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Improving students' representational flexibility in linear-function problems: an intervention

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This study evaluates the effects of an intervention aimed at improving representational flexibility in linear-function problems. Forty-nine students aged 13–16 participated in the study. A pretest–intervention–posttest design with an experimental and control group was used. At pretest, both groups solved a choice test, where they could freely select a table, a graph, or a formula, and three no-choice tests, where they used predetermined representations. Twenty-five students in the experimental group were exposed to the intervention, where they learnt to fine-tune their choices both to the task at hand and to their own characteristics as representational users. The control group was not exposed to any intervention. The posttest was similar to the choice pretest. A flexibility score was calculated per student based on Siegler and Lemaire's score. At posttest, the experimental group became more flexible than the control group. The impact of increased flexibility on problem-solving accuracy and speed is discussed.

Keywords: mathematics; linear functions; representational choice; choice/no-choice method; representational flexibility

Theoretical background

According to Goldin and Shteingold (2001), a representation is 'a sign or a configuration of signs, characters or objects. The important thing is that it can stand for (symbolise, depict, encode or represent) something other than itself' (p. 5). A distinction is often made between internal and external representations. Internal representations are intangible, and they become more elaborate and more abstract with experience (Pape & Tchoshanov, 2001). In the context of mathematics, internal representations encompass, among others, visual, spatial, tactile and kinaesthetic perceptions, problem-solving heuristics and strategies and meanings that each individual assigns to conventional mathematical notations (Goldin, 2003). External representations, on the other hand, are notational systems that 'act as stimuli on the senses' (Janvier, Girardon, & Morand, 1993, p. 81) which help us to express, describe, manipulate and communicate mathematical ideas. Vergnaud (1997) argues that they are inherent to the discipline of mathematics, since a characteristic of mathematical concepts is that they can be represented using symbolic, linguistic or

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graphic representations. More importantly, mathematical concepts can *only* be accessed through their external representations (Duval, 2006). For example, a linear function can only be thought of and communicated when it is represented using a graph, a table, a formula, a verbal description of the relationship between the variables, or a combination of these forms.

Not all authors agree that the dichotomy internal/external representations should be taken for granted. Nemirovsky and Noble (1997), for example, challenge the idea that representations are either internal (i.e. located in the mind) or external (i.e. located on paper, on a screen, etc.). For them, representations can also be ‘neither inside or outside, or both inside and outside at the same time’ (p. 102). They argue that the meaning of symbols (e.g. representations) cannot be understood if we focus exclusively on the specific objects that they represent, or on the thoughts that they express. One can only ascribe meaning to symbols if both their internal and external facets are taken into account (Nemirovsky, Tierney, & Wright, 1998). In the same line, Pérez Echeverría and Scheuer (2009) speak about ‘the phantom of dualism’ and warn us about the dangers of establishing an ‘absolute frontier between outer and inner worlds’ (p. 7). They go on to explain that the terms internal/external representations are sometimes used to distinguish between representations that can be perceived by means of our five senses (i.e. external representations) and representations that can only be observed or recorded using neurobiological measurements (i.e. internal representations). When we use the term ‘external representations’ in this article, we do it with the latter differentiation in mind, and not to stress the theoretical distinction between external and internal representations.

In the last decades, teaching students how to interact with multiple external representations (MERs) has become a key objective of school mathematics worldwide. Promoting the use of MERs in order to improve students’ learning in the mathematical class was put forward as an important goal during the reform movement advocated in the 80s and 90s (e.g. National Council of Teachers of Mathematics, 1989), and it is still a key objective today (National Council of Teachers of Mathematics, 2000).

The way MERs are used in the classroom has changed greatly in the past decades. Before the advent of information technology (IT), MERs were introduced by means of teaching units involving activities where students (with the guidance of their teacher) created and/or manipulated different types of representations of a given concept in order to solve tasks (e.g. Brenner et al., 1997). With the introduction of IT in the classroom, nowadays students are often exposed to computerised, multirepresentational environments which are assumed to help them gain a deeper understanding of mathematical concepts and improve their problem-solving skills. Some studies show that MERs – including graphical, tabular, algebraic and verbal representations – allow students to gain a broad and deep knowledge of mathematical concepts (Elia, Panaoura, Eracleous, & Gagatsis, 2007; Moschkovich, Schoenfeld, & Arcavi, 1993). MERs can also facilitate mathematical problem solving (Duval, 2006; Even, 1998; Gagatsis & Shiakalli, 2004; Kaput, 1992; Yerushalmy, 2006).

Despite all the possible advantages of using MERs, some studies show that providing students with MERs does not necessarily result in improved learning and/or problem-solving performance (e.g. Grüber, Graf, Mandl, Renkl, & Stark, 1995; Tabachneck, Leonardo, & Simon, 1994; Yerushalmy, 1991). As Ainsworth (2006) explains, ‘the benefits of an appropriate representation do not come for free’

(p. 185). She argues that there are a number of cognitive tasks that students need to perform when they are faced with MERs, and that not all students are equipped to deal with such tasks successfully. For example, students need to understand the notations used in the representations and understand the links between the representation and the concept it represents. These tasks alone already place a heavy burden on students' cognitive resources. In some instances, students are also expected to make a choice among representations in order to complete learning and/or problem-solving tasks. Making representational choices *also* consumes cognitive resources. While some students can deal with the complex cognitive tasks associated with using MERs, others experience cognitive overload (Paas, 2004), which could explain why certain students fail to benefit from using MERs.

Research on students' (in)ability to make representational choices while problem solving has shown that, when students have to make a choice among different external representations, they do not always know *when* to apply each representation or *why* it is more appropriate to use a particular representation than the others (Uesaka & Manalo, 2006). In other words, students often lack what is known as representational flexibility or adaptivity (Heinze, Star, & Verschaffel, 2009). For the sake of simplicity, in this article we use the terms *representational flexibility* and *representational adaptivity* as synonyms. We have discussed the different conceptualisations of these terms elsewhere (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009).

