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Note on the article:
**Maximum entropy analysis of the $M^{[x]}/M/1$ queueing system
with multiple vacations and server breakdowns***

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Abstract

Wang et al. (2007) elaborate on an interesting approach to estimate the equilibrium distribution for the number of customers in the $M^{[x]}/M/1$ queueing model with multiple vacations and server breakdowns. Their approach consists of maximizing an entropy function subject to constraints, where the constraints are formed by some known exact results. By a comparison between the exact expression for the expected delay time and an approximate expected delay time based on the maximum entropy estimate, they argue that their maximum entropy estimate is sufficiently accurate for practical purposes. In this note, we show that their maximum entropy estimate is easily rejected by simulation. We propose a minor modification of their maximum entropy method that significantly improves the quality of the estimate.

Keywords: Queueing model; Batch arrival; Multiple vacation; Server breakdowns; Maximum entropy; Simulation

1 Model description and exact results

For clarity, we briefly repeat the description of the queueing model and adopt the notations of Wang et al. (2007).

The (single) server serves customers in the order of their arrivals and in an exponentially distributed time with mean $1/\mu$. The state of the server is denoted by i . The server is either on vacation ($i = 0$), busy ($i = 1$) or defect ($i = 2$). If the system is empty at the instant of a service completion, the server takes a vacation during an exponentially distributed time with mean $1/\nu$. If no customers are present when the server returns from vacation, the server takes another vacation during an exponentially distributed time with mean $1/\nu$; otherwise, it starts its service. During a busy period, the server can break down with a constant failure rate that is denoted by α . It takes an exponentially distributed time with mean $1/\beta$ to repair the defective server. If the server is repaired, it resumes the unfinished service.

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Customers arrive in groups. Groups of customers arrive according to a (modified) Poisson process whose rates λ_0 , λ_1 and λ_2 are allowed to depend on the state of the server. The sizes of the groups are independent and identically distributed, and represented by the positive integer-valued random variable A .

Because all the interims are exponentially distributed, this queueing system can be described by a Markov chain with state space $\{(0, 0)\} \cup \{(i, n): i = 0, 1, 2; n = 1, 2, \dots\}$. In the following, only the equilibrium properties of this Markov chain will be considered. Let $P_i(n)$ be the (equilibrium) probability of the event where there are n customers present in the system *and* the server is in state i . The probability for the server to reside in state i is denoted by a_i , where

$$a_0 = \sum_{n=0}^{\infty} P_0(n), \quad a_1 = \sum_{n=1}^{\infty} P_1(n) \quad \text{and} \quad a_2 = \sum_{n=1}^{\infty} P_2(n).$$

The probability distribution of the number of customers present in the system is $\{P(n): n \geq 0\}$, with $P(0) = P_0(0)$ and $P(n) = P_0(n) + P_1(n) + P_2(n)$ if $n \geq 1$.

Wang et al. (2007) use the word *when* in their description of $P_i(n)$. This suggests erroneously that $P_i(n)$ is a conditional probability. The probability of the event where there are n customers present in the system *when* the server is in state i , is $P_i(n)/a_i$.

Wang et al. (2007) introduce the probability generating functions

$$G_0(z) = \sum_{n=0}^{\infty} z^n P_0(n), \quad G_1(z) = \sum_{n=1}^{\infty} z^n P_1(n), \quad G_2(z) = \sum_{n=1}^{\infty} z^n P_2(n)$$

and $G(z) = G_0(z) + G_1(z) + G_2(z)$, and succeed in obtaining an explicit expression for $G_i(z)$ ($i = 0, 1, 2$); see their equations (8), (12) and (13). By noting that $a_i = G_i(1)$, they conclude that

$$a_0 = P(0) \frac{1}{1 - \rho_v}, \tag{1}$$

$$a_1 = P(0) \frac{\beta \rho_0}{\theta(1 - \rho_v)}, \tag{2}$$

$$a_2 = P(0) \frac{\alpha \rho_0}{\theta(1 - \rho_v)}. \tag{3}$$

Further, because $G(1) = a_0 + a_1 + a_2 = 1$, they find that

$$P(0) = \frac{(1 - \rho_v)\theta}{\theta + (\alpha + \beta)\rho_0}. \tag{4}$$

In (1)–(4), the following notations are used:

$$\rho_v = \frac{\lambda_0}{\lambda_0 + \nu}, \quad \theta = \beta(1 - \rho_1) - \alpha\rho_2 \quad \text{and} \quad \rho_i = \frac{\lambda_i}{\mu} E(A) \quad (i = 0, 1, 2).$$

