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Abstract

In this paper, we introduce a two state homogeneous Markov chain and define a geometric distribution related to this Markov chain. We define also the negative binomial distribution similar to the classical case and call it NB related to interrupted Markov chain. The new binomial distribution is related to the interrupted Markov chain. Some characterization properties of the Geometric distributions are given. Recursion formulas and probability mass functions for the NB distribution and the new binomial distribution are derived.

Key words: Homogeneous Markov chain, Interrupted Markov chain, Geometric distribution related to Markov chain, Inflated negative binomial distribution, New binomial distribution.

AMS subject classifications: 60E05; 62P05.

1 Introduction

It is known that many basic counting distributions are defined on a sequence of independent and identically distributed Bernoulli random variables. Many other distributions are defined by compounding and mixing. Another way of obtaining new discrete distributions is to define the counting distributions related to some Markov chain, see Viveros et al. (1994) and Ome² et al. (2008). Assuming some dependency in the sequence of Bernoulli variables, gives an additional parameter, and could be a more realistic model in practice, see Ome² and Van Gulck (2006).

In this paper we introduce some discrete distributions related to two state homogeneous Markov chain. In Section 2, we introduce a homogeneous Markov chain, defined on the sequence of Bernoulli variables. The additional parameter in the Markov chain is interpreted

as a correlation coefficient. We also define a geometric distribution related to this Markov chain. We give some useful properties and characterizations. In Section 3, we define an interrupted Markov chain and a corresponding negative binomial distribution, called inflated negative binomial. We derive the probability mass function (PMF), recursion formulas, central limit theorem, and a limiting distribution. In Section 4, we use an extension of the well known relations between negative binomial and binomial distribution and define a binomial distribution related to the interrupted Markov chain. This distribution is called a new binomial distribution. Some properties and recursions are given.

2 A geometric distribution related to a Markov Chain

Consider a sequence of binary random variables $(X_n, n = 1, 2, \dots)$, where the states 1 and 0 appear with stationary probabilities

$$P(X_n = 1) = \pi \text{ and } P(X_n = 0) = q = 1 - \pi.$$

We call the state 1 a "success" and the state 0 a "failure". We assume that the X_n form a Markov chain with transition probabilities

$$P(X_{n+1} = 1|X_n = 1) = 1 - q(1 - \rho);$$

$$P(X_{n+1} = 0|X_n = 0) = 1 - \pi(1 - \rho),$$

where $\rho \in [0, 1)$ and $n = 1, 2, \dots$. The transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} 1 - \pi(1 - \rho) & \pi(1 - \rho) \\ q(1 - \rho) & 1 - q(1 - \rho) \end{pmatrix}. \quad (1)$$

The correlation coefficient is $\rho(X_n, X_{n+1}) = \rho$, for $n = 1, 2, \dots$ and then the n step transition matrix is given by

$$\mathbf{P}^n = \begin{pmatrix} 1 - \pi(1 - \rho^n) & \pi(1 - \rho^n) \\ q(1 - \rho^n) & 1 - q(1 - \rho^n) \end{pmatrix},$$

see Feller (1971). This Markov chain represents a sequence of correlated Bernoulli random variables.

2.1 Definitions

Starting from $X_1, X_2, \dots$, we define the random variables Y and Y_S as follows. The r.v. Y counts the number of failures in the Markov chain up to the first success. The r.v. Y_S counts the number of transitions up to the first success. Clearly we have $Y_S = Y + 1$. It is clear that the PMF of Y is given by

$$\begin{aligned} P(Y = 0) &= \pi, \\ P(Y = k) &= q(1 - \pi(1 - \rho))^{k-1} \pi(1 - \rho), \quad k = 1, 2, \dots \end{aligned} \quad (2)$$

The probability generating function (PGF) is given by

$$\psi_Y(t) = E(t^Y) = \frac{\pi(1 - \rho t)}{1 - (1 - \pi(1 - \rho))t} \quad (3)$$

Note that the tail of Y is given by

$$P(Y \geq n) = q(1 - \pi(1 - \rho))^{n-1}, \quad n = 1, 2, \dots \quad (4)$$

It is easy to verify that

$$E(Y) = \frac{1 - \pi}{\pi(1 - \rho)} \quad \text{and} \quad \text{Var}(Y) = \frac{E(Y)}{\pi(1 - \rho)} + \frac{\rho}{\pi(1 - \rho)^2}.$$

Definition 2.1 *The r.v. Y defined by (2) and (3) is said to be geometrically distributed related to the Markov chain.*

We use the notation $Y \sim IGe(\pi, \rho)$.