Traditionally, empirical research concerning the flexibility of students' choices was based on the assumption that making a flexible choice means selecting the representation that better fits the task at hand (e.g. Grawemeyer, 2006; Vessey, 1991). However, our recent work on the topic of representations of linear functions (Acevedo Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2010) has shown that, instead of conceiving representational flexibility in terms of task characteristics alone (i.e. 'representation *a* is better for problem type *b*'), it is more appropriate to conceive in terms of (the interaction between) *task, subject and context variables* (i.e. 'representation *a* is better for problem *b* for subject *c* in context *d*'). In the aforementioned study, we used the choice/no-choice method (Siegler & Lemaire, 1997), a method often used to evaluate students' strategy choices (Luwel, Lemaire, & Verschaffel, 2005; Luwel, Verschaffel, Onghena, & De Corte, 2003; Siegler & Lemaire, 1997; Torbeyns, Verschaffel, & Ghesquière, 2004), but we applied it to the field of *representational* choices. In that study, we asked students to solve linear-function problems related to the concepts of slope, intercept and intersection of two linear functions. They solved the problems under a choice (C-) condition, where they could select a table, a formula, or a graph to solve each problem, and three no-choice (NC-) conditions (NC-table, NC-graph, and NC-formula), where students solved all problems using tables, graphs, or formulae, respectively. The C-condition provided us with information regarding students' preferred choices in the different problem types, whereas the NC-conditions provided us with unbiased data concerning students' representational efficiency, i.e. their accuracy and speed with the different representations when solving the different problem types. According to Siegler and Lemaire (1997), these estimates are unbiased because all students solve all problems with all representations. We compared the data from the C- and NC-conditions of each individual student in order to evaluate the flexibility of their personal representational choices. Two conceptualisations of flexibility were used: a task-based conceptualisation and a task $\times$ student-based conceptualisation. We first

applied the task-based conceptualisation. Based on a rational task analysis, we determined which representations could be considered flexible for each problem type (e.g. tables and graphs were considered the most flexible choice for intersection problems because the intersection point was explicitly displayed in both representations, whereas finding the intersection point using the formula required calculations). Each student was given a task-based flexibility score based on whether or not he selected the representations which, according to our rational task analysis, were the best match for the different problem types. When the task $\times$ student-based conceptualisation was used, a choice was considered flexible if, for a given C-condition problem, a student selected a representation with which he had successfully solved a parallel problem in the corresponding NC-condition (e.g. if a student selected the table in a slope problem in the C-condition, and this was also the representation he was most successful with when solving a parallel problem in the NC-conditions, the student's choice was considered flexible). Our data showed that the task $\times$ student-based flexibility score was a stronger predictor of C-condition performance than the task-based flexibility score. We concluded that flexibility is better understood as students' ability to fine-tune their representational choices not only to task characteristics, but also to their subjective characteristics as representational users (e.g. their fluency with the different representations for the different tasks). Similar findings were yielded by a subsequent study where we asked 86 students aged 13–16 to solve slope, intercept and intersection problems similar to the ones used in the previous study, and also under C/NC conditions (Acevedo Nistal, Van Dooren, & Verschaffel, 2012a).

In the current study, the central idea is that, due to the observed impact of flexible representational choices on students' performance (Acevedo Nistal et al., 2010, 2012a), it may be beneficial to stimulate representational flexibility in the mathematics classroom. In our research, we focus on the impact of representational flexibility on students' performance in linear-function problems. Research on the topic of MERs in linear-function tasks is abundant in the literature (e.g. Eisenberg, 1992; Elia et al., 2007; Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2006; Even, 1998; Yerushalmy, 1997). Most studies focus on students' ability to perform *treatments* (Duval, 2006) (i.e. their ability to manipulate the concept of function within a single representation, e.g. Leinhardt, Zaslavsky, & Stein, 1990) and *conversions* (Duval, 2006) (i.e. their ability to translate a linear function from one system of representation to another, e.g. Elia et al., 2007). In the present study, however, we focused on representational *selection*, and more specifically on students' ability to make flexible representational choices in order to solve linear-function problems. We tested whether it is possible to improve students' representational flexibility in linear-function problems as conceived above (i.e. in terms of task and subject characteristics) through instruction. Our hypothesis was that the development of representational flexibility in linear-function problems can be effectively enhanced through instruction that pays systematic attention to the questions *when* to use a representation and *why* it might be more appropriate to use that representation in particular cases; in other words, instruction that aims at developing representational flexibility through the construction of 'meta-knowledge of representations' (de Jong et al., 1998) or 'meta-representational competence' (diSessa & Sherin, 2000).

In order to test this hypothesis, an experimental group of 13–16-year-old students received instruction focused on representational flexibility, while another group served as a control group and did not receive instruction. Based on Siegler

and Lemaire's (1997) model of strategy choice, we focused on predictions about four parameters: (1) representational repertoire (i.e. how many representations students used in the C-condition problems), (2) frequency of representational choice (i.e. how often each representation was chosen in the C-condition problems), (3) representational efficiency (i.e. how accurate and fast students were at solving the problems) and (4) representational flexibility (i.e. whether or not students took into account the characteristics of the task at hand and their own subject characteristics (more specifically, their representational fluency with the different representations) when they made their choices in the C-problems). Our predictions were as follows:

Prediction 1: A change in students' representational repertoire from pretest to posttest will be observed in the experimental group, a change that will be significantly larger than any possible changes observed in the control group.

Prediction 2: A change in students' frequency of representational choice from pretest to posttest will be observed in the experimental group, a change that will be significantly larger than any possible changes observed in the control group.

Prediction 3: The experimental group will become significantly more accurate and faster from the pretest to the posttest as compared to the control group, because thanks to the intervention, they will make better representational choices. More concretely, we expect that a shift in representational choices in the experimental group will result in a significantly stronger improvement in problem-solving accuracy from pretest to posttest (Prediction 3a). Additionally, we expect that the shift in representational choices in the experimental group will result in a stronger decrease in problem-solving times (i.e. how long it takes them to solve the problem after selecting a representation) from pretest to posttest as compared to the control group (Prediction 3b). Finally, the selection time (i.e. the time students require to select a representation before actually starting to solve the problem) might be affected in the opposite way – students in the experimental group will require more time to select a representation after the intervention, as compared to the control group, presumably because they will make their representational choices more thoughtfully (Prediction 3c).

Prediction 4: The experimental group will show a significantly stronger improvement in representational flexibility from pretest to posttest than the control group. We used an individualised conceptualisation of flexibility (Acevedo Nistal et al., 2012a), meaning that we calculated an individualised flexibility score for each student based on their ability to make choices which were not only appropriate for the problem at hand, but which were also in line with their own efficiency with the different types of representations available. Details about how the flexibility scores were calculated are provided in the Results section.