Moreover, Wang et al. (2007) calculate the expected number of customers in the system, which is denoted by L_s . By using the property $L_s = G'(1)$ and the decomposition $G(z) = G_0(z) + G_1(z) + G_2(z)$, this expectation can be written as $L_s = L_s^0 + L_s^1 + L_s^2$, where the $L_s^i = G_i'(1)$ are given by

$$L_s^0 = P(0) \frac{\rho_v E(A)}{(1 - \rho_v)^2} \quad (5)$$

$$L_s^1 = P(0) \left\{ \frac{\beta[\lambda_0\theta + \rho_0(\lambda_1\beta + \lambda_2\alpha)]E(A^2)}{2\mu\theta^2(1 - \rho_v)} + \frac{\beta\rho_0\rho_v E(A)}{\theta(1 - \rho_v)^2} + \frac{\rho_0(2\mu\alpha\rho_2^2 + \beta^2)}{2\theta^2(1 - \rho_v)} \right\} \quad (6)$$

$$L_s^2 = \frac{\alpha}{\beta} L_s^1 + P(0) \frac{\lambda_2\alpha\rho_0 E(A)}{\beta\theta(1 - \rho_v)} \quad (7)$$

In Wang et al. (2007), L_s^i is erroneously called the expected number of customers in the system *when* the server is in state i ; this conditional expectation is L_s^i / a_i .

The expected number of customers in the queue now easily follows from the general single-server formula $L_q = L_s - (1 - P(0))$; see e.g. Gross & Harris (1998), page 12. Finally, the expected delay time W_q can be obtained by applying Little's law,

$$W_q = \frac{L_q}{\lambda E(A)}, \quad \text{where} \quad \lambda = a_0\lambda_0 + a_1\lambda_1 + a_2\lambda_2 \quad (8)$$

is the (long-run) average number of arriving groups per unit of time. In Wang et al. (2007), formula (8) appears with $\lambda = \lambda_0 + \lambda_1 + \lambda_2$, which is a mistake.

2 Maximum entropy estimate

In principle, the probability distribution of the number of customers present in the queueing system can be obtained by repeatedly taking derivatives of $G(z)$ and setting $z = 0$ hereafter. But because such calculations become rapidly too tedious, this strategy is practically unfeasible.

This is why Wang et al. (2007) estimate the exact probabilities $P_i(n)$ by approximate probabilities $\hat{P}_i(n)$. In their method, the approximate probabilities maximize the entropy function

$$H = - \sum_{n=0}^{\infty} P_0(n) \ln P_0(n) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n) - \sum_{n=1}^{\infty} P_2(n) \ln P_2(n), \quad (9)$$

subject to the constraints

$$\sum_{n=0}^{\infty} P_0(n) + \sum_{n=1}^{\infty} P_1(n) + \sum_{n=1}^{\infty} P_2(n) = 1, \quad (10)$$

$$\sum_{n=1}^{\infty} P_1(n) = a_1, \quad (11)$$

$$\sum_{n=1}^{\infty} P_2(n) = a_2, \quad (12)$$

$$\sum_{n=0}^{\infty} nP_0(n) + \sum_{n=1}^{\infty} nP_1(n) + \sum_{n=1}^{\infty} nP_2(n) = L_s. \quad (13)$$

The optimal probabilities are obtained by Lagrange's method. Their solution is

$$\hat{P}_0(n) = \frac{a_0}{a_0 + L_s} \left(1 - \frac{1}{a_0 + L_s} \right)^n \quad \text{for } n \geq 0, \quad (14)$$

$$\hat{P}_1(n) = \frac{a_1}{a_0 + L_s} \left(1 - \frac{1}{a_0 + L_s} \right)^{n-1} \quad \text{for } n \geq 1, \quad (15)$$

$$\hat{P}_2(n) = \frac{a_2}{a_0 + L_s} \left(1 - \frac{1}{a_0 + L_s} \right)^{n-1} \quad \text{for } n \geq 1. \quad (16)$$

