0$ (< 0), then the existing capacity is overutilised (underutilised). Thus, the firm is producing too much (little) output relative to the tangency point compatible with existing fixed inputs. This again leads to the following additive decomposition identity:

$$OE^{i,\tan}(x,y,w) = OE^{i}(x,y,w) + CE^{i,\tan}(x,y,w), \tag{3.19}$$

where the overall efficiency measure ($OE^{i}(x,y,w)$) can be decomposed into a technical ($TE^{i}(x,y)$) and an allocative ($AE^{i}(x,y,w)$) efficiency index (as defined in subsection 2.3). Notice again that the above decomposition could equally well be defined relative to the short-run overall efficiency measure defined in subsection 2.3 and its corresponding technical and allocative efficiency components. This implies defining a short-run capacity efficiency measure as a normalised difference between the short-run cost function and tangency costs.

4 Comparing Profit and Cost Function Tangency Points: A New and General Result

Assume that the outlined procedure yields on the one hand $\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$, the set of all $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency points, and on the other hand $\aleph(w, \bar{p}, \bar{x}^f)$, the set of all $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency points as defined in section 2 for general technologies satisfying the axioms (T.1)-(T.5). These sets of tangency points per observation under evaluation can be determined using non-parametric specifications of technology as in section 3, or these may result from more traditional parametric approaches (e.g., Nelson (1989) or Segerson and Squires (1990)).[9] It is now possible to establish a relation between the characterisation of both these sets of profit and cost tangency points. This result is perfectly general in that it holds true for any specification of technology and it is -to the best of our knowledge- new to the literature.

Proposition 4.1 *Let $w \in \mathbb{R}^n_+$, be an input price vector. Under the assumptions (T.1)-(T.5):*
a) For all output price vector $p \in \mathbb{R}^m_+$, if an input-output vector $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point, then it is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, i.e.:

$$\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f) \subset \aleph(w, \bar{p}, \bar{x}^f).$$

b) If $(x^v, \bar{x}^f, y) \in T$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then there exists an output price vector $p \in \mathbb{R}^m_+$, such that it is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point.

Proof: a) Assume that $(x^v, \bar{x}^f, y^v, \bar{y}^f) \in T$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point. For all $u \in L(y)$, $p.y - w.x \geq p.y - w.u \Leftrightarrow w.x \leq w.u$. Thus, $x = (x^v, \bar{x}^f)$ minimises costs and a) is proven. b) Let us consider the subset $M(y, w) = \{(u, y) \in \mathbb{R}^{n+m} : w.u \leq C(p, y)\}$. Since there are no interior points in T lying in $M(y, w)$, there exists $(w', p) \in \mathbb{R}^{n+m}_+$, such that $M(x, p) \subset \{(u, y) \in \mathbb{R}^{n+m} : p.y - w'.u \leq \Pi(w', p)\}$. Consequently, since $\Pi(w', p) < +\infty$, we deduce that $\inf\{w'.u; (u, y) \in M(y, w)\} > -\infty$. Consequently, from Farkas Lemma, we deduce that $w' = w$. Now, if $(x^v, \bar{x}^f, y) \in T$ is a $(\bar{w}^v, \bar{p}, \bar{x}^f)$-cost tangency point, then $(x^v, \bar{x}^f, y) \in M(y, w) \cap T$. Thus, $p.y - w.u = \Pi(w, p)$, and $(x^v, \bar{x}^f, y)$ is a $(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$-profit tangency point. □

[9] Under standard differentiability assumptions, a parametric approach yields a unique tangency point. Hence, the sets $\Im(\bar{w}^v, \bar{p}^v, \bar{x}^f, \bar{y}^f)$ and $\aleph(w, \bar{p}, \bar{x}^f)$ become singletons.

Thus, the set of profit tangency points is a proper subset of the set of cost tangency points. Furthermore, any cost tangency point can be transformed into a profit tangency point for a particular choice of the vector of output prices. This is just a different way to state that a profit tangency point is always also a cost tangency point, but that the converse need not be true. To some extent this simply follows from the general relation between profit and cost functions: a point on the profit function is also part of the cost function, but not the reverse.

To develop the intuition behind this result, assume for simplicity that the set of fixed outputs is empty ($J_f = \emptyset$). Then, for a given vector of fixed inputs, to find a cost tangency point one may adjust both the output quantity and price vectors such that the systems of inequalities in Proposition 3.3 (or 3.4) are satisfied. In case of the profit function, however, for a given vector of fixed inputs, to find a profit tangency point involves adjusting outputs such that profits remain maximal for the given vector of output prices (i.e., the systems of inequalities in Proposition 3.1 (or 3.2) are satisfied). Obviously, the latter exercise is much more difficult, whence the relationship.

