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SEMI-RADIAL TECHNICAL EFFICIENCY MEASUREMENT^{*}

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ABSTRACT

Semi-radial efficiency measurement combines technical efficiency, as captured by the classical Farrell measure, with an economically meaningful mix efficiency component. The semi-radial evaluation we propose proceeds in two steps. First, we build on the price interpretation of the generally accepted Koopmans efficiency notion to characterize appropriate benchmarks. Next, we present both a quantity-based distance measure and its dual (price-based) equivalent to evaluate the mix efficiency factor. The type of measures we propose may, e.g., be used to provide a price rationale for the Zieschang technical efficiency evaluation procedure.

Key words: Koopmans technical efficiency, Farrell measure, indication problem, mix efficiency, shadow price interpretation.

JEL classification: C61, D21, D24

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1. INTRODUCTION

It is a commonplace that technical efficiency analysis, which only uses quantity information, is an attractive second-best option when economic efficiency analysis, which uses both quantity and (reliable) price information, is practically impossible. This statement is true insofar as technical efficiency, the absence of waste in production, is a necessary condition for economic efficiency. But it is also slightly misleading because technical efficiency analysis itself builds on underlying assumptions about *shadow* prices, which may enable one to draw minimal inferences about economic efficiency without necessitating price information. Building on this close relationship between technical and economic efficiency, this paper sets out an economically consistent framework for technical efficiency evaluation.

Throughout, we take it that technical efficiency refers to the well-established and uncontested Koopmans (1951) definition. Koopmans technical efficiency means that there are no *slacks*; it is impossible to decrease an input or increase an output without increasing another input or decreasing another output. We will next make the distinction between (a) technical inefficiency as captured by the classical radial measure introduced by Farrell (1957) (see also Debreu, 1951), and (b) technical inefficiency that can be considered as “implicit allocative” or “mix” inefficiency. Our use of the Farrell measure (be it here as one component of an overall technical efficiency measure) is inspired by its interpretation in terms of shadow prices. Still, when used exclusively, this measure may select references that are not Koopmans efficient. This shortcoming is commonly known as the *indication problem* of the Farrell efficiency measure. The indication problem (or equivalently, the Koopmans efficiency conditions) can *also* be cast in terms of shadow prices. Using this price interpretation, this paper points out how the attractive price-based interpretation of the Farrell measure can be preserved while overcoming at the same time its indication problem. More specifically, we suggest *semi-radial* technical efficiency measurement, which combines the Farrell measure with a complementary measure of mix inefficiency. At this point, it is worth emphasizing that the idea of spelling out theoretical foundations for fully Koopmans-consistent technical efficiency evaluation *methods* is not very new (cf. Charnes *et al.*, 1985). But we extend this line of research by looking at the economic interpretation of the efficiency *measures* that quantify deviations from full Koopmans-efficiency.¹

Section 2 briefly recaptures the indication problem and shadow price interpretation of the Farrell efficiency measure, and provides an intuitive account of the complementary mix efficiency concept that we propose as a solution to the indication problem. Sections 3 and 4 are the analytical core of the paper. Basically, they each capture a different dimension of a Koopmans-consistent technical efficiency evaluation process: (a) identify a suitable Koopmans efficient reference; (b) evaluate the distance between this efficient benchmark and the vector under evaluation, using a measure that has an economically meaningful interpretation in price terms. Specifically, in Section 3 we characterize the set of Koopmans efficient references

¹ A similar remark holds regarding the concept and measurement of mix efficiency: Such measures have been introduced before, e.g. by Cooper *et al.* (1999, 2000) and Portela and Thanassoulis (2005), but also here the originality of our approach lies in the fact that we institute an explicit price representation of the mix efficiency measure (i.e., as a measure of dominance in price space; see further).

that preserve the shadow cost legitimisation associated with the Farrell measure. Section 4 then presents the implicated measure of mix inefficiency, and shows that this measure can equivalently be interpreted as a measure of dominance in price space. The overall semi-radial measure is then easily derived. In Section 5 we show that the Zieschang (1984) efficiency evaluation procedure can readily be incorporated into this semi-radial framework. Finally, section 6 concludes. The proofs of our propositions can be found in the Appendix.

2. THE INDICATION PROBLEM AND THE NEED FOR MIX EFFICIENCY EVALUATION

In practice, technical efficiency analysis is mostly input- or output-oriented. For example, because outputs are exogenously fixed, the focus of the analysis is on whether inputs are used in a technically efficient way. In the following we will concentrate on input technical efficiency, as this has most often been analyzed in the literature.² For that purpose, we will use the equivalent representation of technological production possibility sets in terms of input correspondences (see Shephard (1970))

$$L(\mathbf{u}) = \{\mathbf{x} | \mathbf{x} \in \mathfrak{R}_+^n \setminus \{\mathbf{0}_n\} \text{ can produce } \mathbf{u} \in \mathfrak{R}_+^m \setminus \{\mathbf{0}_m\}\},$$

with $\mathbf{x} = (x_1, \dots, x_n)$ an input vector and $\mathbf{u}$ an output vector. We will consider $L(\mathbf{u})$ that are (L1) non-empty, (L2) closed, (L3) convex and (L4) have a finite number of extreme points. The properties (L1)-(L3) are standard. Imposing the property (L4) simplifies our discussion as it allows us to exclude peculiar special cases. We emphasize, however, that our results can be extended to settings that do not meet (L4). Still, the following price characterizations of technical efficiency would become more ambiguous. Most notably, (L4) guarantees that marginal rates of substitution/transformation do not smoothly go to zero at a technical efficient point, which conveniently allows for our price characterization of Koopmans technical efficiency in Proposition 1. See for example Mass-Colell et al. (1995, pp. 149-152) for the price characterization of Koopmans technical efficiency when a property like (L4) does not hold.³

An input vector $\mathbf{x} \in L(\mathbf{u})$ is called *Koopmans input efficient* if and only if⁴

$$(1) \quad \{\mathbf{x}' \in L(\mathbf{u}) | \mathbf{x}' \leq \mathbf{x}\} = \emptyset.$$

² Extensions of our results towards output technical efficiency are straightforward. In addition, for treating technical efficiency analysis that is not input or output oriented, a useful starting point may be the directional distance function framework for technical efficiency measurement as introduced by Chambers et al. (1998). For example, the McFadden measure function (that can be fitted in the directional distance function framework) has a dual interpretation in terms of profit levels (see McFadden, 1978).

³ Property (L4) does not seem very restrictive, especially as regards empirical application. For example, it is satisfied by all activity analysis models (see Von Neumann (1945), Koopmans (1951) and Shephard (1970)) and by all models that are summarized under the denomination Data Envelopment Analysis (see e.g. Cooper et al. (2000) for a survey).

⁴ In the following, we use $>$ ($<$) to denote “strictly bigger (smaller) than”, $\geq$ ($\leq$) to denote “bigger (smaller) than or equal to” and $\mathbf{y} \geq \mathbf{z}$ ($\mathbf{y} \leq \mathbf{z}$) when $\mathbf{y} \geq \mathbf{z}$ ($\mathbf{y} \leq \mathbf{z}$) and $\mathbf{y} \neq \mathbf{z}$. In addition, $\mathfrak{R}_{++}^z = \{\mathbf{y} \in \mathfrak{R}_+^z | \mathbf{y} > \mathbf{0}\}$.

