KU LEUVEN

Arenberg Doctoral School of Science,
Engineering & Technology
Faculty of Sciences
Department of Mathematics

Novikov, LR- and post-Lie algebra structures, and their relation to NIL-affine crystallographic actions

Kim Vercammen

Supervisor:
Prof. dr. Karel Dekimpe

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Sciences

May 2013

Novikov, LR- and post-Lie algebra structures, and their relation to NIL-affine crystallographic actions

Kim VERCAMMEN

Jury:
Prof. dr. Paul Igodt, chair
Prof. dr. Karel Dekimpe, supervisor
Prof. dr. Wim Malfait
Prof. dr. Raf Cluckers
Prof. dr. Dietrich Burde
(Universität Wien)
Prof. dr. Pasha Zusmanovich
(Tallinn University of Technology, Estonia)

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Sciences

May 2013

Geel Huis, Kasteelpark Arenberg 11, B-3001 Leuven (Heverlee) (Belgium)

D/2013/10.705/38
ISBN 978-90-8649-623-5

Dankwoordje

Graag wil ik een aantal mensen bedanken die er elk op hun eigen manier mee voor hebben gezorgd dat deze thesis op een succesvolle manier tot stand kon komen.

Hierbij denk ik in de eerste plaats aan mijn promotor Prof. dr. Karel Dekimpe. Bedankt voor de jarenlange samenwerking die altijd in een aangename en toffe sfeer kon verlopen, voor de kansen die ik kreeg en alle hulp en steun bij het schrijven van deze thesis. Mede dankzij jouw ervaring, raadgevingen en ondersteuning is deze thesis er kunnen komen.

Ook Prof. dr. Dietrich Burde verdient een woordje van dank. Thank you Dietrich, it was a great pleasure working with you.

Daarnaast wil ik ook graag Prof. dr. Paul Igodt bedanken, voor het enthousiasme en de bemoedigende woorden doorheen de voorbije jaren.

Verder wil ik ook alle andere leden van de jury, Wim Malfait, Raf Cluckers en Pasha Zusmanovich, bedanken, voor het nalezen en beoordelen van deze thesis.

Ik wil ook graag al mijn (ex-)collega's bedanken waarmee ik doorheen de jaren heb samengewerkt. In het bijzonder Sandra en Kelly, bedankt voor de toffe sfeer (tijdens de werkuren maar ook zeker daarbuiten) en de goede vriendschap.

Bijzondere dank gaat uit naar mijn familie, in het bijzonder mijn mama, grootouders en zussen. Dank jullie voor alle kansen die ik gekregen heb en om steeds paraat te staan als het nodig was.

Ook al mijn vrienden verdienen een plaatsje op deze pagina voor de niet aflatende morele steun en de vele momenten van ontspanning.

En tenslotte kan ik natuurlijk mijn gezinnetje niet vergeten, mijn man Tim, die ik mee genomen heb naar het verre Kortrijk om aan dit doctoraat te werken. Bedankt voor het vertrouwen dat je steeds bleef hebben en alle steun en hulp de

afgelopen jaren. En voor mijn prachtige dochtertjes Febe en Ella, een hele dikke knuffel, bedankt voor jullie guitige lachjes en grappige uitspraken die mama's gedachten steeds weer konden verzetten...

Kim

Preface

In this thesis we study Novikov structures, LR-structures and post-Lie algebra structures on Lie algebras. These structures all appear in the study of NIL-affine crystallographic actions on Lie groups.

Novikov algebras form a subclass of the more general left-symmetric algebras:

> A left-symmetric algebra is an algebra for which the product satisfies
>
> $$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z.$$
>
> Novikov algebras are left-symmetric algebras having the additional property that all right multiplications commute:
>
> $$(x \cdot y) \cdot z = (x \cdot z) \cdot y.$$

We also study LR-algebras:

> An LR-algebra is an algebra for which all left and all right multiplications commute:
>
> $$x \cdot (y \cdot z) = y \cdot (x \cdot z),$$
>
> $$(x \cdot y) \cdot z = (x \cdot z) \cdot y.$$

Both of these algebras are Lie-admissible algebras since the commutator $[x, y] = x \cdot y - y \cdot x$ defines a Lie bracket. The associated Lie algebra is now said to admit a Novikov or LR-structure.

All algebras we consider in this thesis are assumed to be finite dimensional over a field k of characteristic 0, unless specifically stated otherwise.

A given Lie algebra can admit different Novikov and LR-structures, but not every Lie algebra will admit such structures. Moreover, any Lie algebra admitting a Novikov structure is solvable and any Lie algebra admitting an LR-structure is 2-step solvable. However, not every solvable Lie algebra admits a Novikov structure and not every 2-step solvable Lie algebra admits an LR-structure.

These structures are said to be complete if all right multiplications are nilpotent.

The motivation for studying these structures lies in the study of NIL-affine crystallographic actions on real connected, simply connected, nilpotent Lie groups N. Closely related to these actions are simply transitive NIL-affine actions of a connected, simply connected nilpotent Lie group G on another such Lie group N.

There is a one-one correspondence between the complete left-symmetric structures on a nilpotent Lie algebra $\mathfrak{g}$ and the simply transitive affine actions of its corresponding Lie group G. On the other hand, there is a one-one correspondence between the complete LR-structures on a nilpotent Lie algebra $\mathfrak{n}$ and the simply transitive abelian NIL-affine actions on its Lie group N.

More details on this background can be found in chapter 2.

Since left-symmetric structures and LR-structures arose for a special setting of the general class of simply transitive NIL-affine actions, we considered possible generalizations of these notions.

A general simply transitive NIL-affine action for which G is any real connected, simply connected nilpotent Lie group corresponds to a complete NIL-affine structure of the corresponding Lie algebra:

> A complete NIL-affine structure of a nilpotent Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : x \mapsto (t(x), D(x))$, for which $t : \mathfrak{g} \to \mathfrak{n}$ is bijective and for which $D(x)$ is nilpotent for all $x \in \mathfrak{g}$.

We started investigating such (complete) NIL-affine structures. Using the bijective map t we can identify the underlying vector spaces of $\mathfrak{g}$ and $\mathfrak{n}$ and denote this vector space by V. We discovered that a NIL-affine structure is closely related to the concept of a post-Lie algebra:

A post-Lie algebra is a vector space equipped with a bilinear product $x \cdot y$ and a Lie bracket $\{x, y\}$, which satisfy the following relations:

$$\{x, y\} \cdot z = (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z).$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}.$$

We can associate a second Lie bracket to the given post-Lie algebra, defined by $[x, y] = x \cdot y - y \cdot x + \{x, y\}$. It follows that there are two Lie algebras associated to a post-Lie algebra: $\mathfrak{n} = (V, \{,\})$ and $\mathfrak{g} = (V, [,])$. We say that the product $x \cdot y$ is a post-Lie algebra structure on the pair $(\mathfrak{g}, \mathfrak{n})$. We call such a structure complete if all left multiplications are nilpotent.

In chapter 5 we prove that these post-Lie algebra structures arise very naturally in the study of NIL-affine actions:

> Let G and N be real connected, simply connected nilpotent Lie groups with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$. Then there exists a simply transitive NIL-affine action of G on N if and only if there is a Lie algebra $\mathfrak{g}' \simeq \mathfrak{g}$, with the same underlying vector space as $\mathfrak{n}$, such that the pair of Lie algebras $(\mathfrak{g}', \mathfrak{n})$ admits a complete post-Lie algebra structure.

Although Novikov, LR- and post-Lie algebra structures arose in the context of Lie algebras over the field $\mathbb{R}$, we work in our thesis over an arbitrary field k of characteristic 0.

In chapter 3 we study Novikov algebras and Novikov structures. We present some structure theory of Novikov algebras and consider the existence question for Novikov structures.

We prove the following property for Novikov algebras, which is not true for left-symmetric structures in general:

> If I and J are ideals of a Novikov algebra, then $I \cdot J$ and $[I, J]$ are also ideals of the Novikov algebra.

This is a very strong and helpful structural property of Novikov algebras, implying that all terms of the lower central series and of the derived series are invariant for the product. Using this property one can reduce the existence question substantially, since we know much more about the possible structure of the product.

Moreover, in this chapter, we give an example of a 3-step nilpotent Lie algebra not admitting a Novikov structure and prove that all free 3-step nilpotent Lie algebras admit a Novikov structure and that all these structures are complete in the case of at least 3 generators. We also show that free nilpotent Lie algebras of class at least 4 do not admit any Novikov structure and we study the existence of Novikov structures on triangular matrix algebras.

To end this chapter we construct a family of Lie algebras of arbitrary high solvability class all admitting a Novikov structure, which shows that there is no restriction on the solvability class of Novikov algebras.

In chapter 4 we study LR-algebras and LR-structures. We present some structure theory and consider the existence question.

Also for LR-algebras the important structural property concerning ideals, which we proved for Novikov structures, was proved in [20]. Again this leads to a reduction of the existence question.

For nilpotent Lie algebras, Mizuhara showed in [46] that any complex nilpotent Lie algebra admitting a left-symmetric structure, also admits a complete left-symmetric structure. In this chapter we prove an analogue of this result for LR-structures. In fact, we were able to prove an even more general property, not restricting to the class of nilpotent Lie algebras:

> Any Lie algebra admitting an LR-structure also admits a complete LR-structure.

This result simplifies the existence question ”which Lie algebras admit an LR-structure” considerably.

As is known, any Lie algebra admitting an LR-structure is 2-step solvable. We show in this chapter how specific LR-structures on some quotient of a 2-step solvable Lie algebra $\mathfrak{g}$ can be lifted to an LR-structure on $\mathfrak{g}$ itself. Furthermore we show that any 2-step solvable Lie algebra on 2 generators admits an LR-structure.

Also in the case of LR-structures we study existence on free nilpotent 2-step solvable Lie algebras. We find that all these Lie algebras admit an LR-structure and that for nilpotency class at least 3, all these structures are complete. To end this chapter we study the existence of LR-structures on triangular matrix algebras.

In chapter 5 we study post-Lie algebras and post-Lie algebra structures.

We show that post-Lie algebra structures generalize both left-symmetric structures and LR-structures and show their relation with simply transitive NIL-affine actions.

We present some examples and classify the complex 2-dimensional post-Lie algebras.

Furthermore we consider the existence question for post-Lie algebra structures. We prove the following important one-one correspondences:

> There is a one-one correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and embeddings $\varphi\colon \mathfrak{g} \to \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$ with the identity map on the first factor.

> There is a one-one correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and the subalgebras $\mathfrak{h}$ of $\mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$ for which the projection $p_1 : \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) \to \mathfrak{n}$ onto the first factor induces a Lie algebra isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.

In the case of a semisimple Lie algebra $\mathfrak{n}$, we get the following one-one correspondence:

> There is a one-one correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and the subalgebras $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ for which the map $p_1 - p_2 : \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n} : (x, y) \mapsto x - y$ induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.

To finish the chapter we study how the existence of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ imposes certain algebraic conditions on $\mathfrak{g}$ and $\mathfrak{n}$. In particular, the algebraic structures of $\mathfrak{g}$ and $\mathfrak{n}$ depend on each other in a certain way. We prove, among others, that the existence of a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ implies:

- If $\mathfrak{n}$ is 2-step nilpotent, then $\mathfrak{g}$ admits a left-symmetric structure and hence can not be semisimple.
- If $\mathfrak{n}$ is solvable, non-nilpotent, then $\mathfrak{g}$ is not perfect.
- If $\mathfrak{g}$ is nilpotent, then $\mathfrak{n}$ is solvable.

We also give a classification of all Lie algebras $\mathfrak{g}$ for which $(\mathfrak{g}, \mathfrak{sl}_2(\mathbb{C}))$ admits a post-Lie algebra structure.

Finally, in the last chapter, we discuss some interesting open questions.

Before explaining the details on our results in chapter 3, 4 and 5, we start with a first chapter recalling some basic definitions and known results needed throughout this thesis.

Kim Vercammen
Kortrijk, March 2013

Abstract

The study of simply transitive and crystallographic NIL-affine actions on the Lie algebra level leads to different concepts, including Novikov, LR- and post-Lie algebra structures, which are studied in this thesis. In our research we can distinguish three aspects: construction, existence and structure. In the construction aspect, we search for examples by using different techniques as the lifting of such structures, using theoretical considerations and using computer experiments. In the existence aspect, we try to answer the question which Lie algebras admit such structures. In the structure aspect, we study the algebraic structure of Lie algebras admitting a Novikov, LR- or post-Lie algebra structure. The results obtained here, are of great importance in the construction and existence aspect. For Novikov and LR-structures we try to find out what we can say about ideals, quotients, subalgebras,... For post-Lie algebra structures we study how the existence of such a structure imposes certain algebraic conditions on the Lie algebras involved and how the algebraic structures of these Lie algebras depend on each other.

Beknopte samenvatting

De studie van enkelvoudig transitieve en kristallografische NIL-affiene acties leidt op het niveau van Lie algebra's tot verschillende concepten, waaronder Novikov, LR- en post-Lie algebra structuren, welke bestudeerd worden in dit doctoraat. In ons onderzoek vinden we drie aspecten terug: constructie, existentie en structuur. Bij het eerste aspect gaan we op zoek naar voorbeelden van deze structuren, gebruik makend van verschillende technieken, zoals het liften van structuren, gebruik van theoretische beschouwingen en computer experimenten. Bij het tweede aspect proberen we een antwoord te formuleren op de vraag welke Lie algebras een dergelijke structuur toelaten. In het laatste aspect bestuderen we de algebraïsche structuur van Lie algebra's die een Novikov, LR- of post-Lie algebra structuur toelaten. Wat betreft Novikov en LR-structuren onderzoeken we wat we kunnen zeggen over idealen, quotiënten, deelalgebras,... Voor post-Lie algebra structuren bestuderen we hoe het bestaan van dergelijke structuur bepaalde algebraïsche voorwaarden oplegt aan de betrokken Lie algebras en hoe hun algebraïsche structuur hierdoor door elkaar beïnvloed wordt.

Contents

Chapter 1

Basic notions and results about Lie algebras and Lie groups

The purpose of this chapter is to recall the relevant notions and results needed throughout this thesis. These are all well known and will be stated without any proof.

In the first section we discuss some basic concepts about general Lie algebras, their homomorphisms and ideals. We specialize to solvable and nilpotent Lie algebras and shortly recall some facts about cohomology and extensions of Lie algebras.

In the second section we give some definitions concerning Lie groups and link them to Lie algebras. At the end of this section we recall the Mal'cev completion of a finitely generated torsion-free nilpotent group.

Throughout this chapter, k denotes a field of characteristic 0.

1.1 Lie algebras

1.1.1 Lie algebras, homomorphisms and ideals

Definition 1.1.1 (Lie algebra). *A Lie algebra* $\mathfrak{g}$ *over a field* k *is a vector space over* k *together with a bilinear operation* $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}:(x,y)\mapsto[x,y]$*, called the Lie bracket, such that*

$$0=[x,y]+[y,x]\,\textit{(anti-commutativity)},$$

$$0=[[x,y],z]+[[y,z],x]+[[z,x],y]\,\textit{(Jacobi identity)}$$

for all $x,y,z\in\mathfrak{g}$*.*

We consider finite dimensional Lie algebras.

A Lie algebra is called *abelian* if the Lie bracket is always zero.

A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is a *Lie subalgebra* if $[x,y]\in\mathfrak{a}$ for all $x,y\in\mathfrak{a}$.

A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if for all $x\in\mathfrak{g}$ and all $y\in\mathfrak{a}$ it holds that $[x,y]\in\mathfrak{a}$.
For any two ideals $\mathfrak{a}$ and $\mathfrak{b}$, the subspace $[\mathfrak{a},\mathfrak{b}]$, which is the vector space spanned by all Lie brackets $[x,y]$ where $x\in\mathfrak{a}$ and $y\in\mathfrak{b}$, is again an ideal.

A non-abelian Lie algebra $\mathfrak{g}$ having only trivial ideals is called *simple*.

The *center* of a Lie algebra $\mathfrak{g}$ is the ideal $Z(\mathfrak{g})$ defined by

$$Z(\mathfrak{g})=\{x\in\mathfrak{g}\mid[x,y]=0\text{ for all }y\in\mathfrak{g}\}.$$

A Lie algebra $\mathfrak{g}$ is called *perfect* if $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$.

In fact a simple Lie algebra is perfect and has trivial center.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over a field k. A linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$ is a *(Lie algebra) homomorphism* if it preserves the Lie bracket.
If in addition φ is bijective, it is called a *(Lie algebra) isomorphism*. If $\mathfrak{g}=\mathfrak{h}$, then an isomorphism is called an *automorphism*. We denote the group of automorphisms of $\mathfrak{g}$ by $\mathrm{Aut}(\mathfrak{g})$.

Note that the kernel $\mathrm{Ker}\,\varphi$ of a homomorphism $\varphi:\mathfrak{g}\to\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and that the image $\mathrm{im}(\varphi)$ is a subalgebra of $\mathfrak{h}$.

Let V be a finite dimensional vector space over a field k. The vector space $\mathrm{End}(V)$ can be seen as a Lie algebra, denoted by $\mathfrak{gl}(V)$, with Lie bracket the commutator of elements.

A *(Lie algebra) representation* of a Lie algebra $\mathfrak{g}$ over k is a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ where V is a vector space over k.

A *derivation* of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$D[x, y] = [Dx, y] + [x, Dy]$$

for all $x, y \in \mathfrak{g}$. The set of derivations of $\mathfrak{g}$, denoted by $\mathrm{Der}(\mathfrak{g})$, forms a subalgebra of $\mathfrak{gl}(\mathfrak{g})$ and hence is a Lie algebra itself with Lie bracket $[D, D'] = D \circ D' - D' \circ D$. E.g. $\mathrm{Der}(\mathbb{R}^n) = \mathfrak{gl}_n(\mathbb{R}) := \mathfrak{gl}(\mathbb{R}^n)$.

For each element $x \in \mathfrak{g}$ we have a map

$$\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g} : y \mapsto [x, y].$$

This is in fact a derivation of $\mathfrak{g}$. We call derivations of this form inner and denote the subspace of inner derivations by $\mathrm{ad}(\mathfrak{g})$. The map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) : x \mapsto \mathrm{ad}(x)$$

is called the *adjoint representation* of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called *complete* if all derivations are inner and if the center is trivial , so $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$ and $Z(\mathfrak{g}) = 0$.

Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras over a field k and $\varphi : \mathfrak{g}_2 \to \mathrm{Der}(\mathfrak{g}_1)$ is a Lie algebra homomorphism. The *semidirect product* $\mathfrak{g}_1 \rtimes \mathfrak{g}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is defined to be the vector space $\mathfrak{g}_1 \times \mathfrak{g}_2$ equipped with the Lie bracket given by

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1] + \varphi(x_2)(y_1) - \varphi(y_2)(x_1), [x_2, y_2]),$$

for all $x_1, y_1 \in \mathfrak{g}_1$ and all $x_2, y_2 \in \mathfrak{g}_2$. Now $\mathfrak{g}_1$ can be seen as an ideal of this semidirect product and $\mathfrak{g}_2$ can be seen as a subalgebra.

1.1.2 Solvable and nilpotent Lie algebras

Solvable Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. We define the derived series of $\mathfrak{g}$ inductively by the ideals

$$\mathfrak{g}^{(1)} = \mathfrak{g} \text{ and } \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}].$$

This is a descending series

$$\mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots .$$

The element $\mathfrak{g}^{(2)}$ is called the *derived algebra* of $\mathfrak{g}$.

Definition 1.1.2 (Solvable Lie algebra)**.** *We say that the Lie algebra* $\mathfrak{g}$ *is solvable if* $\mathfrak{g}^{(n+1)} = 0$ *for some* $n \geq 0$*. Let* n *be the minimal integer such that* $\mathfrak{g}^{(n+1)} = 0$*, then we say that* $\mathfrak{g}$ *is* n*-step solvable.*

By $\mathrm{Rad}(\mathfrak{g})$, the *radical* of $\mathfrak{g}$, we denote the unique maximal solvable ideal of the Lie algebra $\mathfrak{g}$.

A *semisimple* Lie algebra is a Lie algebra $\mathfrak{g}$ for which $\mathfrak{g} \neq \emptyset$ and such that $\mathrm{Rad}\,\mathfrak{g} = 0$.
For a semisimple Lie algebra it is known that $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) = \mathfrak{g}$.

Nilpotent Lie algebras

For a Lie algebra $\mathfrak{g}$ we define the *lower (or descending) central series* inductively by the ideals

$$\gamma_1(\mathfrak{g}) = \mathfrak{g} \text{ and } \gamma_{i+1}(\mathfrak{g}) = [\mathfrak{g}, \gamma_i(\mathfrak{g})].$$

This gives a descending series

$$\gamma_1(\mathfrak{g}) \supseteq \gamma_2(\mathfrak{g}) \supseteq \dots.$$

For all integers p and q we have $[\gamma_p(\mathfrak{g}), \gamma_q(\mathfrak{g})] \subseteq \gamma_{p+q}(\mathfrak{g})$.

Definition 1.1.3 (Nilpotent Lie algebra)**.** *We say that the Lie algebra* $\mathfrak{g}$ *is nilpotent if* $\gamma_{n+1}(\mathfrak{g}) = 0$ *for some* $n \geq 0$*. Let* n *be the minimal integer such that* $\gamma_{n+1}(\mathfrak{g}) = 0$*, then we say that* $\mathfrak{g}$ *is* n*-step nilpotent.*

Note that the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of a solvable Lie algebra $\mathfrak{g}$ is nilpotent.

By $\mathrm{nil}(\mathfrak{g})$, the *nilradical* of $\mathfrak{g}$, we denote the unique maximal nilpotent ideal of the Lie algebra $\mathfrak{g}$.

If a Lie algebra $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}(x)$ is nilpotent for every $x \in \mathfrak{g}$. Also the inverse is true:

Theorem 1.1.4 (Engel)**.** *If in a Lie algebra* $\mathfrak{g}$ *all operators* $\mathrm{ad}(x)$ *are nilpotent, then* $\mathfrak{g}$ *is nilpotent.*

We also have the following proposition:

Proposition 1.1.5. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra, then also every image of* $\mathfrak{g}$ *under a Lie algebra homomorphism is nilpotent.*

We can consider some special examples of nilpotent Lie algebras.

A lie algebra of dimension n which is $(n-1)-$step nilpotent is called a *filiform* Lie algebra. The *standard filiform nilpotent Lie algebra* of dimension n has a basis $x_1, \dots, x_n$ and Lie brackets $[x_1, x_i] = x_{i+1}$ for $i = 2, \dots, n-1$.

The *free p-step nilpotent Lie algebra* on n generators $x_1, x_2, \dots, x_n$ is a p-step nilpotent Lie algebra $\mathfrak{f}$, together with an embedding $i : \{x_1, x_2, \dots, x_n\} \hookrightarrow \mathfrak{f}$, which is characterized by the following universal property: for any p-step nilpotent Lie algebra $\mathfrak{h}$ and any map $f : \{x_1, x_2, \dots, x_n\} \to \mathfrak{h}$, there exists a unique Lie algebra morphism $\varphi_f : \mathfrak{f} \to \mathfrak{h}$ such that $\varphi_f \circ i = f$.

Such a free p-step nilpotent Lie algebra $\mathfrak{f}$ is uniquely determined, up to isomorphism. Free Lie algebras are those Lie algebras where the only relations between the basis vectors are those induced by the anti-commutativity of the Lie bracket and the Jacobi identity.

We have the following:

Theorem 1.1.6. *For any p-step nilpotent Lie algebra $\mathfrak{g}$ on n generators, with n minimal, we have that*

$$g \cong \frac{\mathfrak{f}}{I},$$

where I is an ideal of $\mathfrak{f}$, and $I \subseteq [\mathfrak{f}, \mathfrak{f}]$.

1.1.3 Cohomology of Lie algebras

Definition 1.1.7 (Left-module over a Lie algebra)**.** *A left-module over a Lie algebra $\mathfrak{g}$ over a field k is a vector space V over k, together with a map $\mathfrak{g} \times V \to V : (x, u) \mapsto x \cdot u$, which satisfies the following conditions:*

$$x \cdot (u + v) = x \cdot u + x \cdot v,$$

$$(x + y) \cdot u = x \cdot u + y \cdot u,$$

$$\alpha(x \cdot u) = (\alpha x) \cdot u = x \cdot (\alpha u),$$

$$[x, y] \cdot u = x \cdot (y \cdot u) - y \cdot (x \cdot u),$$

for all $\alpha \in k$, $x, y \in \mathfrak{g}$ and $u, v \in V$.

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module.

The n-dimensional *cochains* are the n-linear skew-symmetric functions on $\mathfrak{g}^n$ with image in V. These cochains make up a vector space $C^n(\mathfrak{g}, V)$.

The *coboundary operator* is a linear map $d_n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ defined by

$$d_n(f)(x_1, \dots, x_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} x_k \cdot f(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$$

$$+ \sum_{r<s} (-1)^{r+s} f([x_r, x_s], x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_{n+1}).$$

We have that $d_n d_{n-1} = 0$.