$P(n)$ is estimated by $\hat{P}(n)$, where $\hat{P}(0) = \hat{P}_0(0)$ and $\hat{P}(n) = \hat{P}_0(n) + \hat{P}_1(n) + \hat{P}_2(n)$ if $n \geq 1$:

$$\hat{P}(n) = \begin{cases} \frac{a_0}{a_0 + L_s} & \text{if } n = 0, \\ \frac{L_s}{(a_0 + L_s)^2} \left(1 - \frac{1}{a_0 + L_s} \right)^{n-1} & \text{if } n \geq 1. \end{cases} \quad (17)$$

Figure 1 illustrates the maximum entropy estimate (17) in the case where A is uniformly distributed on $\{1, 2, 3, 4, 5\}$ (notation: $A \sim \text{Un}(1, 5)$) and with the following choice of parameters:

$$\lambda_0 = 0.3, \lambda_1 = 0.5, \lambda_2 = 0.4, \mu = 2, \alpha = 0.05, \beta = 3 \text{ and } \nu = 2. \quad (18)$$

The parameter choice (18) is identical to the baseline of the numerical examples in Wang et al. (2007). Moreover, they also consider the case $A \sim \text{Un}(1, 5)$ in their numerical examples. Remark that $E(A) = 3$ and $E(A^2) = 11$. Hence, from (1)–(8) it can be calculated that

$$\begin{aligned} a_0 &\cong 0.3441, & a_1 &\cong 0.6452, & a_2 &\cong 0.0108, \\ P(0) &\cong 0.2992, & L_s &\cong 6.8422, & W_q &\cong 4.7596. \end{aligned}$$

In figure 1, the small value of $\hat{P}(0)$ attracts our attention. This value $\hat{P}(0) \cong 0.0479$ is a lot smaller than the exact value $P(0) \cong 0.2992$. This is the reason why we will propose an improved maximum entropy estimate for $\{P(n): n \geq 0\}$. Moreover, our result for W_q deviates from the value $W_q \cong 4.0560$ which appears in Wang et al. (2007) for this example.

Figure 1

3 Improved maximum entropy estimate

Because the exact result for $P(0)$ is known, cf. (4), we see no reason to estimate it. So, in our improved maximum entropy method we replace constraint (10) by

$$\sum_{n=1}^{\infty} P_0(n) + \sum_{n=1}^{\infty} P_1(n) + \sum_{n=1}^{\infty} P_2(n) = 1 - \gamma, \quad (19)$$

where the notation $\gamma = P(0) = P_0(0)$ is introduced. By following the same steps as in Wang et al. (2007), we arrive for $n \geq 1$ at the improved maximum entropy estimates:

$$\bar{P}_0(n) = \frac{(a_0 - \gamma)(1 - \gamma)}{L_s - (1 - \gamma)} \left(1 - \frac{1 - \gamma}{L_s} \right)^n, \quad (20)$$

$$\bar{P}_1(n) = \frac{a_1(1 - \gamma)}{L_s - (1 - \gamma)} \left(1 - \frac{1 - \gamma}{L_s} \right)^n, \quad (21)$$

$$\bar{P}_2(n) = \frac{a_2(1 - \gamma)}{L_s - (1 - \gamma)} \left(1 - \frac{1 - \gamma}{L_s} \right)^n. \quad (22)$$

By adding (20)–(22), the improved maximum entropy estimate for $P(n)$, $n \geq 1$, is obtained:

$$\bar{P}(n) = \frac{(1-\gamma)^2}{L_s - (1-\gamma)} \left(1 - \frac{1-\gamma}{L_s}\right)^n. \quad (23)$$

Remark that $L_q = L_s - (1 - \gamma)$, showing that the estimate (23) decreases exponentially with decaying factor L_q/L_s . In summary, our improved maximum entropy estimate for $\{P(n): n \geq 0\}$ is given by

$$\bar{P}(n) = \begin{cases} \gamma & \text{if } n = 0, \\ \frac{(1-\gamma)^2}{L_q} \left(\frac{L_q}{L_s}\right)^n & \text{if } n \geq 1. \end{cases} \quad (24)$$

Figure 2 illustrates (24) for the same case as in figure 1.