This proposition provides an example of a special LeChatelier-type of principle (as introduced by Samuelson (1947)). While the LeChatelier principle is rather well-known in an optimisation context (see Milgrom and Roberts (1996)), additional results have been developed that are global rather than local and apply to many other equilibrium systems (see, e.g., Roberts (2006)). The generalised LeChatelier principle is for a maximum condition of economic equilibrium: when all unknowns of a function are independently variable, additional constraints reduce the response to any parameter change. This is done by just-binding the constraints while leaving initial equilibrium unchanged. Thus, factor-demand and commodity-supply elasticities are hypothesised to be lower in the short-run than in the long-run in case a constraint is added in the short-run. In the case above, the cost-tangency case is not subject to an extra constraint rationing revenues. Hence, since the introduction of each new constraint makes demand more inelastic, the set of profit-tangency points is encompassed in the set of cost-tangency points.

5 Empirical Illustration

To illustrate how the tangency profit notion can be used to assess the evolution on profits, components of the identity (3.13) are computed for a small sample of 16 Chilean hydro-electric power generation plants observed on a monthly basis (see Atkinson and Dorfman (2009)). Limiting ourselves to the observations for the year 1997 and, assuming no technical change, we specify an inter-temporal frontier resulting in a total of 192 units. It is well-known that Chile was one of the first countries deregulating its electricity market and that hydro-power was a dominant source of energy during the 90s.[10]

These hydro-power plants generate one output (electricity) using three inputs: labour, capital, and water. Also the price per unit of output and the prices of all three inputs are available in current Chilean pesos. Except for the fixed input capital, the remaining flow variables are expressed in physical units. Table 1 presents basic descriptive statistics for the inputs and the single output. Observe that the minimum price for water is zero, which corresponds to the 11 power plants located on a river (run-of-river plants). The capacity of run-of-river plants goes from 13 to 160 Megawatts. For the 5 reservoir plants, the price of water equals the marginal cost of fossil-fuelled generation. Reservoir plants have larger capacities ranging between 101 and 500 Megawatts. More details on the data are available in Atkinson and Dorfman (2009).[11]

[Table 1 around here]

The activity of power plants located on a river depends on the flow of water passing through. In comparison with reservoir power plants, run-of-river plants are smaller, require lower investments, and have less environmental impact (no need to inundate land close to the plant). Obviously, it is precisely the capacity to accumulate water that facilitates reservoir plants to adjust energy generation to demand fluctuations over time.

Table 2 reports descriptive statistics of the additive components of the identity (3.13). Paying attention to the trimmed mean, it is worth noting that the absolute value of the capacity efficiency measure ($CE^{\tan}(x, y, w, p)$) is almost equivalent to the overall efficiency measure ($OE\,(x, y, w, p)$). These results serve to underscore the importance of the capacity component when factors affecting the tangency overall profit efficiency are estimated. All descriptive statistics corresponding to the distribution of $CE^{\tan}(x, y, w, p)$ are negative, which implies that firms are producing excessive output levels relative to the tangency point. Thus, the general conclusion is that capacity is overutilised.

[Table 2 around here]

[10]Notice that the role of hydro-power has changed during the deregulation period in that demand growth has started outpacing reserve capacity triggering questions about supply security.

[11]In Chile there are in total 35 hydroelectric power plants, 26 located on a river (run-of-river plants) and 9 located on reservoirs. Notice that we maintain all observations rather than opting for a preliminary screening looking for any potential outliers.

Just to complete this overall picture, Table 3 exhibits the information on $CE^{\tan}(x, y, w, p)$ depending on its sign. The conclusion is that overutilisation is the dominant situation in terms of the number of cases (with 146 cases of overutilisation compared to only 46 of underutilisation) and, irrespective on the specific point of the distribution one considers, with a bigger impact in terms of the magnitude of the inefficiency caused. Obviously, in case capacity is underutilised, the shadow price for the fixed input is zero in system ($\star$): this affects about 24% of plants.

[Table 3 around here]

Knowing these descriptive results, the following question to be answered is to what extent the overutilisation affects the two types of power plants. Intuitively speaking, in a situation where overutilisation dominates, one could expect that smaller power plants are operating at a more distant production point with respect to the tangency profits. Results confirm this intuition: the average value of $CE^{\tan}(x, y, w, p)$ for reservoir plants is smaller than the average corresponding to run-of-river plants for every single month of 1997 (see Figure 1). It is worth noting that the capacity efficiency measure for run-of-river plants increases steadily during the winter season (from May to September: impact of more rainfall), while the dry season (from December to May) provides the lowest capacity efficiency measures. Reservoir plants, however, exhibit less volatility over the year, demonstrating their ability to better manage the desired level of production to match the capacity efficiency measure maximising tangency profits.