Remark 2.1 *The distribution of Y coincides with the inflated - parameter geometric distribution as in Minkova (2001, 2002). In Omei et al. (2008), this PMF was studied for general transition matrices. If $\rho = 0$, then $IGe(\pi, \rho)$ corresponds to the usual geometric distribution.*

Remark 2.2 *We can also consider $-1 < \rho < 0$ as long as $(1 - \pi)(1 - \rho) < 1$ and $\pi(1 - \rho) < 1$, i.e.*

$$\frac{-\rho}{1 - \rho} < \pi < \frac{1}{1 - \rho}.$$

2.2 Characterizations of the $IGe(\pi, \rho)$

We have the following result.

Theorem 2.1 *(i) Let Y be a non-negative, integer-valued random variable satisfying $P(Y = 0) = \pi$, where $\pi \in (0, 1)$ and let $\rho \in [0, 1)$. Then $Y \sim IGe(\pi, \rho)$ if and only if*

$$(1 - \pi)P(Y \geq k + l) = (1 - \pi(1 - \rho))P(Y \geq k)P(Y \geq l) \quad (5)$$

for every $k, l = 1, 2, \dots$

(ii) Let Y be a non-negative, integer-valued random variable satisfying

$$P(Y = 0) = \pi \text{ and } P(Y = 1) = (1 - \pi)\pi(1 - \rho), \quad (6)$$

where $\pi \in (0, 1)$ and $\rho \in [0, 1)$. Then $Y \sim IGe(\pi, \rho)$ if and only if

$$(1 - \pi)P(Y > k + l) = P(Y > k)P(Y > l), \quad (7)$$

for every $k, l = 1, 2, \dots$

Proof. (i) If (2) holds, then (4) shows that (5) holds. Conversely, suppose that (5) holds. Using $P(Y \geq 1) = 1 - \pi$ we have

$$(1 - \pi)P(Y \geq k + 1) = u(1 - \pi)P(Y \geq k), \quad k \geq 1,$$

where $u = 1 - \pi(1 - \rho)$. It follows that

$$P(Y \geq k + 1) = uP(Y \geq k),$$

so that

$$P(Y \geq k) = u^{k-1}P(Y \geq 1) = (1 - \pi)u^{k-1}.$$

This is (5).

(ii) From (4) we have that

$$P(Y > n) = q(1 - \pi(1 - \rho))^n, \quad n = 1, 2, \dots,$$

where $q = 1 - \pi$, and then (7) follows. Conversely, if (7) holds, then

$$\begin{aligned} qP(Y > n + 1) &= P(Y > n)P(Y > 1) \\ &= P(Y > n)(1 - \pi - (1 - \pi)\pi(1 - \rho)) \\ &= quP(Y > n), \end{aligned}$$

where $u = 1 - \pi(1 - \rho)$. It follows that

$$P(Y > n + 1) = uP(Y > n), \quad n = 1, 2, \dots,$$

and again (5) follows. □

Theorem 2.2 Suppose that X is a r.v. with $P(X \leq x) = 1 - \theta e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$, $0 < \theta < 1$. Let $|X| = k$ if and only if $k - 1 < X \leq k$, $k \geq 0$. Then $|X|$ has an $IGe(\pi, \rho)$ distribution with $\pi = 1 - \theta$ and $\rho = (e^{-\lambda} - \theta)/(1 - \theta)$.