As it can be noted, Predictions 1 and 2 do not specify in which direction the changes will occur. The aim of our intervention was to teach students to fine-tune their representational choices to their fluency with the different representations to solve the various types of problems. Based on this, we predicted that, in the posttest, the students from the experimental group would select less often the representations they were less fluent with, and more often the representations they were more fluent with for the different problem types. We could not predict whether students' repertoire would become wider or narrower, since this would depend on their fluency with the different representations, and we did not have any data concerning students' fluency prior to the experiment. For the same reason, we could not predict

either which representations would be selected more/less frequently by the students in the experimental group in the posttest, since this also depended on their representational fluency.

Design

Participants

Three full classes from a public secondary school in Valladolid (Spain) participated in the study. A total of 55 students took part in the pretest, but six students were absent either during the intervention or during the posttest; so the final sample size was 49. Seventeen students were aged 13–14, 16 students were aged 14–15 and 16 students were aged 15–16. Fifty-five per cent of the students were female.

Through the course of their studies, all students had been explicitly taught how to solve linear-function problems using external representations such as tables, graphs and formulae. The concept of linear function is a fundamental topic of the compulsory mathematics curriculum that is taught in Spain up to the age of 14 (Boletín Oficial de Castilla y León, 2007). In the school where the study was conducted, students are first taught about linear functions when they are 13–14. They learn how to represent linear functions using formulae, tables and graphs. Towards the end of the year, they are also taught how to solve simple linear-function problems. The following year (when students turn 14–15), they are introduced to concepts such as slope, intercept and intersection of linear functions. They also learn the basics about other types of functions such as quadratic, radical and exponential. Students aged 15–16 review the topic of linear functions at the beginning of the year, but they focus mainly on the study of quadratic, exponential, rational, radical, trigonometric and logarithmic functions. Students are taught how to use Excel spreadsheets to study functions when they are 14–15, but their use in class is very sporadic. In all grades, linear functions are taught with a very strong focus on manual construction of representations. Students are only allowed to use graphing calculators when they start dealing with more complex functions at the age of 15–16.

In the Spanish educational system, when students turn 14 they can choose to continue studying mathematics or to stop studying that subject. Approximately half of the students in our sample aged 14–16 had opted not to take mathematics, whereas the other half did take mathematics.

Task

Students were asked to solve linear-function problems using three different representations – tables, graphs and formulae. The problems dealt with the concepts of slope, y -intercept and intersection between two functions. The functions used in the different problems had comparable slopes and intercepts. The problems were similar to those used in a previous study (Acevedo Nistal et al., 2012a), except for one difference: in the current study, problems and representations were constructed so that the answer could never be read off from the representation; meaning that in order to solve the problems, students had to carry out some operations with the information provided in the representations. Further details about the different problem types, as well as concrete examples, are provided later on in the Procedure section, since some background information about the C/NC method is needed to understand how the problems differed from each other.

Procedure

The study had a pretest/intervention/posttest design with an experimental group and a control group. All students participated in the pretest and posttests. Twenty-five students were in the experimental group, and participated in the intervention. The remaining 24 students acted as a control group. The computerised pretest and posttests were programmed using Macromedia Flash Player. The pretest, intervention and posttest were all administered within a week (pretest/2 days/intervention/3 days/posttest).

Pretest

The pretest was an individualised, computerised C/NC test. Based on Siegler and Lemaire's (1997) C/NC method, students solved problems in each of the following conditions:

- (1) A C-condition, where they could decide which representation they used for each problem. The programme only allowed students to select one representation per problem, and choices could not be reconsidered. Students were not allowed to go back to previous problems either. Like in our previous computerised C/NC study (Acevedo Nistal et al., 2010), students made their choices by clicking on a button reading 'table', 'graph', or 'formula'. The students only saw the actual representation after they clicked on one of the buttons. An example of a slope problem from the C-condition is provided in Figure 1.
- (2) Three NC-conditions, where students solved all problems using predetermined representations – tables in the NC-table condition, graphs in the NC-graph condition, and formulae in the NC-formula condition. An example of an intercept problem from the NC-table condition is provided in Figure 2.

Maria es socia de una piscina cubierta a la que acude con regularidad después de trabajar para relajarse. Maria paga un precio fijo por entrar, más una tarifa por cada hora de estancia. Las siguientes representaciones expresan el precio de cada visita en relación con el número de horas que Maria pasa allí. Determina la tarifa por hora que cobra la piscina.

Figure 1. Slope problem from the C-condition: Maria has a membership for a swimming pool. Maria pays a fixed rate to access the pool, plus a fee for every hour that she stays in the pool premises. The following representations express how much she pays in relation to the number of hours that she spends at the pool. Determine the fee that she pays per hour. Buttons read 'table', 'algebraic expression' and 'graph'. Button on lower right hand corner reads 'next'.

An example of an intersection problem from the NC-formula condition is provided in Figure 3.

In the C-condition, students solved nine problems (three slope, three intercept and three intersection). In each NC-condition, students solved six problems (two slope, two intercept and two intersection). The problems were parallel across conditions with regards to the slopes and intercepts used in the functions. As suggested by Siegler and Lemaire (1997), students were exposed to the C-condition first to avoid recency effects from the NC-conditions to the C-condition. The order of the NC-conditions was randomised across students. The representations provided with each problem could not be manipulated in the computer. Students were given paper and pencil for calculations. There were no time limitations.

In the C-condition, the following data were logged: representational choice, answer, selection time (i.e. how long it took the student to decide which representation to use) and problem-solving time (i.e. how long it took the student to solve the problem with the chosen representation). In the NC-conditions, answers and problem-solving times were logged. The C-condition data provided us with information about students' choices. The data from the NC-conditions, on the other hand, yielded estimates of students' ability and speed to use the different representations (i.e. their representational efficiency).

Intervention

The intervention was presented to students individually using PowerPoint slides tailor-made for each student. They also received a paper booklet in which the

Sergio es fontanero. Cuando le llaman para hacer un trabajo un sábad o un domingo, Sergio cobra un suplemento por ser fin de semana más una tarifa por cada hora de trabajo. La siguiente tabla expresa cuánto cobra Sergio por una reparación en fin de semana en relación con el número de horas que le lleve el trabajo. ¿Cuánto cobra Sergio de suplemento de fin de semana?

Horas	Precio
1	14
2	18
3	22
4	26
5	30

Sergio cobra

euros de suplemento de fin de semana

Siguiente

Figure 2. Intercept problem from the NC-table condition: Sergio is a plumber. When he is called to make a repair on a Saturday or Sunday, Sergio charges a weekend supplement plus an hourly rate. The following table expresses how much Sergio charges in the weekend in relation to how long each repair takes. How much does Sergio charge as a weekend supplement? Button on lower right hand corner reads ‘next’.