[Figure 1 around here]

Figure 2 shows the evolution of short-run (dashed line) and long-run (solid line) profit in terms of the deviation Δw^{f*} of the shadow price of the fixed input variable from its optimal value. First, notice that the long-run profit is always greater than the short-run profit for all possible deviations Δw^{f*}, except for the tangency point itself where both are equal. Second, it is useful to consider variations around the optimal shadow price for the fixed input. On the one hand, Figure 2a shows this evolution for plant 33, whose optimal shadow price of the fixed input is non-zero. This allows both negative and positive deviations Δw^{f*}. On the other hand, Figure 2b shows the short- and long-run profit for plant 56, a plant with a zero optimal shadow price for the fixed input. For the latter case, a negative deviation Δw^{f*} is not allowed since this would yield negative prices.

[Figure 2 around here]

6 Concluding Comments

To conclude, it is worthwhile mentioning that the analysis could eventually be extended into several directions. First, it is obvious to determine tangency points for the case of a revenue function. Another issue could be to derive tangency notions of capacity in the case of indirect technologies where output maximising production is, e.g., subject to a budget constraint (see Ray, Mukherjee and Wu (2006) for another capacity notion proposal in this context). Also, we have assumed the presence of competitive product markets for homogeneous goods. Other cases (e.g., a monopolist offering a non-storable homogeneous good with periodic demand (i.e., a peak-load problem)) would require some modifications worthwhile pursuing. Furthermore, alternative definitions of capacity utilisation measures could be developed based on optimal and observed quantities (e.g., outputs) if the ensuing problems could be handled (e.g., how to define a primal scalar measure in the multiple output case). Finally, we have ignored any problems of statistical inference related to the estimation of tangency points at the intersection of short- and long-run profit frontiers (see, e.g., Holland and Lee (2002) on capacity estimation or Simar and Wilson (2000) for frontier estimation in general).

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Table 1: Descriptive statistics for 1997

Variable	Trimmed Mean†	Minimum	Maximum
Output (thousands of kWh)	46.79	0.40	353.70
Variable input (billions of m^3 of water)	126.80	0.49	1347.47
Variable input (# workers)	15.62	2.00	52.86
Fixed input (billions)	0.47	0.04	5.98
Output price (per kWh)	12.94	11.31	13.70
Price of water (per m^3 of water)	0.00	0.00	0.00
Price of labour (millions per worker)	1.26	1.23	1.28
Price of capital (estimated cost of capital)	0.70	0.63	0.77

†10% trimming level.

Table 2: Descriptive statistics for identity (3.13)

	$OE^{\tan}(x, y, w, p)$	$OE(x, y, w, p)$	$CE^{\tan}(x, y, w, p)$
Trimmed Mean†	0.1399	14.8124	−13.6338
10th Percentile	−1.1226	1.5414	−40.9226
1st Quartile	−0.6523	4.7273	−20.7913
Median	−0.4013	10.1188	−10.0051
3rd Quartile	0.6833	21.3197	−3.0803
90th Percentile	3.2565	41.7609	0.0005

†10% trimming level.

Table 3: Descriptive statistics for $CE^{\tan}(x, y, w, p)$

	Underutilisation	Overutilisation
Number of cases	46	146
Trimmed Mean	0.0009	−18.8491
10th Percentile	0.0002	−51.4229
1st Quartile	0.0003	−28.4328
Median	0.0004	−12.4979
3rd Quartile	0.0010	−8.2522
90th Percentile	0.0020	−5.3792

†10% trimming level.