We define $Z^n(\mathfrak{g}, V)$ as the vector space of *cocycles*, this is, as the kernel of d_n and $B^n(\mathfrak{g}, V)$ as the vector space of *coboundaries*, this is, as the image of d_{n-1}. We can now define the *n-th cohomology* by

$$H^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V)/B^n(\mathfrak{g}, V).$$

1.1.4 Extensions of Lie algebras

Definition 1.1.8. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras over k. A Lie algebra $\mathfrak{g}$ over k is called an extension of $\mathfrak{b}$ by $\mathfrak{a}$ if there exists a short exact sequence of Lie algebras of the form*

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0.$$

From now on, suppose that $\mathfrak{a}$ is abelian. Hence the short exact sequence induces a Lie algebra homomorphism $\varphi : \mathfrak{b} \to \mathfrak{gl}(\mathfrak{a})$ such that

$$\varphi(x)(a) = i^{-1}([\pi^{-1}(x), i(a)]) \text{ for all } x \in \mathfrak{b} \text{ and } a \in \mathfrak{a}.$$

This makes $\mathfrak{a}$ into a $\mathfrak{b}$-module.

An extension $\mathfrak{g}$ of $\mathfrak{b}$ by $\mathfrak{a}$ will be determined by a 2-cohomology class in $H^2(\mathfrak{b}, \mathfrak{a})$ (see [38]).

When this 2-cohomology class is trivial, there exists a splitting of the short exact sequence above. In this case $\mathfrak{g}$ is the semidirect product of $\mathfrak{a}$ and $\mathfrak{b}$, denoted by $\mathfrak{a} \rtimes \mathfrak{b}$. This is the vector space $\mathfrak{a} \times \mathfrak{b}$ with Lie bracket

$$[(a, x), (b, y)] = (\varphi(x)(b) - \varphi(y)(a), [x, y]).$$

We can see $\mathfrak{b}$ as a subalgebra of $\mathfrak{g}$.

1.2 Lie groups

Definition 1.2.1 (Lie group)**.** *A Lie group G is a manifold which is equipped with a group structure such that the map*

$$G \times G \to G : (g, h) \mapsto gh^{-1}$$

is C^∞.

A map $\varphi : G \to H$ is called a *Lie group homomorphism* if φ is both C^∞ and a group homomorphism of the abstract groups. In fact, every continuous group homomorphism between two Lie groups G and H is C^∞, thus a Lie group homomorphism.

We call $\varphi : G \to H$ an *isomorphism* if in addition φ is a diffeomorphism (a C^∞ map with C^∞ inverse). If $G = H$ this isomorphism is called an *automorphism*. We denote the group of such (continuous) automorphisms of G by $\mathrm{Aut}(G)$.

1.2.1 Lie groups and Lie algebras

We consider real Lie groups.

To any real Lie group G we can associate a real Lie algebra $\mathfrak{g}$. There is a one-one correspondence if we restrict to connected, simply connected Lie groups.

There also is a one-one correspondence between connected Lie subgroups of a Lie group and subalgebras of its Lie algebra. Moreover, we have the following correspondence:

Theorem 1.2.2. *Let $A \subseteq G$ be a connected Lie subgroup of a connected Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G. Then A is a normal subgroup of G if and only if the Lie algebra $\mathfrak{a}$ of A is an ideal in $\mathfrak{g}$.*

A *solvable Lie group* is one whose Lie algebra $\mathfrak{g}$ is solvable.

For connected Lie groups this is equivalent to saying that G itself is a solvable group. If $g, h \in G$, then we define the commutator of g and h as $[g, h] = g^{-1}h^{-1}gh$. Consider the derived group $G' = [G, G]$ of a group G. Solvable groups are defined via the derived series of G, which is defined recursively by $G^{(1)} = G$ and for all $i \geq 1$ we define $G^{(i+1)} = [G^{(i)}, G^{(i)}] = \big(G^{(i)}\big)'$. We can consider the derived series

$$G = G^{(1)} \supseteq G^{(2)} \supseteq G^{(3)} \supseteq \dots .$$

Definition 1.2.3. *A group G is called solvable if there exists an integer c for which $G^{(c+1)} = 1$. Let c be the minimal integer such that $G^{(c+1)} = 1$, then we say that G is c-step solvable.*

The subgroups $G^{(i)}$ of the derived series are Lie subgroups of G and the Lie algebra of each $G^{(i)}$ is $\mathfrak{g}^{(i)}$.

A Lie group is said to be *nilpotent* if its associated Lie algebra is nilpotent.

For connected Lie groups this is equivalent to saying that G itself is a nilpotent group. Nilpotent groups are defined via the lower central series. The lower central series of G is defined recursively by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ for all $i \geq 1$. The lower central series is a decreasing series

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \dots.$$

Definition 1.2.4. *A group G is called nilpotent if there exists an integer c for which $\gamma_{c+1}(G) = 1$. Let c be the minimal integer such that $\gamma_{c+1}(G) = 1$, then we say that G is c-step nilpotent.*

The subgroups $\gamma_i(G)$ of the lower central series are Lie subgroups of G. The Lie algebra of the subgroup $\gamma_i(G)$ is $\gamma_i(\mathfrak{g})$.

1.2.2 Nilpotent Lie groups and Lie algebras

From now on let G be a connected, simply connected nilpotent Lie group and let $\mathfrak{g}$ be the corresponding Lie algebra. We know that $\mathfrak{g}$ and G have the same dimension.

There exists a so-called *exponential map* between $\mathfrak{g}$ and G

$$\exp : \mathfrak{g} \to G,$$

which is a diffeomorphism. We denote its inverse by log.

Let H be another connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{h}$, then we have the following properties.

For any Lie group homomorphism $\varphi : G \to H$, there exists a unique Lie algebra homomorphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$ (the *differential* of φ), making the following diagram commutative

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \exp \uparrow\downarrow \log & & \exp \uparrow\downarrow \log \\ \mathfrak{g} & \xrightarrow[\mathrm{d}\varphi]{} & \mathfrak{h} \end{array} \tag{1.1}$$

Conversely, for any Lie algebra homomorphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group homomorphism $\varphi : G \to H$, making the above diagram commutative. As exp is a diffeomorphism, we find that $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$.

The images of the adjoint representation of $\mathfrak{g}$ are derivations. The corresponding *adjoint representation* on the Lie group G is the group homomorphism

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) : g \mapsto da_g,$$

where

$$a_g : G \to G : h \mapsto ghg^{-1}.$$

1.2.3 The Mal'cev completion of a finitely generated torsion-free nilpotent group

Let us recall the notion of Mal'cev completion of a finitely generated torsion-free nilpotent group.

Definition 1.2.5 (Uniform Lattice)**.** *Let G be a connected, simply connected, nilpotent Lie group. A uniform lattice Γ of G is a discrete and uniform subgroup, i.e. a discrete subgroup of G with compact quotient space $\Gamma\backslash G$.*

Mal'cev described all possible uniform lattices in a connected, simply connected, nilpotent Lie group (see [42]):

Theorem 1.2.6 (Mal'cev completion)**.** *Any uniform lattice Γ of a connected, simply connected, nilpotent Lie group G is a finitely generated torsion-free nilpotent group.*
Conversely, for any torsion-free finitely generated nilpotent group Γ there exists (up to isomorphism) exactly one connected, simply connected, nilpotent Lie group G containing Γ as a uniform lattice. We refer to this G as the Mal'cev completion of Γ.

Gorbacevič proved the following theorem in [31]:

Theorem 1.2.7 (V.V. Gorbacevič)**.** *Let* Γ *be a uniform lattice of a connected, simply connected, nilpotent Lie group* G *and let* H *be an arbitrary connected, simply connected, nilpotent Lie group. Then any homomorphism* $\varphi : \Gamma \to H$ *extends uniquely to a homomorphism of* G *into* H*.*

Chapter 2

Background

The Novikov, LR- and post-Lie algebra structures mentioned in the title of this thesis find their motivation mainly in differential geometry. Although we will be working mainly in the algebraic setting, this thesis also contains a contribution to this geometric aspect (in the chapter on post-Lie algebras). Therefore, we find it useful to present an outline of this geometrical context here.

In the seventies J. Milnor posed a famous question in [45], whether or not any connected, simply connected, solvable Lie group G would admit a representation $\rho : G \to \mathrm{Aff}(\mathbb{R}^n)$, letting G act simply transitively on $\mathbb{R}^n$. This question received a lot of attention, including several articles trying to prove a positive answer. However, finally it turned out that the answer was negative (see [11], [24]).

The main ingredient in this study was the notion of left-symmetric algebras. It is known that a given connected, simply connected Lie group G admits a representation $\rho : G \to \mathrm{Aff}(\mathbb{R}^n)$ letting G act simply transitively on $\mathbb{R}^n$ if and only if the Lie algebra $\mathfrak{g}$ of G admits a complete left-symmetric structure (see [29], [28], [37]). In particular the question of which Lie groups G admit a simply transitive and affine action reduces to the question of which Lie algebras $\mathfrak{g}$ (over $\mathbb{R}$) admit a complete left-symmetric structure.

By a result of L. Auslander (see [6]), such a Lie group G, and hence also the Lie algebra $\mathfrak{g}$, has to be solvable. There was the hope that conversely every connected, simply connected solvable Lie group would admit a simply transitive and affine action. As mentioned above, the answer turned out to be negative.

Because of the counter examples mentioned, one has broadened the geometric context and studied affine actions on connected, simply connected nilpotent Lie groups N.

In contrast to the fact that there are connected, simply connected solvable Lie groups G not allowing a simply transitive affine action, all of them do admit a simply transitive NIL-affine action on a nilpotent Lie group (see [25]).

This result was, among other things, a motivation to study pairs of Lie groups (G, N) where G acts simply transitively on N via NIL-affine transformations.

Just as in the usual affine setting, the translation of the problem to the Lie algebra level plays a crucial role.

In section 2.1 we recall the basic definitions concerning the actions and groups we are interested in.

In section 2.2 we present the questions of Milnor and Auslander and put them into context. Also their geometric significance is discussed. We give a sketch of some translations of Milnor's question in the nilpotent case. We make a link to Novikov structures and state a Lie group version of the questions of Auslander and Milnor.

In section 2.3 we sketch the generalization of this affine case, namely the NIL-affine case. We also point out how LR-structures arise in this study.

In section 2.4 we give an overview of what we will do in this thesis. These new results will be explained in detail in the following chapters.

In this chapter we will always assume we are working over the field $\mathbb{R}$ unless stated otherwise.

2.1 Preliminaries

We start this chapter by recalling some basic notions concerning the groups and actions we are interested in.

Definition 2.1.1. *Let Γ be a group acting on a locally compact topological space X. The action is said to be*

1. free *if no element $\gamma \neq 1$ has a fixed point.*
2. properly discontinuous *if for any compact subset $K \subseteq X$, the set $\{\gamma \in \Gamma \mid {}^{\gamma}K \cap K \neq \emptyset\}$ is finite.*
3. crystallographic *if it is properly discontinuous and the orbit space $\Gamma \backslash X$ is compact.*

4. *simply transitive if for any two elements $x, y \in X$ there exists exactly one $\gamma \in \Gamma$ such that ${}^{\gamma}x = y$.*

If X is simply connected, then a properly discontinuous action of a group Γ on X is free if and only if Γ is a torsion-free group.

We are interested in actions of polycyclic-by-finite groups.

Definition 2.1.2 (Polycyclic-by-finite group). *A group Γ is called polycyclic-by-finite if it has a normal subgroup Γ_0 of finite index which is polycyclic, that is, admits a finite descending series*

$$\Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_k = \{1\}$$

such that each Γ_{i+1} is a normal subgroup of Γ_i and any quotient Γ_i/Γ_{i+1} is cyclic.

Examples of polycyclic-by-finite groups are the finitely generated nilpotent groups.

2.2 The affine case

In the affine case we consider group actions via affine motions on $\mathbb{R}^n$.

The *affine group* is the semidirect product

$$\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$$

and consists of the invertible affine transformations of $\mathbb{R}^n$. Any element of $\mathrm{Aff}(\mathbb{R}^n)$ can be uniquely written as a pair (a, A) consisting of a translational part $a \in \mathbb{R}^n$ and a linear part $A \in \mathrm{GL}_n(\mathbb{R})$.

The affine group $\mathrm{Aff}(\mathbb{R}^n)$ can be thought of as being embedded, as a subgroup, in $\mathrm{GL}_{n+1}(\mathbb{R})$:

$$\mathrm{Aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{GL}_n(\mathbb{R}), a \in \mathbb{R}^n \right\}.$$

The affine group acts on $\mathbb{R}^n$ by affine transformations via the action given by

$$\forall a, x \in \mathbb{R}^n, \forall A \in \mathrm{GL}_n(\mathbb{R}) : \; {}^{(a,A)}x = a + Ax.$$

By an *affine action* we mean a representation $\rho : \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$.

2.2.1 Geometrical background

In this section we sketch the geometrical background of the so called affine structures on polycyclic-by-finite groups we are interested in. These actions have their origin in differential geometry. For more details see e.g. [39], [45], [52], [53].

Let us give some definitions first.

In this thesis, whenever we use the term manifold, we will mean a smooth manifold.

Definition 2.2.1 (Affinely flat manifold). *An n-dimensional manifold M is said to be an affinely flat manifold if it has an atlas of local coordinate charts*

$$f_i : U_i \to \mathbb{R}^n$$

satisfying:

1. *The U_i are open subsets of M covering M.*
2. *The maps f_i are homeomorphisms of U_i onto open subsets $f_i(U_i)$ of $\mathbb{R}^n$.*
3. *Any two local coordinate systems of this atlas are affine, i.e. whenever $U_i \cap U_j \neq \emptyset$ each transition map $f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ is locally a restriction of an affine map of $\mathbb{R}^n$.*

Since the changes of coordinates are given by affine maps of $\mathbb{R}^n$, which are diffeomorphisms preserving the standard affine connection of $\mathbb{R}^n$, the manifold M inherits the affine connection of $\mathbb{R}^n$.

Hence it makes sense to talk about geodesics in an affinely flat manifold M. A *geodesic* in M is an injective map $\gamma :]a, b[\subseteq \mathbb{R} \to M : t \mapsto \gamma(t)$, which, when composed with any local coordinate system of the given atlas of M, gives a map from $]a, b[$ to $\mathbb{R}^n$ which is locally the restriction of affine maps.

A *complete geodesic* in M is a geodesic defined on all of $\mathbb{R}$.

Definition 2.2.2 (Complete manifold). *An affinely flat manifold M is called complete if every geodesic can be extended to a complete geodesic $\gamma : \mathbb{R} \to M$, or equivalently if the universal covering manifold is affinely diffeomorphic to $\mathbb{R}^n$.*

In order to actually construct a complete affinely flat manifold with fundamental group a given group Γ, it is usually easiest to start with an action of Γ on $\mathbb{R}^n$

by affine transformations. If this action is free and properly discontinuous then the quotient $M = \Gamma\backslash\mathbb{R}^n$ is the required manifold.

In fact, this construction yields all connected, complete affinely flat manifolds, as was shown by Auslander and Markus in 1955 (see [7] and [53, page 45]):

Theorem 2.2.3 (Auslander and Markus)**.** *The connected, complete affinely flat manifolds are exactly the quotients $\Gamma\backslash\mathbb{R}^n$, where Γ acts properly discontinuous, freely and via affine motions on $\mathbb{R}^n$.*

In 1977 Milnor tried to decide which groups could occur as fundamental group of complete affinely flat manifolds. He proved (see [45]) that any torsion-free polycyclic-by-finite group Γ admits a properly discontinuous and affine action on some $\mathbb{R}^n$, and hence can be realized as the fundamental group of a connected, complete affinely flat manifold M.

On the other hand, it was already known that such a group Γ could be realized as the fundamental group of a connected, compact manifold.

Milnor asked whether these two results could be combined, so he formulated the following question, widely known as Milnor's question:

Question 2.2.4 (Milnor's question)**.** *Let Γ be any torsion-free polycyclic-by-finite group. Is it true that Γ can be realized as the fundamental group of a connected, compact, complete affinely flat manifold M?*

Let us explain why we are so interested in the class of polycyclic-by-finite groups.

In 1964, Auslander proved that the fundamental group of any connected, complete affinely flat manifold is polycyclic-by-finite (see [5]). However, the proof given by Auslander turned out to be wrong.
Moreover, Margulis (see [43],[44]) constructed examples of connected, complete affinely flat manifolds having a free (non-abelian) fundamental group. This shows that even the statement of Auslander was incorrect.
However, the examples which Margulis constructed are non-compact, so the statement of Auslander is still open in the compact case. This statement is widely known as Auslander's conjecture:

Conjecture 2.2.5 (Auslander's conjecture)**.** *The fundamental group of a connected, compact, complete affinely flat manifold is polycyclic-by-finite.*

The answer to Auslander's conjecture is not known in general. So far only positive answers have been obtained. Most people believe that this conjecture holds and it has been proved in dimensions $1 \leq n \leq 6$. Therefore, it is natural to consider the class of polycyclic-by-finite groups in the study of compact, complete affinely flat manifolds.

Virtual solvability

The question of Milnor and the conjecture of Auslander are sometimes stated in terms of virtual solvability instead of polycyclic-by-finiteness. But it turns out that in the context we work in, these concepts are actually the same.

For a certain group theoretic property P, a group is called *P-by-finite* or *virtually P*, if it admits a normal finite index subgroup with property P.

Polycyclic groups are examples of solvable groups, but not all solvable groups are polycyclic.
However, it follows from [47, Proposition 3.8], that for discrete linear groups, these are the same:

Theorem 2.2.6. *Every discrete solvable subgroup of* $\mathrm{GL}_m(\mathbb{R})$ $(m \geq 1)$ *is polycyclic.*

This applies to $\mathrm{Aff}(\mathbb{R}^n)$ and its subgroups since we can think of $\mathrm{Aff}(\mathbb{R}^n)$ as being embedded, as a subgroup, in $\mathrm{GL}_{n+1}(\mathbb{R})$.

We also have the Tits alternative (see [50]):

Theorem 2.2.7 (Tits alternative)**.** *Let* Γ *be a subgroup of* $\mathrm{GL}_n(\mathbb{C})$*. Then* Γ *is either virtually solvable or contains a free non-abelian subgroup.*

This theorem puts the questions of Milnor and Auslander in perspective, since they really ask to which of the two types of groups described by the Tits alternative our groups belong.

2.2.2 Affine crystallographic actions

Milnor's question 2.2.4 can be formulated as: do all torsion-free polycyclic-by-finite groups Γ admit a properly discontinuous and affine action on some space $\mathbb{R}^n$, which has compact quotient.

Such an action is called an affine structure on Γ or an affine crystallographic action:

Definition 2.2.8 (Affine structure, Affine crystallographic action, Affine crystallographic group)**.**
An affine crystallographic action of a group Γ *on* $\mathbb{R}^n$ *is a representation* $\rho : \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$*, letting* Γ *act crystallographically on* $\mathbb{R}^n$*.*
We call such an action an affine structure on Γ *and* $\rho(\Gamma)$ *is called an affine crystallographic group.*

In this algebraic setting, Milnor's question can be translated into:

Question 2.2.9 (Milnor - algebraic version)**.** *Is it true that any torsion-free polycyclic-by-finite group* Γ *can be realized as an affine crystallographic group?*

We can also translate Auslander's conjecture into this algebraic setting:

Conjecture 2.2.10 (Auslander - algebraic version)**.** *Any affine crystallographic group is polycyclic-by-finite.*

A positive answer to both questions would imply a complete understanding of the affine crystallographic groups.

For some time, most people were convinced that the answer to Milnor's question was positive. The question received a lot of attention, including several articles trying to prove a positive answer. However, finally it turned out that the answer was negative: Y. Benoist announced a counter-example to Milnor's question in 1992 (see [11]):

Theorem 2.2.11 (Benoist)**.** *There exists a finitely generated torsion-free nilpotent group, which cannot be realized as an affine crystallographic group.*

To construct this nilpotent counterexample, Benoist used a translation into the Lie algebra level of Milnor's question, which will be explained in the following section. More precisely, Benoist described an 11-dimensional, 10-step nilpotent Lie algebra not admitting a complete affine structure.

This counterexample was checked by an alternative method and generalized to a family of counterexamples (all of them still 11-dimensional, 10-step nilpotent) by Dietrich Burde and Fritz Grunewald, (see [24]). They also used the translation into the Lie algebra level. In 1996 Burde could reduce both the dimension (to 10) as the nilpotency class (to 9) (see [12]).

In some sense, the work of Benoist, Burde and Grunewald settles Milnor's question completely. The setting of affine crystallographic groups is nevertheless very interesting (e.g. geometrically), and has been studied a lot. One can e.g. search now for exactly which groups can be realized as an affine crystallographic group.

Another approach suggested by the negative answer to Milnor's question is to consider more general classes of crystallographic groups. It is, to seek for some sort of generalization of affine crystallographic groups, to fill this lack of affine structures. One such a setting is the setting of NIL-affine actions.

Concerning the Auslander conjecture, a number of positive results have been obtained in particular cases (by restricting linear parts), a general positive answer is known up to dimension 6. The question in general is still open, and it seems that specialists expect this question to have a positive answer. For more details, see e.g. [1], [2], [3], [4].

2.2.3 Some translations of Milnor's question in the nilpotent case

A first approach to Milnor's question was to restrict it to the class of finitely generated torsion-free nilpotent groups:

Question 2.2.12 (Milnor's question - nilpotent case)**.** *Is it true that any finitely generated torsion-free nilpotent group* Γ *can be realized as an affine crystallographic group?*

In this case there are many equivalent formulations of Milnor's question in the theory of Lie groups and Lie algebras. Remember that it was using one of these translations that Benoist was able to find his counterexample.

Let us list some of these translations.

From affine crystallographic actions to simply transitive affine actions

Let Γ be a torsion-free and finitely generated nilpotent group and let G be its Mal'cev completion, as defined in section 1.2.3.

The existence of an affine crystallographic action of Γ appears to be equivalent to the existence of a simply transitive affine action of G.

Assume that $\rho : \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$ defines an affine crystallographic action of Γ on $\mathbb{R}^n$.

By Fried, Goldman and Hirsch ([28], Theorem 4.3), we have that $\rho(\Gamma)$ is a unipotent group (i.e. every element can be seen as a unipotent automorphism in $\mathrm{GL}_{n+1}(\mathbb{R})$). Now there exists a basis in $\mathbb{R}^n$ such that $\rho(\Gamma)$ consists of upper unitriangular matrices, so elements of $Tr_1(n+1,\mathbb{R})$.

By theorem 1.2.7 the map ρ uniquely extends to a representation $\tilde{\rho} : G \to Tr_1(n+1,\mathbb{R}) \subseteq \mathrm{Aff}(\mathbb{R}^n)$. This action of G on $\mathbb{R}^n$ is simply transitive as proved by Fried, Goldman and Hirsch (see [28], Theorem 7.1).

The converse is also true: let G be a connected, simply connected, nilpotent Lie group. Given a simply transitive affine action $\tilde{\rho}$ of G on $\mathbb{R}^n$, then its restriction ρ to a uniform lattice Γ of G determines an affine crystallographic action of Γ.

We have the following conclusion:

Conclusion 2.2.13. *There is a one-one correspondence between the affine crystallographic actions of a finitely generated, torsion-free nilpotent group* Γ *and the simply transitive affine actions of its Mal'cev completion* G.

Hence in the nilpotent case we can state Milnor's question as follows:

Question 2.2.14 (Milnor's question - nilpotent case)**.** *Let* G *be a connected, simply connected, nilpotent Lie group. Does* G *admit a simply transitive affine action* $\rho : G \to \mathrm{Aff}(\mathbb{R}^n)$*?*

From simply transitive affine actions to complete affine structures

In the case of nilpotent Lie groups, Milnor's question can also be formulated at the Lie algebra level.

Let G be a connected, simply connected, nilpotent Lie group and let $\mathfrak{g}$ be the corresponding nilpotent Lie algebra.

The existence of a simply transitive affine action of G appears to be equivalent to the existence of a certain Lie algebra representation of $\mathfrak{g}$.

A Lie group homomorphism $\tilde{\rho} : G \to \mathrm{Aff}(\mathbb{R}^n)$ can be described by specifying both the linear part $\tilde{\rho}_l$ and the translational part $\tilde{\rho}_t$. Such a homomorphism

$$\tilde{\rho} : G \to \mathrm{Aff}(\mathbb{R}^n) : \; g \mapsto \begin{pmatrix} \tilde{\rho}_l(g) & \tilde{\rho}_t(g) \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

determines a simply transitive affine action of G on $\mathbb{R}^n$ if and only if $\tilde{\rho}_t : G \to \mathbb{R}^n$ is a bijective map.

At the Lie algebra level, we have to consider the affine Lie algebra $\mathfrak{aff}(\mathbb{R}^n)$. This is the Lie algebra corresponding to the affine group $\mathrm{Aff}(\mathbb{R}^n)$ and $\mathfrak{aff}(\mathbb{R}^n)$ can be seen as the Lie subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{R})$, given by:

$$\mathfrak{aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}_n(\mathbb{R}), a \in \mathbb{R}^n \right\}.$$

We take the differential of $\tilde{\rho}$. This is the unique Lie algebra homomorphism $\mathrm{d}\tilde{\rho} : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ such that the following diagram commutes (see (1.1)):

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\rho}} & \mathrm{Aff}(\mathbb{R}^n) \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{g} & \xrightarrow[\mathrm{d}\tilde{\rho}]{} & \mathfrak{aff}(\mathbb{R}^n) \end{array}$$

It can be seen as the Lie algebra homomorphism

$$\mathrm{d}\tilde{\rho} : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n) : \; x \mapsto \begin{pmatrix} (\mathrm{d}\tilde{\rho})_l(x) & (\mathrm{d}\tilde{\rho})_t(x) \\ 0 & 0 \end{pmatrix},$$

with translational part $(\mathrm{d}\tilde{\rho})_t$ and linear part $(\mathrm{d}\tilde{\rho})_l$.