Sara ayuda a dos ejecutivos, Carlos y Esteban, en las tareas administrativas de sus oficinas. Los dos le pagan una tarifa cada vez que trabaja para ellos, más un precio por hora. Las siguientes expresiones algebraicas expresan cuánto le paga cada uno en relación con el número de horas que trabaja. Determina cuántas horas tendría que trabajar para que los dos le pagaran lo mismo.

Carlos $f(x) = 4x + 8$
 Esteban $g(x) = 6x + 4$

Tendría que trabajar horas

Siguiente

Figure 3. Intersection problem from the NC-formula condition: Sara helps two executives, Carlos and Esteban, in the administrative tasks of their offices. Both pay her a fixed rate every time she works for them, plus a rate per hour. The following algebraic expressions express how much each one of them pays her in relation to the number of hours that she works. Determine how many hours she would have to work for both of them to pay her the same. Button on lower right hand corner reads 'next'.

assignments that were shown on the slides were to be answered. The intervention aimed to improve students' representational flexibility (understood as their ability to fine-tune their representational choices to task and subject characteristics) by:

- (1) Exposing students to their own performance in particular tasks using specific representations, extracted from their own NC-pretest.
- (2) Exposing students to their own representational choices, extracted from the C-pretest.
- (3) Presenting students with examples of different types of justifications for representational choices. The examples were extracted from a previous study where we interviewed students while they made choices to solve similar problems to the ones used in the current study (Acevedo Nistal, Van Dooren, & Verschaffel, 2012b).

We explicitly confronted students with the fact that not all problems are the same, and as a result different types of problems might be easier/more difficult to solve with one representation than with the others. Using concrete examples, we also showed them that not everybody is equally proficient with all representations, and for that reason different choices might be appropriate for different people. We wanted students to become aware of the fact that task-related and subject-related characteristics can influence not only the representational choices they make, but also the success (in terms of performance) of their choices. We expected that this awareness would help them become better (i.e. more flexible) in their choices.

The intervention consisted of six sections:

- (1) General introduction: Participants were told that in the past we have asked students their age to solve similar problems to the ones they have just solved (in the C-pretest), and that we asked these students to explain their choices. We pointed out that these students' explanations concerning their choices might also be of interest to them, since they might have mentioned issues that they never considered and that might help them to make better choices in the future.
- (2) Introduction to subject-related characteristics: Students were explained that not everybody is equally proficient with tables, graphs and formulae, and that one's own ability to use different representations can influence how often you choose them to solve problems. In more technical terms, we explained to them that their own representational efficiency, a subject-related characteristic, can influence their choices. In order to illustrate this idea, we presented students with three justifications given by other students who used their (lack of) representational efficiency to explain why they chose or did not choose a table, a graph and a formula to solve a slope problem. In order to encourage students to actively engage in the intervention, they were asked to compare their own reasons (not) to use a particular representation with the reasons from the students in the examples, and to write their reflections in their booklet.
- (3) Confrontation with students' own choices (based on their C-condition data) and with their own representational efficiency (based on their NC-conditions data). Table 1 illustrates how this information was presented to students.

Students were given some time to reflect about their own data. Then, each student was asked a tailor-made question concerning his or her data. For example, the student whose data are displayed in Table 1 was asked: 'The representations you performed best with were the table and the graph. However, the representation that you chose the most was the formula. Could you give any reasons why you selected more often the formula?' Students answered their questions in their booklet.

- (4) Introduction to task-related characteristics: Here, we explicitly told students that the problems that they solved belong to three different categories: slope, intercept and intersection problems. Then, we showed them examples of students who referred to task characteristics to explain why they chose a table, a graph, or a formula to solve a slope, an intercept and an intersection

Table 1. A student is confronted with his own representational choices and representational efficiency.

	Your results
Table	<ul style="list-style-type: none">• You selected the table in 22% of cases• Your performance with the table was 90%
Graph	<ul style="list-style-type: none">• You selected the graph in 11% of cases• Your performance with the graph was 100%
Formula	<ul style="list-style-type: none">• You selected the formula in 67% of cases• Your performance with the formula was 50%

problem (total: 9 problems, 3 types of problems $\times$ 3 representations). Once again, in order to maximise students' engagement in thinking about the types of problems and their own representational choices, for each example we asked them to compare their own reasons (not) to choose a particular representation with the reasons provided by the students in the examples.

- (5) Confrontation with students' own representational efficiency and representational choices per problem type: As a last step in the intervention, students were also confronted with their own efficiency with the different representations in the different problem types (extracted from the NC-conditions of the pretest), and also with their choices per problem type (extracted from the C-condition of the pretest). This is the step where we combined task and subject characteristics. Each student answered questions in his/her booklet which encouraged him/her to compare his/her NC-and C-data per problem type and to evaluate whether his/her choices were in line or not with his/her representational efficiency per problem type.
- (6) Conclusions and recommendations: The intervention concluded with a brief text encouraging students to use not only what they had just learnt from other students' problem-solving experiences and explanations, but also what they learnt about themselves as representational users, to improve their choices in the future. In other words, we tried to encourage them to use their (newly acquired) meta-representational competence to make better choices.

The intervention consisted of a single session, and it took place at the school's computer lab. All students from the experimental group were exposed to the intervention at the same time. There were no time limitations – students were allowed to explore each problem, students' justifications and their own efficiency and choice data for as long as they wanted. They were also allowed to re-explore previous examples. Frequent reminders were inserted at different stages in the intervention to encourage students to examine *all* examples and answer *all* questions. Hyperlinked overview tables were used to facilitate the exploration of the different problems. Students took an average of 1 h 5 min to go through the intervention. An experimenter was onsite to ensure that students understood what they were expected to do and to supervise the intervention. As soon as students were given the go-ahead to start, the experimenter did not interact with the students in any way. Students were not allowed to interact with each other either. The students from the control group stayed in class with their math teacher and worked on an activity which was not directly related to the intervention.

Posttest

The posttest was an individualised, computerised C-test where students solved three slope, three intercept and three intersection problems. The problems were new to the students in the sense that the scenarios were different from the ones used in the pretest. However, the problems were parallel to those in the C-pretest with regards to their underlying structure. Also, the functions used were comparable to those used in the C-pretest problems. Both groups took the same posttest.