Proof. For $k = 0$ we have

$$P(|X| = 0) = P(X \leq 0) = 1 - \theta.$$

For $k \geq 1$, we have

$$\begin{aligned} P(|X| = k) &= P(k - 1 < X \leq k) \\ &= \theta e^{-\lambda(k-1)} - \theta e^{-\lambda k} \\ &= \theta(e^{-\lambda})^{k-1}(1 - e^{-\lambda}). \end{aligned}$$

Comparing with (2), we have $\pi = 1 - \theta$ and $(1 - \pi)\pi(1 - \rho) = \theta(1 - e^{-\lambda})$. It follows that

$$1 - \rho = \frac{1 - e^{-\lambda}}{\pi},$$

and then also that

$$\rho = 1 - \frac{1 - e^{-\lambda}}{\pi} = \frac{\pi - 1 + e^{-\lambda}}{\pi} = \frac{e^{-\lambda} - \theta}{1 - \theta}.$$

Clearly we have $1 - \pi(1 - \rho) = e^{-\lambda}$ as required. □

Remark 2.3 If $\theta = e^{-\lambda}$, then $P(X \leq x) = 1 - e^{-\lambda(1+x)}$ and $X + 1 \sim EXP(\lambda)$. The relation between an exponential distribution and the geometric distribution is well known, see Johnson et al. (2005).

Remark 2.4 If $\theta = 1$, then $P(X \leq x) = 1 - e^{-\lambda x}$ and $X \sim EXP(\lambda)$, see Deng and Chhikara (1990), for example. Now we find that

$$P(|X| = 0) = 0,$$

$$P(|X| = k) = (e^{-\lambda})^{k-1}(1 - e^{-\lambda}), \quad k \geq 1.$$

3 Inflated Negative Binomial Distribution

3.1 Definition

Let $Y_1, Y_2, \dots, Y_r$ denote independent r.v. with the same $IGe(\pi, \rho)$ distribution, and let $N(r) = Y_1 + Y_2 + \dots + Y_r$. The distribution of $N(r)$ is called an inflated negative binomial distribution with parameters $\pi > 0$, $\rho > 0$ and $r \in \mathbb{N}_0$. Notation $N(r) \sim INB(\pi, \rho, r)$.

The PGF of $N(r)$ is given by

$$\psi_{N(r)}(t) = \left(\frac{\pi(1 - \rho t)}{1 - (1 - \pi(1 - \rho))t} \right)^r.$$

The random variables Y_i are assumed to be independent. As a consequence, when a success appears, the Markov chain starts again at the origin and the initial PMF is given by $P(X_1 = 1) = \pi$, $P(X_1 = 0) = 1 - \pi$. Let the sequence (X_n) denote a Markov chain with the following transition matrix $\mathbf{P}^*$:

$$\mathbf{P}^* = \begin{pmatrix} 1 - \pi(1 - \rho) & \pi(1 - \rho) \\ 1 - \pi & \pi \end{pmatrix}.$$

This Markov chain will be called an *interrupted Markov chain*. The r.v. $N(r)$ counts the number of failures up to the r th success in this Markov chain, see Omey et al. (2008), section 2.3.

The n step transition matrix is given by

$$\mathbf{P}^{*n} = \begin{pmatrix} 1 - \frac{\pi(1 - \rho)}{1 - \pi\rho} (1 - (\pi\rho)^n) & \frac{\pi(1 - \rho)}{1 - \pi\rho} (1 - (\pi\rho)^n) \\ \frac{1 - \pi}{1 - \pi\rho} (1 - (\pi\rho)^n) & 1 - \frac{1 - \pi}{1 - \pi\rho} (1 - (\pi\rho)^n) \end{pmatrix}.$$

3.2 Properties of $N(r)$

In our first result we show how the pdf of $N(r)$ can be obtained recursively.

Lemma 3.1 *Let $p_r(k) = P(N(r) = k)$ and $p_{r+1}(k) = P(N(r+1) = k)$. The following relation holds:*

$$\begin{aligned} p_{r+1}(0) &= \pi p_r(0); \\ p_{r+1}(k) &= \pi p_r(k) + (1 - \pi)\pi(1 - \rho) \sum_{j=0}^{k-1} p_r(j) u^{k-j-1}, \quad k = 1, 2, \dots, \end{aligned} \tag{8}$$

where $u = 1 - \pi(1 - \rho)$.