The *efficient subset* of $L(\mathbf{u})$ (see Shephard (1970)) contains the Koopmans input efficient points and is defined as

$$Eff L(\mathbf{u}) = \{\mathbf{x} \in L(\mathbf{u}) \mid \mathbf{x}' \leq \mathbf{x} \Rightarrow \mathbf{x}' \notin L(\mathbf{u})\}.$$

As indicated in the Introduction, this technical efficiency criterion can be given a nice interpretation in terms of shadow prices, so revealing the link between technical and economic efficiency (*in casu* cost efficiency):

Proposition 1: Let $L(\mathbf{u})$ satisfy (L1)-(L4). Then
 $\mathbf{x} \in Eff L(\mathbf{u}) \Leftrightarrow \exists \mathbf{p} \in \mathfrak{R}_{++}^n : \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}' \quad \forall \mathbf{x}' \in L(\mathbf{u}).$

Thus, Koopmans input efficiency of $(\mathbf{x}, \mathbf{u})$ is equivalent to $\mathbf{x}$ being cost minimizing over $L(\mathbf{u})$ for a strictly positive input price vector. This is a well-known result in the literature and directly relates to the analogy between the Koopmans technical efficiency notion and the general Pareto efficiency concept.

In practice, deviations from Koopmans efficiency are frequently observed. Such observations are based on technical efficiency measures. The most well-known and frequently employed technical input efficiency measure is the Farrell measure. We will consider an extended version of the original Farrell measure, i.e. for $\mathbf{x} \in L(\mathbf{u})$

$$E_F(\mathbf{x}, \mathbf{u}) = \min\{\rho \in \mathfrak{R} \mid \rho \mathbf{x} \in M(L(\mathbf{u}))\} \text{ with } M(L(\mathbf{u})) = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{x}' \text{ for } \mathbf{x}' \in L(\mathbf{u})\}.^5$$

Obviously, $E_F(\mathbf{x}, \mathbf{u}) \leq 1$ for $\mathbf{x} \in L(\mathbf{u})$. The extended Farrell measure compares $\mathbf{x}$ to $M(L(\mathbf{u}))$, the monotone hull of $L(\mathbf{u})$. Note that using $M(L(\mathbf{u}))$ instead of $L(\mathbf{u})$ in (1) does not affect the Koopmans efficiency condition, i.e. $Eff L(\mathbf{u}) = Eff M(L(\mathbf{u}))$ (see also Färe (1975)).

In view of the shadow price interpretation of Koopmans technical efficiency, it is desirable to give an equally sensible economic interpretation in price terms for inefficient observations. In point of fact, the Farrell measure has such a convenient interpretation, which undoubtedly helps to explain its benchmark status in the literature:

Proposition 2: Let $L(\mathbf{u})$ satisfy (L1)-(L4). Then

$$E_F(\mathbf{x}, \mathbf{u}) = \max_{\mathbf{p} \in \mathfrak{R}_{++}^n} \min_{\mathbf{x}' \in L(\mathbf{u})} \left\{ \frac{\mathbf{p} \cdot \mathbf{x}'}{\mathbf{p} \cdot \mathbf{x}} \mid \mathbf{p} \cdot \mathbf{x} > 0 \right\}.$$

Thus, the Farrell input measure provides a natural upper bound to the degree of economic/cost efficiency (i.e. the ratio of minimal to actual cost); Debreu (1951) originally stated a similar result. In words, the measure applies “benefit-of-the-doubt

⁵ Färe and Grosskopf (1983) and Färe et al. (1983), who called it the “weak (input) measure of technical efficiency”, first introduced the technical efficiency measure $E_F(\mathbf{x}, \mathbf{u})$. The denomination “extended Farrell measure” is due to Russell (1988, 1990).

pricing” for assessing the degree of technical inefficiency. Several authors (e.g. Russell (1985)) have emphasized this price feature to justify using the Farrell measure (in a setting, to recall, that typically is quantity-based).

There exists a one-to-one relationship between the Farrell measure and the *isoquant* of $M(L(\mathbf{u}))$ (see Shephard, 1970)

$$Isoq M(L(\mathbf{u})) = \{\mathbf{x} \in M(L(\mathbf{u})) | \rho \in (0,1) \Rightarrow \rho \mathbf{x} \notin M(L(\mathbf{u}))\}.$$

Obviously, $E_F(\mathbf{x}, \mathbf{u}) = 1 \Leftrightarrow \mathbf{x} \in Isoq M(L(\mathbf{u}))$. However, while $Eff L(\mathbf{u}) \subseteq Isoq M(L(\mathbf{u}))$ both sets do generally not coincide for the class of technologies that we consider. In the presence of slacks, the Farrell efficiency measure does not *indicate* Koopmans efficiency. This *indication problem* has received much attention in the axiomatic literature on technical efficiency measures (after Färe and Lovell, 1978; see Russell and Schworm, 2005, for a recent account of the axiomatic literature). In this paper we deviate from this axiomatic literature precisely by explicitly upholding Koopmans’ (and Debreu’s) own economic (shadow price) perspective when solving the indication problem. From this perspective the indication problem can be given a specific price interpretation, as from Proposition 2 we have

Corollary 1: Let $L(\mathbf{u})$ satisfy (L1)-(L4). Then
 $\mathbf{x} \in Isoq M(L(\mathbf{u})) \Leftrightarrow \exists \mathbf{p} \in \mathfrak{R}_+^n : 0 < \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}' \quad \forall \mathbf{x}' \in L(\mathbf{u}).$

In fact, for $\mathbf{x} \in Isoq M(L(\mathbf{u}))$ it suffices to be cost minimizing over $L(\mathbf{u})$ for a price vector that may contain zero prices, whereas $\mathbf{x} \in Eff L(\mathbf{u})$ requires cost minimization over $L(\mathbf{u})$ for a strictly positive price vector.

Example. The potential conflict between Koopmans and Farrell input efficiency is illustrated for a two-input situation in Figure 1, where the input efficiency of a vector $(\mathbf{x}, \mathbf{u})$ is to be evaluated. In the diagram $Eff L(\mathbf{u})$ corresponds to the facet $\overline{AB}$ whereas $Isoq M(L(\mathbf{u}))$ also contains the vertical and horizontal facets $\overline{AC}$ and $\overline{BD}$. The measure $E_{DF}(\mathbf{x}, \mathbf{u}) = \|\mathbf{x}_e\| / \|\mathbf{x}\|$ clearly has a benefit of the doubt interpretation in price terms. For example, it exceeds $\|\mathbf{x}'_e\| / \|\mathbf{x}\|$ and –to an even greater extent– $\|\mathbf{x}''_e\| / \|\mathbf{x}\|$, where $\mathbf{x}'_e$ and $\mathbf{x}''_e$ respectively lie on the supporting (“isocost”) hyperplanes of the isoquant facets $\overline{AC}$ and $\overline{BD}$. However, the Farrell reference vector $\mathbf{x}_e$ does not belong to $Eff L(\mathbf{u})$, although it would be labeled Farrell efficient. In price terms, it is only cost minimizing over $L(\mathbf{u})$ when a zero price is accorded to the second input.

The example itself hints at a way to solve the indication problem: it clearly reveals that the part of Koopmans inefficiency which is not captured by the Farrell efficiency measure pertains to the input *mixes*. Hence, one can call it “mix” inefficiency. This denomination refers to the primal variables of analysis, i.e. the physical input quantities. Alternatively, referring to its shadow price interpretation, we can call it “implicit cost” inefficiency, or “implicit allocative” inefficiency when using Farrell’s

concept of allocative efficiency (i.e. the ratio of economic/cost efficiency over technical efficiency). The denomination “implicit allocative (in)efficiency” hence refers to the dual interpretation of technical efficiency in price terms.