Figure 2

4 Simulation

Wang et al. (2007) test the quality their maximum entropy estimate (17) by comparing the exact value W_q for the expected waiting time in the queue (obtained by applying Little's law) with a corresponding approximate value W_q^* that is based on (17). They develop a formula for W_q^* by dividing customers into three groups: those who arrive when the server is on vacation ($i = 0$), those who arrive when the server is busy ($i = 1$) and those who arrive when the server is defect ($i = 2$). We find their expression for W_q^* questionable, because they ignore possible service interruptions due to server breakdowns (the breakdown rate α nowhere appears) and because they abuse the PASTA property (by applying Bayes rule, as e.g. in Gross & Harris (1998), it is easily seen that the arrival time probabilities are given by $P_i(n) \times \lambda_i / \lambda$).

In any case, we are convinced that a more direct approach to test the quality of the maximum entropy estimate of $\{P(n): n \geq 0\}$ consists of comparing the estimate with an empirical distribution obtained by simulation. The $M^{[x]}/M/1$ queueing model with multiple vacations and server breakdowns can easily be simulated. We performed this simulation in Microsoft Excel and our simulation is available on demand addressed to the corresponding author.

In the numerical examples of Wang et al. (2007), A is either uniformly distributed on $\{1, 2, 3, 4, 5\}$ (denoted by $A \sim \text{Un}(1, 5)$) or geometrically distributed with parameter 0.25 (denoted by $A \sim \text{Geo}(0.25)$). The first case already appeared in figures 1 and 2, and will further be explored here. In the latter case, our interpretation of the notation $A \sim \text{Geo}(0.25)$ is

$$P(A = n) = 0.25 \times 0.75^{n-1} \quad (n = 1, 2, \dots). \quad (25)$$

The reader can verify that, according to (4), $P(0) < 0$ for the case where (18) and (25) hold, which implies that this case is unstable; the traffic intensity when the server is busy, amounts to $\rho_1 = 1$. We have also performed simulations in the case where A is geometrically distributed. We content ourselves by reporting only on the case where A is uniformly distributed, because the case where A is geometrically distributed leads to the same conclusion. The following results report on a simulation with (18) as the choice for the parameters and where $A \sim \text{Un}(1, 5)$.

In figure 3, a possible realization of $(i(t), n(t))$ is shown for the first 300 time units ($0 \leq t \leq 300$) and with initial state $(i(0), n(0)) = (0, 0)$. Figure 3 illustrates that the breakdown state $i = 2$ is seldom visited and $i(t)$ most often resides in the busy state $i = 1$. Moreover, figure 3 shows that $n(t)$ strongly fluctuates and that large excursions of $n(t)$ are possible. By comparing figure 3 and other realizations of $(i(t), n(t))$ with the exact equilibrium results ($a_0 \cong 0.3441$, $a_1 \cong 0.6452$, $a_2 \cong 0.0108$, $P(0) \cong 0.2992$, $L_s \cong 6.8422$), we are confident that in the present example the system has sufficiently equilibrated at $t = 300$.

We observed $(i(300), n(300))$ 1000 times in our simulation. The exact values of a_0 , a_1 , a_2 , $P(0)$ and L_s lie well within the 95 % confidence intervals obtained with our simulation; these intervals are respectively given by $[0.3077, 0.3663]$, $[0.6215, 0.6805]$, $[0.0053, 0.0187]$, $[0.2677, 0.3243]$ and $[6.1790, 7.2110]$. Hence, we are inclined to strengthen our belief that the equilibrium distribution $\{P(n): n \geq 0\}$ is well represented by the distribution of $n(300)$.

The empirical distribution of $n(300)$ is compared in figure 4 with the maximum entropy estimates (17) and (24), which already appeared in figures 1 and 2. From figure 4, it can be concluded that the maximum entropy estimate (17) is a bad approximation of the true distribution $\{P(n): n \geq 0\}$, while the improved maximum entropy estimate (24) seems to be close to $\{P(n): n \geq 0\}$.