Suppose we have a simply transitive affine action $\tilde{\rho} : G \to \mathrm{Aff}(\mathbb{R}^n)$ as in (2.1). Then the Lie algebra $(\mathrm{d}\tilde{\rho})_l(\mathfrak{g})$ consists of nilpotent matrices and $(\mathrm{d}\tilde{\rho})_t : \mathfrak{g} \to \mathbb{R}^n$ is a linear isomorphism. (More details can be found in [49] and [28].)

Such a Lie algebra homomorphism $\mathrm{d}\tilde{\rho}$ is called a complete affine structure on $\mathfrak{g}$:

Definition 2.2.15 (Complete affine structure on a Lie algebra $\mathfrak{g}$)**.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra. An affine structure on $\mathfrak{g}$ is a Lie algebra homomorphism*

$$\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n) : x \mapsto \begin{pmatrix} \varphi_l(x) & \varphi_t(x) \\ 0 & 0 \end{pmatrix},$$

where $\varphi_t : \mathfrak{g} \to \mathbb{R}^n$ is a bijective map.
This affine structure is called complete if all the matrices $\varphi_l(x)$ are nilpotent.

Also the converse is true, any complete affine structure $\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ on a nilpotent Lie algebra $\mathfrak{g}$ is the differential $\mathrm{d}\tilde{\rho}$ of a simply transitive affine action $\tilde{\rho} : G \to \mathrm{Aff}(\mathbb{R}^n)$ of the corresponding Lie group G.

We have the following conclusion:

Conclusion 2.2.16. *There is a one-one correspondence between simply transitive affine actions $\tilde{\rho}$ of a connected, simply connected nilpotent Lie group G and complete affine structures $\mathrm{d}\tilde{\rho}$ on its Lie algebra $\mathfrak{g}$.*

From complete affine structures to complete left-symmetric structures

Suppose $\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ is a complete affine structure on a nilpotent Lie algebra $\mathfrak{g}$. So the linear part $\varphi_l(\mathfrak{g})$ consists of nilpotent matrices and the translational part $\varphi_t : \mathfrak{g} \to \mathbb{R}^n$ is bijective.

We can define a bilinear product $\bullet : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, as follows

$$\forall x, y \in \mathfrak{g} : x \bullet y := (\varphi_t^{-1} \circ \varphi_l(x) \circ \varphi_t)(y).$$

This product satisfies the following two criteria for all $x, y, z \in \mathfrak{g}$:

$$[x, y] \bullet z = x \bullet (y \bullet z) - y \bullet (x \bullet z),$$

$$[x, y] = x \bullet y - y \bullet x.$$

Moreover, since $\varphi_l(x)$ is nilpotent for all $x \in \mathfrak{g}$, all left multiplications are nilpotent.

Such a product is called a complete left-symmetric structure on $\mathfrak{g}$:

Definition 2.2.17 (Complete left-symmetric structure)**.** *A left-symmetric structure on a nilpotent Lie algebra* $\mathfrak{g}$ *is a bilinear product* $\bullet : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying*

$$[x, y] \bullet z = x \bullet (y \bullet z) - y \bullet (x \bullet z),$$

$$[x, y] = x \bullet y - y \bullet x.$$

This structure is called complete if all left multiplications are nilpotent.

Conversely, let $\bullet : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a complete left-symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$. We can identify $\mathfrak{g}$ with some $\mathbb{R}^n$ by choosing a basis in $\mathfrak{g}$. If we use $\varphi_l(x) \in \mathfrak{gl}_n(\mathbb{R})$ to denote the matrix representation of the left multiplication map $L(x)$ w.r.t. the chosen basis, and use $\varphi_t(x)$ to denote the coordinate of the element x, then we find that the map

$$\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n) : x \mapsto \begin{pmatrix} \varphi_l(x) & \varphi_t(x) \\ 0 & 0 \end{pmatrix}$$

determines a complete affine structure on $\mathfrak{g}$.

We have the following conclusion:

Conclusion. *There is a one-one correspondence between complete affine structures on a nilpotent Lie algebra* $\mathfrak{g}$ *and complete left-symmetric structures on* $\mathfrak{g}$*.*

Within the class of left-symmetric structures, there is a special subclass of structures having the additional property that the right multiplications commute. These are called Novikov structures, which on itself arise in several contexts in mathematics and physics.

They were firstly introduced in the study of Hamiltonian operators concerning integrability of certain nonlinear partial differential equations (see [30]). They also appear in connection with Poisson brackets of hydrodynamic type, operator Yang-Baxter equations (see [9]) and vertex algebras (see [27]). In particular, Novikov algebras bijectively correspond to a special class of Lie conformal algebras. This shows the importance of Novikov algebras in theoretical physics.

Our own main results about Novikov structures are stated in 2.4.1 and worked out in chapter 3.

Summary

The next theorem summarizes all steps we have dealt with, so different levels on which Milnor's question can be formulated:

Theorem 2.2.18. *Let Γ be a finitely generated torsion-free nilpotent group with Mal'cev completion G. Denote the Lie algebra of G by $\mathfrak{g}$. Then, there are one-one correspondences between the following four sets of objects:*

1. *The affine crystallographic actions of the group Γ (= The affine structures on the group Γ).*
2. *The simply transitive affine actions of G.*
3. *The complete affine structures on $\mathfrak{g}$.*
4. *The complete left-symmetric structures on $\mathfrak{g}$.*

2.2.4 The Lie group version of Auslander and Milnor

We saw that, in the nilpotent case, we could translate affine crystallographic actions into simply transitive affine actions at the Lie group level.

It might be interesting to put this Lie group version in the bigger context of solvable groups. Actually, Milnor already stated this question in 1977 at the solvable Lie group level (see [45]):

Question 2.2.19 (Milnor's question - Lie group version)**.** *Let G be a connected, simply connected, solvable Lie group. Does G admit a simply transitive affine action $\rho : G \to \mathrm{Aff}(\mathbb{R}^n)$?*

Auslander showed, also in 1977, that the converse is always true (see [6]):

Theorem 2.2.20 (Auslander - Lie group version). *Suppose G is a Lie group acting simply transitively on $\mathbb{R}^n$ via affine motions, then G must be solvable.*

A positive answer to the question of Milnor would have provided a geometric characterization of the class of connected, simply connected, solvable Lie groups.

But we know that the answer to Milnor's question is negative, by the nilpotent counterexample of Benoist. Our generalization to NIL-affine actions will solve this item.

As in the nilpotent case, we also have the following equivalence for general solvable Lie groups (see [37]):

Theorem 2.2.21. *A (solvable) Lie group G admits a simply transitive affine action if and only if its Lie algebra $\mathfrak{g}$ admits a complete left-symmetric structure.*

We note that in the general case of solvable Lie algebras a left-symmetric structure is called complete if all right multiplications commute. This is compatible with definition 2.2.17 since for nilpotent Lie algebras all right multiplications are nilpotent if and only if all left multiplications are nilpotent (see [37]).

2.3 The NIL-affine case

Because of the counter examples mentioned, one has broadened the geometric context and studied affine actions on connected, simply connected nilpotent Lie groups.

Let N be a connected, simply connected, nilpotent Lie group, with corresponding nilpotent Lie algebra $\mathfrak{n}$.

To N we can associate its group $\text{Aff}(N)$ of affine transformations, called the *affine group* of N, it is the semidirect product

$$\text{Aff}(N) = N \rtimes \text{Aut}(N),$$

where $\text{Aut}(N)$ consists of the continuous automorphisms of N.

Note that this really is a generalization of the usual affine group $\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}_n(\mathbb{R})$, where $\text{GL}_n(\mathbb{R})$ is the group of continuous automorphisms of the abelian, connected, simply connected Lie group $\mathbb{R}^n$.

This group acts on N by so called NIL-affine transformations via the action given by

$$\forall m, n \in N, \forall \alpha \in \operatorname{Aut}(N) : \ {}^{(m,\alpha)}n = m \cdot \alpha(n).$$

The corresponding Lie algebra of the group $\operatorname{Aff}(N)$ is given by

$$\mathfrak{aff}(\mathfrak{n}) = \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$$

and its Lie bracket is given by

$$[(x, D), (x', D')] = ([x, x'] + Dx' - D'x, [D, D'])\,.$$

By a *NIL-affine action* we mean a representation $\rho : \Gamma \to \operatorname{Aff}(N)$. Also here we are considering crystallographic actions:

Definition 2.3.1 (NIL-affine structure, NIL-affine crystallographic action, NIL-affine crystallographic group)**.**
A NIL-affine crystallographic action of a group Γ *on* N *is a representation* $\rho : \Gamma \to \operatorname{Aff}(N)$*, letting* Γ *act crystallographically on* N*.*
We call such an action a NIL-affine structure on Γ *and* $\rho(\Gamma)$ *is called a NIL-affine crystallographic group.*

2.3.1 Geometrical background

Studying NIL-affine crystallographic groups is really a very natural generalization of studying affine crystallographic groups since $\operatorname{Aff}(N)$ is actually the group of connection preserving diffeomorphisms of N for any left invariant affine connection on N (see e.g. [36]).

As we deal with crystallographic actions, there is also a great interest from the geometrical point of view.

Suppose $\Gamma \subseteq \operatorname{Aff}(N)$ is a subgroup of the affine group of N which acts freely and properly discontinuously on N, then the quotient space $M = \Gamma \backslash N$ is called a NIL-affinely flat manifold, with Γ as its fundamental group. If moreover this Γ is a crystallographic group, the quotient manifold M is compact.

Geodesics are defined via the connection on N, and also here we obtain a complete manifold, in the sense that every partial geodesic $\gamma : I \to M$ can be extended to the whole real line.

2.3.2 The analogue of Milnor's question and the generalized Auslander conjecture

In the NIL-affine setting the analogue of Milnor's question does hold:

Theorem 2.3.2 ([25],[10])**.** *Let* Γ *be a torsion-free polycyclic-by-finite group. Then there exists a connected, simply connected, nilpotent Lie group* N *and an embedding* $\rho : \Gamma \to \mathrm{Aff}(N)$*, such that* $\rho(\Gamma)$ *is a crystallographic subgroup of* $\mathrm{Aff}(N)$*.*

Hence we already know that any torsion-free polycyclic-by-finite group appears as the fundamental group of a compact and complete NIL-affinely flat manifold. This shows that it is a good choice to consider these NIL-affine actions as a possible alternative for the failing affine actions.

Just as in the case of affine crystallographic groups, it is very natural to ask whether every NIL-affine crystallographic group is polycyclic-by-finite:

Conjecture 2.3.3 (The Generalized Auslander conjecture)**.** *Let* N *be a connected, simply connected, nilpotent Lie group and let* $\Gamma \subseteq \mathrm{Aff}(N)$ *be a group acting crystallographically on* N*. Then* Γ *is polycyclic-by-finite.*

The answer is expected to be positive since the question is closely related to the original Auslander problem. Note that a positive answer to this generalized Auslander conjecture would imply a complete algebraic description of the class of NIL-affine crystallographic groups, or stated otherwise, would provide a complete geometric description of the class of polycyclic-by-finite groups.

It has been proven by Burde, Dekimpe and Deschamps in [18] that this conjecture holds for all N up to dimension 5. In [26] it is proven to hold up to dimension 6 by Dekimpe and Petrosyan.

Also in [18] they proof that the generalized Auslander conjecture reduces to the original one if N is two-step nilpotent:

Proposition 2.3.4. *Let* N *be a connected, simply connected, 2-step nilpotent Lie group. Assume that* $\Gamma \leq \mathrm{Aff}(N)$*, then* Γ *acts crystallographically on* N *if and only if it also acts crystallographically and affinely on* $\mathbb{R}^n$*.*

2.3.3 Translations in the nilpotent case

Also in the NIL-affine case we can restrict to the class of finitely generated torsion-free nilpotent groups and get translations into the theory of Lie groups and Lie algebras. This translation of the problem to the Lie algebra level plays a crucial role.

From NIL-affine crystallographic actions to simply transitive NIL-affine actions

Like in the affine case, also in the NIL-affine case we have a one-one correspondence between the NIL-affine crystallographic actions of a finitely generated, torsion-free nilpotent group Γ and the simply transitive NIL-affine actions of its Mal'cev completion G.

As a first approach towards the study of simply transitive NIL-affine actions one concentrated on this situation, namely the situation where both G and N are nilpotent.

From simply transitive NIL-affine actions to complete NIL-affine structures

Also in the NIL-affine setting everything can be translated to the Lie algebra level. More exactly we can completely translate the notion of a simply transitive NIL-affine action of a connected, simply connected nilpotent Lie group G into a notion on the Lie algebra level, which we call a complete NIL-affine structure on the corresponding Lie algebra $\mathfrak{g}$.

Recall that the Lie algebra corresponding to the semidirect product $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ is equal to the semidirect product $\mathfrak{aff}(\mathfrak{n}) = \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$.

Definition 2.3.5 (Complete NIL-affine structure on a Lie algebra $\mathfrak{g}$)**.** *Let $\mathfrak{g}$ and $\mathfrak{n}$ be nilpotent Lie algebras.*
A NIL-affine structure on $\mathfrak{g}$ is a Lie algebra homomorphism

$$\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{n}) = \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : x \mapsto \varphi(x) = (t(x), D(x)),$$

such that $t : \mathfrak{g} \to \mathfrak{n} : x \mapsto t(x)$ is bijective.
It is called complete if for every $x \in \mathfrak{g}$ we have that $D(x)$ is nilpotent.

This is again a very natural generalization of the complete affine structures in the affine situation.

Let G and N be connected, simply connected, nilpotent Lie groups with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$.

Let $\rho : G \to \mathrm{Aff}(N)$ be a representation of Lie groups. We know that there exists a unique homomorphism $\mathrm{d}\rho$, the differential of ρ, of their respective Lie algebras $\mathfrak{g}$ and $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ making the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & \mathrm{Aff}(N) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[\ d\rho\ ]{} & \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \end{array}$$

Conversely, any Lie algebra homomorphism $\mathrm{d}\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ can be seen as the differential of a Lie group homomorphism $\rho : G \to \mathrm{Aff}(N)$.

Then by Burde, Dekimpe, Deschamps we have the following theorem (see [19]):

Theorem 2.3.6. *Let G and N be connected, simply connected nilpotent Lie groups and let $\mathfrak{g}$ and $\mathfrak{n}$ be the corresponding Lie algebras. Let $\rho : G \to \mathrm{Aff}(N)$ be a representation, with corresponding differential $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$. Then we have that $\rho : G \to \mathrm{Aff}(N)$ induces a simply transitive NIL-affine action of G on N if and only if $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is a complete NIL-affine structure on $\mathfrak{g}$.*

As mentioned in conclusion 2.2.16, this theorem is known to hold in the usual affine case ($N = \mathbb{R}^n$) and so we really obtain a very natural generalization.

It is still an open problem to determine for a given G all connected, simply connected, nilpotent Lie groups N, on which G acts simply transitively via NIL-affine motions.

Using the translation into the Lie algebra level, it has been proven by Burde, Dekimpe, Deschamps in [19] that for any two connected, simply connected, nilpotent Lie groups G and N of the same dimension n with $1 \leq n \leq 5$, there exists a representation $\rho : G \to \mathrm{Aff}(N)$ which induces a simply transitive NIL-affine action of G and N.

In the same article they also showed that this does not hold in dimension 6.

From abelian complete NIL-affine structures to complete LR-structures

Thus far we have been looking at simply transitive actions $\rho : G \to \mathrm{Aff}(N)$ where both G and N are arbitrary connected, simply connected, nilpotent Lie groups.

As already pointed out in 2.2.3, the case where $N = \mathbb{R}^n$ has been well studied and is equivalent to the study of complete left-symmetric structures on the Lie algebra $\mathfrak{g}$, corresponding to the Lie group G.

In order to get a better understanding of simply transitive NIL-affine actions $\rho : G \to \mathrm{Aff}(N)$, one specialized to the situation where N is arbitrary and G is abelian.

Even for the case $G = \mathbb{R}^n$ the problem of determining all connected, simply connected, nilpotent Lie groups N, on which G acts simply transitively via NIL-affine motions, is non-trivial and interesting.

Also in this case a translation to the Lie algebra level is possible and it turns out that such a simply transitive abelian NIL-affine action on N corresponds to a particular Lie compatible bilinear product on te Lie algebra $\mathfrak{n}$ of N, which is called a complete LR-structure:

Definition 2.3.7 (Complete LR-structure on a Lie algebra $\mathfrak{n}$)**.** *An LR-structure on a nilpotent Lie algebra $\mathfrak{n}$ is a bilinear product $(x, y) \mapsto x \cdot y$ which satisfies*

$$x \cdot (y \cdot z) = y \cdot (x \cdot z),$$

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y,$$

$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y, z \in \mathfrak{n}$.
The LR-structure is said to be complete if all left multiplications are nilpotent.

By Burde, Dekimpe, Deschamps (see [19]) we now have:

Theorem 2.3.8. *Let N be a connected, simply connected, nilpotent Lie group of dimension n with corresponding Lie algebra $\mathfrak{n}$. Then there exists a simply transitive NIL-affine action of $\mathbb{R}^n$ on N via a representation $\rho\colon \mathbb{R}^n \to \mathrm{Aff}(N)$ if and only if the Lie algebra $\mathfrak{n}$ of N admits a complete LR-structure.*

This result shows that a deeper study of (complete) LR-structures is required in order to obtain a good understanding of simply transitive abelian and NIL-affine actions on nilpotent Lie groups.

The first steps in this direction were taken in [20]. We will continue this research in chapter 4. An overview of our main results can be found in section 2.4.2.

2.3.4 The Lie group version of the generalized Auslander and Milnor question

In the NIL-affine setting, we also get a positive answer to the generalized Lie group version of Milnor, unlike in the affine setting. In contrast to the fact that there are connected, simply connected, solvable Lie groups G not allowing a simply transitive affine action, all of them do admit a simply transitive affine action on a nilpotent Lie group.

We also get a positive answer to the generalized Lie group version of Auslander in this setting.

Theorem 2.3.9. *Let G be a connected, simply connected, solvable Lie group. Then there exists a connected, simply connected, nilpotent Lie group N and a representation $\rho : G \to \mathrm{Aff}(N)$ letting G act simply transitively on N.*

Let N be a connected, simply connected, nilpotent Lie group and assume that $\rho : G \to \mathrm{Aff}(N)$ denotes a simply transitive action, then G is solvable.

Once more this indicated that the NIL-affine setting is really a good choice.

2.4 New results about Novikov, LR- and post-Lie algebras and their corresponding structures

In this section we give an overview of our own research, by stating the main results, without any proof. All details can be found in the following chapters.

Although the algebras and structures we study arose when working over $\mathbb{R}$, our own research will be over an arbitrary field k of characteristic zero.

2.4.1 Novikov algebras and Novikov structures

In chapter 3 we study ideals in Novikov algebras and look at the existence of Novikov structures on particular Lie algebras.

We prove that all free 3-step nilpotent Lie algebras admit a Novikov structure and show that there exists a 3-step nilpotent Lie algebra on 4 generators not admitting a Novikov structure.

Furthermore we prove that all Novikov structures on the free 3-step nilpotent Lie algebra on at least 3 generators are complete. For 2 generators this is not the case.

We take a look at the free p-step nilpotent Lie algebras with $p \geq 4$, where it seems that Novikov structures do not exist at all.

We also treat the existence question for Novikov structures on triangular matrix algebras and construct a family of filiform nilpotent Lie algebras of arbitrary high solvability class all admitting a complete Novikov structure.

2.4.2 LR-algebras and LR-structures

In the affine case it was asked whether a complete left-symmetric structure would exist automatically, once there was any such structure at all. Mizuhara showed that in the nilpotent case this is indeed true:

Theorem 2.4.1. *[46] Let $\mathfrak{g}$ be a complex nilpotent Lie algebra. If $\mathfrak{g}$ admits a left-symmetric structure, then $\mathfrak{g}$ also admits a complete left-symmetric structure.*

One goal in chapter 4 is to obtain the analogue of Mizuhara's result for LR-algebras. We will show that if a Lie algebra $\mathfrak{g}$ admits any LR-structure, then $\mathfrak{g}$ also admits a complete LR-structure.

First we give a proof in the nilpotent case, afterwards we prove this for general Lie algebras. We do this by considering the nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g}/\mathfrak{g}^{\infty}$ where $\mathfrak{g}^{\infty} = \bigcap_{i=1}^{\infty} \gamma_i(\mathfrak{g})$, which will then admit a complete LR-structure, and lift this structure to a complete LR-structure on $\mathfrak{g}$.

With some extra condition we will show how specific LR-structures on $\mathfrak{n}$ can be lifted to complete LR-structures on $\mathfrak{g}$.

We construct complete LR-structures on all 2-step solvable Lie algebras on 2 generators and all free p-step nilpotent 2-step solvable Lie algebras.
We also prove that all LR-structures on the free p-step nilpotent 2-step solvable Lie algebras are complete for nilpotency class at least 3. For nilpotency class 2 this seems not to be the case.

To end this chapter we treat the existence question for LR-structures on triangular matrices.

2.4.3 Post-Lie algebras and post-Lie algebra structures

Post-Lie algebras and post-Lie algebra structures recently have been introduced by Valette in [51] in connection with homology of partition posets and the study of Koszul operads. Moreover, they have been discussed in several articles of Loday, see for example [40] and the references given therein. Furthermore, post-Lie algebras also turned up in relation with the classical Yang-Baxter equation in [8].

In this thesis we will show that these algebras also appear in yet another, quite different, context, namely in connection with NIL-affine actions of Lie groups.

We will prove the following fact in chapter 5:

Let G and N be real connected, simply connected nilpotent Lie groups with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$. Then there exists a simply transitive NIL-affine action of G on N if and only if there is a Lie algebra $\mathfrak{g}' \simeq \mathfrak{g}$, with the same underlying vector space as $\mathfrak{n}$, such that the pair of Lie algebras $(\mathfrak{g}', \mathfrak{n})$ admits a complete post-Lie algebra structure.

We will show that post-Lie algebra structures generalize both LR-structures and left-symmetric structures.

We prove that we have a one-one correspondence between the post-Lie algebra structures on a pair $(\mathfrak{g}, \mathfrak{n})$ and embeddings $\mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ with the identity map on the first factor. We also have a one-one correspondence with the subalgebras $\mathfrak{h}$ of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ for which the projection $p_1 \colon \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \to \mathfrak{n}$ onto the first factor induces a Lie algebra isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.

We investigate this correspondence in the particular case of $\mathfrak{n}$ being semisimple.

We also give a classification of the complex two-dimensional post-Lie algebras and study the existence of post-Lie algebra structures in terms of algebraic conditions of $\mathfrak{g}$ and $\mathfrak{n}$.

Chapter 3

Novikov algebras and Novikov structures

In this chapter we study ideals of Novikov algebras and Novikov structures on finite-dimensional Lie algebras. We present the first example of a 3-step nilpotent Lie algebra which does not admit a Novikov structure. On the other hand we show that any free 3-step nilpotent Lie algebra admits a Novikov structure and that all these Novikov structures are complete when there are at least 3 generators. We study the existence question also for the free p-step nilpotent Lie algebras for $p \geq 4$ and for Lie algebras of triangular matrices. Finally we show that there are families of Lie algebras of arbitrary high solvability class which admit Novikov structures.

In section 3.1 we recall the definition of a Novikov algebra and a Novikov structure.

In section 3.2 we present some structure theory concerning ideals in Novikov algebras.

In section 3.3 we show that not every 3-step nilpotent Lie algebra admits a Novikov structure, however every free 3-step nilpotent Lie algebra does.
We give an explicit Novikov structure on the free 3-step nilpotent Lie algebras and construct a 3-step nilpotent Lie algebra on 4 generators that does not admit a Novikov structure.
In this section we also prove that all Novikov structures on the free 3-step nilpotent Lie algebras on at least 3 generators are complete. For 2 generators this is not the case.

In section 3.4 we prove that there do not exist Novikov structures on the free p-step nilpotent Lie algebras for $p \geq 4$.

In section 3.5 we look at some specific solvable and nilpotent Lie algebras, namely the Lie algebras of the (strictly) upper triangular matrices. We show that for these Lie algebras Novikov structures only exist in very small dimensions.

In chapter 3.6 we construct a family of filiform nilpotent Lie algebras of arbitrary high solvability class which all admit a complete Novikov structure.

All algebras we consider are assumed to be finite dimensional over a field k of characteristic 0.

Many results of this chapter are presented in [21].

3.1 Novikov algebras and Novikov structures

We start by recalling some definitions. Novikov algebras and, more generally, left-symmetric algebras are defined as follows:

Definition 3.1.1. *An algebra $(A, \cdot)$ over k with product $(x, y) \mapsto x \cdot y$ is called a left-symmetric algebra (LSA), if the product is left-symmetric, i.e., if the identity*

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \tag{3.1}$$

is satisfied for all $x, y, z \in A$. The algebra is called Novikov, if in addition

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{3.2}$$

is satisfied.

Denote by $L(x), R(x)$ the left, respectively right multiplication operator in the algebra $(A, \cdot)$. Then an LSA is a Novikov algebra if the right multiplications commute:

$$[R(x), R(y)] = 0.$$

It is well known that LSAs, and hence also Novikov algebras, are Lie-admissible algebras: the commutator

$$[x, y] = x \cdot y - y \cdot x$$

defines a Lie bracket. The associated Lie algebra is denoted by $\mathfrak{g}_A$. The adjoint operator can be expressed by $\operatorname{ad}(x) = L(x) - R(x)$.