Figure 1

The evaluation of mix efficiency essentially proceeds in two steps: select an appropriate mix efficient reference input vector and, subsequently, evaluate the mix distance from the mix inefficient reference to this efficient benchmark. These two steps will be analyzed rigorously in the next two sections. Before doing so, we use the example in Figure 1 to sketch the main ideas. First, we suggest vector A as an appropriate benchmark for the evaluated vector x . Essentially, as we formally motivate in the next section, an appropriate reference vector preserves the shadow price interpretation of the Farrell efficiency measure and, in addition, belongs to the relevant $Eff L(u)$. Second, for the given reference input vector, an appropriate mix efficiency measure is defined as the norm ratio $\|A\|/\|x_e\|$. In Section 4 we show that this ratio can be interpreted as a measure of dominance in price space. Interestingly, this interpretation naturally complements the price-based interpretation of the Koopmans efficiency notion and the Farrell efficiency measure.

In addition, the norm ratio specification of the mix efficiency measure logically institutes the ratio $\|A\|/\|x\|$ as a mix-adjusted Farrell efficiency measure, which is thus obtained as the product of the original Farrell efficiency measure ($\|x_e\|/\|x\|$) and the proposed mix efficiency measure ($\|A\|/\|x_e\|$). This provides a comprehensive semi-radial measure for overall technical efficiency. Section 5 discusses this mix-adjusted Farrell measure in greater detail. *Inter alia*, this will provide an economic rationale for the Zieschang (1984) technical efficiency evaluation procedure.

3. BENEFIT-OF-THE-DOUBT PRICING AND KOOPMANS EFFICIENCY

In the following we will use $P(\mathbf{x}, \mathbf{u})$ to denote the correspondence capturing all (non-negative) shadow price vectors under which $\mathbf{x} \in Isoq M(L(\mathbf{u}))$ is cost minimizing over $L(\mathbf{u})$, i.e.

$$P(\mathbf{x}, \mathbf{u}) = \left\{ \mathbf{p} \in \mathfrak{R}_+^n \mid 0 < \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}' \quad \forall \mathbf{x}' \in L(\mathbf{u}) \right\} \text{ for all } \mathbf{x} \in Isoq M(L(\mathbf{u})).$$

$P(\mathbf{x}, \mathbf{u})$ defines the (set of) supporting hyperplane(s) tangent to $M(L(\mathbf{u}))$ in $\mathbf{x}$.⁶ In addition, we will use the following definitions for notational convenience:

$$\begin{aligned} \mathbf{x}_F(\mathbf{x}, \mathbf{u}) &= E_F(\mathbf{x}, \mathbf{u})\mathbf{x}, \\ P_F(\mathbf{x}, \mathbf{u}) &= P(\mathbf{x}_F(\mathbf{x}, \mathbf{u}), \mathbf{u}). \end{aligned}$$

Proposition 2 provides a (shadow) cost efficiency characterization for $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$. However, as argued in the previous section, it may still be that $\mathbf{x}_F(\mathbf{x}, \mathbf{u}) \notin Eff L(\mathbf{u})$. We now show that there always exist references that preserve both the Farrell benefit-of-the-doubt interpretation *and* pass the test for Koopmans efficiency. Specifically, these “pivotal” references belong to

$$X_D(\mathbf{x}, \mathbf{u}) = \{\mathbf{x}' \mid \mathbf{x}' \in Eff L(\mathbf{u}) \wedge \mathbf{x}' = \mathbf{x}_F(\mathbf{x}, \mathbf{u})\}.$$
⁷

Trivially, $P_F(\mathbf{x}, \mathbf{u}) \subseteq P(\mathbf{x}_D, \mathbf{u})$ for all $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$. Hence, all $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ preserve the straightforward economic interpretation of $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ (see Proposition 2). But this point can be strengthened by

Proposition 3: Let $L(\mathbf{u})$ satisfy (L1)-(L4). Then

$\forall \mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u}), \mathbf{x}' \in Eff L(\mathbf{u}) \setminus X_D(\mathbf{x}, \mathbf{u}), \mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u}), \mathbf{p}' \in P(\mathbf{x}', \mathbf{u})$:

$$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}} \geq \frac{\mathbf{p}' \cdot \mathbf{x}'}{\mathbf{p}' \cdot \mathbf{x}}.$$

⁶ Our definition of the correspondence $P(\mathbf{x}, \mathbf{u})$ is directly related to the single-valued cost function concept (see Shephard (1970)).

⁷ In the efficiency measurement literature there have been several proposals that always yield references in $Eff L(\mathbf{u})$ (i.e. that “solve” the indication problem of DF measures). But some of these proposals may select references that do not necessarily belong to $X_D(\mathbf{x}, \mathbf{u})$, e.g. those introduced by Färe and Lovell (1978) and Charnes *et al.* (1985). Other proposals always select a reference in $X_D(\mathbf{x}, \mathbf{u})$, e.g. those introduced by Zieschang (1984), Banker *et al.* (1984), Coelli (1998) and Cherchye and Van Puyenbroeck (2001). Loosely stated, this concerns a class of multi-stage procedures which first apply the Farrell projection method “to get on the production frontier” and subsequently adjust for any remaining zero shadow prices “by moving along the frontier” (i.e. by selecting an appropriate supporting price hyperplane characterized by prices $\mathbf{p} \in \mathfrak{R}_{++}^n$).

Thus, for each $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ the associated implicit cost ratio will never be lower than for any $\mathbf{x}' \in \text{Eff } L(\mathbf{u}) \setminus X_D(\mathbf{x}, \mathbf{u})$. The interpretation is analogous to the one of Proposition 2. The important difference is that Proposition 3 is defined with reference to $\text{Eff } L(\mathbf{u})$, whereas Proposition 2 concerns $\text{Isoq } L(\mathbf{u})$. Hence, in view of Proposition 1, the result is appealing since it allows us to use strictly positive price vectors in comparisons. Stated otherwise, Proposition 3 reveals that input technical efficiency evaluation of a vector $(\mathbf{x}, \mathbf{u})$ by comparing it to a vector $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ is consistent with both Koopmans efficiency and the convenient price interpretation of the Farrell input measure.

Example. We take again Figure 1. In this example the set $X_D(\mathbf{x}, \mathbf{u}) = \{A\}$. Cost ratios for $\mathbf{x}$ as obtained from using any $\mathbf{p} \in P(A, \mathbf{u})$ are not below $\|\mathbf{x}'_e\|/\|\mathbf{x}\|$: one can always find a supporting hyperplane through A that crosses the radial somewhere between $\mathbf{x}'_e$ and $\mathbf{x}_e$. Obviously, no other element belonging to the facet $\overline{AB}$ would yield a strictly higher ratio value.