Since our intervention was *not* aimed at improving students' representational efficiency (in fact, during the intervention students were not asked to actually solve any problems, neither were they exposed to worked-out solutions of problems), we

decided not to expose students to another NC-test at the posttest moment but to use the pretest NC-data to compute the posttest flexibility score (see further). Given the nature of our intervention, we assumed that the efficiency data collected during the NC-pretest would not be affected by the intervention. A check was performed to ensure that our assumption was correct.

Results

Based on Siegler and Lemaire's (1997) model of strategy choice, we structured our predictions and the description of our results around four parameters: (1) representational repertoire, (2) frequency of representational choice, (3) representational efficiency and (4) representational flexibility. An individualised conceptualisation of flexibility was used (Acevedo Nistal et al., 2012a). This conceptualisation acknowledges that both task and subject features play a role in determining students' choices. It takes into account individualised measures of students' efficiency with the available representations (a subject characteristic) to solve different types of problems (task characteristics).

Before turning to the results of the intervention as such, we will present some preliminary analyses. These analyses focus on parameters 1, 2 and 3, and are aimed at answering the question: were both groups comparable at pretest with regards to repertoire, frequency, and efficiency? We will also conduct a test to check the assumption that the intervention did not affect accuracy and speed using particular representations as such.

After these preliminary analyses, we will turn to the effects of the intervention, which will be discussed in terms of shifts in the same parameters: changes in repertoire and frequency, increased efficiency as a consequence of making better representational choices, and increased flexibility, each of which will allow us to test the predictions mentioned above.

Preliminary analyses

Some preliminary analyses were conducted to test whether the experimental group and control group were comparable at the pretest moment. Students were assigned to the experimental ($n=25$) and control group ($n=24$) on the basis of their age, pretest performance and representational choices. Both the experimental and control group included a comparable number of students with low (beyond 1 SD below average), average (between ± 1 SD) and high (beyond 1 SD above average) pretest accuracy scores, and with slow (beyond 1 SD above average), average (between ± 1 SD) and fast (beyond 1 SD below average) selection and problem-solving speeds. The two groups also included students with different representation selection patterns (e.g. in both groups, there was approximately the same number of students who selected always the same representation for all the C-pretest problems). Data concerning the C/NC conditions of the pretest are provided in Table 2.

In what follows, we will test whether both groups were comparable at pretest with regards to repertoire, frequency and efficiency, and check the assumption that the intervention did not affect accuracy and speed using particular representations as such.

Table 2. Choice (C-) and no-choice (NC-) pretest data from the experimental and control group.

	C-pretest					NC-pretest		
	Representational choices (%)			<i>M</i> speed (SD) (s)		<i>M</i> accuracy (SD) (%)		
	Table	Graph	Formula	Selection speed	Solving speed	Table	Graph	Formula
Experimental	53.3	16.9	29.8	37.6 (15.8)	92.3 (51.3)	74.0 (14.4)	64.8 (38.7)	42.0 (38.9)
Control	59.7	16.7	23.6	36.8 (18.1)	91.7 (56.2)	70.0 (17.2)	69.2 (30.2)	45.8 (33.1)

Representational repertoire

A chi-square test determined that both groups did not differ from each other at pretest with regards to their representational repertoire, $\chi^2(4, n = 49) = 0.52, p = .969$. The groups were comparable with regards to the number of students who selected one, two or three representations to solve the C-pretest problems. Approximately, 12% of students in each group used only one representation to solve all problems, around 35% used two representations and around 53% used the three representations.

Frequency of representational choice

Whereas in the previous section we studied the *number* of representations that students used in the C-pretest, in the current section we analysed the *frequency* with which each representation was chosen by each group. Students' choices were coded as a binary variable (*representational choice*) where 1 meant that a representation was chosen and 0 that a representation was not chosen. Students' choices are displayed in Table 3.

Before reporting the results of our analyses, two practical clarifications need to be made: first, the data used for the analyses were dichotomous, and the responses correlated. For that reason, the repeated measures logistic regression analyses reported in this paper were done using the Generalised Estimating Equations approach in Statistical Package for the Social Sciences, which does not rely on any normality or independence of assumptions (Liang & Zeger, 1986). Second, for purposes of readability, statistical details are only included in text for the main effects and interaction effects. In contrast analyses, we only report whether differences were statistically significant ($p < .05$) or not, without mentioning exact p -values.

A repeated-measures logistic regression analysis was used to determine if students displayed any representational preferences at pretest, and if such preferences differed between groups. *Representational choice* was the dependent variable, and

Table 3. Students' representational choices in the choice pretest (in % of total).

	Table	Graph	Formula
Experimental	53.3	16.9	29.8
Control	59.7	16.7	23.6

Table 4. Students' representational choices in the choice pretest by problem type (in % of total) (T=table, G=graph, F=formula).

	Slope			Intercept			Intersection			Total		
	T	G	F	T	G	F	T	G	F	T	G	F
Experimental	49.3	16.0	34.7	57.3	16.0	26.7	53.3	18.7	28.0	53.2	16.8	29.7
Control	65.3	11.1	23.6	70.8	16.7	12.5	43.1	22.2	34.7	59.7	16.7	23.6

group (experimental vs. control) and *representation* (table, graph and formula) the independent variables.

Representation did have a main effect on representational choice, $\chi^2(2, n=49) = 17.69, p < .001$. The interaction between *representation* and *group* was not significant – in both groups, the preferred representation was the table, followed by the formula and the graph. The table was chosen significantly more often than the formula, and the formula significantly more often than the graph.

Students' choices in the pretest were also analysed per problem type (slope, intercept and intersection). Table 4 provides an overview of the results.

A repeated-measure logistic regression analysis was conducted to determine if students' choices were influenced by problem type, and if so whether there was a difference between the groups. The independent variables were *type of problem*, *group* and *representation*, and the dependent variable was *representational choice*.

The interaction between *type of problem* and *representation* was significant, $\chi^2(4, n=49) = 18.87, p = .001$, whereas the interaction between *type of problem*, *group* and *representation* was not, $\chi^2(4, n=49) = 2.33, p = .679$. The table was selected significantly more often than the other two representations by both groups in the three problem types. In slope problems, both groups chose the formula significantly more often than the graph. In intercept and intersection problems, there were no significant differences between the formula and the graph.

Representational efficiency

We evaluated representational efficiency at the pretest by analysing the accuracy and speed data collected in the correct trials of the three NC-conditions. As Siegler and Lemaire (1997) explain, the data from the NC-conditions provide unbiased estimates of how fluent students are with the different representations, since all students solve all types of problems with all available representations. NC-data from the experimental and control group were compared to find out if the groups were comparable at pretest with regards to accuracy and speed measures.