Proof. Using the PGF, we see that $N(r+1) \stackrel{d}{=} N(r) + Y$. It follows that

$$p_{r+1}(k) = \sum_{j=0}^k p_r(j) P(Y = k - j).$$

From here and using (2), it follows that $p_{r+1}(0) = p_r(0)\pi$ and for $k \geq 1$ that (8) holds. $\square$

Remark 3.1 *If $\rho = 0$, (8) can be simplified and we find $p_{r+1}(0) = \pi p_r(0)$ and*

$$p_{r+1}(k) = \pi \sum_{j=0}^k p_r(j) (1 - \pi)^{k-j}, \quad k = 1, 2, \dots$$

Using the PGF, we can also obtain the pdf of $N(r)$ explicitly. Taking $u = 1 - \pi(1 - \rho)$, we find that

$$\begin{aligned} \psi_{N(r)}(t) &= \pi^r (1 - \rho t)^r \frac{1}{(1 - ut)^r} \\ &= \pi^r \sum_{k=0}^r \binom{r}{k} (-1)^k \rho^k t^k \sum_{i=0}^{\infty} \binom{r+i-1}{i} u^i t^i. \end{aligned}$$

It follows that

$$\psi_{N(r)}(t) = \pi^r \sum_{k=0}^r \sum_{j=k}^{\infty} \binom{r}{k} \binom{r+j-k-1}{j-k} (-1)^k \rho^k u^{j-k} t^j.$$

For $0 \leq n \leq r-1$, we find that

$$\begin{aligned} P(N(r) = n) &= \pi^r \sum_{k=0}^n \binom{r}{k} \binom{r+n-k-1}{n-k} (-1)^k \rho^k u^{n-k} \\ &= \pi^r \sum_{i=0}^n \binom{r}{n-i} \binom{r+i-1}{i} (-1)^{n-i} \rho^{n-i} u^i. \end{aligned} \tag{9}$$

For $n \geq r$ we find that

$$\begin{aligned}
P(N(r) = n) &= \pi^r \sum_{k=0}^r \binom{r}{k} \binom{r+n-k-1}{n-k} (-1)^k \rho^k u^{n-k} \\
&= \pi^r \sum_{i=n-r}^n \binom{r}{n-i} \binom{r+i-1}{i} (-1)^{n-i} \rho^{n-i} u^i.
\end{aligned} \tag{10}$$

Summarizing, we have the following result.

Proposition 3.1 *For the PMF of $N(r)$ we have*

$$P(N(r) = n) = \pi^r \sum_{i=\max(0, n-r)}^n \binom{r}{n-i} \binom{r+i-1}{i} (-1)^{n-i} \rho^{n-i} u^i. \tag{11}$$

A different approach to the PMF of $N(r)$ was studied in Minkova (2001, 2002). The PMF of $N(r)$ is given by

$$\begin{aligned}
P(N(r) = n) &= \begin{cases} \pi^r, & n = 0 \\ \pi^r \sum_{i=1}^n \binom{n-1}{i-1} \binom{r+i-1}{i} [(1-\pi)(1-\rho)]^i \rho^{n-i}, & n = 1, 2, \dots \end{cases}
\end{aligned} \tag{12}$$

The proof of the next lemma is given in the Appendix.

Lemma 3.2 *The PMF in (11) is equivalent to the PMF given in (12).*

Remark 3.2 *Note that for $\rho = 0$, we have*

$$P(N(r) = n) = \binom{r+n-1}{n} \pi^r (1-\pi)^n, \quad n \geq 0,$$

as expected.

Remark 3.3 *Using the PGD of $N(r)$, we also have the following alternative representation. We have*

$$\begin{aligned}
\psi_{N(r)}(t) &= \left(\frac{1-\rho t}{1-\rho} \right)^r \left(\frac{\pi(1-\rho)}{1-(1-\pi(1-\rho))t} \right)^r \\
&= \left(\frac{1}{1-\rho} + \frac{-\rho}{1-\rho} t \right)^r \left(\frac{\pi(1-\rho)}{1-(1-\pi(1-\rho))t} \right)^r.
\end{aligned}$$

If $\rho < 0$, the first term is the PGF of a binomial r.v. $B(r) \sim \text{BIN}(r, \delta)$ with parameter $\delta = \frac{-\rho}{1-\rho}$. The second term is the PGF of an usual negative binomial r.v. $N^*(r)$ with parameter $\pi(1-\rho)$. We obtain that $N(r) \stackrel{d}{=} B(r) + N^*(r)$. This representation can be useful for simulation purposes.