While proposition 3 indicates that an *appropriate* benchmark can be found in $X_D(\mathbf{x}, \mathbf{u})$ it does not identify a *unique* benchmark. Of course, if $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u}) \in \text{Eff } L(\mathbf{u})$ then $X_D(\mathbf{x}, \mathbf{u}) = \{\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})\}$. Also, even if $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u}) \notin \text{Eff } L(\mathbf{u})$ the set $X_D(\mathbf{x}, \mathbf{u})$ will always be a singleton if $n = 2$ (see the example). However, with more dimensions it could be that $X_D(\mathbf{x}, \mathbf{u})$ contains more elements; e.g., consider the case that the radial Farrell projection is characterized by slacks in two dimensions (see Cherchye and Van Puyenbroeck (2001a, pp. 94-95) for an example). The following proposition demonstrates that, in such cases, each input vector $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ is justifiable from a Koopmans efficiency perspective:

Proposition 4: Let $L(\mathbf{u})$ satisfy (L1)-(L4). Then

$$\forall \mathbf{x}_D, \mathbf{x}'_D \in X_D(\mathbf{x}, \mathbf{u}), \mathbf{x}_D \neq \mathbf{x}'_D : \forall \mathbf{p}'_D \in P(\mathbf{x}'_D, \mathbf{u}), \mathbf{p}'_D > \mathbf{0}_n : \exists \mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u}), \mathbf{p}_D > \mathbf{0}_n : \\ \frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}} \geq \frac{\mathbf{p}'_D \cdot \mathbf{x}'_D}{\mathbf{p}'_D \cdot \mathbf{x}}.$$

As Proposition 4 applies to all $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ there is no general (economic) reason to further discriminate within the set $X_D(\mathbf{x}, \mathbf{u})$ in terms of implicit cost efficiency, at least when one is only concerned with a consistent characterization of technical efficiency *per se*.

So far we have shown that vectors $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ have a sound economic grounding as input benchmarks for $(\mathbf{x}, \mathbf{u})$. However, we have not yet related this to an efficiency measure. The attractiveness of the Farrell measure does not straightforwardly extend when using $X_D(\mathbf{x}, \mathbf{u})$. In particular, for $\mathbf{x}_D \neq \mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})$ the correspondence $P(\mathbf{x}_D, \mathbf{u})$ will typically be multi-valued. In turn, this leads to a multitude of possible implicit cost ratios.

However, the preceding discussion suggested (a) that there is still room for using the Farrell measure to rationalize a part of observed Koopmans technical inefficiency, and

(b) that the remaining part of inefficiency pertains to the implicit allocative or mix properties of the evaluated vector (which is present as soon as $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u}) \notin \text{Eff } L(\mathbf{u})$). The question is then whether one can give an economically meaningful interpretation to this second component. In the next section we propose a way to evaluate this complementary dimension of suboptimal behavior that mirrors the dual price interpretation of technical efficiency.

4. THE MEASUREMENT OF MIX EFFICIENCY

To measure mix efficiency, we first construct the vector of directional cosines, which is defined as follows for a (nonzero) vector $\mathbf{x} \in L(\mathbf{u})$

$$\mathbf{cos}_{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

In $\mathbf{cos}_{\mathbf{x}} = (\cos_{x_1}, \dots, \cos_{x_n}) \in \mathfrak{R}_+^n$ each $\cos_{x_l} \in [0, 1] (l = 1, \dots, n)$ gives the cosine of the angle between the vector $\mathbf{x}$ and the l -th input axis (see Charnes and Cooper, 1961, pp. 162-164).⁸

Example. To clarify the concept we return to our previous illustration as recaptured in Figure 2. Using $A = (a_1, a_2)$ and $\mathbf{x}_e = (x_{e1}, x_{e2})$ the directional cosines corresponding to A and $\mathbf{x}_e$ are respectively $\cos_{a_l} = \cos \alpha_l = a_l / \|A\|$ and $\cos_{x_{el}} = \cos \beta_l = x_{el} / \|\mathbf{x}_e\| (l = 1, 2)$.

Hence, the vector $\mathbf{cos}_{\mathbf{x}}$ contains dimensional characterizations of the mix of $\mathbf{x}$: the proportions between each pair of row entries of $\mathbf{x}$ can equivalently be represented in terms of directional cosines as $x_i / x_j = \cos_{x_i} / \cos_{x_j} \forall i, j \in \{1, \dots, n\}$. Two input vectors are collinear if and only if they share the same vector of directional cosines. Formally,

$$(2) \quad \mathbf{x}, \mathbf{x}' \in L(\mathbf{u}) : \mathbf{x} = \rho \mathbf{x}' (\rho \in (0, +\infty)) \Leftrightarrow \mathbf{cos}_{\mathbf{x}} = \mathbf{cos}_{\mathbf{x}'}.$$

Evidently, given (2), when evaluating the input mix efficiency of the vector $(\mathbf{x}, \mathbf{u})$ we can confine attention to its radial input projection $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$.

In this and the following sections we will assume that an input reference $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$ has been selected for $(\mathbf{x}, \mathbf{u})$. Slightly abusing notation, we denote the directional cosine vectors corresponding to $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})$ by respectively $\mathbf{cos}_{\mathbf{x}_f}$ and $\mathbf{cos}_{\mathbf{x}_D^*}$. Each ratio $\cos_{x_{f,l}} / \cos_{x_{D,l}^*} (l = 1, \dots, n)$ gives an angular representation of mix deviation (e.g. in Figure 2 these are $(\cos_{x_{e1}} / \cos_{x_{a1}}) < 1$ and $(\cos_{x_{e2}} / \cos_{x_{a2}}) > 1$). To obtain an overall mix efficiency measure these dimension-

⁸ Directional cosines are more commonly labeled 'normalized vectors'. Still, in the present context we prefer the denomination 'directional cosines' as this makes explicit our use of the concept for characterizing input *mixes*.

specific values should be combined. In analogy to the Farrell technical efficiency measure we take the ratio of price-weighted sums of the different row elements of $\mathbf{cos}_{\mathbf{x}_{DF}}$ and $\mathbf{cos}_{\mathbf{x}_D^*}$, using $\mathbf{p} \in P_F(\mathbf{x}, \mathbf{u})$ as the weighting vector. The overall mix efficiency measure is hence defined as⁹

$$(3) \quad E_{ME}(\mathbf{x}, \mathbf{u}) = \frac{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}_F}}{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}_D^*}} \quad \forall \mathbf{p} \in P_F(\mathbf{x}, \mathbf{u}).$$

Of course, in view of (2) we can rewrite (3) as

$$(4) \quad E_{ME}(\mathbf{x}, \mathbf{u}) = \frac{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}}}{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}_D^*}} \quad \forall \mathbf{p} \in P_F(\mathbf{x}, \mathbf{u}).$$

which links $E_{ME}(\mathbf{x}, \mathbf{u})$ directly to the input mix properties of $(\mathbf{x}, \mathbf{u})$.

Figure 2

The fact that we use $P_F(\mathbf{x}, \mathbf{u})$ rather than e.g. $P(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), \mathbf{u})$ as the set of admissible shadow price weighting vectors in (3) allows for a convenient third definition of $E_{ME}(\mathbf{x}, \mathbf{u})$. As $\mathbf{p} \cdot \mathbf{x}_F(\mathbf{x}, \mathbf{u}) = \mathbf{p} \cdot \mathbf{x}_D^*(\mathbf{x}, \mathbf{u})$ for all $\mathbf{p} \in P_F(\mathbf{x}, \mathbf{u})$ and recalling the definition of $\mathbf{cos}_{\mathbf{x}}$ we get

$$(5) \quad E_{ME}(\mathbf{x}, \mathbf{u}) = \frac{\|\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})\|}{\|\mathbf{x}_F(\mathbf{x}, \mathbf{u})\|},$$

⁹ It will become clear in what follows why the equality in (3) holds for all $\mathbf{p} \in P_F(\mathbf{x}, \mathbf{u})$.

from which it is immediate that $E_{ME}(\mathbf{x}, \mathbf{u}) \in (0, 1]$ and that $E_{ME}(\mathbf{x}, \mathbf{u}) = 1$ if and only if $\mathbf{x}_F(\mathbf{x}, \mathbf{u}) \in \text{Eff } L(\mathbf{u})$. The norm ratio formulation (5) clearly reveals the analogy with the way the Farrell technical efficiency measure is commonly presented.