The associated Lie algebra is then said to admit a left-symmetric structure, respectively Novikov structure:

Definition 3.1.2. *A left-symmetric structure on a Lie algebra* $\mathfrak{g}$ *over* k *is a left-symmetric product* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying*

$$[x, y] = x \cdot y - y \cdot x \tag{3.3}$$

for all $x, y \in \mathfrak{g}$.
If the product is Novikov, we say that $\mathfrak{g}$ *admits a Novikov structure.*

Definition 3.1.3 (Complete)**.** *A left-symmetric structure (and hence a Novikov structure) is called* complete *if all right multiplications* $R(x)$ *are nilpotent.*

Remark 3.1.4. *It has been proved by Kim (see [37]) that for nilpotent Lie algebras this is equivalent to all left multiplications* $L(x)$ *being nilpotent.*

We have the following important property of Lie algebras admitting a Novikov structure (for a proof see [16]):

Proposition 3.1.5. *Any Lie algebra admitting a Novikov structure is solvable.*

A given solvable Lie algebra need not admit a Novikov structure, or a left-symmetric structure. The existence question for left-symmetric structures is very hard in general. It is more accessible for Novikov structures. For results, background and references see, for example, [11] and [16].

If A is a Novikov algebra, then we obtain, by expanding the condition $0 = [R(x), R(y)] = [L(x) - \mathrm{ad}(x), L(y) - \mathrm{ad}(y)]$, the following operator identity:

$$L([x, y]) + \mathrm{ad}([x, y]) - [\mathrm{ad}(x), L(y)] - [L(x), \mathrm{ad}(y)] = 0. \tag{3.4}$$

3.2 Ideals in Novikov algebras

In this section we study ideals in Novikov algebras. For related results in this direction see also [9] and [54].

In our study of ideals we will use the following two identities, which are similar to the Jacobi identity for Lie algebras. For a proof, see [13].

Lemma 3.2.1. *Let* $(A, \cdot)$ *be a Novikov algebra. Then we have for all* $x, y, z \in A$*:*

$$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0,$$

$$x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] = 0.$$

Next we show that the product of two ideals is again an ideal:

Lemma 3.2.2. *Let $(A, \cdot)$ be a Novikov algebra and let I, J be 2-sided ideals of A. Then $I \cdot J$ is also a 2-sided ideal of A.*

Proof. Let $a \in A$, $x \in I$ and $y \in J$. Then identity (3.1) gives

$$a \cdot (x \cdot y) = (a \cdot x) \cdot y + x \cdot (a \cdot y) - (x \cdot a) \cdot y$$

and this shows that $a \cdot (x \cdot y) \in I \cdot J$. Because of identity (3.2) we have $(x \cdot y) \cdot a = (x \cdot a) \cdot y$ and hence $(x \cdot y) \cdot a \in I \cdot J$. □

We have the following identity in Novikov algebras, which will be useful later on:

Lemma 3.2.3. *Let $(A, \cdot)$ be a Novikov algebra, then we have for all $x, y, z \in A$*

$$[x, y] \cdot z = \frac{1}{2}[x \cdot z, y] + \frac{1}{2}[x, y \cdot z].$$

Proof. The operator identity (3.4) implies that

$$\begin{aligned}
0 &= [x, y] \cdot z + [[x, y], z] - [x, y \cdot z] + y \cdot [x, z] - x \cdot [y, z] + [y, x \cdot z] \\
&= [x, y] \cdot z + [[x, y], z] - [x, y \cdot z] + [y, x \cdot z] \\
&\qquad + (y \cdot [x, z] + x \cdot [z, y] + z \cdot [y, x]) - z \cdot [y, x] \\
&= [x, y] \cdot z + [[x, y], z] - [x, y \cdot z] + [y, x \cdot z] + z \cdot [x, y] \\
&= [x, y] \cdot z + [[x, y], z] - [x, y \cdot z] + [y, x \cdot z] + [x, y] \cdot z + [z, [x, y]] \\
&= 2[x, y] \cdot z - [x \cdot z, y] - [x, y \cdot z].
\end{aligned}$$

The term in brackets above vanishes because of lemma 3.2.1. This gives the required identity. □

We can now show that the commutator of two ideals is again an ideal:

Lemma 3.2.4. *Let $(A, \cdot)$ be a Novikov algebra and assume that I, J are 2-sided ideals of A. Then $[I, J]$ is also a 2-sided ideal of A.*

Proof. Let $a \in A$, $x \in I$ and $y \in J$. From the above lemma we deduce

$$[x, y] \cdot a = \frac{1}{2}[x \cdot a, y] + \frac{1}{2}[x, y \cdot a] \in [I, J],$$

$$a \cdot [x, y] = [x, y] \cdot a + [a, [x, y]] \in [I, J],$$

which was to be shown. □

Denote by

$$\gamma_1(A) = \gamma_1(\mathfrak{g}_A) = A,$$

$$\gamma_{i+1}(A) = \gamma_{i+1}(\mathfrak{g}_A) = [A, \gamma_i(A)]$$

the terms of the lower central series of A (and $\mathfrak{g}_A$). Furthermore, denote by

$$A^{(1)} = \mathfrak{g}_A^{(1)} = A,$$

$$A^{(i+1)} = \mathfrak{g}_A^{(i+1)} = [A^{(i)}, A^{(i)}]$$

the terms of the derived series of A (and $\mathfrak{g}_A$).

The above lemma immediately implies the following result concerning the terms of the lower central series and the derived series of a Novikov algebra A:

Corollary 3.2.5. *Let $(A, \cdot)$ be a Novikov algebra. Then all $\gamma_i(A)$ and all $A^{(i)}$ are 2-sided ideals of A.*

The ideals of the lower central series satisfy the following property:

Lemma 3.2.6. *Let $(A, \cdot)$ be a Novikov algebra. Then we have*

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+1}(A)$$

for all $i, j \geq 0$.

Proof. We will show this by induction on $i \geq 0$. The case $i = 0$ follows from the fact that $\gamma_{j+1}(A)$ is an ideal in A for all $j \geq 0$, see corollary 3.2.5.

Assume now that $\gamma_k(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{k+j}(A)$ for all $k = 1, \ldots, i$ and for all $j \geq 0$. Let $x \in \gamma_1(A)$, $y \in \gamma_i(A)$ and $z \in \gamma_{j+1}(A)$. We have to show that $[x, y] \cdot z \in \gamma_{i+j+1}(A)$.

The identity of lemma 3.2.3 says that

$$[x, y] \cdot z = \frac{1}{2}[x \cdot z, y] + \frac{1}{2}[x, y \cdot z].$$

By the induction hypothesis we have that $x \cdot z \in \gamma_{j+1}(A)$ and $y \cdot z \in \gamma_{i+j}(A)$, hence $[x \cdot z, y]$ and $[x, y \cdot z]$ are both elements of $\gamma_{i+j+1}(A)$ and we can conclude that so is $[x, y] \cdot z$. □

We also have the following interesting lemma about the ideals of the lower central series:

Lemma 3.2.7. *Let $(A, \cdot)$ be a Novikov algebra. If*

$$\gamma_1(A) \cdot \gamma_1(A) \subseteq \gamma_2(A),$$

we have

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+2}(A)$$

for all $i, j \geq 0$.

Proof. First we will proof the case $j = 0$ by induction on $i \geq 0$. The case $i = 0$ is given.

Assume now that $\gamma_k(A) \cdot \gamma_1(A) \subseteq \gamma_{k+1}(A)$ for all $k = 1, \ldots, i$. Let $x \in \gamma_1(A)$, $y \in \gamma_i(A)$ and $z \in \gamma_1(A)$. We have to show that $[x, y] \cdot z \in \gamma_{i+2}(A)$.

We use the identity from lemma 3.2.3:

$$[x, y] \cdot z = \frac{1}{2}[x \cdot z, y] + \frac{1}{2}[x, y \cdot z].$$

From the induction hypothesis it follows that $x \cdot z \in \gamma_2(A)$ and that $y \cdot z \in \gamma_{i+1}(A)$. Now we have that both $[x \cdot z, y]$ and $[x, y \cdot z]$ are in $\gamma_{i+2}(A)$ and hence so should $[x, y] \cdot z$.

Next we will proof the case $i = 0$. Let $x \in \gamma_1(A)$ and $y \in \gamma_{j+1}(A)$. We have to show that $x \cdot y \in \gamma_{j+2}(A)$. This follows directly from the identity

$$x \cdot y = [x, y] + y \cdot x$$

and the proof of the case $j = 0$.

Finally we will proof the general case $i, j \geq 0$ by induction on $i \geq 0$. The case $i = 0$ we just proved above.

Assume that $\gamma_k(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{k+j+1}(A)$ for all $k = 1, \ldots, i$ and all j. Let $x \in \gamma_1(A)$, $y \in \gamma_i(A)$ and $z \in \gamma_{j+1}(A)$. We have to prove that $[x, y] \cdot z \in \gamma_{i+j+2}(A)$.

We again use the following identity:

$$[x, y] \cdot z = \frac{1}{2}[x \cdot z, y] + \frac{1}{2}[x, y \cdot z].$$

From the induction hypothesis it follows that $x \cdot z \in \gamma_{j+2}(A)$ and $y \cdot z \in \gamma_{i+j+1}(A)$. Now we have that $[x \cdot z, y]$ and $[x, y \cdot z]$ are both elements of $\gamma_{i+j+2}(A)$ and hence we can conclude that also $[x, y] \cdot z \in \gamma_{i+j+2}(A)$. □

Denote the center of a Novikov algebra A by

$$Z(A) = \{x \in A \mid x \cdot y = y \cdot x \text{ for all } y \in A\}.$$

Note that $Z(A)$ is also the center of the associated Lie algebra $\mathfrak{g}_A$.

We prove the following lemma concerning this center:

Lemma 3.2.8. *Let $(A, \cdot)$ be a Novikov algebra. Then $Z(A) \cdot [A, A] = [A, A] \cdot Z(A) = 0$.*

Proof. Let $a, b \in A$ and $z \in Z(A)$. By lemma 3.2.1 we have

$$z \cdot [a, b] + a \cdot [b, z] + b \cdot [z, a] = 0.$$

Since z is also in the center of the associated Lie algebra of A we obtain $z \cdot [a, b] = 0$. Furthermore we have

$$0 = [z, [b, a]] = z \cdot [b, a] - [b, a] \cdot z = [a, b] \cdot z.$$

□

Using this lemma, we can prove the following result:

Lemma 3.2.9. *Let $(A, \cdot)$ be a Novikov algebra. Then $Z(A)$ is a 2-sided ideal of A.*

Proof. Let $z \in Z(A)$. Since the product is Novikov and z is also in the center of the associated Lie algebra, we have for any $b \in A$:

$$[L(b), L(z)] = L([b, z]) = 0,$$

$$[R(b), R(z)] = 0.$$

Because $z \in Z(A)$ we have $R(z) = L(z)$. Hence we also have $[L(b), R(z)] = 0$, so that

$$0 = [L(b) - R(b), R(z)] = [\mathrm{ad}(b), R(z)].$$

In particular it follows that $[b, a \cdot z] - [b, a] \cdot z = 0$ for all $a \in A$.

By Lemma 3.2.8 we have $[b, a] \cdot z = 0$, hence $[b, a \cdot z] = 0$. Because this is true for every $b \in A$, we can conclude that $a \cdot z \in Z(A)$. Since $z \in Z(A)$ we also have $z \cdot a \in Z(A)$. □

Let $Z_1(A) = Z(A)$ and define $Z_{i+1}(A)$ by the identity $Z_{i+1}(A)/Z_i(A) = Z(A/Z_i(A))$. Note that the $Z_i(A)$ are the terms of the upper central series of the associated Lie algebra $\mathfrak{g}_A$.

As an immediate consequence of the previous lemma, we obtain the following corollary:

Corollary 3.2.10. *Let $(A, \cdot)$ be a Novikov algebra. Then all terms $Z_i(A)$ of the upper central series of A are 2-sided ideals of A.*

Denote by $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ the associator of three elements in A. We can prove the following lemma:

Lemma 3.2.11. *Let A be a Novikov algebra and suppose that one of the elements x, y, z is in $Z(A)$. Then $(x, y, z) = 0$.*

Proof. In any LSA we have the identity

$$(x, y, z) = x \cdot [y, z] + [z, x \cdot y] + [x, z] \cdot y.$$

If $z \in Z(A)$, then this immediately implies $(x, y, z) = 0$. If $y \in Z(A)$ then also $x \cdot y \in Z(A)$ by lemma 3.2.9, and $[x, z] \cdot y = 0$ by lemma 3.2.8. Hence the above identity implies $(x, y, z) = 0$. The same argument shows the claim for $x \in Z(A)$. □

3.3 Novikov structures on 3-step nilpotent Lie algebras

In [16, Remark 4.11] it was questioned whether or not there exists a 3-step nilpotent Lie algebra not admitting a Novikov structure. This is a natural question since all 3-step nilpotent Lie algebras admit a left-symmetric structure. This is a result of Scheuneman (see [48]) and was alternatively proved in [16].

In the same paper it was shown that a Novikov structure does exist when the 3-step nilpotent Lie algebra $\mathfrak{g}$ can be generated by at most 3 elements.

In this section we will show that for more than 3 generators, there does not always exist a Novikov structure.

The existence of a Novikov structure on a 3-step nilpotent Lie algebra on 3 generators was obtained by first considering a Novikov structure on the free 3-step nilpotent Lie algebra $\mathfrak{f}$ on 3 generators and then it was shown that $\mathfrak{g}$

could be realized as a quotient $\mathfrak{g} = \mathfrak{f}/I$, where I is an ideal of $\mathfrak{f}$ seen as a Novikov algebra.

Having this in mind we first study the free 3-step nilpotent case and construct a Novikov structure on this Lie algebra. At the end of this section we will also prove that any Novikov structure on the free 3-step nilpotent Lie algebra on at least 3 generators is complete. For 2 generators, this seems not to be the case.

Proposition 3.3.1. *Let $\mathfrak{g}$ be the free 3-step nilpotent Lie algebra on n generators $x_1, x_2, \ldots, x_n$. Then $\mathfrak{g}$ admits a (complete) Novikov structure.*

Proof. As a vector space, $\mathfrak{g}$ has a basis

$$x_1, x_2, \ldots, x_n,$$

$$y_{i,j} = [x_i, x_j],\ (1 \leq i < j \leq n),$$

$$z_{i,j,k} = [x_i, y_{j,k}],\ (1 \leq j < k \leq n,\ 1 \leq i \leq k \leq n). \tag{3.5}$$

Note that in the case $i > k > j$, we have

$$\begin{aligned} z_{i,j,k} &= [x_i, y_{j,k}] \\ &= [x_i, [x_j, x_k]] \\ &= -[x_j, [x_k, x_i]] - [x_k, [x_i, x_j]] \\ &= -z_{j,k,i} + z_{k,j,i}. \end{aligned} \tag{3.6}$$

The following product defines a Novikov structure on $\mathfrak{g}$:

- If $1 \leq j < i \leq n$, then $x_i \cdot x_j = -y_{j,i}$.

- If $1 \leq i \leq j < k \leq n$, then $x_i \cdot y_{j,k} = \frac{z_{i,j,k}}{2}$.
  If $1 \leq j < i < k \leq n$, then $x_i \cdot y_{j,k} = -\frac{z_{j,i,k}}{2} + z_{i,j,k}$.
  If $1 \leq j < k \leq i \leq n$, then $x_i \cdot y_{j,k} = z_{i,j,k}$.

- If $1 \leq i \leq j < k \leq n$, then $y_{j,k} \cdot x_i = -\frac{z_{i,j,k}}{2}$.
  If $1 \leq j < i < k \leq n$, then $y_{j,k} \cdot x_i = -\frac{z_{j,i,k}}{2}$.

All other products are zero.

Let us prove that this indeed defines a Novikov structure on $\mathfrak{g}$.

By considering each of the above cases it is easy to see that the identity $[a, b] = a \cdot b - b \cdot a$ holds for all basis elements a and b. We have to show the following two other identities:

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) - (b \cdot a) \cdot c + b \cdot (a \cdot c) = 0,$$

$$(a \cdot b) \cdot c - (a \cdot c) \cdot b = 0,$$

for all basis elements a, b and c.

It is clear that we only have to consider the case where $a = x_i$, $b = x_j$ and $c = x_k$: otherwise the two identities will be trivially satisfied, because any product of the form $y_{i,j} \cdot y_{k,l}$ is zero, and any product that involves an element $z_{i,j,k}$ is also zero.

For the first condition we can assume that $i < j$. In the case $1 \leq k < i < j \leq n$, we have

$$\begin{aligned}
&(x_i \cdot x_j) \cdot x_k - x_i \cdot (x_j \cdot x_k) - (x_j \cdot x_i) \cdot x_k + x_j \cdot (x_i \cdot x_k) \\
&= -x_i \cdot (-y_{k,j}) - (-y_{i,j}) \cdot x_k + x_j \cdot (-y_{k,i}) \\
&= -\frac{z_{k,i,j}}{2} + z_{i,k,j} - \frac{z_{k,i,j}}{2} - z_{j,k,i} \\
&= -z_{k,i,j} + z_{i,k,j} + z_{k,i,j} - z_{i,k,j} \\
&= 0.
\end{aligned}$$

In the case $1 \leq i \leq k < j \leq n$, we have

$$\begin{aligned}
&(x_i \cdot x_j) \cdot x_k - x_i \cdot (x_j \cdot x_k) - (x_j \cdot x_i) \cdot x_k + x_j \cdot (x_i \cdot x_k) \\
&= -x_i \cdot (-y_{k,j}) - (-y_{i,j}) \cdot x_k \\
&= \frac{z_{i,k,j}}{2} - \frac{z_{i,k,j}}{2} \\
&= 0.
\end{aligned}$$

Finally in the case $1 \leq i < j \leq k \leq n$, we have

$$\begin{aligned}
&(x_i \cdot x_j) \cdot x_k - x_i \cdot (x_j \cdot x_k) - (x_j \cdot x_i) \cdot x_k + x_j \cdot (x_i \cdot x_k) \\
&= y_{i,j} \cdot x_k \\
&= 0.
\end{aligned}$$

For the second condition we may assume that $j < k$. First we consider the case $1 \leq i \leq j < k \leq n$. We have

$$(x_i \cdot x_j) \cdot x_k - (x_i \cdot x_k) \cdot x_j = 0.$$

In the case $1 \leq j < i \leq k \leq n$, we have

$$(x_i \cdot x_j) \cdot x_k - (x_i \cdot x_k) \cdot x_j = -y_{j,i} \cdot x_k = 0.$$

Finally, in the case $1 \leq j < k < i \leq n$, we find

$$\begin{aligned}(x_i \cdot x_j) \cdot x_k - (x_i \cdot x_k) \cdot x_j &= -y_{j,i} \cdot x_k + y_{k,i} \cdot x_j \\ &= \frac{z_{j,k,i}}{2} - \frac{z_{j,k,i}}{2} \\ &= 0.\end{aligned}$$

It follows that the product defines a Novikov structure on $\mathfrak{g}$.

By the definition of the product it is immediately clear that the structure is complete. □

As a motivation for what follows, we provide a detailed description in the four generator case:

Example 3.3.2. *Let* $\mathfrak{g}$ *be the free 3-step nilpotent Lie algebra on* 4 *generators. Then* $\dim \mathfrak{g} = 30$*. The nonzero Lie brackets and Novikov products are given as follows:*

$[x_1, x_2] = y_{1,2}$
$[x_1, x_3] = y_{1,3}$
$[x_1, x_4] = y_{1,4}$
$[x_2, x_3] = y_{2,3}$
$[x_2, x_4] = y_{2,4}$
$[x_3, x_4] = y_{3,4}$
$[x_1, y_{1,2}] = z_{1,1,2}$
$[x_1, y_{1,3}] = z_{1,1,3}$
$[x_1, y_{1,4}] = z_{1,1,4}$
$[x_1, y_{2,3}] = z_{1,2,3}$
$[x_1, y_{2,4}] = z_{1,2,4}$
$[x_1, y_{3,4}] = z_{1,3,4}$
$[x_2, y_{1,2}] = z_{2,1,2}$
$[x_2, y_{1,3}] = z_{2,1,3}$
$[x_2, y_{1,4}] = z_{2,1,4}$
$[x_2, y_{2,3}] = z_{2,2,3}$
$[x_2, y_{2,4}] = z_{2,2,4}$
$[x_2, y_{3,4}] = z_{2,3,4}$
$[x_3, y_{1,2}] = -z_{1,2,3} + z_{2,1,3}$
$[x_3, y_{1,3}] = z_{3,1,3}$
$[x_3, y_{1,4}] = z_{3,1,4}$
$[x_3, y_{2,3}] = z_{3,2,3}$
$[x_3, y_{2,4}] = z_{3,2,4}$
$[x_3, y_{3,4}] = z_{3,3,4}$
$[x_4, y_{1,2}] = -z_{1,2,4} + z_{2,1,4}$
$[x_4, y_{1,3}] = -z_{1,3,4} + z_{3,1,4}$
$[x_4, y_{1,4}] = z_{4,1,4}$
$[x_4, y_{2,3}] = -z_{2,3,4} + z_{3,2,4}$
$[x_4, y_{2,4}] = z_{4,2,4}$
$[x_4, y_{3,4}] = z_{4,3,4}$

$$x_2 \cdot x_1 = -y_{1,2}$$
$$x_3 \cdot x_1 = -y_{1,3}$$
$$x_3 \cdot x_2 = -y_{2,3}$$
$$x_4 \cdot x_1 = -y_{1,4}$$
$$x_4 \cdot x_2 = -y_{2,4}$$
$$x_4 \cdot x_3 = -y_{3,4}$$
$$x_1 \cdot y_{1,2} = \frac{z_{1,1,2}}{2}$$
$$x_1 \cdot y_{1,3} = \frac{z_{1,1,3}}{2}$$
$$x_1 \cdot y_{1,4} = \frac{z_{1,1,4}}{2}$$
$$x_1 \cdot y_{2,3} = \frac{z_{1,2,3}}{2}$$
$$x_1 \cdot y_{2,4} = \frac{z_{1,2,4}}{2}$$
$$x_1 \cdot y_{3,4} = \frac{z_{1,3,4}}{2}$$
$$x_2 \cdot y_{1,2} = z_{2,1,2}$$
$$x_2 \cdot y_{1,3} = \frac{-z_{1,2,3}}{2} + z_{2,1,3}$$
$$x_2 \cdot y_{1,4} = \frac{-z_{1,2,4}}{2} + z_{2,1,4}$$
$$x_2 \cdot y_{2,3} = \frac{z_{2,2,3}}{2}$$
$$x_2 \cdot y_{2,4} = \frac{z_{2,2,4}}{2}$$
$$x_2 \cdot y_{3,4} = \frac{z_{2,3,4}}{2}$$
$$x_3 \cdot y_{1,2} = -z_{1,2,3} + z_{2,1,3}$$
$$x_3 \cdot y_{1,3} = z_{3,1,3}$$
$$x_3 \cdot y_{1,4} = \frac{-z_{1,3,4}}{2} + z_{3,1,4}$$
$$x_3 \cdot y_{2,3} = z_{3,2,3}$$
$$x_3 \cdot y_{2,4} = \frac{-z_{2,3,4}}{2} + z_{3,2,4}$$
$$x_3 \cdot y_{3,4} = \frac{z_{3,3,4}}{2}$$
$$x_4 \cdot y_{1,2} = -z_{1,2,4} + z_{2,1,4}$$
$$x_4 \cdot y_{1,3} = -z_{1,3,4} + z_{3,1,4}$$
$$x_4 \cdot y_{1,4} = z_{4,1,4}$$
$$x_4 \cdot y_{2,3} = -z_{2,3,4} + z_{3,2,4}$$
$$x_4 \cdot y_{2,4} = z_{4,2,4}$$
$$x_4 \cdot y_{3,4} = z_{4,3,4}$$
$$y_{1,2} \cdot x_1 = -\frac{z_{1,1,2}}{2}$$
$$y_{1,3} \cdot x_1 = -\frac{z_{1,1,3}}{2}$$
$$y_{1,3} \cdot x_2 = -\frac{z_{1,2,3}}{2}$$
$$y_{1,4} \cdot x_1 = -\frac{z_{1,1,4}}{2}$$
$$y_{1,4} \cdot x_2 = -\frac{z_{1,2,4}}{2}$$
$$y_{1,4} \cdot x_3 = -\frac{z_{1,3,4}}{2}$$
$$y_{2,3} \cdot x_1 = -\frac{z_{1,2,3}}{2}$$
$$y_{2,3} \cdot x_2 = -\frac{z_{2,2,3}}{2}$$
$$y_{2,4} \cdot x_1 = -\frac{z_{1,2,4}}{2}$$
$$y_{2,4} \cdot x_2 = -\frac{z_{2,2,4}}{2}$$
$$y_{2,4} \cdot x_3 = -\frac{z_{2,3,4}}{2}$$
$$y_{3,4} \cdot x_1 = -\frac{z_{1,3,4}}{2}$$
$$y_{3,4} \cdot x_2 = -\frac{z_{2,3,4}}{2}$$
$$y_{3,4} \cdot x_3 = -\frac{z_{3,3,4}}{2}$$

When trying to find an example of a 3-step nilpotent Lie algebra without a Novikov structure, we know from [16] that such an example must have at least 4 generators.

Any 3-step nilpotent Lie algebra on 4 generators is a quotient $\mathfrak{g}/I$ of the Lie algebra $\mathfrak{g}$ described in example 3.3.2, where I is an ideal of $\mathfrak{g}$. So, in order to find such a Lie algebra $\mathfrak{g}/I$ without Novikov structure, we have to choose an ideal I which is certainly not an ideal of $\mathfrak{g}$, seen as a Novikov algebra.