Figure 3

Figure 4

We finally applied Kolmogorov-Smirnov's test in order to compare statistically the maximum entropy estimates (17) and (24) with the true distribution $\{P(n): n \geq 0\}$, which is supposed to be well represented by our simulation. The test value KS in the

Kolmogorov-Smirnov test is the maximum absolute difference between the empirical distribution function (obtained in our simulation) and a theoretical distribution function (corresponding either to (17) or (24)). This test value KS is compared with the 5 % significance critical value for the corresponding test statistic, which is given by

$$\sqrt{-\frac{\ln(0.05/2)}{2 \times 1000}} \cong 0.0429.$$

If the test value KS is larger than this critical value, the theoretical distribution is rejected at a confidence level of 95 %.

	maximum entropy	improved maximum entropy
test value (KS)	0.2481	0.0230
p-value	6.732×10^{-54}	0.6663

Table 1

It can be observed from table 1 that the maximum entropy estimate (17) is rejected, while the improved maximum entropy estimate (24) is not rejected. To further highlight the difference in quality between the estimates (17) and (24), the p-values also appear in table 1; these p-values are calculated with the formula

$$2 \times \sum_{k=1}^{\infty} (-1)^{k+1} \times e^{-2 \times 1000 \times (k \times \text{KS})^2}.$$

As expected, the maximum absolute difference between the empirical distribution function and the theoretical distribution function is attained at $n = 0$ for the estimate (17). This property has repeatedly been observed and shows that the major drawback of the estimate (17) is due to $\hat{P}(0)$.

5 Conclusions

We have shown that the maximum entropy estimate of Wang et al. (2007) is deficient in accurately approximating the true distribution of the number of customers present in the system and that a significant improvement can be obtained by including the exact probability of the empty system in the constraints of their method.

6 Appendix: further improvements

In our simulation of section 4, we also calculated the 95 % confidence intervals for $P(1)$ and $P(2)$. These intervals are respectively given by $[0.0382, 0.0658]$ and $[0.0533, 0.0847]$. The improved maximum entropy estimate (24) for $P(1)$ and $P(2)$ are, respectively, $\bar{P}(1) \cong 0.0718$ and $\bar{P}(2) \cong 0.0644$. Hence, $\bar{P}(1)$ is not contained in the

95 % confidence interval for $P(1)$. Moreover, Kolmogorov-Smirnov's test value $KS \cong 0.0230$ for (24) is attained at $n = 1$. Hence, there seems to be room for further improvements of the maximum entropy estimate (24), in particular by taking the exact value of $P(1)$ into account.

In an extra effort to improve the maximum entropy estimate of Wang et al. (2007), the exact expressions for $P(0), P(1), \dots, P(k)$, with $k \geq 1$, can be used as additional constraints in Lagrange's method. These probabilities can be found by applying $P(n) = G^{(n)}(0)/n!$, where $G^{(n)}$ denotes the n^{th} derivative of the probability generating function G . For $n = 1$ and $n = 2$, we obtain

$$P(1) = P(0) \left(\sigma_1 \rho_v + \frac{\lambda_0}{\mu} + \frac{\lambda_0 \alpha}{\mu(\lambda_2 + \beta)} \right) \quad (26)$$

and

$$P(2) = P(0) \left(\sigma_2 \rho_v + \sigma_1^2 \rho_v^2 + \sigma_1 \frac{\lambda_0 \lambda_2 \alpha}{\mu(\lambda_2 + \beta)^2} + \frac{\lambda_0}{\mu^2} \left(1 + \frac{\alpha}{\lambda_2 + \beta} \right) \left(\lambda_1 + \mu + \alpha - \sigma_1 \mu(1 - \rho_v) - \frac{\alpha \beta}{\lambda_2 + \beta} \right) \right). \quad (27)$$

In (26) and (27), the notation $\{\sigma_n; n \geq 1\}$ of Wang et al. (2007) for the probability distribution of A has been used. For the example where $A \sim \text{Un}(1, 5)$ and with (18) as the choice for the parameters, we find $P(1) \cong 0.0533$ and $P(2) \cong 0.0572$.

In analogy with Wang et al. (2007), a further improved estimate $\{\tilde{P}(n); n \geq 0\}$ for $\{P(n); n \geq 0\}$ can be found by maximizing the entropy function (9) according to Lagrange's method, where the constraints (10)–(13) are supplemented with the constraints

$$P(0) = \gamma_0 \quad \text{and} \quad P_i(n) = \gamma_{in} \quad (i = 0, 1, 2; n = 1, \dots, k). \quad (28)$$

In (28), γ_0 and γ_{in} denote the exact values for $P(0)$ and $P_i(n)$ ($i = 0, 1, 2; n = 1, \dots, k$). This procedure would first lead to estimates $\tilde{P}_i(n)$ for $P_i(n)$ ($i = 0, 1, 2; n = 1, 2, \dots$). The estimates for $P(n)$ are successively obtained by $\tilde{P}(n) = \tilde{P}_0(n) + \tilde{P}_1(n) + \tilde{P}_2(n)$ ($n = 1, 2, \dots$).