Observe that definitions (3), (4) and (5) apply to any $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$ (compare with the “indifference” result in Proposition 4). Conversely, equivalence of the three characterizations crucially depends on the fact that $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$.

The use of $P_F(\mathbf{x}, \mathbf{u})$ implies that zero weights are assigned to those input dimensions in which $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ exhibits slack. In general, when $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ exhibits slack in a particular dimension the corresponding directional cosine value will be lower for $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})$ than for a vector belonging to $X_D(\mathbf{x}, \mathbf{u})$. As this cannot be viewed as mix *dominance* it seems natural to accord a zero weight to these cosines.

Example. The intuition can be sharpened by means of Figure 2. Only the directional cosines with respect to the first axis are weighted positively as for $\mathbf{x}_e$ the implicit price for the second input equals zero (and $\cos_{x_{e2}} > \cos_{x_{a2}}$). The mix efficiency measure equals $(\cos_{x_{e1}} / \cos_{x_{a1}}) = (\|A\| / \|\mathbf{x}_e\|)$.

There also exists a *dual* characterization of $E_{ME}(\mathbf{x}, \mathbf{u})$. As we will now show, the proposed mix efficiency measure can as well be interpreted, rather conveniently, as a measure of dominance in price space.

By definition, the cost level associated with $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})$ is the same for all $\mathbf{p} \in P_F(\mathbf{x}, \mathbf{u})$. Let us then reverse the picture and consider all price vectors that imply the same cost level (arbitrarily fixed at unity) for both input vectors. Obviously, this dual perspective allows us to consider strictly positive price vectors to compare $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})$. Specifically, we consider the following two hyperplanes in price space

$$\begin{aligned} H(\mathbf{x}_F(\mathbf{x}, \mathbf{u}), 1) &= \{\pi_F \in \mathfrak{R}_+^n \mid \pi_F \cdot \mathbf{x}_F(\mathbf{x}, \mathbf{u}) = 1\}, \\ H(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), 1) &= \{\pi_D^* \in \mathfrak{R}_+^n \mid \pi_D^* \cdot \mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) = 1\}. \end{aligned}$$

Throughout the domain $\mathfrak{R}_+^n$ isocost hyperplanes associated with $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$ are located at least as far away from the origin as those corresponding to $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$. Mix efficiency decreases the more these distances differ.

Example. We present these hyperplanes for input vectors $\mathbf{x}_e$ and A of our illustration in Figure 3. At the shadow price vector corresponding to $\mathbf{x}_e$ (with zero weight for the second input) both hyperplanes intersect. The implicit allocative inefficiency of $\mathbf{x}_e$ is revealed by the dominance of the hyperplane associated with A over the one associated with $\mathbf{x}_e$: because it uses strictly less of the second input, observation A

allows higher (positive) second input prices to imply the same cost level for each possible price assigned to the first input.

Figure 3

A natural way to evaluate this dominance in price space is to compare the Euclidean distances from these hyperplanes to the origin. These distances are given by

$$d_{\mathbf{x}_F} = \min_{\pi_F \in \mathbb{R}_+^n} \{ \|\pi_F\| \mid \pi_F \cdot \mathbf{x}_F(\mathbf{x}, \mathbf{u}) = 1 \},$$

$$d_{\mathbf{x}_D^*}(\mathbf{x}, \mathbf{u}) = \min_{\pi_D^* \in \mathbb{R}_+^n} \{ \|\pi_D^*\| \mid \pi_D^* \cdot \mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) = 1 \}.$$

and we have

$$(6) \quad E_{ME}(\mathbf{x}, \mathbf{u}) = \frac{d_{\mathbf{x}_{DF}}(\mathbf{x}, \mathbf{u})}{d_{\mathbf{x}_D^*}(\mathbf{x}, \mathbf{u})},$$

The equivalence of this price-based dominance characterization with the quantity-based characterization in (5) follows directly from the fact that the distance from a hyperplane to the origin can be retrieved by using its normal. In this case, substituting

$$d_{\mathbf{x}_F}(\mathbf{x}, \mathbf{u}) = \frac{1}{\|\mathbf{x}_F(\mathbf{x}, \mathbf{u})\|} \quad \text{and} \quad d_{\mathbf{x}_D^*}(\mathbf{x}, \mathbf{u}) = \frac{1}{\|\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})\|}$$

in (6) immediately yields (5).

Finally, we show that characterization (6) provides a rationale for our particular choice of $\mathbf{p} \in P_{DF}(\mathbf{x}, \mathbf{u})$ as the weighting vector in (3). For this purpose we switch to the Hesse normal form representations of $H(\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u}), 1)$ and $H(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), 1)$:

$$H^n(\mathbf{x}_F(\mathbf{x}, \mathbf{u}), 1) = \{\pi_F \in \mathfrak{R}_+^n \mid \pi_F \cdot \mathbf{cos}_{\mathbf{x}_F} = d_{\mathbf{x}_F}(\mathbf{x}, \mathbf{u})\},$$

$$H^n(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), 1) = \{\pi_D^* \in \mathfrak{R}_+^n \mid \pi_D^* \cdot \mathbf{cos}_{\mathbf{x}_D^*} = d_{\mathbf{x}_D^*}(\mathbf{x}, \mathbf{u})\}.$$

Using this representation it can be seen that $E_{ME}(\mathbf{x}, \mathbf{u}) = [\pi_F \cdot \mathbf{cos}_{\mathbf{x}_F}] / [\pi_D^* \cdot \mathbf{cos}_{\mathbf{x}_D^*}]$ for $\pi_F \in H^n(\mathbf{x}_F(\mathbf{x}, \mathbf{u}), 1)$ and $\pi_D^* \in H^n(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), 1)$. The *only* legitimate way to employ a common price-weighting vector in the numerator and the denominator, as is done in (3), is to take vectors that lie in the intersection of $H^n(\mathbf{x}_F(\mathbf{x}, \mathbf{u}), 1)$ and $H^n(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), 1)$. These vectors are indeed all $\mathbf{p} \in P_F(\mathbf{x}, \mathbf{u})$ for which $\mathbf{p} \cdot \mathbf{x}_F(\mathbf{x}, \mathbf{u}) = \mathbf{p} \cdot \mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) = 1$.¹⁰

5. MIX-ADJUSTED FARRELL MEASURES AND UNITS INVARIANCE

As we have indicated in the introduction, our aim was to preserve the commonly employed Farrell measure of technical efficiency in view of its economic intuition while at the same time making it consistent with the well-established notion of Koopmans technical efficiency. Our discussion makes clear that both the Farrell and the mix efficiency measures capture separate information about two connected dimensions of technical efficiency performance. Taken one by one, they reveal the fundamental two-stage nature of the reference selection process very clearly: first radially projecting on the isoquant, and next moving along this isoquant towards the efficient subset. However, a semi-radial measure that merges all this information in a single statistic is sometimes useful. Such a measure can readily be obtained by combining the measures $E_F(\mathbf{x}, \mathbf{u})$ and $E_{ME}(\mathbf{x}, \mathbf{u})$ into the following *mix-adjusted Farrell measure*

$$(7) \quad E_{MA}(\mathbf{x}, \mathbf{u}) = E_F(\mathbf{x}, \mathbf{u}) \cdot E_{ME}(\mathbf{x}, \mathbf{u}).$$