The following proposition uses such an example, more precisely it takes I to be the Lie algebra ideal generated by the following elements:

$y_{1,3}$,	$z_{1,1,2}$,	$z_{2,2,4}$,	$z_{2,2,3} - z_{2,1,2}$,
$y_{3,4} + y_{1,2}$,	$z_{1,1,3}$,	$z_{3,1,3}$,	$z_{3,3,4} - z_{1,2,3}$,
	$z_{1,3,4}$,	$z_{3,1,4}$,	$z_{4,1,4} - z_{3,2,4}$,
	$z_{2,1,3}$,	$z_{3,2,3}$,	$z_{4,3,4} - z_{1,2,4}$,
	$z_{2,1,4}$,	$z_{4,2,4}$,	$z_{2,3,4} + z_{2,1,2}$.

Note that this indeed is not an ideal for the Novikov product since

$$x_2 \cdot y_{1,3} = \frac{-z_{1,2,3}}{2} + z_{2,1,3} \notin I.$$

We have the following proposition:

Proposition 3.3.3. *Consider the following* 3*-step nilpotent Lie algebra* $\mathfrak{g}$ *on* 4 *generators of dimension* 13*, with basis* $(x_1, \dots, x_{13})$ *and non-trivial Lie brackets*

$$\begin{array}{lll}
[x_1, x_2] = x_5, & [x_2, x_3] = x_7, & [x_3, x_5] = -x_{11}, \\
[x_1, x_4] = x_6, & [x_2, x_4] = x_8, & [x_3, x_8] = x_9, \\
[x_1, x_6] = x_{10}, & [x_2, x_5] = x_{13}, & [x_4, x_5] = -x_{12}, \\
[x_1, x_7] = x_{11}, & [x_2, x_7] = x_{13}. & [x_4, x_6] = x_9, \\
[x_1, x_8] = x_{12}, & [x_3, x_4] = -x_5, & [x_4, x_7] = x_9 + x_{13}.
\end{array}$$

This Lie algebra does not admit a Novikov structure.

Note that $\mathfrak{g}$ admits a left-symmetric structure since it is positively graded.

Proof. We will assume that $\mathfrak{g}$ admits a Novikov structure and show that this leads to a contradiction. We use Mathematica to make our computations.

We express the adjoint operators $\mathrm{ad}(x_i)$ and the left (resp. right) multiplication operators $L(x_i)$ (resp. $R(x_i)$) as matrices with respect to the basis $x_1, x_2, \dots, x_{13}$. The adjoint operators $\mathrm{ad}(x_i)$ are given by the Lie brackets of $\mathfrak{g}$, while the left multiplication operators are unknown. We denote the (j,k)-th entry of $L(x_i)$ by

$$L(x_i)_{j,k} = x^i_{j,k}.$$

We use the convention that the j-th column of $L(x_i)$ gives the coordinates of $L(x_i)(x_j)$. Note that once the entries of the left multiplication operators are chosen, the right multiplication operators are given by $R(x_i)_{j,k} = x^k_{j,i}$.

We have to satisfy all relations given by (3.1), (3.2) and (3.3), where x, y and z run over all basis vectors. This leads to a huge system of quadratic equations in the variables $x^i_{j,k}$ for $1 \le i,j,k \le 13$, summing up to a total of $13^3 = 2197$ variables. We need to show that these equations are contradictory.

At first sight, this seems to be a rather hopeless task. However, we can use our knowledge on ideals in a Novikov algebra. Then we find that a lot of the unknowns $x^i_{j,k}$ already have to be zero. In the table below, we list the triples (i,j,k) for which we already know that $x^i_{j,k} = 0$:

$1 \leq i \leq 13$, $1 \leq j \leq 4$, $5 \leq k \leq 13$,
because $x_k \in \gamma_2(\mathfrak{g})$ and this is an ideal.
$1 \leq i \leq 13$, $5 \leq j \leq 8$, $9 \leq k \leq 13$,
because $x_k \in \gamma_3(\mathfrak{g})$ and this is an ideal.
$5 \leq i \leq 13$, $1 \leq j \leq 4$, $1 \leq k \leq 4$,
because $x_i \in \gamma_2(\mathfrak{g})$ and this is an ideal.
$5 \leq i \leq 13$, $5 \leq j \leq 8$, $5 \leq k \leq 8$,
because $x_i, x_k \in \gamma_2(\mathfrak{g})$ and $\gamma_2(\mathfrak{g}) \cdot \gamma_2(\mathfrak{g}) \subseteq \gamma_3(\mathfrak{g})$.
$5 \leq i \leq 13$, $9 \leq j \leq 13$, $9 \leq k \leq 13$,
because $x_i \in \gamma_2(\mathfrak{g}), x_k \in \gamma_3(\mathfrak{g})$ and $\gamma_2(\mathfrak{g}) \cdot \gamma_3(\mathfrak{g}) \subseteq \gamma_4(\mathfrak{g}) = 0$.
$9 \leq i \leq 13$, $5 \leq j \leq 8$, $1 \leq k \leq 4$,
because $x_i \in \gamma_3(\mathfrak{g})$ and this is an ideal.
$9 \leq i \leq 13$, $9 \leq j \leq 13$, $5 \leq k \leq 8$,
because $x_i \in \gamma_3(\mathfrak{g}), x_k \in \gamma_2(\mathfrak{g})$ and $\gamma_3(\mathfrak{g}) \cdot \gamma_2(\mathfrak{g}) \subseteq \gamma_4(\mathfrak{g}) = 0$.

It follows that 1421 of the $x^i_{j,k}$ have to be zero, leaving us with 776 variables.

On the other hand the conditions

$$\mathrm{ad}(x_i) = L(x_i) - R(x_i), \;\; 1 \leq i \leq 13$$

yield a (large but very simple) system of linear equations; allowing us to determine 352 variables $x^i_{j,k}$ in dependence of the remaining $776 - 352 = 424$ ones.

To get a further reduction we use that

$$x_i \cdot [x_j, x_k] + x_j \cdot [x_k, x_i] + x_k \cdot [x_i, x_j] = 0, \quad 1 \leq i < j < k \leq 13,$$

which is the same as

$$L(x_i)(\mathrm{ad}(x_j)x_k) + L(x_j)(\mathrm{ad}(x_k)x_i) + L(x_k)(\mathrm{ad}(x_i)x_j) = 0,$$

for $1 \leq i < j < k \leq 13$. Again this leads to a system of linear equations, this time specifying 156 unknowns in terms of the other ones, leaving $424 - 156 = 268$ variables.

Now, we consider the operator identity (3.4), i.e.,

$$L([x_i, x_j]) + \mathrm{ad}([x_i, x_j]) - [\mathrm{ad}(x_i), L(x_j)] - [L(x_i), \mathrm{ad}(x_j)] = 0, \tag{3.7}$$

for $1 \leq i < j \leq 13$. Note that for any pair (i, j), we can write $[x_i, x_j]$ as a linear combination of the x_k, $1 \leq k \leq 13$. Hence we can also write $L([x_i, x_j])$ as the corresponding linear combination of the $L(x_k)$. Doing this, we obtain another system of linear equations, determining 210 extra variables, leaving $268 - 210 = 58$ free variables.

Finally, we use that the right multiplications have to commute, i.e.,

$$R(x_i)R(x_j) - R(x_j)R(x_i) = 0, \quad 1 \leq i < j \leq 13.$$

This yields a system of quadratic equations, which is immediately contradictory. In fact, when taking $i = 1$ and $j = 2$, one obtains the equation $0 = \frac{1}{8}$, which is the desired contradiction.

More precisely, after solving (3.7) we find

$$\begin{aligned}
x_3 \cdot x_1 &\in \gamma_3(\mathfrak{g}), \\
x_3 \cdot x_2 &= \alpha x_5 - \frac{1}{2} x_7 + b \text{ with } \alpha \in k, b \in \gamma_3(\mathfrak{g}), \\
\gamma_3(\mathfrak{g}) \cdot x_1 &= 0, \\
\gamma_3(\mathfrak{g}) \cdot x_2 &= 0, \\
x_5 \cdot x_1 &= 0, \\
x_7 \cdot x_1 &= -\frac{1}{4} x_{11}.
\end{aligned}$$

This implies that $(x_3 \cdot x_2) \cdot x_1 - (x_3 \cdot x_1) \cdot x_2$ is equal to $\frac{1}{8} x_{11}$ and hence cannot be 0, which gives a contradiction. □

To end this section, we will prove that any Novikov structure on the free 3-step nilpotent Lie algebra on at least 3 generators is complete:

Proposition 3.3.4. *Let $\mathfrak{g}$ be the free 3-step nilpotent Lie algebra on n generators $x_1, x_2, \ldots, x_n$ ($n \geq 3$). Then any Novikov structure on $\mathfrak{g}$ is complete.*

Proof. Consider the basis of $\mathfrak{g}$ as in the proof of proposition 3.3.1.

For every element $x \in \mathfrak{g}$ we define $\operatorname{Sup}(x)$ as the set of all basis elements of $\mathfrak{g}$ that have nonzero coefficient in the representation of x as linear combination of the basis elements.

We introduce the following notation for this proof. If $i \neq j$, then we denote by

$$\pm y_{i,j} \quad \text{the element} \quad \begin{array}{ll} y_{i,j} & \text{if } i < j, \\ y_{j,i} & \text{if } j < i. \end{array}$$

Let $j \neq k$, then we denote by

$$\pm z_{i,j,k} \quad \text{the element} \quad \begin{array}{ll} z_{i,j,k} & \text{if } j < k, \\ z_{i,k,j} & \text{if } k < j. \end{array}$$

Considering all multiple Lie brackets (up to anti-commutativity) that can be produced by the generators, we find that the basis element ($i \neq j, k$):

$$\begin{array}{lll} y_{i,j} & \text{only belongs to} & \operatorname{Sup}([x_i, x_j]), \\ z_{k,j,k} & & \operatorname{Sup}([x_k, [x_j, x_k]]), \\ z_{j,j,k} & & \operatorname{Sup}([x_j, [x_j, x_k]]), \\ z_{i,j,k} & & \operatorname{Sup}([x_i, [x_j, x_k]]) \text{ and to } \operatorname{Sup}([x_k, [x_i, x_j]]). \end{array} \tag{3.8}$$

Now suppose that $(A, \cdot)$ is a Novikov structure on $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, to prove completeness it is enough to show that all left multiplications are nilpotent because of lemma 3.1.4. We shall prove that all left multiplication matrices are strictly lower triangular.

We start by proving the following fact which will be used later on:

$$\pm y_{j,k} \notin \operatorname{Sup}(y_{i,j} \cdot x_k) \quad \text{if } 1 \leq i, j, k \leq n,\ i, j, k \text{ mutually different.} \tag{3.9}$$

Suppose that $1 \leq i, j, k \leq n$ with i, j, k mutually different. By repeatedly using identity (3.3) and the identities of lemma 3.2.1 and lemma 3.2.3 we obtain from the fact that $\mathfrak{g}$ is 3-step nilpotent that

$$\begin{aligned} 0 = [[x_k, [x_k, x_i]], x_j] &= [x_k, [x_k, x_i]] \cdot x_j - x_j \cdot [x_k, [x_k, x_i]] \\ &= [x_k, [x_k, x_i]] \cdot x_j + x_j \cdot [[x_k, x_i], x_k] \\ &= \frac{1}{2}[x_k \cdot x_j, [x_k, x_i]] + \frac{1}{2}[x_k, [x_k, x_i] \cdot x_j] \\ &\quad - [x_k, x_i] \cdot [x_k, x_j] - x_k \cdot [x_j, [x_k, x_i]]. \end{aligned}$$

We can rewrite this as

$$\begin{aligned} 0 = &\frac{1}{2}[x_k \cdot x_j, [x_k, x_i]] - \frac{1}{2}[x_k, [x_i, x_j] \cdot x_k] \\ &- \frac{1}{2}[x_k, [x_j, x_k] \cdot x_i] - \frac{1}{2}[x_k \cdot [x_k, x_j], x_i] \\ &- \frac{1}{2}[x_k, x_i \cdot [x_k, x_j]] - x_k \cdot [x_j, [x_k, x_i]]. \end{aligned}$$

Again using the fact that $\mathfrak{g}$ is 3-step nilpotent we find

$$\begin{aligned}
0 &= \frac{1}{2}[x_k \cdot x_j, [x_k, x_i]] - \frac{1}{2}[x_k, [x_i, x_j] \cdot x_k] \\
&\quad - \frac{1}{2}[x_k, x_i \cdot [x_j, x_k]] - \frac{1}{2}[x_k \cdot [x_k, x_j], x_i] \\
&\quad - \frac{1}{2}[x_k, x_i \cdot [x_k, x_j]] - [x_j, [x_k, x_i]] \cdot x_k \\
&= \frac{1}{2}[x_k \cdot x_j, y_{k,i}] - \frac{1}{2}[x_k, y_{i,j} \cdot x_k] - \frac{1}{2}[x_k \cdot y_{k,j}, x_i] - [x_j, y_{k,i}] \cdot x_k. \qquad (3.10)
\end{aligned}$$

Since i, j and k are mutually different, it follows from (3.8) that $\pm z_{k,j,k} \notin \bigcup_{a\in\mathfrak{g}} \mathrm{Sup}([x_i, a])$, $\pm z_{k,j,k} \notin \bigcup_{a\in\mathfrak{g}} \mathrm{Sup}([x_j, a])$ and $\pm z_{k,j,k} \notin \bigcup_{a\in\mathfrak{g}} \mathrm{Sup}([y_{k,i}, a])$. So using

$$[x_j, y_{k,i}] \cdot x_k = \frac{1}{2}[x_j \cdot x_k, y_{k,i}] + \frac{1}{2}[x_j, y_{k,i} \cdot x_k],$$

we find that $\pm z_{k,j,k} \notin \mathrm{Sup}([x_j, y_{k,i}] \cdot x_k)$.

Hence from (3.10) we get that $\pm z_{k,j,k} \notin \mathrm{Sup}([x_k, y_{i,j} \cdot x_k])$ and hence $\pm y_{j,k} \notin \mathrm{Sup}(y_{i,j} \cdot x_k)$. We can conclude that (3.9) is true.

We will now prove the following four claims:

- $x_j \notin \mathrm{Sup}(x_i \cdot x_j)$ and $x_i \notin \mathrm{Sup}(x_i \cdot x_j) \quad$ if $1 \le i, j \le n,\ i \ne j$,
- $x_k \notin \mathrm{Sup}(x_i \cdot x_j) \quad$ if $1 \le i, j, k \le n,\ i \ne k, j \ne k$,
- $x_i \notin \mathrm{Sup}(x_i \cdot x_i) \quad$ if $1 \le i < n$,
- $x_n \notin \mathrm{Sup}(x_n \cdot x_n)$.

Once this is proven, we will have that $x_i \cdot x_j \in \gamma_2(\mathfrak{g})$ for all $1 \le i, j \le n$. Then, by lemma 3.2.6 , we have $\gamma_1(A) \cdot \gamma_1(A) \subseteq \gamma_2(A)$ and according to lemma 3.2.7 this means that $\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+2}(A)$ for all $i, j \ge 0$. This exactly says that the left multiplication matrices are strictly lower triangular and hence nilpotent.

For the first claim, suppose that $1 \le i, j \le n$ with $i \ne j$. Take $1 \le k \le n$ such that $k \ne i$ and $k \ne j$, this is possible since $n \ge 3$.

From (3.9) we get that $\pm y_{k,j} \notin \mathrm{Sup}(y_{i,k} \cdot x_j)$ since i, j, k are mutually different. We also have $\pm y_{k,j} \notin \bigcup_{a \in \mathfrak{g}} \mathrm{Sup}([x_i, a])$ by (3.8). Hence it follows from

$$y_{i,k} \cdot x_j = \frac{1}{2}[x_i \cdot x_j, x_k] + \frac{1}{2}[x_i, x_k \cdot x_j]$$

that $\pm y_{k,j} \notin \mathrm{Sup}([x_i \cdot x_j, x_k])$ and this is only possible if $x_j \notin \mathrm{Sup}(x_i \cdot x_j)$.

From this result we can also derive that $x_i \notin \mathrm{Sup}(x_j \cdot x_i)$. Now since

$$x_i \cdot x_j = [x_i, x_j] + x_j \cdot x_i = y_{i,j} + x_j \cdot x_i,$$

we also have $x_i \notin \mathrm{Sup}(x_i \cdot x_j)$. We can conclude that

$$x_j \notin \mathrm{Sup}(x_i \cdot x_j) \text{ and } x_i \notin \mathrm{Sup}(x_i \cdot x_j) \quad \text{if } 1 \leq i, j \leq n,\ i \neq j. \tag{3.11}$$

To prove the second claim, suppose that $1 \leq i, j, k \leq n$ with $i \neq k, j \neq k$. Take $1 \leq l \leq n$ such that $l \neq i$ and $l \neq k$. This is possible since $n \geq 3$.

Now we have from (3.8) that $\pm z_{k,k,l} \notin \bigcup_{a \in \mathfrak{g}} \mathrm{Sup}([x_i, a])$, $\pm z_{k,k,l} \notin \bigcup_{a \in \mathfrak{g}} \mathrm{Sup}([x_j, a])$ and $\pm z_{k,k,l} \notin \bigcup_{a \in \mathfrak{g}} \mathrm{Sup}([x_l, a])$. Hence from the fact that $\mathfrak{g}$ is 3-step nilpotent we get

$$y_{j,k} \cdot y_{i,l} = y_{i,l} \cdot y_{j,k} = \frac{1}{2}[x_i \cdot y_{j,k}, x_l] + \frac{1}{2}[x_i, x_l \cdot y_{j,k}]$$

and it follows that $\pm z_{k,k,l} \notin \mathrm{Sup}(y_{j,k} \cdot y_{i,l})$. Again since $\mathfrak{g}$ is 3-step nilpotent we have

$$\begin{aligned} y_{j,k} \cdot y_{i,l} &= \frac{1}{2}[x_j \cdot y_{i,l}, x_k] + \frac{1}{2}[x_j, x_k \cdot y_{i,l}] \\ &= \frac{1}{2}[y_{i,l} \cdot x_j, x_k] + \frac{1}{2}[x_j, x_k \cdot y_{i,l}] \end{aligned}$$

and so we get $\pm z_{k,k,l} \notin \mathrm{Sup}([y_{i,l} \cdot x_j, x_k])$. This implies that $\pm y_{k,l} \notin \mathrm{Sup}(y_{i,l} \cdot x_j)$.

We also have $\pm y_{k,l} \notin \bigcup_{a \in \mathfrak{g}} \mathrm{Sup}([x_i, a])$. Now it follows from

$$y_{i,l} \cdot x_j = \frac{1}{2}[x_i \cdot x_j, x_l] + \frac{1}{2}[x_i, x_l \cdot x_j]$$

that $\pm y_{k,l} \notin \mathrm{Sup}([x_i \cdot x_j, x_l])$ and this is only possible if $x_k \notin \mathrm{Sup}(x_i \cdot x_j)$. We can conclude that

$$x_k \notin \mathrm{Sup}(x_i \cdot x_j) \quad \text{if } 1 \leq i, j, k \leq n,\ i \neq k, j \neq k.$$

For the third claim, suppose that $1 \leq i < n$. Take $1 \leq j < n$ with $i \neq j$, this is possible since $n \geq 3$.

We have $y_{j,n} \notin \mathrm{Sup}(y_{i,n} \cdot x_j)$ because of (3.9). So it follows from (3.8) that $z_{i,j,n} \notin \mathrm{Sup}([y_{i,n} \cdot x_j, x_i])$. We also have $z_{i,j,n} \notin \bigcup_{a \in \mathfrak{g}} \mathrm{Sup}([x_j, a])$, so since $\mathfrak{g}$ is 3-step nilpotent we have

$$y_{i,n} \cdot y_{j,i} = y_{j,i} \cdot y_{i,n} = \frac{1}{2}[x_j \cdot y_{i,n}, x_i] + \frac{1}{2}[x_j, x_i \cdot y_{i,n}]$$

$$= \frac{1}{2}[y_{i,n} \cdot x_j, x_i] + \frac{1}{2}[x_j, x_i \cdot y_{i,n}]$$

and we get $z_{i,j,n} \notin \mathrm{Sup}(y_{i,n} \cdot y_{j,i})$.

Again from (3.9) we get $y_{j,n} \notin \mathrm{Sup}(y_{j,i} \cdot x_n)$. Now

$$x_n \cdot y_{j,i} = [x_n, y_{j,i}] + y_{j,i} \cdot x_n = z_{n,j,i} + y_{j,i} \cdot x_n$$

implies that $y_{j,n} \notin \mathrm{Sup}(x_n \cdot y_{j,i})$. Hence we have $z_{i,j,n} \notin \mathrm{Sup}([x_i, x_n \cdot y_{j,i}])$.

From

$$y_{i,n} \cdot y_{j,i} = \frac{1}{2}[x_i \cdot y_{j,i}, x_n] + \frac{1}{2}[x_i, x_n \cdot y_{j,i}]$$

we deduce that $z_{i,j,n} \notin \mathrm{Sup}([x_i \cdot y_{j,i}, x_n])$, so by (3.8) we have $\pm y_{j,i} \notin \mathrm{Sup}(x_i \cdot y_{j,i})$.

Now it follows from

$$x_j \cdot y_{j,i} = [x_i, y_{j,i}] + y_{j,i} \cdot x_i = z_{i,j,i} + y_{j,i} \cdot x_i$$

that $\pm y_{j,i} \notin \mathrm{Sup}(y_{j,i} \cdot x_i)$.

From (3.11) it follows that $x_j \notin \mathrm{Sup}(x_j \cdot x_i)$ since $i \neq j$, so $\pm y_{j,i} \notin \mathrm{Sup}([x_j \cdot x_i, x_i])$. Now

$$y_{j,i} \cdot x_i = \frac{1}{2}[x_j \cdot x_i, x_i] + \frac{1}{2}[x_j, x_i \cdot x_i]$$

implies that $\pm y_{j,i} \notin \mathrm{Sup}([x_j, x_i \cdot x_i])$, and hence $x_i \notin \mathrm{Sup}(x_i \cdot x_i)$. We can conclude that

$$x_i \notin \mathrm{Sup}(x_i \cdot x_i) \quad \text{if } 1 \leq i < n.$$

Finally we will proof that $x_n \notin \mathrm{Sup}(x_n \cdot x_n)$. Take $1 \leq i < j < n$, this is possible since $n \geq 3$.

From (3.9) it follows that $y_{i,j} \notin \operatorname{Sup}(y_{i,n} \cdot x_j)$ since i, j, n are mutually different. Hence we have $z_{i,j,n} \notin \operatorname{Sup}([y_{i,n} \cdot x_j, x_n])$.

We also have $z_{i,j,n} \notin \bigcup_{a \in \mathfrak{g}} \operatorname{Sup}([x_j, a])$, so since $\mathfrak{g}$ is 3-step nilpotent we get

$$y_{i,n} \cdot y_{j,n} = y_{j,n} \cdot y_{i,n} = \frac{1}{2}[x_j \cdot y_{i,n}, x_n] + \frac{1}{2}[x_j, x_n \cdot y_{i,n}]$$

$$= \frac{1}{2}[y_{i,n} \cdot x_j, x_n] + \frac{1}{2}[x_j, x_n \cdot y_{i,n}]$$

and thus $z_{i,j,n} \notin \operatorname{Sup}(y_{i,n} \cdot y_{j,n})$.

Again from (3.9) it follows that $y_{i,j} \notin \operatorname{Sup}(y_{j,n} \cdot x_i)$ since i, j, n are mutually different. Hence

$$x_i \cdot y_{j,n} = [x_i, y_{j,n}] + y_{j,n} \cdot x_i = z_{i,j,n} + y_{j,n} \cdot x_i$$

implies that $y_{i,j} \notin \operatorname{Sup}(x_i \cdot y_{j,n})$ and hence $z_{i,j,n} \notin \operatorname{Sup}([x_i \cdot y_{j,n}, x_n])$.

From

$$y_{i,n} \cdot y_{j,n} = \frac{1}{2}[x_i \cdot y_{j,n}, x_n] + \frac{1}{2}[x_i, x_n \cdot y_{j,n}]$$

it now follows that $z_{i,j,n} \notin \operatorname{Sup}([x_i, x_n \cdot y_{j,n}])$ and this means that $y_{j,n} \notin \operatorname{Sup}(x_n \cdot y_{j,n})$.

Now

$$y_{j,n} \cdot x_n = [y_{j,n}, x_n] + x_n \cdot y_{j,n} = -z_{n,j,n} + x_n \cdot y_{j,n}$$

implies $y_{j,n} \notin \operatorname{Sup}(y_{j,n} \cdot x_n)$.

From (3.11) we get $x_j \notin \operatorname{Sup}(x_j \cdot x_n)$ since $j \neq n$. This means that $y_{j,n} \notin \operatorname{Sup}([x_j \cdot x_n, x_n])$.

Finally we have

$$y_{j,n} \cdot x_n = \frac{1}{2}[x_j \cdot x_n, x_n] + \frac{1}{2}[x_j, x_n \cdot x_n]$$

and so we get that $y_{j,n} \notin \operatorname{Sup}([x_j, x_n \cdot x_n])$, and we can conclude that $x_n \notin \operatorname{Sup}(x_n \cdot x_n)$. This finishes the proof. □

Remark that we can not improve this proposition in any way. It will not be true for 2 generators nor for nilpotency class 2, this is shown in the following two examples. In the next section we will prove that Novikov structures do not even exist for nilpotency class bigger than 3, so the previous proposition really includes all cases.