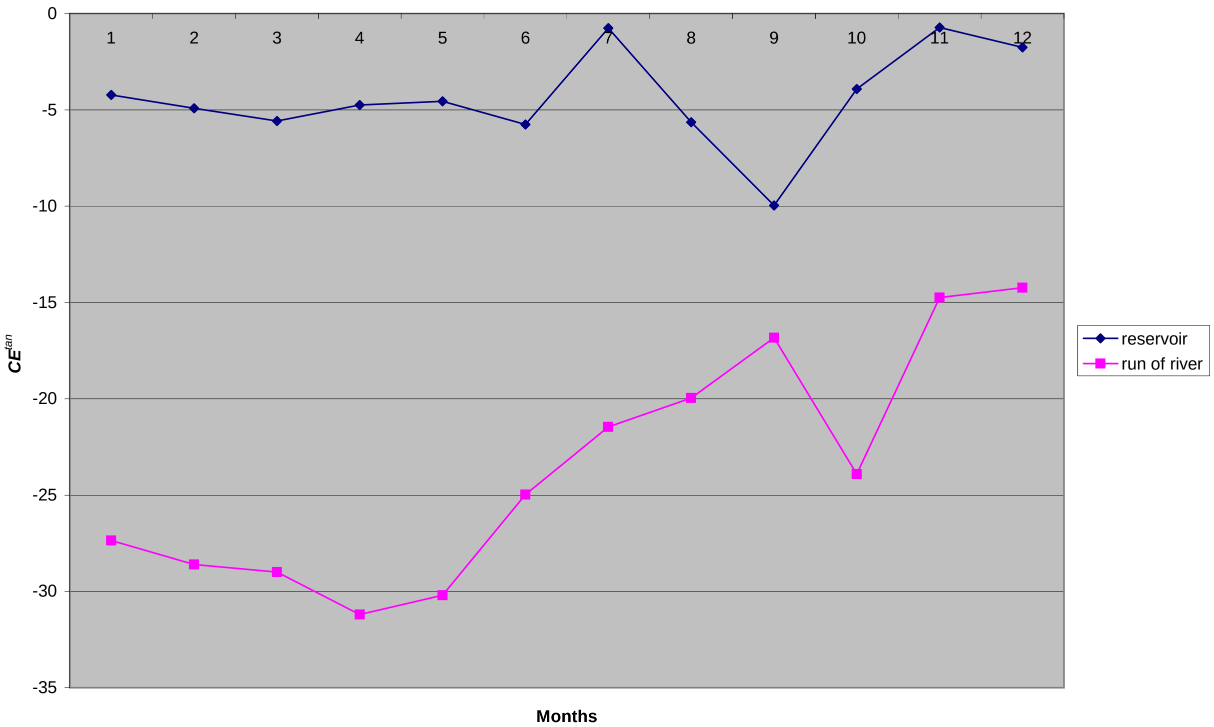

Figure 1: Capacity efficiency measure for run-of-river and reservoir plants

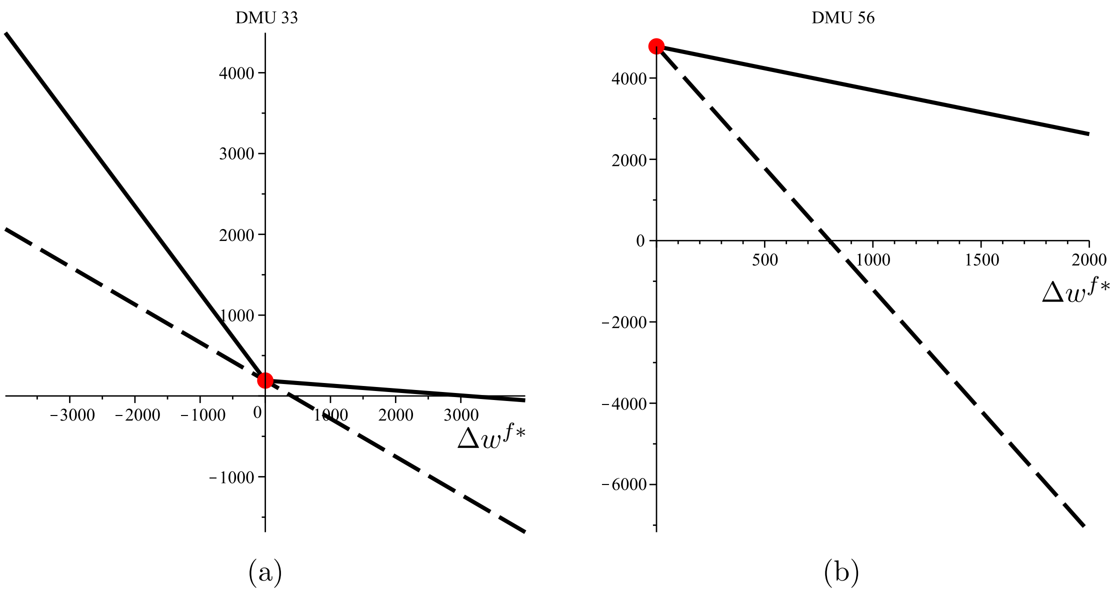

Figure 2: Evolution of short-run (dashed line) and long-run (solid line) profit in terms of deviation of the shadow price of the fixed input from its optimal value. Case (a): Non-zero shadow price. Case (b): Zero shadow price.