In view of definition (5) and given that $E_F(\mathbf{x}, \mathbf{u}) = \|\mathbf{x}_F(\mathbf{x}, \mathbf{u})\| / \|\mathbf{x}\|$ we can rewrite (7) as

$$E_{MA}(\mathbf{x}, \mathbf{u}) = \frac{\|\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})\|}{\|\mathbf{x}\|}$$

which indicates that the measure $E_{MA}(\mathbf{x}, \mathbf{u})$ can also be computed directly. The semi-radial measure $E_{MA}(\mathbf{x}, \mathbf{u})$ then captures the relative distance from $\mathbf{x}$ to $Eff L(\mathbf{u})$. Moreover, it is well justified by economic/price considerations. Its radial component $E_F(\mathbf{x}, \mathbf{u})$ gives the traditional “benefit-of-the-doubt” shadow cost efficiency estimate for $(\mathbf{x}, \mathbf{u})$ and measures the distance from $\mathbf{x}$ to $Isoq L(\mathbf{u})$. It does not only yield the (maximum) cost efficiency ratio of $(\mathbf{x}, \mathbf{u})$ with respect to $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ but also with respect to $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u})$ (as $P_F(\mathbf{x}, \mathbf{u}) = P(\mathbf{x}_F(\mathbf{x}, \mathbf{u}), \mathbf{u}) \subseteq P(\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}), \mathbf{u})$ by definition). On the other hand, $E_{ME}(\mathbf{x}, \mathbf{u})$ deals with the presence of any slack in (or, alternatively, zero implicit prices associated with) $\mathbf{x}_F(\mathbf{x}, \mathbf{u})$ and pertains to the mix properties of $\mathbf{x}$.

¹⁰ It is immediate that the validity of (6) does not depend on the fact that we considered a cost level of unity. This also implies that this legitimization of (3) holds for all $\mathbf{p} \in P_F(\mathbf{x}, \mathbf{u})$.

To conclude, it is worth to emphasize that $E_{MA}(\mathbf{x}, \mathbf{u})$ as defined in (7) coincides with $E_F(\mathbf{x}, \mathbf{u})$ when this last measure correctly indicates Koopmans efficiency (so that $E_{ME}(\mathbf{x}, \mathbf{u}) = 1$), i.e. when zero prices are not needed to reconstruct $E_F(\mathbf{x}, \mathbf{u})$. In effect, the mix-adjusted Farrell efficiency evaluation approach completely encompasses the standard Farrell approach.

One important point is still in order: the above mix-adjusted Farrell efficiency measure is not invariant to the units in which the different input quantities are measured. Russell (1988) has especially advocated the units invariance property. Specifically, just as Russell (1988), we focus on invariance up to a scalar multiplication of measurement units, a condition that has also been referred to as ‘commensurability’ in the literature.¹¹ To obtain units invariance of $E_{ME}(\mathbf{x}, \mathbf{u})$ one needs to define the efficiency measure with respect to rescaled input data: dividing each original input quantity by a value expressed in the same measurement unit indeed implies units invariance of this newly defined mix efficiency measure.¹²

Elaborating on this point, we now show that so accounting for the commensurability property institutes a price-based rationale for the technical efficiency evaluation procedure that was originally proposed by Zieschang (1984). This special case is particularly interesting since this measure has attractive axiomatic properties (under the technological properties (L1)-(L4)), which are well-documented in the axiomatic literature on technical efficiency measures (initiated by Färe and Lovell, 1978).

The norm representation as given in definition (5) provides a convenient point of departure. Using $\mathbf{x}_F(\mathbf{x}, \mathbf{u}) = (x_{F1}, \dots, x_{Fn})$ and $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) = (x_{D1}^*, \dots, x_{Dn}^*)$, we re-express it as:

$$(8) \quad E_{ME}(\mathbf{x}, \mathbf{u}) = \left[\frac{\sum_{l=1}^n (x_{Dl}^*)^2}{\sum_{l=1}^n (x_{Fl})^2} \right]^{1/2}.$$

Consider then the replacement of all input entries x_l ($l = 1, \dots, n$) by rescaled vectors $\hat{x}_l = (x_l / \sqrt{x_l x_{DFl}})$ (i.e. we divide each input value by the geometric mean of itself and the corresponding value in $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})$). Substituting in (8) yields

¹¹ See however Salnykov and Zelenyuk (2005) for a recent account of alternative commensurability notions.

¹² This is, e.g., also done by Cooper *et al.* (2000, p. 96-98) in order to obtain their mix efficiency measure. Note that so far we have assumed the projection $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$ to be given. A first point of interest then concerns the reference selection procedure itself. The Zieschang (1984) procedure for selecting $\mathbf{x}_D^*(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$, on which we consequently focus, satisfies the units invariance property (see Russell (1988)).

$$(9) \quad \hat{E}_{ME}(\mathbf{x}, \mathbf{u}) = \left[\frac{\sum_{l=1}^n (\hat{x}_{Dl}^*)^2}{\sum_{l=1}^n (\hat{x}_{DFl}^*)^2} \right]^{1/2} = \left[\sum_{l=1}^n \left(\frac{x_{Dl}^*}{x_{DFl}^*} \right) / n \right]^{1/2}.$$

This is a monotonic transformation of the arithmetic mean of the original input proportions.¹³ It is precisely this arithmetic mean which is minimized in the second stage of the Zieschang (1984) reference selection procedure. It therefore follows that, for this particular rescaling, the Zieschang reference selection procedure is one that minimizes mix inefficiency.¹⁴

Note finally that the rescaled mix efficiency measure is expressed in terms of the *ratios of the (original) evaluated and reference input vector elements*. Now these are precisely the ratios of the intercepts of the respective associated price hyperplanes, which –as noted in the previous section (see e.g. Figure 3)- reveal information on the dominance of the reference vector in price space. Therefore, the rescaling that leads to expression (9) seems particularly appealing for solving the units invariance problem.

6. CONCLUSION

We have presented a two-stage procedure for semi-radial technical efficiency measurement that builds on the price interpretation of the generally accepted Koopmans technical efficiency notion. The first step characterizes appropriate benchmarks that are consistent with this Koopmans efficiency notion. The second step measures the distance from the evaluated production vector to the technically efficient benchmark. The corresponding semi-radial efficiency measurement combines the classical Farrell measure for technical efficiency with an economically meaningful mix efficiency measure that can also be interpreted as a measure of dominance in price space. Finally, this suggests a mix-adjusted Farrell measure, and provides an interesting price-based rationale for the Zieschang technical efficiency evaluation procedure.

APPENDIX: PROOFS

PROOF OF PROPOSITION 1: First, $\{\mathbf{x}' \in L(\mathbf{u}) | \mathbf{x}' \leq \mathbf{x}\} = \emptyset \Leftrightarrow$

$$(10) \quad \max_{(\mathbf{x}') \in L(\mathbf{u}), \mathbf{s} \in \mathfrak{R}_+^n} \{\mathcal{E}(\mathbf{e}_n \cdot \mathbf{s}) | \mathbf{s} \leq \mathbf{x} - \mathbf{x}'\} = 0,$$

where $\mathbf{e}_i$ ($i = n, m$) is the i -dimensional unit vector and $0 < \varepsilon < +\infty$ can be chosen arbitrarily small. The right hand side is well defined because of (L1) and (L2).

¹³ Strictly speaking the possibility of zero input values is excluded. However, Färe et al. (1983) proposed a way to circumvent this problem (see also Zieschang (1984)).