Example 3.3.5. *The proposition is not true for 2 generators. Indeed, the following Novikov structure on the free 3-step nilpotent Lie algebra on 2 generators is not complete:*

$x_1 \cdot x_1 = x_1,$
$x_1 \cdot x_2 = x_2 + y_{1,2},$
$x_1 \cdot y_{1,2} = y_{1,2} + z_{1,1,2},$
$x_1 \cdot z_{1,1,2} = z_{1,1,2},$
$x_1 \cdot z_{2,1,2} = z_{2,1,2},$
$x_2 \cdot x_1 = x_2,$
$x_2 \cdot y_{1,2} = \frac{1}{2} z_{2,1,2},$
$x_2 \cdot z_{1,1,2} = \frac{1}{2} z_{2,1,2},$
$y_{1,2} \cdot x_1 = y_{1,2},$
$y_{1,2} \cdot x_2 = -\frac{1}{2} z_{2,1,2},$
$y_{1,2} \cdot y_{1,2} = -\frac{1}{2} z_{2,1,2},$
$z_{1,1,2} \cdot x_1 = z_{1,1,2},$
$z_{1,1,2} \cdot x_2 = \frac{1}{2} z_{2,1,2},$
$z_{2,1,2} \cdot x_1 = z_{2,1,2}.$

Example 3.3.6. *The proposition also does not hold for the free 2-step nilpotent Lie algebras. The following Novikov structure on the free 2-step nilpotent Lie algebra on 3 generators is not complete:*

$x_1 \cdot x_2 = x_1,$
$x_1 \cdot y_{2,3} = \frac{1}{2} y_{1,3},$
$x_2 \cdot x_1 = x_1 - y_{1,2},$
$x_2 \cdot x_2 = x_2,$
$x_2 \cdot x_3 = x_3 + y_{2,3},$
$x_2 \cdot y_{1,2} = y_{1,2},$
$x_2 \cdot y_{1,3} = y_{1,3},$
$x_2 \cdot y_{2,3} = y_{2,3},$
$x_3 \cdot x_1 = -y_{1,3},$
$x_3 \cdot x_2 = x_3,$
$x_3 \cdot y_{1,2} = \frac{1}{2} y_{1,3},$
$y_{1,2} \cdot x_2 = y_{1,2},$
$y_{1,2} \cdot x_3 = \frac{1}{2} y_{1,3},$
$y_{1,3} \cdot x_2 = y_{1,3},$
$y_{2,3} \cdot x_1 = \frac{1}{2} y_{1,3},$
$y_{2,3} \cdot x_2 = y_{2,3}.$

3.4 The non existence of Novikov structures on free nilpotent Lie algebras of class at least 4

In the previous section we proved that all free 3-step nilpotent Lie algebras admit a Novikov structure. We will prove in this section that for higher nilpotency class no Novikov structures exist at all:

Proposition 3.4.1. *Let $\mathfrak{g}$ be the free p-step nilpotent Lie algebra ($p \geq 4$) on n generators $x_1, \ldots, x_n$. Then there does not exist a Novikov structure on $\mathfrak{g}$.*

Proof. This proposition will be proved in several steps. The non existence of Novikov structures in the general case will be deduced from the non existence for $p = 4$. This special case will be proved on the one hand for $n = 2$ and more generally for $n \geq 3$.

Free 4-step nilpotent on 2 generators

In this section we prove the proposition for $p = 4$ and $n = 2$.

Denote the basis of $\mathfrak{g}$ by

$$
\begin{aligned}
&x_1, x_2, \\
&x_3 = y_{1,2} = [x_1, x_2], \\
&x_4 = z_{1,1,2} = [x_1, [x_1, x_2]], && x_5 = z_{2,1,2} = [x_2, [x_1, x_2]], \\
&x_6 = w_{1,1,1,2} = [x_1, [x_1, [x_1, x_2]]], && x_7 = w_{1,2,1,2} = [x_1, [x_2, [x_1, x_2]]] \\
&x_8 = w_{2,2,1,2} = [x_2, [x_2, [x_1, x_2]]], && \quad = w_{2,1,1,2} = [x_2, [x_1, [x_1, x_2]]].
\end{aligned}
$$

As in the proof of proposition 3.3.3 it can be proved by direct calculations that no Novikov structure exists on $\mathfrak{g}$. We again use Mathematica to make our computations.

Suppose that we do have a Novikov structure on $\mathfrak{g}$. Following the same steps as in that proof, we start from initially 512 variables and reduce them successively to 149, 85, 74 and in the end 18 variables.

Now it only has to be checked whether the right multiplications commute.

At this moment the following facts are known (with $a, b, c \in k$):

$$
\begin{aligned}
&x_1 \cdot x_1 = bx_3 + A && \text{with } A \in \gamma_3(\mathfrak{g}), \\
&x_2 \cdot x_1 = (-1 + a)x_3 + B && \text{with } B \in \gamma_3(\mathfrak{g}), \\
&x_2 \cdot x_2 = cx_3 + C && \text{with } C \in \gamma_3(\mathfrak{g}), \\
&x_1 \cdot x_3 = \left(\frac{1}{2} + \frac{1}{2}a\right)x_4 - \frac{1}{2}bx_5 + D && \text{with } D \in \gamma_4(\mathfrak{g}), \\
&x_2 \cdot x_3 = \frac{1}{2}cx_4 + \left(1 - \frac{1}{2}a\right)x_5 + E && \text{with } E \in \gamma_4(\mathfrak{g}),
\end{aligned}
$$

$$
\begin{aligned}
\gamma_4(\mathfrak{g}) \cdot x_1 &= 0, \\
\gamma_4(\mathfrak{g}) \cdot x_2 &= 0, \\
\gamma_3(\mathfrak{g}) \cdot x_3 &= 0.
\end{aligned}
$$

The coefficient of x_6 in	$x_4 \cdot x_1$	is given by	$-\frac{1}{4}+\frac{1}{4}a$,
x_7 in	$x_4 \cdot x_1$	is given by	$-\frac{1}{4}b$,
x_8 in	$x_4 \cdot x_1$	is given by	0,
The coefficient of x_8 in	$x_4 \cdot x_2$	is given by	0,
The coefficient of x_7 in	$x_5 \cdot x_1$	is given by	$-\frac{1}{4}+\frac{1}{4}a$,
x_8 in	$x_5 \cdot x_1$	is given by	$-\frac{1}{4}b$,
The coefficient of x_8 in	$x_5 \cdot x_2$	is given by	$-\frac{1}{4}a$,
The coefficient of x_7 in	$x_3 \cdot x_3$	is given by	$\frac{1}{4}-\frac{1}{2}a$,
x_8 in	$x_3 \cdot x_3$	is given by	$\frac{1}{4}b$,

From this, it follows that

$$(x_1 \cdot x_3) \cdot x_1 - (x_1 \cdot x_1) \cdot x_3 = \Big(\frac{1}{2}+\frac{1}{2}a\Big)x_4 \cdot x_1 - \frac{1}{2}bx_5 \cdot x_1 - bx_3 \cdot x_3.$$

The coefficient of x_8 in this expression is

$$\frac{1}{8}b^2 - \frac{1}{4}b^2.$$

Since this should be 0 for the right multiplications to commute we find that $b = 0$.

Knowing this, the coefficient of x_6 in the above expression is

$$\Big(\frac{1}{2}+\frac{1}{2}a\Big)\Big(-\frac{1}{4}+\frac{1}{4}a\Big) = \frac{1}{8}(1+a)(-1+a).$$

From this it follows that a should be $+1$ or -1 for the right multiplications to commute.

Now let us take a look at the following identity, using the above facts:

$$\begin{aligned}&(x_2 \cdot x_3) \cdot x_1 - (x_2 \cdot x_1) \cdot x_3\\&= \frac{1}{2}cx_4 \cdot x_1 + \Big(1 - \frac{1}{2}a\Big)x_5 \cdot x_1 - (-1+a)x_3 \cdot x_3.\end{aligned}$$

The coefficient of x_7 is

$$\begin{aligned}&\Big(1 - \frac{1}{2}a\Big)\Big(-\frac{1}{4} + \frac{1}{4}a\Big) - (-1+a)\Big(\frac{1}{4} - \frac{1}{2}a\Big)\\&= \frac{1}{8}((2-a)(-1+a) - (-1+a)(2-4a))\\&= \frac{3}{8}(-1+a)a.\end{aligned}$$

This implies, together with the restrictions on a found above, that $a = 1$ if we want the right multiplications to commute.

Finally, look at the following identity

$$\begin{aligned}&(x_2 \cdot x_3) \cdot x_2 - (x_2 \cdot x_2) \cdot x_3\\&= \frac{1}{2}cx_4 \cdot x_2 + \frac{1}{2}x_5 \cdot x_2 - cx_3 \cdot x_3.\end{aligned}$$

The coefficient of x_8 is $-\frac{1}{8}$. Since this is nonzero, the right multiplications do not commute and hence we get a contradiction.

We can conclude that no Novikov structure exists on the free 4-step nilpotent Lie algebra on 2 generators.

Free 4-step nilpotent on at least 3 generators

In this section we will prove the proposition for $p = 4$ and $n \geq 3$.

Given a basis of $\mathfrak{g}$, we will use, as in the proof of proposition 3.3.4, $\mathrm{Sup}(x)$ to denote the set of all basis elements of $\mathfrak{g}$ that have nonzero coefficient in the representation of x as linear combination of the basis elements.

The following elements form a basis of $\mathfrak{g}$:

$$
\begin{aligned}
&x_1, x_2, \ldots, x_n, \\
y_{i,j} &= [x_i, x_j] && \text{for } 1 \le i < j \le n, \\
z_{i,j,k} &= [x_i, [x_j, x_k]] && \text{for } 1 \le i, j, k \le n; j < k; i \ge j, \\
w_{i,j,k,l} &= [x_i, [x_j, [x_k, x_l]]] && \text{for } 1 \le i, j, k, l \le n; k < l; i \ge j \ge k, \\
t_{i,j,k,l} &= [[x_i, x_j], [x_k, x_l]] && \text{for } 1 \le i, j, k, l \le n; i < j; k < l \text{ and} \\
& && \quad \text{either } i = k; j < l \text{ or } i < k.
\end{aligned}
$$

For the following three basis elements we write down for which nonzero multiple Lie brackets (up to anti-commutativity) they belong to $\mathrm{Sup}(x)$ and with what coefficient:

$$
\begin{aligned}
t_{1,2,1,3} &\in \mathrm{Sup}(t_{1,2,1,3}) \text{ with coefficient } 1, \\
&\in \mathrm{Sup}(w_{1,1,2,3}) \text{ with coefficient } 2, \\
&\in \mathrm{Sup}(w_{1,2,1,3}) \text{ with coefficient } 1, \\
&\in \mathrm{Sup}(w_{1,3,1,2}) \text{ with coefficient } -1, \qquad (3.12) \\
t_{1,2,2,3} &\in \mathrm{Sup}(t_{1,2,2,3}) \text{ with coefficient } 1, \\
&\in \mathrm{Sup}(w_{1,2,2,3}) \text{ with coefficient } 2, \\
&\in \mathrm{Sup}(w_{2,1,2,3}) \text{ with coefficient } 1, \\
&\in \mathrm{Sup}(w_{2,3,1,2}) \text{ with coefficient } -1, \qquad (3.13) \\
t_{1,3,2,3} &\in \mathrm{Sup}(t_{1,3,2,3}) \text{ with coefficient } 1, \\
&\in \mathrm{Sup}(w_{1,3,2,3}) \text{ with coefficient } 1, \\
&\in \mathrm{Sup}(w_{2,3,1,3}) \text{ with coefficient } -1. \qquad (3.14)
\end{aligned}
$$

Suppose that $(A, \cdot)$ defines a Novikov structure on $\mathfrak{g}$.

In what follows we shall denote the coefficient of $y_{p,q}$ in $x_i \cdot x_j$ by $x[i\|p,q\|j]$ and similarly for all other products and coefficients.

Remark 3.4.2. *Let $1 \leq i < j \leq n$. Since $[x_i, x_j] = y_{i,j}$ we have that $x[j\|i,j\|i] = -1 + x[i\|i,j\|j]$.*

By (3.3), lemma 3.2.1 and lemma 3.2.3 we have

$$y_{i,j} \cdot y_{k,l} = y_{i,j} \cdot [x_k, x_l] = -x_k \cdot [x_l, y_{i,j}] - x_l \cdot [y_{i,j}, x_k]$$

$$= -[x_k, [x_l, y_{i,j}]] - [x_l, y_{i,j}] \cdot x_k + [x_l, [x_k, y_{i,j}]] + [x_k, y_{i,j}] \cdot x_l$$

$$= -w_{k,l,i,j} - \frac{1}{2}[x_l \cdot x_k, y_{i,j}] - \frac{1}{2}[x_l, y_{i,j} \cdot x_k] + w_{l,k,i,j}$$

$$+ \frac{1}{2}[x_k \cdot x_l, y_{i,j}] + \frac{1}{2}[x_k, y_{i,j} \cdot x_l]$$

$$= -w_{k,l,i,j} - \frac{1}{2}[x_l \cdot x_k, y_{i,j}] - \frac{1}{4}[x_l, [x_i \cdot x_k, x_j]] - \frac{1}{4}[x_l, [x_i, x_j \cdot x_k]]$$

$$+ w_{l,k,i,j} + \frac{1}{2}[x_k \cdot x_l, y_{i,j}] + \frac{1}{4}[x_k, [x_i \cdot x_l, x_j]] + \frac{1}{4}[x_k, [x_i, x_j \cdot x_l]]$$

$$= -w_{k,l,i,j} + w_{l,k,i,j} + \frac{1}{2}[y_{k,l}, y_{i,j}] - \frac{1}{4}[x_l, [x_i \cdot x_k, x_j]]$$

$$- \frac{1}{4}[x_l, [x_i, x_j \cdot x_k]] + \frac{1}{4}[x_k, [x_i \cdot x_l, x_j]] + \frac{1}{4}[x_k, [x_i, x_j \cdot x_l]].$$

On the other hand, we have by (3.3) and lemma 3.2.3

$$y_{i,j} \cdot y_{k,l} = [x_i, x_j] \cdot y_{k,l} = \frac{1}{2}[x_i \cdot y_{k,l}, x_j] + \frac{1}{2}[x_i, x_j \cdot y_{k,l}]$$

$$= \frac{1}{2}[[x_i, y_{k,l}], x_j] + \frac{1}{2}[y_{k,l} \cdot x_i, x_j] + \frac{1}{2}[x_i, [x_j, y_{k,l}]] + \frac{1}{2}[x_i, y_{k,l} \cdot x_j]$$

$$= -\frac{1}{2}w_{j,i,k,l} + \frac{1}{4}[[x_k \cdot x_i, x_l], x_j] + \frac{1}{4}[[x_k, x_l \cdot x_i], x_j]$$

$$+ \frac{1}{2}w_{i,j,k,l} + \frac{1}{4}[x_i, [x_k \cdot x_j, x_l]] + \frac{1}{4}[x_i, [x_k, x_l \cdot x_j]].$$

These two equalities give us

$$\begin{aligned}
&- 4w_{k,l,i,j} + 4w_{l,k,i,j} + 2[y_{k,l}, y_{i,j}] - [x_l, [x_i \cdot x_k, x_j]] \\
&- [x_l, [x_i, x_j \cdot x_k]] + [x_k, [x_i \cdot x_l, x_j]] + [x_k, [x_i, x_j \cdot x_l]] \\
&= -2w_{j,i,k,l} + [[x_k \cdot x_i, x_l], x_j] + [[x_k, x_l \cdot x_i], x_j] \\
&\quad + 2w_{i,j,k,l} + [x_i, [x_k \cdot x_j, x_l]] + [x_i, [x_k, x_l \cdot x_j]].
\end{aligned}$$

When we look at this equality for $i = 1, j = 2, k = 1, l = 3$ we get

$$\begin{aligned}
&- 4w_{1,3,1,2} + 4w_{3,1,1,2} + 2[y_{1,3}, y_{1,2}] - [x_3, [x_1 \cdot x_1, x_2]] \\
&- [x_3, [x_1, x_2 \cdot x_1]] + [x_1, [x_1 \cdot x_3, x_2]] + [x_1, [x_1, x_2 \cdot x_3]] \\
&= -2w_{2,1,1,3} + [[x_1 \cdot x_1, x_3], x_2] + [[x_1, x_3 \cdot x_1], x_2] \\
&\quad + 2w_{1,2,1,3} + [x_1, [x_1 \cdot x_2, x_3]] + [x_1, [x_1, x_3 \cdot x_2]].
\end{aligned}$$

The coefficient of $t_{1,2,1,3}$ is given by

$$4 - 2 - x[1\|1, 3\|3] + 2x[2\|2, 3\|3] = 2 + x[1\|1, 2\|2] + 2x[3\|2, 3\|2],$$

or by remark 3.4.2

$$-x[1\|1, 3\|3] + 2 = x[1\|1, 2\|2]. \tag{3.15}$$

When we look at the equality for $i = 1, j = 2, k = 2, l = 3$ we get

$$\begin{aligned}
&- 4w_{2,3,1,2} + 4w_{3,2,1,2} + 2[y_{2,3}, y_{1,2}] - [x_3, [x_1 \cdot x_2, x_2]] \\
&- [x_3, [x_1, x_2 \cdot x_2]] + [x_2, [x_1 \cdot x_3, x_2]] + [x_2, [x_1, x_2 \cdot x_3]] \\
&= -2w_{2,1,2,3} + [[x_2 \cdot x_1, x_3], x_2] + [[x_2, x_3 \cdot x_1], x_2] \\
&\quad + 2w_{1,2,2,3} + [x_1, [x_2 \cdot x_2, x_3]] + [x_1, [x_2, x_3 \cdot x_2]].
\end{aligned}$$

The coefficient of $t_{1,2,2,3}$ is given by

$$4 - 2 + x[2\|2, 3\|3] = -2 - x[2\|1, 2\|1] + 4 + 2x[3\|2, 3\|2],$$

or by remark 3.4.2

$$x[2\|2, 3\|3] = 1 - x[1\|1, 2\|2] - 2 + 2x[2\|2, 3\|3],$$

or

$$x[2\|2,3\|3] = 1 + x[1\|1,2\|2]. \tag{3.16}$$

When we look at the equality for $i = 1, j = 3, k = 2, l = 3$ we get

$$\begin{aligned}&-4w_{2,3,1,3} + 4w_{3,2,1,3} + 2[y_{2,3}, y_{1,3}] - [x_3, [x_1 \cdot x_2, x_3]]\\ &-[x_3, [x_1, x_3 \cdot x_2]] + [x_2, [x_1 \cdot x_3, x_3]] + [x_2, [x_1, x_3 \cdot x_3]]\\ &= -2w_{3,1,2,3} + [[x_2 \cdot x_1, x_3], x_3] + [[x_2, x_3 \cdot x_1], x_3]\\ &\quad + 2w_{1,3,2,3} + [x_1, [x_2 \cdot x_3, x_3]] + [x_1, [x_2, x_3 \cdot x_3]].\end{aligned}$$

The coefficient of $t_{1,3,2,3}$ is given by

$$4 - 2 + x[1\|1,3\|3] = 2 - x[2\|2,3\|3],$$

or simplified

$$x[1\|1,3\|3] = -x[2\|2,3\|3]. \tag{3.17}$$

When we fill in equality (3.17) in (3.15) we get

$$x[2\|2,3\|3] + 2 = x[1\|1,2\|2]$$

and this contradicts (3.16).

Hence a Novikov structure can not exist on $\mathfrak{g}$.

General case

In this section we will use the previous two sections to prove by induction that the free p-step nilpotent Lie algebra on n generators does not admit any Novikov structure when $p \geq 4$.

We already have proved this for $p = 4$. Now suppose we know this is true for some $p \geq 4$ and let $\mathfrak{g}$ be the free $(p+1)$-step nilpotent Lie algebra on n generators.

Suppose that $\mathfrak{g}$ does admit a Novikov structure. Then we know from lemma 3.2.5 that $\gamma_{p+1}(\mathfrak{g})$ is a 2-sided ideal for this product. Hence the Novikov structure on $\mathfrak{g}$ will induce a Novikov structure on the quotient $\mathfrak{g}/\gamma_{p+1}(\mathfrak{g})$.

Now remark that this quotient is actually the free p-step nilpotent Lie algebra on n generators. This immediately contradicts our induction hypothesis. □

3.5 The (non) existence of Novikov structures on triangular matrix algebras

One of the most fundamental examples of solvable, resp. nilpotent Lie algebras are the Lie algebras of upper triangular, resp. strictly upper triangular matrices of size n over a field k, which we denote by $\mathfrak{t}(n,k)$, resp. $\mathfrak{n}(n,k)$. It is therefore natural to ask, which of those Lie algebras admit a Novikov structure. It turns out that such structures exist only in very small dimensions.

We first study the case of strictly upper triangular matrices:

Proposition 3.5.1. *The Lie algebra $\mathfrak{n}(n,k)$ admits a Novikov structure if and only if $n \leq 4$.*

Proof. If $n \leq 4$, the Lie algebra $\mathfrak{n}(n,k)$ is abelian ($n = 2$), 2-step nilpotent ($n = 3$) or 3-step nilpotent and generated by 3 elements ($n = 4$). In any of these cases, we know that a Novikov structure exists. Indeed, in [16] it is proved that any 2-step nilpotent Lie algebra and any 3-step nilpotent Lie algebra on 3 generators admits a Novikov structure.

Now let $n > 4$ and suppose that $\mathfrak{n}(n,k)$ admits a Novikov structure. Denote by $e_{i,j}$ the elementary matrices, which have a 1 on the (i,j)-th position and a zero elsewhere. The $e_{i,j}$ with $1 \leq i < j \leq n$ form a basis of $\mathfrak{n}(n,k)$. The Lie bracket is given by

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j}.$$

Assume that $(A, \cdot)$ defines a Novikov structure on $\mathfrak{n}(n,k)$. Then some easy calculations, using lemma 3.2.1, yield:

$$e_{1,2} \cdot [e_{3,4}, e_{4,5}] + e_{3,4} \cdot [e_{4,5}, e_{1,2}] + e_{4,5} \cdot [e_{1,2}, e_{3,4}] = 0$$

$$\Rightarrow \; e_{1,2} \cdot e_{3,5} = 0,$$

$$e_{3,4} \cdot [e_{4,5}, e_{1,3}] + e_{4,5} \cdot [e_{1,3}, e_{3,4}] + e_{1,3} \cdot [e_{3,4}, e_{4,5}] = 0$$

$$\Rightarrow \; e_{1,3} \cdot e_{3,5} = -e_{4,5} \cdot e_{1,4},$$

$$e_{3,4} \cdot [e_{4,5}, e_{2,3}] + e_{4,5} \cdot [e_{2,3}, e_{3,4}] + e_{2,3} \cdot [e_{3,4}, e_{4,5}] = 0$$

$$\Rightarrow \; e_{2,3} \cdot e_{3,5} = -e_{4,5} \cdot e_{2,4},$$

$$e_{1,2} \cdot [e_{4,5}, e_{2,4}] + e_{4,5} \cdot [e_{2,4}, e_{1,2}] + e_{2,4} \cdot [e_{1,2}, e_{4,5}] = 0$$

$$\Rightarrow \; e_{1,2} \cdot e_{2,5} = -e_{4,5} \cdot e_{1,4} = e_{1,3} \cdot e_{3,5},$$

$$e_{1,2} \cdot [e_{2,3}, e_{3,5}] + e_{2,3} \cdot [e_{3,5}, e_{1,2}] + e_{3,5} \cdot [e_{1,2}, e_{2,3}] = 0$$

$$\Rightarrow \ -e_{3,5} \cdot e_{1,3} = e_{1,2} \cdot e_{2,5} = e_{1,3} \cdot e_{3,5}.$$

Identity (3.3) yields

$$e_{1,5} = [e_{1,3}, e_{3,5}] = e_{1,3} \cdot e_{3,5} - e_{3,5} \cdot e_{1,3} = -2e_{3,5} \cdot e_{1,3},$$

or else

$$e_{1,5}/2 = -e_{3,5} \cdot e_{1,3} = e_{1,3} \cdot e_{3,5}.$$

It also gives

$$e_{1,5} = [e_{1,4}, e_{4,5}] = e_{1,4} \cdot e_{4,5} - e_{4,5} \cdot e_{1,4}.$$

By the previous identities we get

$$-2e_{4,5} \cdot e_{1,4} = 2e_{1,3} \cdot e_{3,5} = e_{1,5} = e_{1,4} \cdot e_{4,5} - e_{4,5} \cdot e_{1,4}.$$

Hence we find

$$-e_{4,5} \cdot e_{1,4} = e_{1,4} \cdot e_{4,5} = e_{1,5}/2.$$

Applying the operator identity (3.4) for $x = e_{1,2}$ and $y = e_{2,3}$ to $z = e_{3,5}$, and using the above computations, we find

$$\begin{aligned}
0 &= \big(L([e_{1,2}, e_{2,3}]) + \mathrm{ad}[e_{1,2}, e_{2,3}] - [L(e_{1,2}), \mathrm{ad}(e_{2,3})] \\
&\quad - [\mathrm{ad}(e_{1,2}), L(e_{2,3})]\big)(e_{3,5}) \\
&= \big(L(e_{1,3}) + \mathrm{ad}(e_{1,3}) - [L(e_{1,2}), \mathrm{ad}(e_{2,3})] - [\mathrm{ad}(e_{1,2}), L(e_{2,3})]\big)(e_{3,5}) \\
&= e_{1,3} \cdot e_{3,5} + [e_{1,3}, e_{3,5}] - e_{1,2} \cdot [e_{2,3}, e_{3,5}] + [e_{2,3}, e_{1,2} \cdot e_{3,5}] \\
&\quad - [e_{1,2}, e_{2,3} \cdot e_{3,5}] + e_{2,3} \cdot [e_{1,2}, e_{3,5}] \\
&= e_{1,3} \cdot e_{3,5} + e_{1,5} - e_{1,2} \cdot e_{2,5} - [e_{1,2}, e_{2,3} \cdot e_{3,5}] \\
&= e_{1,5} + [e_{1,2}, e_{4,5} \cdot e_{2,4}] \\
&= e_{1,5} + [e_{1,2}, e_{2,4} \cdot e_{4,5} + [e_{4,5}, e_{2,4}]] \\
&= [e_{1,2}, e_{2,4} \cdot e_{4,5}].
\end{aligned}$$

Again applying operator identity (3.4), but now for $x = e_{1,2}$ and $y = e_{2,4}$ to $z = e_{4,5}$, we find

$$\begin{aligned}
0 &= \big(L([e_{1,2}, e_{2,4}]) + \mathrm{ad}[e_{1,2}, e_{2,4}] - [L(e_{1,2}), \mathrm{ad}(e_{2,4})] \\
&\qquad - [\mathrm{ad}(e_{1,2}), L(e_{2,4})]\big)(e_{4,5}) \\
&= \big(L(e_{1,4}) + \mathrm{ad}(e_{1,4}) - [L(e_{1,2}), \mathrm{ad}(e_{2,4})] - [\mathrm{ad}(e_{1,2}), L(e_{2,4})]\big)(e_{4,5}) \\
&= e_{1,4} \cdot e_{4,5} + [e_{1,4}, e_{4,5}] - e_{1,2} \cdot [e_{2,4}, e_{4,5}] + [e_{2,4}, e_{1,2} \cdot e_{4,5}] \\
&\qquad - [e_{1,2}, e_{2,4} \cdot e_{4,5}] + e_{2,4} \cdot [e_{1,2}, e_{4,5}] \\
&= e_{1,4} \cdot e_{4,5} + e_{1,5} - e_{1,2} \cdot e_{2,5} + [e_{2,4}, e_{1,2} \cdot e_{4,5}] - [e_{1,2}, e_{2,4} \cdot e_{4,5}] \\
&= e_{1,5} + [e_{2,4}, e_{1,2} \cdot e_{4,5}].
\end{aligned}$$

However, this is impossible, since $e_{1,5} \notin [e_{2,4}, \mathfrak{n}(n,k)]$. This contradiction shows that there is no Novikov structure on $\mathfrak{n}(n,k)$ when $n \geq 5$. $\square$

As a consequence we can easily prove the following result concerning upper triangular matrices:

Proposition 3.5.2. *The Lie algebra* $\mathfrak{t}(n,k)$ *admits a Novikov structure if and only if* $n \leq 2$.