Accuracy. A repeated-measures logistic regression analysis was conducted with the accuracy from the three NC-conditions as dependent variable, and *group* (experimental vs. control), *representation* (table, graph and formula) and *type of problem* (slope, intercept and intersection) as independent variables. The variable *group* did not have a significant effect on NC-accuracy, meaning that both groups were comparable in terms of their overall representational efficiency at the pretest (experimental group 64.9% correct and control group 59.1% correct).

The variable *representation* did have a main effect on NC-accuracy, $\chi^2(2, n=49) = 41.82, p < .001$. Students displayed significantly higher accuracy in the NC-table and NC-graph conditions than in the NC-formula condition (NC-table

72.9% correct; NC-graph 67.0% correct and NC-formula 42.9% correct). The difference in performance between the NC-table and NC-graph conditions was not significant.

With regards to *type of problem*, a main effect was also found, $\chi^2(2, n=49) = 25.65, p < .001$. Students performed significantly better in the slope problems than in the intercept and intersection problems, with no difference between the latter two (slope 78.0% correct, intercept 54.9% correct and intersection 48.9% correct).

The only significant interaction was between *type of problem* and *representation*, $\chi^2(4, n=49) = 73.04, p < .001$. In intercept problems, all representations yielded comparable accuracies (table = .58; graph = .51; and formula = .57). In slope problems, students were significantly more accurate with the table than with the graph, and significantly more accurate with the graph than with the formula (table = .96; graph = .84; and formula = .27). In intersection problems, the graph was significantly more accurate than both the table and the formula (graph = .61; table = .39; and formula = .47).

Speed. As far as selection times are concerned, *t*-tests showed that the two groups were comparable at pretest (experimental 37.6s, control 36.8s, $t(727) = .36, p = .357$). Concerning problem-solving times, a repeated-measures linear regression analysis was conducted using the *problem-solving times* extracted from the correct trials in the NC-conditions as the dependent variable, and *group* (experimental vs. control), *representation* (table, graph and formula) and *type of problem* (slope, intercept and intersection) as the independent variables.

The variable *group* did not have an effect on speed – the experimental group took an average of 92.3 s to solve each problem, while the control took 91.7 s. None of the interactions involving the variable *group* were significant, which shows that both groups were also comparable at pretest in terms of speed. The variable *representation*, on the other hand, did have a significant effect on speed, $\chi^2(2, n=49) = 42.57, p < .001$. Times in the NC-graph condition were significantly shorter than in the NC-table condition (NC-graph 74.5s and NC-table 90.6s), which, in turn, were significantly shorter than in the NC-formula condition (NC-formula 109.9 s).

The variable *type of problem* also had a significant effect on speed, $\chi^2(2, n=49) = 157.21, p < .001$. While slope and intercept problems took more or less the same time (average of 68 and 69.8 s, respectively), intersection problems took significantly longer (137.3 s).

The only significant interaction was between *type of problem* and *representation*, $\chi^2(4, n=49) = 29.22, p < .001$. In slope problems, the formula was significantly slower than the other representations. In intercept problems, all representations yielded comparable speeds. In intersection problems, the graph was significantly faster than the other representations.

Representational efficiency after the intervention

The preliminary analyses allowed us to conclude that the experimental and control groups were comparable at the pretest on all relevant aspects. We conducted an additional test to check whether students' efficiency (in terms of accuracy and speed) using the different representations was not affected by the intervention. We compared the accuracy and speed displayed by both groups in trials where students selected the

same representation in the parallel C-pretest and posttest items. In 12.0% of these trials, students in the experimental group performed better (in terms of accuracy) in the posttest than in the pretest, while this was the case for 9.7% of trials in the control condition. A *t*-test revealed that the increased accuracy was not significantly different for the experimental and the control group, $t(1321)=1.33, p=.090$.

We also compared the problem-solving speeds for the same trials, but only in cases where students obtained a correct answer. The experimental group was 34.8s faster in the posttest than in the pretest, whereas the control group was 31.6s faster. There was no significant difference between the two, $\chi^2(1, n=49)=0.16, p<.690$, showing that problem-solving speed as such improved equally in the experimental and control group.

Given that both groups showed similar improvements in efficiency (both in terms of accuracy and speed) from pretest to posttest in trials where they used the same representation, we can conclude that these improvements were not due to the intervention but to practice and/or retest effects.

Effects of the intervention

After the preliminary analyses (which revealed that both groups were comparable at the pretest and showed a similar increase in efficiency from pretest to posttest), we now turn to the effects of our experimental intervention as such. The results that are of principal interest relate to the fourth parameter in Siegler and Lemaire's (2007) model, i.e. the increase in representational flexibility. However, we also formulated predictions related to the other three parameters.

Effect on representational repertoire (Prediction 1)

A significant shift was observed from pretest to posttest, and this shift was applicable to both groups (experimental $\chi^2(6, n=25)=20.5, p=.002$; control $\chi^2(5, n=24)=11.4, p=.037$). In both groups, the number of students who used only one or two representations went up from pretest to posttest (from around 12% to around 28% in the case of one representation and from around 35% to around 53% in the case of two representations), whereas the number of students who used the three representations went down from pretest to posttest (from around 53% to around 18%).

The groups did not differ from each other at posttest, $\chi^2(6, n=49)=7.82, p=.022$. At posttest, students seemed to prefer using only one or two representations (especially, table+graph, graph only and table only). Further details are provided in Table 5. Given that both groups' representational repertoire changed similarly, our Prediction 1 was not confirmed.

Effect on frequency of representational choice (Prediction 2)

A repeated-measures logistic regression analysis using *group* (experimental versus control), *representation* (table, graph, formula) and *test moment* (pretest versus posttest) as independent variables showed that the interaction between *test moment* and *representation* was significant, $\chi^2(2, n=49)=10.31, p=.006$. However, the interaction between *group*, *test moment* and *representation* was not. Students' representational choices changed from pretest to posttest, and they changed in the same way in both groups. In the C-posttest, the table was still the preferred representation.

Table 5. Students' representational repertoire in the choice pretest and posttest (in % of total).

		Table only	Graph only	Formula only	Table + graph	Table + formula	Graph + formula	Table + graph + formula
Experimental	Pretest	8.0	0.0	4.0	12.0	24.0	0.0	52.0
	Posttest	16.0	4.0	0.0	52.0	0.0	8.0	20.0
Control	Pretest	8.3	0.0	4.1	16.7	16.7	0.0	54.2
	Posttest	16.7	12.5	8.3	37.5	8.3	0.0	16.7

However, the second preferred representation changed from the formula in the pretest to the graph in the posttest. In the posttest, the table was chosen significantly more often than the graph and the graph significantly more often than the formula.