¹⁴ Other rescaling options could evidently lead to provide an economic rationale for other mix-adjusting efficiency measures. For instance, it is easy to check that using $\hat{x}_i = (x_i / \sqrt{x_i x_{Dl}^*})$ eventually yields a mix-efficiency measure based on the harmonic mean of the original input proportions. On this ‘inverse Zieschang procedure’, see Cherchye and Van Puyenbroeck (2001b).

Because of (L3) and (L4) we have, for V the set of extreme points of $L(\mathbf{u})$,

$$L(\mathbf{u}) = \left\{ \mathbf{x} \mid \mathbf{x} \geq \sum_{\mathbf{x}' \in V} \lambda_{\mathbf{x}'} \mathbf{x}'; \sum_{\mathbf{x}' \in V} \lambda_{\mathbf{x}'} = 1; \lambda_{\mathbf{x}'} \in \mathfrak{R}_+ \quad \forall \mathbf{x}' \in V \right\},$$

so that (10) can be re-expressed as

$$\max_{\lambda_{\mathbf{x}'} \in \mathfrak{R}_+, \forall \mathbf{x}' \in V, \mathbf{s} \in \mathfrak{R}_+^n} \left\{ \mathcal{E}(\mathbf{e}_n \cdot \mathbf{s}) \mid \mathbf{s} \leq \mathbf{x} - \sum_{\mathbf{x}' \in V} \lambda_{\mathbf{x}'} \mathbf{x}'; \sum_{\mathbf{x}' \in V} \lambda_{\mathbf{x}'} = 1 \right\} = 0,$$

a linear programming problem. Its dual can be written as

$$\min_{\mathbf{p} \in \mathfrak{R}_+^n, q \in \mathfrak{R}} \{ \mathbf{p} \cdot \mathbf{x} - q \mid \mathbf{p} \cdot \mathbf{x}' \geq q \quad \mathbf{x}' \in V; \mathbf{p} \geq \mathcal{E} \mathbf{e}_n \} = 0.$$

Because of (L3), (L4) (given the standard result that the maximum value of a linear function over a convex set is always achieved in an extreme point of that set) and $0 < \mathcal{E} < +\infty$ this gives the result.

Q.E.D.

PROOF OF PROPOSITION 2: Because of (L1) and (L2) $E_F(\mathbf{x}, \mathbf{u})$ is defined for $\mathbf{x} \in L(\mathbf{u})$.

Further, under (L3) and (L4) we can redefine $E_F(\mathbf{x}, \mathbf{u})$ as

$$E_F(\mathbf{x}, \mathbf{u}) = \min_{\gamma_{\mathbf{x}'} \in \mathfrak{R}_+, \forall \mathbf{x}' \in E} \left\{ \rho \in \mathfrak{R} \mid \rho \mathbf{x} \geq \sum_{\mathbf{x}' \in E} \gamma_{\mathbf{x}'} \mathbf{x}'; \sum_{\mathbf{x}' \in E} \gamma_{\mathbf{x}'} = 1 \right\},$$

a linear programming problem. Its dual is

$$E_F(\mathbf{x}, \mathbf{u}) = \max_{\mathbf{q} \in \mathfrak{R}_+^n} \left\{ r \in \mathfrak{R} \mid \mathbf{q} \cdot \mathbf{x} = 1; \mathbf{q} \cdot \mathbf{x}' \geq r \quad \forall \mathbf{x}' \in E \right\},$$

from which the result follows immediately (given (L3), (L4) and the standard result that the minimum value of a linear function over a convex set is always achieved in a vertex point of that set).

Q.E.D.

PROOF OF PROPOSITION 3: For $\mathbf{x}_F(\mathbf{x}, \mathbf{u}) \in \text{Eff } L(\mathbf{u})$ the result follows directly from Proposition 4. Let us then consider $\mathbf{x}_F(\mathbf{x}, \mathbf{u}) \notin \text{Eff } L(\mathbf{u})$. We have to compare input vectors $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}' \in \text{Eff } L(\mathbf{u}) \setminus X_D(\mathbf{x}, \mathbf{u})$. To facilitate exposition we consider scaled shadow price vectors $\mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u})$ and $\mathbf{p}' \in P(\mathbf{x}', \mathbf{u})$ such that $\mathbf{p}_D \cdot \mathbf{x}_D = \mathbf{p}' \cdot \mathbf{x}' = 1$ in the following. We are led to proof that for all pairs $(\mathbf{p}_D, \mathbf{p}')$

$$(11) \quad \frac{1}{\mathbf{p}_D \cdot \mathbf{x}} \geq \frac{1}{\mathbf{p}' \cdot \mathbf{x}} \quad \text{or} \quad \mathbf{p}' \cdot \mathbf{x} \geq \mathbf{p}_D \cdot \mathbf{x}.$$

Multiplying both sides of the second inequality in (11) by $E_F(\mathbf{x}, \mathbf{u})$ we obtain

$$(12) \quad \mathbf{p}' \cdot \mathbf{x}_F \geq \mathbf{p}_D \cdot \mathbf{x}_F,$$

where $\mathbf{x}_F = \mathbf{x}_F(\mathbf{x}, \mathbf{u})$ to save on notation. Of course, from Proposition 4, (12) holds for $\mathbf{p}_D \in P_F(\mathbf{x}, \mathbf{u})$. Let us therefore focus on $\mathbf{p}_D \notin P_F(\mathbf{x}, \mathbf{u})$. As a preliminary step recall that $P_F(\mathbf{x}, \mathbf{u}) \subseteq P(\mathbf{x}_D, \mathbf{u})$ while $P_F(\mathbf{x}, \mathbf{u}) \cap P(\mathbf{x}', \mathbf{u}) = \emptyset$ by construction (as $\mathbf{x}_{DF}$ and $\mathbf{x}'$ lie on different hyperplanes supporting the set $L(\mathbf{u})$). Thus, for $\mathbf{p}_F \in P_F(\mathbf{x}, \mathbf{u})$ (with again $\mathbf{p}_F \cdot \mathbf{x}_F = 1$ for simplicity) we get

$$(13) \quad \mathbf{p}' \cdot \mathbf{x}_F > 1 = \mathbf{p}_F \cdot \mathbf{x}_F.$$

Define $\mathbf{p}_\lambda = \lambda \mathbf{p}_F + (1 - \lambda) \mathbf{p}_D$. We have $\mathbf{p}_\lambda \in P(\mathbf{x}_D, \mathbf{u}) \quad \forall \lambda \in [0, 1]$. Indeed, $\forall \mathbf{x} \in L(\mathbf{u}) : \mathbf{p}_F \cdot \mathbf{x}_D \leq \mathbf{p}_F \cdot \mathbf{x} \wedge \mathbf{p}_D \cdot \mathbf{x}_D \leq \mathbf{p}_D \cdot \mathbf{x} \Rightarrow \mathbf{p}_\lambda \cdot \mathbf{x}_D \leq \mathbf{p}_\lambda \cdot \mathbf{x} \quad \forall \lambda \in [0, 1]$.