If $\rho > 0$, we have

$$\left(\frac{1-\rho}{1-\rho t}\right)^r \psi_{N(r)}(t) = \psi_{N^*(r)}(t).$$

The first term on the left hand side corresponds to the PGF of an usual negative binomial r.v. $B^*(r)$ with parameter ρ . Now we obtain that $B^*(r) + N(r) \stackrel{d}{=} N^*(r)$.

Since $N(r)$ is the sum of i.i.d. random variables, we can apply the central limit theorem here.

Theorem 3.1 As $r \rightarrow \infty$, we have

$$\frac{N(r) - r\mu}{\sqrt{r\sigma^2}} \xrightarrow{d} Z,$$

where μ and σ^2 are given in Section 2 and where Z denotes a standard normal r.v.

Another limit theorem is given in the next result

Theorem 3.2 Suppose r, π and ρ vary in such a way that $r \rightarrow \infty$, $r(1-\pi) \rightarrow a > 0$ and $\rho \rightarrow c \geq 0$. Then as $r \rightarrow \infty$, we have

$$\psi_{N(r)}(t) \rightarrow \psi(t) = e^{-a(1-K(t))},$$

where $K(t) = \frac{(1-c)t}{1-ct}$.

Proof. Let $u = 1 - \pi(1 - \rho)$. We have

$$\begin{aligned} \psi_{N(r)}(t) &= \left(1 - \left(1 - \frac{\pi(1-\rho t)}{1-ut}\right)\right)^r \\ &= \left(1 - \frac{r(1-\pi)(1-t)}{r(1-ut)}\right)^r \\ &\rightarrow \exp\left(-a\left(\frac{1-t}{1-ct}\right)\right), \end{aligned}$$

where we used the fact that $\pi \rightarrow 1$ and $u \rightarrow c$. Clearly the limit is of the form $e^{-a(1-K(t))}$, with $K(t) = \frac{(1-c)t}{1-ct}$. □

The limit in the previous theorem has the following interpretation. Let $A, A_1, A_2, \dots$ denote i.i.d. r.v. with the following geometric distribution:

$$P(A = n) = (1-c)c^n, \quad n = 0, 1, \dots$$

and let $A_0 = 0$. Also, let $V(a)$ denote a r.v. with a Poisson distribution with parameter a :

$$P(V(a) = n) = \frac{1}{n!} a^n e^{-a}.$$

Using these notations, it follows that $\psi(t)$ is the PGF of the compound Poisson r.v. N , given by

$$N = \sum_0^{V(a)} A_i.$$

4 The ρ - Binomial distribution

4.1 Definition

For $r \geq 1$, let $Y(r) = N(r) + r$ denote the number of experiments up to the r^{th} success and let S_n denote the number of successes in the sequence $X_1, X_2, \dots, X_n$. Clearly we have the following relations:

$$P(S_n \geq r) = P(Y(r) \leq n);$$

$$P(Y(r) = n) = P(S_n = r, X_n = 1).$$

For $r = 0$, we have $P(S_n \geq 0) = 1$. We say that the r.v. S_n has a *binomial distribution related to the interrupted Markov chain* with transition matrix $\mathbf{P}^*$. We call it the ρ -binomial distribution. Notation: $S_n \sim RHOBI(n, \pi, \rho)$. Note that S_n is the renewal process associated with $Y(r)$, i.e.,

$$S_n = \sup \{r : Y(r) \leq n\}.$$

4.2 Properties of S_n

We clearly have

$$\begin{aligned} P(S_n = r) &= P(Y(r) \leq n) - P(Y(r+1) \leq n) \\ &= P(N(r) \leq n - r) - P(N(r+1) \leq n - r - 1). \end{aligned} \tag{13}$$

Let us denote $f_{S_n}(r) = P(S_n = r)$, $r = 0, 1, \dots, n$, the PMF of S_n .