A second repeated-measure logistic regression analysis was performed to find out if students' choices changed from pretest to posttest *in the different problem types*, and if so, to determine whether such change was affected by the intervention. The interaction between *type of problem*, *test moment* and *representation* was significant, $\chi^2(17, n=49)=473.05, p<.001$. However, the interaction between *type of problem*, *group*, *test moment* and *representation* was again not significant, meaning that the change in representational choices observed from pretest to posttest was the same in both groups. In all problem types, the formula lost popularity from pretest to posttest. The graph, on the other hand, became more popular. The table became less popular except in the case of slope problems, where it was chosen in a comparable number of instances in the pretest and posttest. Although the tendency to switch from the formula to the other two representations was stronger in the experimental group than in the control group (the frequency of choice of the formula went down from pretest to posttest by 22.2% in the experimental group, but only 7.4% in the control group), this difference was not significant. Therefore, our Prediction 2 was not confirmed either.

Effect on representational efficiency (Prediction 3)

In the preliminary analyses, it was already pointed out that the intervention had no impact on the accuracy and problem-solving speed *when students made the same representational choice at pretest and posttest*. The preceding sections have also shown that – against Predictions 1 and 2 – the intervention showed impact neither on students' representational repertoire, nor in the frequency of representational choices. Still, as formulated in Prediction 3, it is possible that students in the experimental group solve problems faster and more accurately when they have a choice of representations if, as a consequence of the intervention, they manage to make better representational choices. This can be shown by comparing the improvement in accuracy and speed from pretest to posttest in the C-tests for the data set as a whole.

Accuracy. A *t*-test showed that the experimental group improved significantly more than the control group from pretest to posttest, $t(49)=2.47, p=.008$. In line with Prediction 3a, the data provided in Table 6 show that, while the experimental group gained about 11% in accuracy, the experimental group gained about 28%.

Table 6. Improvement in accuracy (in % correct) from choice (C-) pretest to C-posttest in the experimental and control group.

	C-pretest accuracy	C-posttest accuracy	Improvement
Experimental	47.2	75.0	27.8
Control	48.1	59.0	10.9

Speed. We compared each individual student's selection times as well and problem-solving times in the items of the C-pretest with the parallel items in the C-posttest. Only items where students obtained correct answers were included in this analysis. Students in both groups became faster choosers and faster problem solvers. Interestingly, the control group made their choices 16.8s faster in the posttest than in the pretest, whereas the experimental group made their choices only 10.1s faster. A *t*-test revealed that this difference between conditions was significant, $t(336)=1.73$, $p=.041$. In line with Prediction 3c, this difference suggests that students in the experimental group spent more time selecting a representation, presumably because they selected more thoughtfully. In line with Prediction 3b, the reverse pattern was observed with regards to problem-solving times, and differences were much more pronounced here: whereas the control group solved problems only 16.1 s faster from pretest to posttest, the experimental group did it 44.8 s faster. A paired *t*-test showed that this difference between conditions was also significant, $t(336)=2.51$, $p=.006$.

Effect on representational flexibility (Prediction 4)

Given that the experimental group became considerably more accurate than the control group as a consequence of the intervention, while the representational repertoire and the frequency of choices in both groups evolved very similarly in both groups, it seems likely that the gain in accuracy in the C-test was due to gains in flexibility in the experimental group.

In order to compare changes in representational flexibility, we calculated flexibility scores per student and per test moment. Each student's representational flexibility was evaluated by comparing his performance in the NC-conditions with his choices in the C-pretest and posttest. A student's representational choice was considered flexible if, for a given C-condition problem, he chose the representation with which he had obtained the highest accuracy in that type of problem in the NC-conditions (Acevedo Nistal et al., 2012a, 2012b based on Siegler & Lemaire, 1997). Let us imagine that, in slope problems, a student obtained 100% accuracy in the NC-table condition, 67.0% accuracy in the NC-graph condition and 33.0% accuracy in the NC-formula condition. If he selected the formula in a slope problem in the C-condition, his choice would be labelled as inflexible (and given a score of -1) because he selected the representation with which he had the lowest chances of finding the correct answer. If he selected the table, his choice would be considered flexible (and given a score of 1) because he selected the representation most likely to lead him to the correct answer. Students who obtained 100% accuracy in a particular type of problem in all three NC-conditions were given a 1 flexibility score regardless of their choice, since all three representations were equally likely to lead them to the correct answer. Students with 0 accuracy in a specific type of problem in all three NC-conditions were given a flexibility score of 0 for that

Table 7. Average flexibility scores (range -1 to 1) and standard deviations of the experimental and control group at pretest and posttest.

	Pretest	Posttest
Experimental	0.22 (0.48)	0.64 (0.41)
Control	0.30 (0.55)	0.38 (0.54)

problem regardless of their choice, since speaking about flexibility is irrelevant if a student is not fluent with any of the representations available.

The average pretest and posttest flexibility scores obtained by both groups are listed in Table 7.

At pretest, *t*-tests showed that the average flexibility score of the control group was slightly higher than that of the experimental group, although the difference was not significant, $t(439)=0.95$, $p=.173$. At posttest, not only did the experimental group make up for the advantage of the control group at pretest, a *t*-test showed that it also became significantly more flexible than the control group, $t(421)=3.41$, $p<.001$. A paired *t*-test showed that the improvement in flexibility from pretest to posttest was significantly higher in the experimental group than in the control group, $t(437)=3.08$, $p=.001$. These data clearly confirm Prediction 4.

Discussion

The purpose of our study was to test the following hypothesis: the development of representational flexibility can be effectively enhanced through instruction that pays systematic attention to the questions *when* to use a representation and *why* it might be more appropriate to use that representation in particular cases; in other words, instruction that aims at developing representational flexibility through the construction of 'meta-knowledge of representations' (de Jong et al., 1998) or 'meta-representational competence' (diSessa & Sherin, 2000).