Proof. It is easy to construct a Novikov structure on $\mathfrak{t}(1,k) \cong k$ and on $\mathfrak{t}(2,k)$. On $\mathfrak{t}(1,k)$ the zero product will be a Novikov structure. For $\mathfrak{t}(2,k)$ one can easily check that the following product defines a Novikov structure:

$$e_{1,1} \cdot e_{1,2} = e_{1,2}, \qquad e_{2,2} \cdot e_{1,2} = -e_{1,2}.$$

For $n \geq 5$ we can use the above proposition to show that there does not exist a Novikov structure. Assume that $\mathfrak{t}(n,k)$ admits a Novikov structure for $n \geq 5$. Then also $[\mathfrak{t}(n,k), \mathfrak{t}(n,k)] = \mathfrak{n}(n,k)$ admits a Novikov structure, in contradiction to our previous proposition.

For $n = 3$ and $n = 4$ it is not difficult to see by direct calculations that $\mathfrak{t}(n,k)$ does not admit a Novikov structure. This method actually works for all $n \geq 3$.

As in the above proof, denote by $e_{i,j}$ the elementary matrices, which have a 1 on the (i,j)-th position and a zero elsewhere. The $e_{i,j}$ with $1 \leq i \leq j \leq n$ form a basis of $\mathfrak{t}(n,k)$. The Lie bracket is given by

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j}.$$

Assume that $(A,\cdot)$ defines a Novikov structure on $\mathfrak{t}(n,k)$. We use lemma 3.2.1 to get the following identities:

$$
\begin{aligned}
& e_{1,1}\cdot[e_{2,2},e_{2,3}]+e_{2,2}\cdot[e_{2,3},e_{1,1}]+e_{2,3}\cdot[e_{1,1},e_{2,2}]=0\\
&\quad\Rightarrow\ e_{1,1}\cdot e_{2,3}=0,\\
& e_{1,3}\cdot[e_{2,2},e_{3,3}]+e_{2,2}\cdot[e_{3,3},e_{1,3}]+e_{3,3}\cdot[e_{1,3},e_{2,2}]=0\\
&\quad\Rightarrow\ e_{2,2}\cdot e_{1,3}=0,\\
& e_{1,2}\cdot[e_{2,2},e_{2,3}]+e_{2,2}\cdot[e_{2,3},e_{1,2}]+e_{2,3}\cdot[e_{1,2},e_{2,2}]=0\\
&\quad\Rightarrow\ e_{1,2}\cdot e_{2,3}-e_{2,2}\cdot e_{1,3}+e_{2,3}\cdot e_{1,2}=0\\
&\quad\Rightarrow\ 2e_{1,2}\cdot e_{2,3}+[e_{2,3},e_{1,2}]=0\\
&\quad\Rightarrow\ e_{1,2}\cdot e_{2,3}=\frac{1}{2}e_{1,3}=-e_{2,3}\cdot e_{1,2},\\
& e_{1,1}\cdot[e_{1,2},e_{2,3}]+e_{1,2}\cdot[e_{2,3},e_{1,1}]+e_{2,3}\cdot[e_{1,1},e_{1,2}]=0\\
&\quad\Rightarrow\ e_{1,1}\cdot e_{1,3}=-e_{2,3}\cdot e_{1,2}=\frac{1}{2}e_{1,3}.
\end{aligned}
$$

Applying the operator identity (3.4) for $x=e_{1,1}$ and $y=e_{1,2}$ to $z=e_{2,3}$, and using the above computations, we find

$$
\begin{aligned}
0 &= \big(L([e_{1,1},e_{1,2}])+\mathrm{ad}[e_{1,1},e_{1,2}]-[L(e_{1,1}),\mathrm{ad}(e_{1,2})]\\
&\qquad -[\mathrm{ad}(e_{1,1}),L(e_{1,2})]\big)(e_{2,3})\\
&= \big(L(e_{1,2})+\mathrm{ad}(e_{1,2})-[L(e_{1,1}),\mathrm{ad}(e_{1,2})]-[\mathrm{ad}(e_{1,1}),L(e_{1,2})]\big)(e_{2,3})\\
&= e_{1,2}\cdot e_{2,3}+[e_{1,2},e_{2,3}]-e_{1,1}\cdot[e_{1,2},e_{2,3}]+[e_{1,2},e_{1,1}\cdot e_{2,3}]\\
&\qquad -[e_{1,1},e_{1,2}\cdot e_{2,3}]+e_{1,2}\cdot[e_{1,1},e_{2,3}]\\
&= \frac{1}{2}e_{1,3}+e_{1,3}-e_{1,1}\cdot e_{1,3}-[e_{1,1},\frac{1}{2}e_{1,3}]\\
&= \frac{1}{2}e_{1,3}+e_{1,3}-\frac{1}{2}e_{1,3}-\frac{1}{2}e_{1,3}\\
&= \frac{1}{2}e_{1,3},
\end{aligned}
$$

which gives a contradiction. This contradiction shows that there is no Novikov structure on $\mathfrak{t}(n,k)$ when $n \geq 3$. $\square$

3.6 Novikov structures on k-step solvable Lie algebras

A natural question is, whether there are families of Lie algebras of solvability class k, which admit Novikov structures for all $k \geq 1$.

The same question for nilpotency class has an easy answer. Here the standard filiform nilpotent Lie algebras with basis $(e_1, \dots, e_n)$ and brackets $[e_1, e_i] = e_{i+1}$ for $i = 2, \dots, n-1$ admit Novikov structures. Hence, they provide examples of nilpotency class $k = n-1$, see [16].

The following result shows that there are indeed filiform nilpotent Lie algebras of arbitrary solvability class, which admit Novikov structures.

Define for every $n \geq 3$ a filiform Lie algebra $\mathfrak{f}_{\frac{9}{10},n}$ of dimension n by

$$[e_1, e_j] = e_{j+1}, \quad 2 \leq j \leq n-1,$$

$$[e_i, e_j] = \frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}} e_{i+j}, \quad 2 \leq i \leq j;\ i+j \leq n.$$

In particular we have

$$[e_2, e_j] = \frac{6(j-2)}{j(j-1)} e_{j+2}, \quad 3 \leq j \leq n-2,$$

$$[e_j, e_{j+1}] = \frac{6(j-1)!(j-2)!}{(2j-1)!} e_{2j+1}, \quad 2 \leq j \leq (n-1)/2.$$

Then $[e_2, e_3] = e_5$, $[e_2, e_4] = e_6$, $[e_2, e_5] = \frac{9}{10} e_7$, etc. Similar Lie algebras were studied in [11].

To verify the Jacobi identity, introduce a new basis $(f_1, \dots, f_n)$ by

$$f_1 = 6e_1,$$

$$f_j = \frac{1}{(j-2)!} e_j, \quad 2 \leq j \leq n.$$

Then the new brackets are given by

$$[f_i, f_j] = 6(j-i)f_{i+j}, \quad 1 \le i \le j;\ i+j \le n.$$

Here the Jacobi identity can easily be checked.

All these Lie algebras admit a complete Novikov structure:

Proposition 3.6.1. *For each $n \ge 3$ the Lie algebra $\mathfrak{f}_{\frac{9}{10},n}$ admits a complete Novikov structure. It is given by the following multiplication:*

$$e_1 \cdot e_j = e_{j+1}, \quad 2 \le j \le n-1,$$

$$e_i \cdot e_j = \frac{6}{j\binom{j+i-2}{i-2}} e_{i+j}, \quad 2 \le i,j \le n;\ i+j \le n.$$

Remark 3.6.2. *Note that $\mathfrak{f}_{\frac{9}{10},n}$ is k-step solvable if $2^k \le n+1 < 2^{k+1}$. Indeed, by induction, one can easily prove that*

$$\mathfrak{g}^{(0)} = \mathfrak{g},$$

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \langle e_3, \ldots, e_n \rangle,$$

$$\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}] = \langle e_{2^{i+1}-1}, \ldots, e_n \rangle.$$

Hence these algebras can have arbitrary high solvability class.

Proof. One can easily check that in the new basis $(f_1, \ldots, f_n)$ the product is given by

$$f_i \cdot f_j = 6(j-1)f_{i+j}, \quad 1 \le i,j \le n,\ i+j \le n.$$

Now it is easy to verify the required identities. We have (using the convention that $f_m = 0$ when $m > n$)

$$\begin{aligned} f_i \cdot f_j - f_j \cdot f_i &= 6(j-1)f_{i+j} - 6(i-1)f_{i+j} \\ &= 6(j-i)f_{i+j} = [f_i, f_j], \end{aligned}$$

so that (3.3) is satisfied. We have

$$(f_i \cdot f_j) \cdot f_k = 36(j-1)(k-1)f_{i+j+k},$$

$$(f_i \cdot f_k) \cdot f_j = 36(k-1)(j-1)f_{i+j+k},$$

so that (3.2) is satisfied. Finally,

$$f_i \cdot (f_j \cdot f_k) - (f_i \cdot f_j) \cdot f_k = 36 \cdot k(k-1) f_{i+j+k},$$

$$f_j \cdot (f_i \cdot f_k) - (f_j \cdot f_i) \cdot f_k = 36 \cdot k(k-1) f_{i+j+k},$$

so that (3.1) is satisfied. □

Chapter 4

LR-algebras and LR-structures

In this chapter we study LR-algebras and LR-structures. As explained in chapter 2 LR-structures arise in the study of simply transitive NIL-affine actions on Lie groups. In particular one is interested in the question which Lie algebras admit a complete LR-structure. The main result of this chapter says that a Lie algebra $\mathfrak{g}$ admits an LR-structure if and only if it admits a complete LR-structure.
We also study the existence question for the 2-step solvable Lie algebras on 2 generators, the free p-step nilpotent 2-step solvable Lie algebras and the Lie algebras of triangular matrices.

In section 4.1 we recall the definition of an LR-algebra and a (complete) LR-structure and state some known results. We also prove some helpful results about LR-algebras that will be useful later on.

In section 4.2 we specialize to the case of nilpotent Lie algebras and prove our main result in this case.

In section 4.3 we go back to study general (solvable) LR-algebras. First of all we prove the main theorem of this chapter in this general case. For this proof we consider the nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g}/\mathfrak{g}^{\infty}$ where $\mathfrak{g}^{\infty}$ is the intersection of the elements of the lower central series of $\mathfrak{g}$. In our proof of the main theorem we will show that the existence of an LR-structure on $\mathfrak{g}$ implies the existence of an LR-structure on $\mathfrak{n}$. We will also prove a partial converse of this result.

In section 4.4 we construct a complete LR-structure on every 2-step solvable Lie algebra on 2 generators.

In section 4.5 we construct a complete LR-structure on all free p-step nilpotent 2-step solvable Lie algebras.
We also prove that in fact all LR-structures on the free p-step nilpotent 2-step solvable Lie algebras are complete if p is at least 3. This is not true for nilpotency class 2.

In section 4.6 we show that for the Lie algebras of (strictly) upper triangular matrices, LR-structures only exist in very small dimensions.

All algebras we consider are assumed to be finite dimensional over a field k of characteristic 0.

Many results of this chapter are presented in [22].

4.1 LR-algebras and LR-structures

We start by recalling the definition of an LR-algebra and a (complete) LR-structure and state some known results that will be needed later on.

Definition 4.1.1. *A vector space A over a field k together with a bilinear product $A \times A \to A$: $(x, y) \mapsto x \cdot y$ is called an LR-algebra, if the product satisfies the identities*

$$x \cdot (y \cdot z) = y \cdot (x \cdot z), \tag{4.1}$$

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{4.2}$$

for all $x, y, z \in A$.

If we denote by $L(x), R(x)$ the left, respectively right multiplication operator then (4.1) and (4.2) are equivalent to the requirement that all left and all right multiplications commute:

$$[L(x), L(y)] = [R(x), R(y)] = 0. \tag{4.3}$$

It is well known that LR-algebras are Lie-admissible algebras: the commutator

$$[x, y] = x \cdot y - y \cdot x$$

defines a Lie bracket. The associated Lie algebra is denoted by $\mathfrak{g}_A$. The adjoint operator can be expressed by $\operatorname{ad}(x) = L(x) - R(x)$.

The associated Lie algebra then is said to admit an LR-structure:

Definition 4.1.2. *An LR-structure on a Lie algebra* $\mathfrak{g}$ *over* k *is an LR-product* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying*

$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y \in \mathfrak{g}$.
The LR-structure is said to be complete if all right multiplications $R(x)$ *are nilpotent.*

Remark 4.1.3. *We note that* $\mathfrak{g}$ *admits an LR-structure such that all right multiplications are nilpotent, if and only if* $\mathfrak{g}$ *admits an LR-structure for which all left multiplications are nilpotent. Indeed, let* $x \cdot y$ *denote a complete LR-structure, then the opposite product* $x \circ y := -y \cdot x$ *defines an LR-structure where all left multiplications are nilpotent and vice versa.*

We state here some important results concerning LR-algebras and LR-structures that will be needed later on in this chapter. Proofs of them can be found in [20].

Lemma 4.1.4. *In any LR-algebra* $(A, \cdot)$ *we have the following symmetry relation*

$$(x \cdot y) \cdot (u \cdot v) = (u \cdot v) \cdot (x \cdot y)$$

for all $x, y, u, v \in A$.

We have the following important property of Lie algebras admitting an LR-structure:

Proposition 4.1.5. *Any Lie algebra admitting an LR-structure is 2-step solvable.*

Another known result that will be needed later on is the following:

Lemma 4.1.6. *Let* $(A, \cdot)$ *be an LR-algebra with associated Lie algebra* $\mathfrak{g}$. *Then all left and right multiplications are derivations of* $\mathfrak{g}$, *i.e., for any* $a, x, y \in A$, *the following identities hold:*

$$a \cdot [x, y] = [a \cdot x, y] + [x, a \cdot y], \tag{4.4}$$

$$[x, y] \cdot a = [x \cdot a, y] + [x, y \cdot a]. \tag{4.5}$$

Denote by

$$\gamma_1(A) = \gamma_1(\mathfrak{g}_A) = A,$$

$$\gamma_{i+1}(A) = \gamma_{i+1}(\mathfrak{g}_A) = [A, \gamma_i(A)]$$

the terms of the lower central series of A, respectively $\mathfrak{g}_A$.

We have the following lemmas:

Lemma 4.1.7. *Let $(A, \cdot)$ be an LR-algebra. Then all $\gamma_i(A)$ are 2-sided ideals of A.*

Lemma 4.1.8. *Let $(A, \cdot)$ be an LR-algebra. Then we have*

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+1}(A)$$

for all $i, j \geq 0$.

We prove the following lemma about the ideals of the lower central series:

Lemma 4.1.9. *Let $(A, \cdot)$ be an LR-algebra. If*

$$\gamma_1(A) \cdot \gamma_1(A) \subseteq \gamma_2(A),$$

we have

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{i+j+2}(A)$$

for all $i, j \geq 0$.

Proof. We proof this by induction on $i \geq 0$.

The case $i = 0$ is proven by induction on $j \geq 0$. The case $j = 0$ is exactly the assumption of our lemma.

Suppose we have $\gamma_1(A) \cdot \gamma_{j+1}(A) \subseteq \gamma_{j+2}(A)$ for some $j \geq 0$. We prove it for $j + 1$. Let $x, y \in \gamma_1(A)$, $z \in \gamma_{j+1}(A)$, then because of (4.4) we have

$$x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z].$$

The first term is in $[\gamma_1(A) \cdot \gamma_1(A), \gamma_{j+1}(A)] \subseteq [\gamma_2(A), \gamma_{j+1}(A)] \subseteq \gamma_{j+3}(A)$ because of our assumption.

The second term is in $[\gamma_1(A), \gamma_1(A) \cdot \gamma_{j+1}(A)] \subseteq [\gamma_1(A), \gamma_{j+2}(A)] \subseteq \gamma_{j+3}(A)$ because of our induction hypothesis. Hence

$$x \cdot [y, z] \in \gamma_{j+3}(A),$$

which was to be proven.

Now suppose we have proven the expression for some $i \geq 0$. We prove the expression for $i + 1$. Let $x \in \gamma_1(A), y \in \gamma_{i+1}(A)$ and $z \in \gamma_{j+1}(A)$, then by (4.5) we have

$$[x, y] \cdot z = [x \cdot z, y] + [x, y \cdot z].$$

The first term is in $[\gamma_1(A) \cdot \gamma_{j+1}(A), \gamma_{i+1}(A)] \subseteq [\gamma_{j+2}(A), \gamma_{i+1}(A)] \subseteq \gamma_{i+j+3}(A)$ by the previous case.

By our induction hypothesis, the second term is in $[\gamma_1(A), \gamma_{i+1}(A) \cdot \gamma_{j+1}(A)] \subseteq [\gamma_1(A), \gamma_{i+j+2}(A)] \subseteq \gamma_{i+j+3}(A)$. Hence

$$[x, y] \cdot z \in \gamma_{i+j+3}(A),$$

which finishes the proof. □

In an LR-algebra several relations involving left and right multiplications hold. We already saw two of them in (4.3). We can derive more of those rules:

Lemma 4.1.10. *Let $(A, \cdot)$ be an LR-algebra. Then the following identities hold in A:*

$$L(x)R(y) = R(x \cdot y), \tag{4.6}$$

$$R(x)L(y) = L(y \cdot x), \tag{4.7}$$

$$L(x)R(y \cdot z) = R(x \cdot (y \cdot z)), \tag{4.8}$$

$$R(x)L(y \cdot z) = L((y \cdot z) \cdot x), \tag{4.9}$$

$$L(x)L(y \cdot z) = L(y \cdot (x \cdot z)), \tag{4.10}$$

$$R(x)R(y \cdot z) = R((y \cdot x) \cdot z). \tag{4.11}$$

Proof. Let $x, y, z, a \in A$. Using (4.1) we have

$$L(x)R(y)(a) = x \cdot (a \cdot y) = a \cdot (x \cdot y) = R(x \cdot y)(a),$$

so that $L(x)R(y) = R(x \cdot y)$. Rewriting y as $y \cdot z$ we obtain (4.8). In the same way, using (4.2), we have

$$R(x)L(y)(a) = (y \cdot a) \cdot x = (y \cdot x) \cdot a = L(y \cdot x)(a),$$

so that $R(x)L(y) = L(y \cdot x)$. This also implies (4.9). Furthermore, using (4.6) and (4.7), we have

$$L(x)L(y \cdot z) = L(x)R(z)L(y) = R(x \cdot z)L(y) = L(y \cdot (x \cdot z)),$$

which shows (4.10). In the same way follows (4.11), we have

$$R(x)R(y \cdot z) = R(x)L(y)R(z) = L(y \cdot x)R(z) = R((y \cdot x) \cdot z).$$

□

We can now prove the following proposition about nilpotency:

Proposition 4.1.11. *Let $x \cdot y$ be an LR-structure on a Lie algebra $\mathfrak{g}$. Then any two of the statements below imply the third one:*

(a) All left multiplications $L(x)$ are nilpotent operators.

(b) All right multiplications $R(x)$ are nilpotent operators.

(c) The Lie algebra $\mathfrak{g}$ is nilpotent.

Proof. Assume that (a) and (b) hold. We will show that (c) holds. For all $n \geq 1$ consider the identity

$$\mathrm{ad}(x)^n = (L(x) - R(x))^n.$$

By assumption all left and right multiplications, and their powers, have zero trace.

We claim that the right hand side of the above identity has zero trace for all $n \geq 1$. For this it is enough to show that all summands between $L(x)^n$ and $R(x)^n$ are of the form $L(y \cdot z)$ or $R(y \cdot z)$, where y, z are certain powers of x, with various bracketings.

We prove this by induction on n. The case $n = 1$ is clear. Assume this for n. Then $\mathrm{ad}(x)^{n+1} = (L(x) - R(x))(L(x) - R(x))^n$. Then, by the induction hypothesis, the inner summands are of the form

$$L(x)R(x)^n,\ R(x)L(x)^n,$$

$$L(x)L(y \cdot z),\ L(x)R(y \cdot z),\ R(x)R(y \cdot z),\ R(x)L(y \cdot z).$$

They are all of the form $L(y \cdot z)$ or $R(y \cdot z)$, this follows from lemma 4.1.10. For the last four summands this is obvious, and for the first two summands we obtain

$$L(x)R(x)^n = R(y \cdot x), \tag{4.12}$$

$$R(x)L(x)^n = L(x \cdot z) \tag{4.13}$$

for certain $y, z \in \mathfrak{g}$. This can be seen by induction on n and lemma 4.1.10. For $n = 1$ we have $L(x)R(x) = R(x \cdot x)$ and $R(x)L(x) = L(x \cdot x)$. Assume we have the identities for n, then because all left and all right multiplications commute, we have

$$\begin{aligned} L(x)R(x)^{n+1} &= L(x)R(x)^n R(x) \\ &= R(y \cdot x)R(x) \\ &= R(x)R(y \cdot x) \\ &= R((y \cdot x) \cdot x), \\ R(x)L(x)^{n+1} &= R(x)L(x)^n L(x) \\ &= L(x \cdot z)L(x) \\ &= L(x)L(x \cdot z) \\ &= L(x \cdot (x \cdot z)). \end{aligned}$$

It follows that $\operatorname{tr}(\operatorname{ad}(x)^n) = 0$ for all $n \geq 1$, so that all $\operatorname{ad}(x)$ are nilpotent, because the field has characteristic zero. By Engel's theorem $\mathfrak{g}$ is nilpotent.

Assume that (b) and (c) hold. We will prove that (a) holds. Now we consider the identity

$$L(x)^n = (\operatorname{ad}(x) + R(x))^n,$$

where $n \geq 1$ and $x \in \mathfrak{g}$.

We claim that the right hand side of this identity has zero trace for all $n \geq 1$. This follows because it can be expressed as a linear combination of the terms

$$\operatorname{ad}(x)^n, R(x)^n, L(y \cdot z), R(y \cdot z), \tag{4.14}$$

where y and z are certain powers of x with various bracketings.

To verify this, one does a very similar calculation as in the previous case. The case $n = 1$ is clear. Assume this for n, then we have $L(x)^{n+1} = (\operatorname{ad}(x) + R(x))(\operatorname{ad}(x) + R(x))^n$. Then, by the induction hypothesis, the inner summands are of the form

$$\begin{aligned} &R(x)\operatorname{ad}(x)^n, \\ \operatorname{ad}(x)R(x)^n &= L(x)R(x)^n - R(x)^{n+1}, \\ \operatorname{ad}(x)L(y \cdot z) &= L(x)L(y \cdot z) - R(x)L(y \cdot z), \\ \operatorname{ad}(x)R(y \cdot z) &= L(x)R(y \cdot z) - R(x)R(y \cdot z), \end{aligned}$$

$$R(x)L(y \cdot z),$$

$$R(x)R(y \cdot z).$$

The last four are linear combinations of terms like (4.14) because of lemma 4.1.10. For the second one this is true because of (4.12).
We proved in the previous part that $\mathrm{ad}(x)^n$ is a linear combination of terms like $L(x)^n$, $R(x)^n$, $L(y \cdot z)$ and $R(y \cdot z)$. Hence $R(x)\,\mathrm{ad}(x)^n$ is a linear combination of terms like

$$R(x)L(x)^n, R(x)^{n+1}, R(x)L(y \cdot z), R(x)R(y \cdot z).$$

Using (4.13) and lemma 4.1.10 these are all of the form (4.14).