Theorem 4.1 *The PMF of S_n is given by*

$$f_{S_n}(r) = \begin{cases} (1 - \pi)[1 - \pi(1 - \rho)]^{n-1}, & r = 0 \\ \pi^r + \pi^r \sum_{k=1}^{n-r} \sum_{i=1}^k \binom{k-1}{i-1} \binom{r+i-1}{i} [(1 - \pi)(1 - \rho)]^i \rho^{k-i} \\ - \pi^{r+1} - \pi^{r+1} \sum_{k=1}^{n-r-1} \sum_{i=1}^k \binom{k-1}{i-1} \binom{r+i}{i} [(1 - \pi)(1 - \rho)]^i \rho^{k-i}, & r = 1, \dots, n-2, \\ \pi^{n-1}(1 - \pi)[1 + (n-1)(1 - \rho)], & r = n-1 \\ \pi^n, & r = n. \end{cases}$$

Proof. For $r = 0$, we have

$$\begin{aligned}
f_{S_n}(0) &= 1 - P(N(1) \leq n-1) = P(N(1) > n-1) \\
&= \pi \sum_{j=n}^{\infty} \sum_{i=1}^j \binom{j-1}{i-1} [(1-\rho)(1-\pi)]^i \rho^{j-i} \\
&= (1-\pi)[1 - \pi(1-\rho)]^{n-1}.
\end{aligned}$$

For $r = 1, 2, \dots, n-2$, the proof follows from (13) and the PMF of $N(r)$ in (12). In the case of $r = n-1$, we have

$$\begin{aligned}
f_{S_n}(n-1) &= P(N(n-1) \leq 1) - P(N(n) \leq 0) \\
&= \pi^{n-1} - \pi^{n-1}(n-1)(1-\pi)(1-\rho) - \pi^n. \\
&= \pi^{n-1}(1-\pi)[1 + (n-1)(1-\rho)].
\end{aligned} \tag{14}$$

For $r = n$, (13) gives $f_{S_n}(n) = P(N(n) \leq 0) = \pi^n$. □

Information about the important moments of S_n is given in the next Lemma, (Omey et al. (2008), Section 2.1).

Lemma 4.1 *For $n \geq 1$, we have*

$$E(S_n) = n \frac{\pi(1-\rho)}{1-\pi\rho} + \frac{\rho\pi(1-\pi)}{(1-\pi\rho)^2} (1 - (\rho\pi)^n)$$

and

$$\begin{aligned}
Var(S_n) &= \frac{n\pi(1-\pi)(1-\rho)(1+\pi\rho)}{(1-\pi\rho)^3} + \frac{2n(1-\pi)\pi\rho}{(1-\pi\rho)^3} [\pi(1-\rho) - (1-\pi)(\pi\rho)^n] + \\
&\quad + \frac{n^2\pi^2(1-\rho)^2}{(1-\pi\rho)^2} + \frac{(1-\pi)\pi\rho}{(1-\pi\rho)^4} [(1+\pi\rho)^2 - 2\pi(1-\rho(1-\pi))] [1 - (\pi\rho)^n].
\end{aligned}$$

Remark 4.1 *The mean of the ρ -binomial distribution is negatively dependent from the correlation coefficient. It follows from the fact that*

$$\frac{\partial}{\partial \rho} ES_n < 0.$$

In the following theorem we give recursion formulas for PMF of the ρ -binomial distribution.

Theorem 4.2 *The PMF of the ρ - binomial distribution satisfies the following recursion:*

$$f_{S_n}(n) = p_n(0),$$

$$f_{S_n}(r) = p_r(n-r) + (1-\pi) \sum_{j=0}^{n-r-1} p_r(j)(1-\pi(1-\rho))^{n-r-j-1}, \quad r = 1, 2, \dots, n-1,$$

$$f_{S_n}(0) = (1-\pi)[1-\pi(1-\rho)]^{n-1},$$

where $p_r(0), p_r(1), \dots, p_r(n)$ are the probabilities of $INB(\pi, \rho, r)$ distributed random variable, $r = 0, 1, \dots, n$.

Proof. The proof follows from (13) and the recursions (8) for the PMF of the INBD. □

Remark 4.2 *In the case of $\rho = 0$ the formulas of Theorem 4.1 simplify to the usual binomial distribution, and we find*

$$f_{S_n}(n) = p_n(0),$$

$$f_{S_n}(r) = p_r(n-r) + (1-\pi) \sum_{j=0}^{n-r-1} p_r(j)(1-\pi)^{n-r-j-1}, \quad r = 1, 2, \dots, n-1,$$

$$f_{S_n}(0) = (1-\pi)^n,$$

where $p_r(0), p_r(1), \dots, p_r(n)$ are the probabilities of $NB(\pi, r)$ distributed random variable, $r = 0, 1, \dots, n$.