Summary of findings

Our preliminary analyses concerning representational repertoire and frequency of representational choice seemed to suggest that our intervention had not been successful, since both groups changed equally from pretest to posttest with regards to these two parameters. In both groups, the number of students who used only one and two representations went up from pretest to posttest, whereas the number of students who used all three representations went down. This meant that our Prediction 1 was not confirmed. Also, both groups displayed the same shift in frequency of representational choice from pretest to posttest, which meant that our Prediction 2 was not confirmed either. However, further analyses concerning representational efficiency in problems involving a representational choice showed that the switches from pretest to posttest done by the experimental group were more successful (both in terms of accuracy and speed) than the switches of the control group. With regards to accuracy, the experimental group improved more than the control group from pretest to posttest, and this confirmed our Prediction 3a. As far as speed is concerned, we predicted that the experimental group would show a stronger decrease in problem-solving times (Prediction 3b) as well as in selection times (Pre-

diction 3c) from pretest to posttest than the control group. While Prediction 3b was confirmed, the decrease in selection times was actually stronger in the *control* group than in the experimental group, meaning that our Prediction 3c was not confirmed. This could be due to the fact that the students in the experimental group became more aware of certain task-related and subject-related features that can influence their choices, and as a result they made a more conscious effort to reflect on these features and evaluate their possibilities before making their final choice. The fact that the experimental group was somewhat slower in making choices than the control group was highly compensated by the fact that they were much faster in the problem-solving phase itself than the control group. Thus, sacrificing a few seconds to make a better choice during the selection stage seems to pay off in terms of speed in the problem-solving stage. More importantly, it also seems to pay off in terms of accuracy – the experimental group became more accurate at posttest than the control group because it made better choices.

In order to test our Prediction 4 (i.e. that the experimental group would show a significantly stronger improvement in representational flexibility from pretest to posttest than the control group), students' choices were assessed using an individualised conceptualisation of flexibility. When pretest and posttest flexibility measures were compared, the data clearly showed that the experimental group had become much more flexible than the control group in the posttest, which confirmed our Prediction 4.

Limitations of the current study

As Verschaffel et al. (2009) point out, recent developments within the field of strategy choice have stressed the fact that students' flexibility is influenced by factors which are not always task- and subject-related. In the past years, contextual factors and their influence on students' strategy choices have started attracting research attention, although their role in the development of flexibility is still not completely clear. Hatano and Oura (2003) argue that mathematics teaching and learning always take place in specific sociocultural contexts, and the characteristics of these contexts can greatly influence the learning process. In the current study, we adopted an approach to representational flexibility that takes into account task and subject-related factors, and not (yet) contextual factors. While we strongly believe that contextual factors are essential in a psychological description of the notions of flexibility and choice (as explained, for example, in Ellis (1997) and Verschaffel et al. (2009)), we doubt whether it is necessary, from an instructional perspective, to include context as something students need to be *taught* to take into consideration when making a representational choice. In the domain of linear functions, for example, formulae are often given considerably more attention in the mathematics classroom than tables and graphs (Yerushalmy & Schwartz, 1993). As a consequence, students who are sensitive to context may often prefer formulae, merely because this is encouraged by the teacher (Acevedo Nistal et al., 2012b). However, from an instructional perspective, is this the type of flexibility that we are aiming to achieve?

Another possible limitation of our study was that we did not include NC-measurements at the posttest moment. We decided not to do so to avoid repeated practice with the test items, as we did not want the intervention to cause an increase in students' efficiency. Our results indicated that the intervention indeed did not lead

to an increase in efficiency, even though this finding needs to be treated cautiously, as it was based on performance data from the C-condition, and as Siegler and Lemaire (1997) explain, performance on C-condition items may be somewhat biased by the representational choices students make. However, the main focus of our study was actually on how students would perform in situations where they actually had a representational choice, and these data were available at pretest and posttest. The fact that the experimental group performed better in the posttest than in the pretest in situations of representational choice (while their fluency was not affected by the intervention as such) is, therefore, a clear indicator of increased representational flexibility as a consequence of the intervention.

The intervention that we used was short (it lasted about an hour), there was no human interaction between the students and the experimenter (once the intervention started, no questions concerning the intervention could be asked), and students did not receive any feedback. Still, the intervention had a remarkable effect on students' flexibility. Without attempting to improve students' efficiency, the intervention considerably improved students' performance in items where a choice was required, because choices were made more flexibly. It seems to be worthwhile to develop interventions which do not target efficiency only, but efficiency in combination with flexibility. A follow-up study could be conducted where different groups of students are exposed to interventions that address efficiency only, flexibility only, or both of them simultaneously, in order to assess the effects of improved efficiency and flexibility in isolation and in combination with each other.

Implications for instruction

Despite the above-mentioned limitations, some important instructional implications can be mentioned. First, an interesting shift in representational choice was observed: students were more likely to select the formula in the pretest than in the posttest. Judging by the results of our previous research (Acevedo Nistal et al., 2012b), this could be closely connected with what is known as the *didactical contract* (Brousseau, 1997), or the *social and sociomathematical norms* which regulate the interactions between teachers and students in the classroom (Yackel & Cobb, 1996). Certain students feel forced to choose the representation that their teacher uses the most in class, in most cases the formula. This could have also been the case in our current study – in the pretest, students might have found it difficult to let go of that contract completely (although in a way they did deviate from the contract, since they used the table as their first choice in many cases), but the formula still remained in a noticeable second place. At posttest, the students in the experimental group replaced the formula with a representation which was more efficient for them, whereas the students in the control group were less able to switch to a more efficient representation for certain problems. While students seem to be able to take the first step on their own (both groups realised that the formula was not always the best choice), explicit support seems to be needed in order to find alternatives that result in improved performance. The results of our study show that flexibility needs to be explicitly targeted for it to develop.

Second, our study targeted secondary school students who had already had some experience with linear functions previous to the experiment, but who could by no means be considered experts in the topic. The fact that our intervention was successful proves that representational flexibility is not only relevant for students

who have reached a high level of expertise, but also for novices in that particular topic.

Third, our intervention was not designed to be implemented in a mathematics classroom as such. Rather, it was designed to experimentally test whether flexibility – conceived as students' ability to choose representations which are not only appropriate for the problem at hand, but which are also in line with their own efficiency with the different types of representations available – can be improved by means of an intervention, and to show the impact that improved flexibility can have on performance. Given that this kind of flexibility can indeed be improved, even in a short intervention, research effort should be invested in developing teaching materials which address the issue of representational flexibility in a systematic way, materials that give students the opportunity to make their own representational choices and that stress the importance of reflecting on one's choices. Policy documents (e.g. Boletín Oficial de Castilla y León, 2007; National Council of Teachers of Mathematics, 2000) encourage teachers to challenge students to make their own representational choices, but in many (traditional) mathematics classrooms students are hardly ever given the opportunity to choose themselves, and this prevents them from developing their representational flexibility.

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