Now suppose (12) is not met so that

$$(14) \quad \mathbf{p}' \cdot \mathbf{x}_F < \mathbf{p}_D \cdot \mathbf{x}_F.$$

From (13) and (14), we have $\exists \bar{\lambda} \in (0,1): \mathbf{p}' \cdot \mathbf{x}_F = \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_F$ or

$$(15) \quad \bar{\lambda} = \frac{\mathbf{p}' \cdot \mathbf{x}_F - \mathbf{p}_D \cdot \mathbf{x}_F}{1 - \mathbf{p}_D \cdot \mathbf{x}_F}$$

Note that $\mathbf{p}_{\bar{\lambda}} \in P(\mathbf{x}_D, \mathbf{u}) \Rightarrow 1 = \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_D = \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_F$. However, under (v) $\mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_D = \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_F \Leftrightarrow 1 = \mathbf{p}' \cdot \mathbf{x}_F$, which contradicts (13). We therefore conclude that (14) does not hold and (12) is indeed satisfied.

Q.E.D.

PROOF OF PROPOSITION 4: We consider two input vectors $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}'_D \in X_D(\mathbf{x}, \mathbf{u}) \setminus \{\mathbf{x}_D\}$. We assume $\mathbf{p}'_D \in P(\mathbf{x}'_D, \mathbf{u})$ with $\mathbf{p}'_D \in \mathfrak{R}_{++}^n$ given. We have to proof that $\mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u})$ with $\mathbf{p}_D \in \mathfrak{R}_{++}^n$ can be constructed such that

$$(16) \quad \frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}} \geq \frac{\mathbf{p}'_D \cdot \mathbf{x}'_D}{\mathbf{p}'_D \cdot \mathbf{x}}.$$

Dividing both sides by $E_{DF}(\mathbf{x}, \mathbf{u})$ gives the equivalent condition

$$(17) \quad \frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}_F} \geq \frac{\mathbf{p}'_D \cdot \mathbf{x}'_D}{\mathbf{p}'_D \cdot \mathbf{x}_F},$$

where $\mathbf{x}_F = \mathbf{x}_F(\mathbf{x}, \mathbf{u})$ to save on notation. We can decompose $\mathbf{x}_F$, $\mathbf{x}_D$ and $\mathbf{x}'_D$ in a slack and non-slack subvector. Let $\mathbf{x}_F = (\bar{\mathbf{x}}_F, \tilde{\mathbf{x}}_F)$, $\mathbf{x}_D = (\bar{\mathbf{x}}_D, \tilde{\mathbf{x}}_D)$ and $\mathbf{x}'_D = (\bar{\mathbf{x}}'_D, \tilde{\mathbf{x}}'_D)$ such that (for the non-slack subvectors) $\bar{\mathbf{x}}_F = \bar{\mathbf{x}}_D = \bar{\mathbf{x}}'_D$ while (for the slack subvectors) $\tilde{\mathbf{x}}_D < \tilde{\mathbf{x}}_F$ and $\tilde{\mathbf{x}}'_D < \tilde{\mathbf{x}}_F$. We restate (17):

$$(18) \quad \frac{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_D}{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_F} \geq \frac{\bar{\mathbf{p}}'_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}'_D \cdot \tilde{\mathbf{x}}'_D}{\bar{\mathbf{p}}'_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}'_D \cdot \tilde{\mathbf{x}}_F}.$$

with corresponding decompositions $\mathbf{p}_D = (\bar{\mathbf{p}}_D, \tilde{\mathbf{p}}_D)$ and $\mathbf{p}'_D = (\bar{\mathbf{p}}'_D, \tilde{\mathbf{p}}'_D)$. Obviously, $\forall \mathbf{p}_D, \mathbf{p}'_D \in \mathfrak{R}_{++}^n$

$$(19) \quad \frac{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_D}{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_F} \geq \frac{\bar{\mathbf{p}}'_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}'_D \cdot \tilde{\mathbf{x}}'_D}{\bar{\mathbf{p}}'_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}'_D \cdot \tilde{\mathbf{x}}_F} \text{ and } \frac{\tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_D}{\tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_F} < 1, \frac{\bar{\mathbf{p}}'_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}'_D \cdot \tilde{\mathbf{x}}'_D}{\bar{\mathbf{p}}'_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}'_D \cdot \tilde{\mathbf{x}}_F}.$$

The left and right hand sides of expression (18) are both strictly smaller than one. Let

$$(20) \quad \frac{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}'_D}{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_F} = 1 - \varepsilon, \text{ with } \varepsilon > 0.$$

We now proceed by constructing $\mathbf{p}_D \in \mathfrak{R}_{++}^n$ such that (18) is satisfied. First note that each $\mathbf{p}_F \in P_F(\mathbf{x}, \mathbf{u})$ can be decomposed similarly as $\mathbf{p}_D$ and $\mathbf{p}'_D$ above so that $\mathbf{p}_F = (\bar{\mathbf{p}}_F, 0)$. As $\mathbf{x}_D \in X_D(\mathbf{x}, \mathbf{u})$ we can define $\mathbf{p}_D^* = (\bar{\mathbf{p}}_F, \tilde{\mathbf{p}}_D^*)$ with $\tilde{\mathbf{p}}_D^*$ appropriately chosen such that $\mathbf{p}_D^* \in P(\mathbf{x}_D, \mathbf{u})$ and $\mathbf{p}_D^* \in \mathfrak{R}_{++}^n$. We have $\forall \mathbf{x} \in L(\mathbf{u})$

$$(21) \quad \bar{\mathbf{p}}_F \cdot \bar{\mathbf{x}}_{DF} = \bar{\mathbf{p}}_F \cdot \bar{\mathbf{x}},$$

$$(22) \quad \bar{\mathbf{p}}_F \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D^* \cdot \tilde{\mathbf{x}}_D = \bar{\mathbf{p}}_F \cdot \bar{\mathbf{x}} + \tilde{\mathbf{p}}_D^* \cdot \tilde{\mathbf{x}},$$

where again we use the decomposition $\mathbf{x} = (\bar{\mathbf{x}}, \tilde{\mathbf{x}})$. Convex combinations of the inequalities in (21) and (22) yield $\forall \kappa \in [0,1]$

$$(23) \quad \bar{\mathbf{p}}_F \cdot \bar{\mathbf{x}}_F + \kappa(\tilde{\mathbf{p}}_D^* \cdot \tilde{\mathbf{x}}_D) = \bar{\mathbf{p}}_F \cdot \bar{\mathbf{x}} + \kappa(\tilde{\mathbf{p}}_D^* \cdot \tilde{\mathbf{x}}).$$

Let $\mathbf{p}_D^\kappa = (\bar{\mathbf{p}}_F, \kappa \tilde{\mathbf{p}}_D)$. From (23) we know that $\mathbf{p}_D^\kappa \in P(\mathbf{x}_D, \mathbf{u}) \quad \forall \kappa \in [0, 1]$. Moreover, $\mathbf{p}_D^\kappa \in \mathfrak{R}_{++}^n \quad \forall \kappa \in (0, 1]$. So, we can shrink the shadow price vector associated with $\tilde{\mathbf{x}}_D$ independently of that associated with $\bar{\mathbf{x}}_F$. In fact, it can be made infinitesimally small while still remaining positive. Using (19) and for a given value of $\varepsilon > 0$, we can always set $\mathbf{p}_D = \mathbf{p}_D^{\bar{\kappa}} \in \mathfrak{R}_{++}^n$ with $\bar{\kappa} \in (0, 1]$ sufficiently small such that

$$(24) \quad \frac{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_D}{\bar{\mathbf{p}}_D \cdot \bar{\mathbf{x}}_F + \tilde{\mathbf{p}}_D \cdot \tilde{\mathbf{x}}_F} \geq 1 - \varepsilon.$$

Combining (21) and (24) yields (16) and thus also (15).

Q.E.D.

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