5 Concluding remarks

Generalizations of the binomial distribution have been studied before. We mention, for example, the paper of Chang and Zeltermann (2002). In this paper, the authors consider Bernoulli variables such that

$$P(X_{n+1} = 1 | X_1, X_2, \dots, X_n) = P(X_{n+1} = 1 | S_n),$$

i.e., the success probability depends on the number of previous successes. In the paper of Drezner and Farnum (1993), the authors study Bernoulli variables such that

$$P(X_{n+1} = 1 | X_1, X_2, \dots, X_n) = (1-\theta)p + \theta \frac{S_n}{n},$$

for parameters p and θ .

In Omei et al. (2008), it is used a different approach in a general case. At first it is defined S_n , as the number of successes in the Markov sequence $X_1, X_2, \dots$, and then the negative binomial r.v. $N(r)$ is defined by inverting of S_n .

In the case of interrupted Markov chain, the recursion formulas, given in Theorem 4.1, together with the recursions (8), could be used in numerical calculations. We plan to study further properties of the r.v. that we introduced in this paper.

Appendix

Proof of Lemma 3.2.

Let us denote $v = (1 - \rho)(1 - \pi)$ such that $u = 1 - \pi(1 - \rho) = v + \rho$. From (9), for $n \leq r - 1$, we obtain

$$\begin{aligned}
 P(N(r) = n) &= \pi^r \sum_{k=0}^n \binom{r}{k} \binom{r+n-k-1}{n-k} (-1)^k \rho^k (v + \rho)^{n-k} \\
 &= \pi^r \sum_{k=0}^n \binom{r}{k} \binom{r+n-k-1}{n-k} (-1)^k \rho^k \sum_{j=0}^{n-k} \binom{n-k}{j} v^j \rho^{n-k-j} \\
 &= \pi^r \sum_{j=0}^n v^j \rho^{n-j} \sum_{k=0}^{n-j} (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} \binom{n-k}{j}.
 \end{aligned} \tag{A1}$$

From Prudnikov et al. (1986), it follows that for $j = 0$

$$\sum_{k=0}^n (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} = 0. \tag{A2}$$

For $j = 1, 2, \dots$, we have

$$\begin{aligned}
 &\sum_{k=0}^{n-j} (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} \binom{n-k}{j} \\
 &= \binom{r+j-1}{j} \sum_{k=0}^{n-j} (-1)^k \binom{r}{k} \binom{r+n-1-k}{r+j-1} \\
 &= \binom{r+j-1}{j} \binom{r+n-1-r}{n-j} = \binom{r+j-1}{j} \binom{n-1}{j-1},
 \end{aligned}$$

where the last equation follows from Prudnikov et al. (1986), and then (A1) is equal to (12) for $n \leq r - 1$.

Similarly, for $n = r, r + 1, \dots$, we have

$$\begin{aligned}
 P(N(r) = n) &= \pi^r \sum_{k=0}^r \binom{r}{k} \binom{r+n-k-1}{n-k} (-1)^k \rho^k \sum_{j=0}^{n-k} \binom{n-k}{j} v^j \rho^{n-k-j} \\
 &= \pi^r \sum_{j=0}^{n-r-1} v^j \rho^{n-j} \sum_{k=0}^r (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} \binom{n-k}{j} \\
 &\quad + \pi^r \sum_{j=n-r}^n v^j \rho^{n-j} \sum_{k=0}^{n-j} (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} \binom{n-k}{j}.
 \end{aligned} \tag{A3}$$

So, for $j = 0$ we obtain

$$\sum_{k=0}^r (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} = \frac{r}{n} \sum_{k=0}^r (-1)^k \binom{r}{k} \binom{r+n-1-k}{n-1} = 0. \quad (\text{A4})$$

For $j = 1, 2, \dots$,

$$\begin{aligned} & \sum_{k=0}^r (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k} \binom{n-k}{j} \\ &= \binom{r+j-1}{j} \sum_{k=0}^r (-1)^k \binom{r}{k} \binom{r+n-k-1}{n-k-j} = \binom{r+j-1}{j} \binom{n-1}{j-1}. \end{aligned} \quad (\text{A5})$$

Substituting (A4) and (A5) in (A3) leads to PMF (12) for $n = r, r+1, \dots$.

□

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