Although the above procedure is certainly tractable, the derivation of $\{\tilde{P}(n); n \geq 0\}$ can be shortened by noting that exactly the same estimate is obtained by maximizing the entropy function

$$H = - \sum_{n=0}^{\infty} P(n) \ln P(n), \quad (29)$$

subject to the constraints

$$\sum_{n=0}^{\infty} P(n) = 1, \quad \sum_{n=0}^{\infty} nP(n) = L_s \quad \text{and} \quad P(n) = \gamma_n \quad (n = 0, \dots, k). \quad (30)$$

In (30), $\gamma_n = \gamma_{0n} + \gamma_{1n} + \gamma_{2n}$ denotes the exact value of $P(n)$ ($n = 1, \dots, k$).

For illustrative purposes, we include here this shortened derivation. The problem formulation (29)–(30) leads to the Langrangian function

$$L = -\sum_{n=0}^{\infty} P(n) \ln P(n) - \theta_1 \left(\sum_{n=0}^{\infty} P(n) - 1 \right) - \theta_2 \left(\sum_{n=0}^{\infty} nP(n) - L_s \right) - \sum_{n=0}^k \tau_n (P(n) - \gamma_n), \quad (31)$$

where $\theta_1, \theta_2, \tau_0, \dots, \tau_k$ are Lagrangian multipliers. Because the estimates $\tilde{P}(n) = \gamma_n$ for $n \leq k$ are supposed to be known, we restrict our attention hereafter to $\tilde{P}(n)$ with $n > k$. The derivative of (31) with respect to $P(n)$ is

$$\frac{\partial L}{\partial P(n)} = -\ln P(n) - 1 - \theta_1 - n\theta_2 \quad \text{if } n > k. \quad (32)$$

By introducing the notations $\phi_1 = e^{-1-\theta_1}$ and $\phi_2 = e^{-\theta_2}$, it follows from (32) that the maximum entropy estimate satisfies

$$\tilde{P}(n) = \phi_1 \phi_2^n \quad \text{if } n > k. \quad (33)$$

The expressions for ϕ_1 and ϕ_2 follow from (33) and the constraints in (30), because

$$1 - \sum_{n=0}^k \gamma_n = \sum_{n=k+1}^{\infty} \tilde{P}(n) = \frac{\phi_1 \phi_2^{k+1}}{1 - \phi_2}, \quad (34)$$

$$L_s - \sum_{n=0}^k n\gamma_n = \sum_{n=k+1}^{\infty} n\tilde{P}(n) = \frac{\phi_1 \phi_2^{k+1} (k+1 - k\phi_2)}{(1 - \phi_2)^2}. \quad (35)$$

To obtain an attractive formula for the maximum entropy estimate, we introduce the notation N for the random variable that denotes the equilibrium number of customers present in the system. Remark that the probability that N exceeds k and the (conditional) expectation of N when N exceeds k are respectively given by

$$P(N > k) = 1 - \sum_{n=0}^k \gamma_n \quad \text{and} \quad E(N | N > k) = \frac{L_s - \sum_{n=0}^k n\gamma_n}{1 - \sum_{n=0}^k \gamma_n}. \quad (36)$$

From (34)–(36) follows that

$$\phi_1 = \frac{P(N > k)}{E(N - k | N > k)} \left(1 - \frac{1}{E(N - k | N > k)} \right)^{-k-1}, \quad (37)$$

$$\phi_2 = 1 - \frac{1}{E(N - k | N > k)}. \quad (38)$$

In summary, the maximum entropy estimate $\{\tilde{P}(n) : n \geq 0\}$ is given by

$$\tilde{P}(n) = \begin{cases} P(N = n) & \text{if } n \leq k, \\ \frac{P(N > k)}{E(N - k | N > k)} \left(1 - \frac{1}{E(N - k | N > k)} \right)^{n-k-1} & \text{if } n > k. \end{cases} \quad (39)$$

Remark that (39) coincides with (24) if $k = 0$.

In figure 5, the maximum entropy estimates (39) for $k = 0$ (which already appeared in figure 2), $k = 1$ and $k = 2$ are compared for the example where $A \sim \text{Un}(1, 5)$ and with (18) as the choice for the parameters.

Figure 5

7 References

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