By assumption all $\mathrm{ad}(x)$ and all $R(x)$ are nilpotent and hence they, and all their powers, have trace zero. Because $L(x) = \mathrm{ad}(x) + R(x)$, all $L(x)$ have zero trace. Then all terms listed in (4.14) have zero trace, so that $L(x)^n$ has zero trace for all $n \geq 1$, and hence all $L(x)$ are nilpotent.

Finally, assume that (a) and (c) hold. We have to show that (b) holds. This case follows from the above case using remark 4.1.3. □

4.2 LR-structures on nilpotent Lie algebras

In this section we concentrate on nilpotent Lie algebras. We will prove the main result of this chapter first in the nilpotent case.

For the study of LR-structures we repeatedly need the following proposition, which can be found in [16, Proposition 5.3]:

Proposition 4.2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let V be a finite dimensional $\mathfrak{g}$-module. Then V can be written as a direct sum of $\mathfrak{g}$-modules $V = V_n \oplus V_0$ where V_n is the unique maximal nilpotent submodule of V and $H^i(\mathfrak{g}, V_0) = 0$, $\forall i \geq 0$.*

In the nilpotent case our main result sounds as follows:

Proposition 4.2.2. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over k. If $\mathfrak{g}$ admits an LR-structure then it also admits a complete LR-structure.*

Proof. Let $(A, \cdot)$ be an LR-structure on $\mathfrak{g}$. We will prove this proposition in two parts. First we prove it assuming k is an algebraically closed field. Afterwards we will prove it for an arbitrary field k of characteristic 0.

Proof for an algebraically closed field of characteristic 0

Assume in this section that the field k is algebraically closed.

We recall the definition of a weight and a weight subspace of a representation:

Definition 4.2.3. *Let $\mathfrak{n}$ be a Lie algebra over k and V a vector space over k. A weight of a representation $\rho : \mathfrak{n} \to \mathfrak{gl}(V)$ is a linear map $\alpha : \mathfrak{n} \to k$ such that the subspace V_α of V, given by*

$$V_\alpha = \{v \in V \mid \forall x \in \mathfrak{g}\, \exists m \in \mathbb{N}_0 \text{ such that } (\rho(x) - \alpha(x))^m v = 0\},$$

is nonzero. The space V_α is called the weight space corresponding to the weight α.

We will use the following proposition:

Proposition 4.2.4 (Weight subspace decomposition [33], [34])**.** *Suppose $\mathfrak{n}$ is a nilpotent Lie algebra over k, V a finite dimensional vector space over k and $\rho : \mathfrak{n} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{n}$. Then there exists a finite number of different weights $\alpha_1, \dots, \alpha_s$ of ρ such that V is a direct sum of the weight spaces A_i and there exists a base of A_i such that for any $x \in \mathfrak{n}$ the restriction $\rho(x)|_{A_i}$ can be represented by the following matrix form with respect to this base*

$$\begin{pmatrix} \alpha_i(x) & & & * \\ & \alpha_i(x) & & \\ 0 & & \ddots & \\ & & & \alpha_i(x) \end{pmatrix}.$$

Denote the image of the linear map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : x \mapsto L(x)$ by $\mathfrak{a} = \mathrm{im}(L)$. Since all left multiplications commute this is the abelian, and hence nilpotent, subalgebra of $\mathfrak{gl}(\mathfrak{g})$ generated by all $L(x)$.

Denote by

$$\rho\colon \mathfrak{a} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$$

the representation of $\mathfrak{a}$ given by inclusion. Since k is algebraically closed we can apply the weight subspace decomposition for ρ. We have

$$A = \bigoplus_{i=1}^{s} A_i,$$

where $\alpha_i \in \mathrm{Hom}(\mathfrak{a}, k)$ are the different weights of ρ and A_i are the (nonzero) weight subspaces corresponding to the weights α_i.

We may identify $\rho(L(x))$ with $L(x)$. Then, in a suitable basis of the vector space A, the operators $L(x)$ have a block matrix, where each block is of the form

$$L(x)|_{A_i} = \begin{pmatrix} \alpha_i(L(x)) & & & * \\ & \alpha_i(L(x)) & & \\ 0 & & \ddots & \\ & & & \alpha_i(L(x)) \end{pmatrix}.$$

A first consequence is that each A_i is a left ideal in $A = A_1 \oplus \cdots \oplus A_s$.

Denote by $\{e_{i,1}, \dots, e_{i,n_i}\}_{i=1,\dots,s}$ a suitable basis for A as mentioned above, where for each $1 \le i \le s$, $(e_{i,1}, \dots, e_{i,n_i})$ is a basis of A_i.

Let $A_{i,k} = \langle e_{i,1}, \dots e_{i,k} \rangle$ for $k = 1, \dots, n_i$. We obtain a filtration of A_i by left ideals:

$$A_i = A_{i,n_i} \supseteq A_{i,n_i-1} \supseteq \cdots \supseteq A_{i,2} \supseteq A_{i,1} \supseteq A_{i,0} = 0.$$

We can write each $x \in A$ uniquely as $x = x_1 + \cdots + x_s$ with $x_i \in A_i$, i.e., as

$$x = \sum_{i=1}^{s} (x_{i,1} e_{i,1} + \ldots + x_{i,n_i} e_{i,n_i}).$$

We can now prove the following lemma about the representation of an element $x \in A$:

Lemma 4.2.5. *Suppose that in the representation of $x \in A$ we have $x_i \in A_{i,n_i-1}$ for some i, i.e., $x_{i,n_i} = 0$. Then $\alpha_i(L(x)) = 0$.*

Proof. Assume that $\alpha_i(L(x)) \neq 0$. We will show that this is impossible.

For $z \in A$ the condition $x_i \in A_{i,n_i-1}$ implies that $L(z)x_i \in A_{i,n_i-1}$, because A_{i,n_i-1} is a left ideal. This means that $(L(z)x)_{i,n_i} = 0$ for all $z \in A$.

We can choose an element $y \in A$ satisfying $y_{i,n_i} = 1$ with respect to its basis representation. Consider the elements $y^{(0)} = y$ and $y^{(k)} = [x, y^{(k-1)}]$ for all $k \geq 1$.

Using induction it is easy to see that $(y^{(k)})_{i,n_i} = (\alpha_i(L(x)))^k$. Indeed, the case $k = 0$ is trivial. Assume this for k, then

$$(y^{(k+1)})_{i,n_i} = (L(x)y^{(k)})_{i,n_i} - (L(y^{(k)})x)_{i,n_i} = (L(x)y^{(k)})_{i,n_i}$$

$$= \alpha_i(L(x))\,(y^{(k)})_{i,n_i} = (\alpha_i(L(x)))^{k+1}.$$

This shows that all elements $y^{(k)}$ are nonzero. In particular, the operator $\mathrm{ad}(x)$ for this element $x \in A$ is not nilpotent. Hence the Lie algebra $\mathfrak{g}$ is not nilpotent, which is a contradiction. □

To proceed, we consider two cases. In the first case there is only one weight. In this case we may assume that the weight is nonzero, otherwise the LR-structure $x \cdot y$ is clearly complete since then all left multiplications are strictly upper triangular in our chosen basis. In the situation of one (nonzero) weight, we will prove that the Lie algebra $\mathfrak{g}$ is abelian and hence the zero product defines a complete LR-structure on $\mathfrak{g}$.
Secondly, we use the first case to form a complete LR-structure in the general case.

Case 1: *One weight.* We first assume that ρ has only a single, nonzero, weight α.

Then there is a basis $(e_1, \ldots, e_n)$ of A such that the operators $L(x)$ are upper triangular with $\alpha(L(x))$ on the diagonal. For $x \in A$ we write $x = x_1 e_1 + \cdots + x_n e_n$.

Lemma 4.2.5 says that $\alpha(L(x)) = 0$ for elements x with $x_n = 0$, i.e., $\alpha(L(e_k)) = 0$ for $k = 1, \ldots, n-1$. Since α is nonzero it follows that $\alpha(L(e_n)) \neq 0$, and we may normalize it to 1.

We may write the operators $L(e_k)$ as follows:

$$L(e_k) = (a_{i,j}^k)_{1 \le i,j \le n}, \quad k = 1, \ldots, n,$$

with the conditions

$$a_{i,j}^k = 0 \text{ for all } 1 \le j \le i \le n, 1 \le k \le n-1,$$

$$a_{i,j}^n = 0 \text{ for all } 1 \le j < i \le n,$$

$$a_{i,i}^n = 1 \text{ for all } 1 \le i \le n.$$

We want to show that in this case A is commutative. For this we need two lemmas.

Lemma 4.2.6. *We have $a_{n-1,n}^k = 0$ for all $k = 1, \ldots, n-2$. In particular, the subspace generated by $e_1, \ldots, e_{n-2}$ is a 2-sided ideal in A.*

Proof. The operators $R(e_j)$ for $1 \leq j \leq n$ are given as follows:

$$R(e_j) = \begin{pmatrix} a_{1,j}^1 & a_{1,j}^2 & \cdots & a_{1,j}^{n-1} & a_{1,j}^n \\ a_{2,j}^1 & a_{2,j}^2 & \cdots & a_{2,j}^{n-1} & a_{2,j}^n \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{j-1,j}^1 & a_{j-1,j}^2 & \cdots & a_{j-1,j}^{n-1} & a_{j-1,j}^n \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Suppose it is not true that $a_{n-1,n}^1 = a_{n-1,n}^2 = \cdots = a_{n-1,n}^{n-2} = 0$. Then there is a minimal $k \leq n-2$ such that $a_{n-1,n}^k \neq 0$, and $a_{n-1,n}^j = 0$ for all $1 \leq j \leq k-1$.

We consider the matrix identity

$$[R(e_k), R(e_n)] = 0.$$

Let us look at the $(n-1, n)$-th entry of the left hand side. Note that the $(n-1)$-th row of $R(e_k)$ is the zero row, that the $(n-1)$-th row of $R(e_n)$ is $(0, \ldots, 0, a_{n-1,n}^k, \ldots, a_{n-1,n}^{n-1}, a_{n-1,n}^n)$ and that the n-th column of $R(e_k)$ has a 1 on the k-th entry, followed by zeros. Hence we find that $a_{n-1,n}^k = 0$, which is a contradiction. Hence we have $a_{n-1,n}^k = 0$ for all $k = 1, \ldots, n-2$.

This implies that $R(e_n)$ maps $\langle e_1, \ldots, e_{n-2} \rangle$ into itself. Together with the particular form of the operators $R(e_j)$ this shows that $\langle e_1, \ldots, e_{n-2} \rangle$ is also a right ideal in A, which finishes the proof. □

The second lemma we need, says there exists a nonzero idempotent element in A:

Lemma 4.2.7. *There is an idempotent $a \neq 0$ in A, i.e., satisfying $a \cdot a = a$.*

Proof. We will use induction on n. For $n = 1$ we have $L(e_1) = (1)$, so that $e_1 \cdot e_1 = e_1$. For $n = 2$ the left multiplications have the form

$$L(e_1) = \begin{pmatrix} 0 & a_{1,2}^1 \\ 0 & 0 \end{pmatrix}, \quad L(e_2) = \begin{pmatrix} 1 & a_{1,2}^2 \\ 0 & 1 \end{pmatrix}.$$

We have $a_{1,2}^1 = 1$, indeed, $\operatorname{tr}(\operatorname{ad}(e_2)) = \operatorname{tr}(L(e_2)) - \operatorname{tr}(R(e_2)) = 1 - a_{1,2}^1$ has to be zero. Now $a := -a_{1,2}^2 e_1 + e_2$ is the desired non-trivial idempotent:

$$\begin{aligned} a \cdot a &= (-a_{1,2}^2 e_1 + e_2) \cdot (-a_{1,2}^2 e_1 + e_2) = -a_{1,2}^2 e_1 - a_{1,2}^2 e_1 + a_{1,2}^2 e_1 + e_2 \\ &= -a_{1,2}^2 e_1 + e_2 = a. \end{aligned}$$

Suppose $n \geq 3$. By lemma 4.2.6 the subspace $B = \langle e_1, \dots, e_{n-2} \rangle$ is an ideal in A such that A/B is an LR-structure on the associated Lie algebra $\mathfrak{g}_{A/B}$, which is again nilpotent. The associated homomorphism $\overline{\rho}$ again has one single nonzero weight.

Since $\dim(A/B) = 2$ there is a non-trivial idempotent $\overline{a} = a + B$ in A/B. We have $a \cdot a + B = \overline{a} \cdot \overline{a} = \overline{a}$, hence $a \cdot a = a + b$ for some $b \in B$. Note that $a \notin B$.

For such an element a let $B_a = \langle B, a \rangle$. As subalgebra of A, this is an LR-algebra with associated nilpotent Lie algebra and homomorphism having again one single nonzero weight. That this weight is nonzero follows from the fact that the coefficient of $a + B$ with $e_n + B$ is 1, see the definition of the idempotent element in the 2-dimensional case. Hence the coefficient of a with e_n is also 1.

Since $\dim B_a < \dim A$ we can apply the induction hypothesis: there is a non-trivial idempotent $e \in B_a$, hence also in A. □

Now we can show that A is commutative and hence the zero product will define a complete LR-structure on $\mathfrak{g}$:

Proposition 4.2.8. *If ρ has only a single nonzero weight α, then A is commutative. Hence the zero product defines a complete LR-structure on $\mathfrak{g}$.*

Proof. Let $a \neq 0$ be an idempotent in A. Then $L(a)$ is not nilpotent. Because $L(a)$ is upper triangular with nonzero diagonal elements $\alpha(L(a))$, the operator $L(a)$ is invertible.

Lemma 4.1.10 implies that $R(a)L(a) = L(a \cdot a) = L(a)$. Multiplying with $L(a)^{-1}$ yields $R(a) = \mathrm{id}$. This means that a is a right identity for A.

Again using lemma 4.1.10 we obtain

$$L(x) = L(x)R(a) = R(x \cdot a) = R(x),$$

so that A is commutative, and $\mathfrak{g}$ is abelian. □

<u>*Case 2:*</u> We now consider the general case, i.e., where $A = A_1 \oplus \cdots \oplus A_s$ is the direct sum of weight spaces A_i corresponding to the weights α_i.

We may assume that all α_i with $i \geq 2$ are nonzero, and that $\alpha_1 = 0$, possibly setting $A_1 = 0$ if there is no zero weight.

The spaces A_i, for $i \geq 2$, are left ideals, hence LR-algebras with associated nilpotent Lie algebra and homomorphism $\rho_{|L(A_i)}$ having one single weight. It is a consequence of lemma 4.2.5 that this weight is nonzero.

Now it follows from the first case that all A_i for $i \geq 2$ are commutative and have non-trivial right identities $a_i \in A_i$. In particular, if $x \in A_i$ and $y \in A_j$ for $i \neq j$ with $2 \leq i \leq s$ and $1 \leq j \leq s$, we have

$$x \cdot y = (x \cdot a_i) \cdot y = (x \cdot y) \cdot a_i \ \in A_i \cap A_j \ = 0, \tag{4.15}$$

this follows from the fact that A_i and A_j are left ideals.

Define a bilinear product $x \circ y$ on A as follows: on basis vectors we set

$$e_{i,k} \circ e_{j,l} = \begin{cases} 0 & \forall \, i = j \geq 2, \ 1 \leq k, l \leq n_i, \\ e_{i,k} \cdot e_{j,l} & \text{in all other cases.} \end{cases}$$

The proof is finished if we can show that this product defines a complete LR-structure on $\mathfrak{g}$.

Note that for $1 \leq k \leq n_i$ and $1 \leq l \leq n_j$ we have, using (4.15),

$$e_{i,k} \circ e_{j,l} = 0 \quad \forall \, i \geq 2, \ \forall j \geq 1, \tag{4.16}$$

$$[e_{i,k}, e_{j,l}] = 0 \quad \forall i, j \geq 2.$$

From this it follows easily that

$$[e_{i,k}, e_{j,l}] = e_{i,k} \circ e_{j,l} - e_{j,l} \circ e_{i,k}.$$

Indeed, if $i, j \geq 2$ then both sides are equal to zero, while in the other cases the product $x \circ y$ coincides with the original LR-product $x \cdot y$.

Next we will show that all left multiplications for this product commute, i.e., that

$$e_{i,p} \circ (e_{j,q} \circ e_{k,r}) = e_{j,q} \circ (e_{i,p} \circ e_{k,r}).$$

If $i \geq 2$ or $j \geq 2$ then both sides are equal to zero by (4.16). In the other case the product $x \circ y$ coincides with the original LR-product $x \cdot y$, which satisfies this identity.

Next we show that all right multiplications commute, i.e., that

$$(e_{i,p} \circ e_{j,q}) \circ e_{k,r} = (e_{i,p} \circ e_{k,r}) \circ e_{j,q}.$$

For $i \geq 2$ both sides are equal to zero by (4.16). Thus, we can assume that $i = 1$ and the identity reduces to

$$(e_{1,p} \cdot e_{j,q}) \circ e_{k,r} = (e_{1,p} \cdot e_{k,r}) \circ e_{j,q}.$$

Then if both $j \geq 2$ and $k \geq 2$ we obtain $0 = 0$, because A_j and A_k are left ideals, and we can apply (4.16). If j or k is equal to 1, we may assume that $j = 1$ because of the symmetry. Then the identity to show is

$$(e_{1,p} \cdot e_{1,q}) \circ e_{k,r} = (e_{1,p} \cdot e_{k,r}) \circ e_{1,q}.$$

Since A_1 is a left ideal and products involving elements of A_1 are always given by the original product, this identity reduces to

$$(e_{1,p} \cdot e_{1,q}) \cdot e_{k,r} = (e_{1,p} \cdot e_{k,r}) \cdot e_{1,q},$$

which holds true.

Finally we prove that the new LR-structure is complete.

Denote by $\ell(x)$ the left multiplications, i.e., $\ell(x)y = x \circ y$, and by $r(x)$ the right multiplications. We show that all $\ell(x)$ are nilpotent.

For $i \geq 2$ we have $\ell(e_{i,p}) = 0$ because of 4.16. For $i = 1$ we have $\ell(e_{1,p}) = L(e_{1,p})$. This last transformation has an upper triangular matrix with on the diagonal $\alpha_j(L(e_{1,p}))$. For $j = 1$ this is zero since $\alpha_1 = 0$. For $j \geq 2$ this is zero because of lemma 4.2.5. Hence $\ell(e_{1,p})$ is a strictly upper triangular matrix.

We obtain that all $\ell(x), x \in A$ are strictly upper triangular and hence nilpotent. Since $\mathfrak{g}$ is nilpotent, proposition 4.1.11 implies that also all right multiplications $r(x)$ are nilpotent. Hence the new LR-structure is complete.

Proof for an arbitrary field of characteristic 0

Finally let us consider the case where $\mathfrak{g}$ is a nilpotent Lie algebra over an arbitrary field k of characteristic 0.

It is still true that $\mathfrak{a} = \mathrm{im}(L)$ is an abelian Lie algebra and that A is an $\mathfrak{a}$-module.

By proposition 4.2.1 we know that A splits as a direct sum of $\mathfrak{a}$-modules: $A = V_n \oplus V_0$ where V_n is the unique maximal nilpotent submodule of A. Stated differently, V_n is the unique maximal left ideal of A on which A acts nilpotently by multiplications from the left.

If we denote the algebraic closure of k by $\bar{k}$, then $\mathfrak{g} \otimes \bar{k}$ is a Lie algebra over $\bar{k}$ having an LR-structure $(A \otimes \bar{k}, \cdot)$. Note that $V_n \otimes \bar{k}$ is the unique maximal left ideal of $A \otimes \bar{k}$ on which $A \otimes \bar{k}$ acts nilpotently by multiplications from the left.

It follows that when we decompose $A \otimes \bar{k}$ as a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_s$ of its weight spaces as we did before, the subspace corresponding to the weight zero is $A_1 = V_n \otimes \bar{k}$ and $A_2 \oplus \cdots \oplus A_s = V_0 \otimes \bar{k}$. So any element $x \in A \otimes \bar{k}$ can be uniquely written as a sum $x = x_n + x_0$, with $x_n \in V_n \otimes \bar{k}$ and $x_0 \in V_0 \otimes \bar{k}$. Moreover, such an element belongs to $A = A \otimes k \subseteq A \otimes \bar{k}$ if and only if $x_n \in V_n = V_n \otimes k$ and $x_0 \in V_0 = V_0 \otimes k$.

Now, let $\circ$ again denote the complete LR-structure on $\mathfrak{g} \otimes \bar{k}$ as constructed above. Using the definition of $\circ$ and (4.16), we see that

$$\forall x_n, y_n \in V_n \otimes \bar{k},\ \forall x_0, y_0 \in V_0 \otimes \bar{k}:\ (x_n + x_0) \circ (y_n + y_0) = x_n \cdot y_n + x_n \cdot y_0.$$

From this, it is obvious that $\circ$ restricts to a k-bilinear product on $A = A \otimes k$, which is then of course a complete LR-structure on A. □

Remark 4.2.9. *For use later on, we note that the above proof shows that the complete LR-product $x \circ y$ on $\mathfrak{g}$, which was constructed from the original LR-product $x \cdot y$, satisfies $\mathfrak{g} \circ \mathfrak{g} \subseteq \mathfrak{g} \cdot \mathfrak{g}$.*

4.3 LR-structures on solvable Lie algebras

In this section we prove our main theorem about LR-structures, namely that the existence of an LR-structure on a Lie algebra implies the existence of a complete LR-structure. The proof of this theorem leads us to another result, which says that if a specific quotient of a 2-step solvable Lie algebra admits an LR-structure which satisfies some extra condition, then this LR-structure can be lifted to an LR-structure on $\mathfrak{g}$.

As before, let $\gamma_i(\mathfrak{g})$ denote the terms of the lower central series of $\mathfrak{g}$. As we always assume that $\mathfrak{g}$ is finite dimensional, this series stabilizes after finitely many steps, with

$$\mathfrak{g}^{\infty} = \bigcap_{i=1}^{\infty} \gamma_i(\mathfrak{g}).$$

Let $\mathfrak{n} := \mathfrak{g}/\mathfrak{g}^{\infty}$. This is a nilpotent Lie algebra.

We have the following result:

Lemma 4.3.1. *Let $\mathfrak{g}$ be a 2-step solvable Lie algebra. Then the extension*

$$0 \to \mathfrak{g}^{\infty} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}^{\infty} \to 0$$

splits, so that $\mathfrak{g} = \mathfrak{g}^{\infty} \rtimes \mathfrak{n}$.

Proof. As by definition $\mathfrak{g}^\infty \subset \gamma_2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, it follows that $[\mathfrak{g}^\infty, \mathfrak{g}^\infty] \subseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, because $\mathfrak{g}$ is 2-step solvable. So $\mathfrak{g}^\infty$ is abelian.

The above short exact sequence induces a Lie algebra homomorphism

$$\varphi\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{g}^\infty) = \mathrm{End}(\mathfrak{g}^\infty).$$

Recall that for all $n \in \mathfrak{n}$ and all $x \in \mathfrak{g}^\infty$ it holds that $\varphi(n)(x) = [\tilde{n}, x]$, where $\tilde{n}$ is any pre-image of n in $\mathfrak{g}$.

Since $\mathfrak{n}$ is nilpotent, we can apply proposition 4.2.1, where we view $\mathfrak{g}^\infty$ as an $\mathfrak{n}$-module via the representation φ. So, we have that $\mathfrak{g}^\infty = V_n \oplus V_0$ where V_n is the unique maximal nilpotent submodule of $\mathfrak{g}^\infty$ and $H^i(\mathfrak{n}, V_0) = 0$, for all $i \geq 0$.

Since $[\mathfrak{g}, \mathfrak{g}^\infty] = \mathfrak{g}^\infty$, the unique maximal nilpotent submodule V_n of $\mathfrak{g}^\infty$ is trivial. Indeed, since V_n and V_0 are submodules, we have that

$$\varphi(\mathfrak{n})(V_n) \subset V_n, \text{ hence } [\mathfrak{g}, V_n] \subset V_n,$$

$$\varphi(\mathfrak{n})(V_0) \subset V_0, \text{ hence } [\mathfrak{g}, V_0] \subset V_0.$$

So, since $[\mathfrak{g}, V_n \oplus V_0] = V_n \oplus V_0$, this implies that $[\mathfrak{g}, V_n] = V_n$, or else $\varphi(\mathfrak{n})(V_n) = V_n$.

Now suppose $V_n \neq 0$. Take some $v \neq 0$ in V_n. There exists $v_1 \in V_n$, $v_1 \neq 0$ and $x_1 \in \mathfrak{g}$ such that $\varphi(x_1 + \mathfrak{g}^\infty)(v_1) = v$. We can continue doing this and hence for every $k \geq 1$ there exists $v_k \in V_n$, $v_k \neq 0$ and $x_1, \ldots, x_k \in \mathfrak{g}$ such that

$$\varphi(x_1 + \mathfrak{g}^\infty) \ldots \varphi(x_k + \mathfrak{g}^\infty)(v_k) = v.$$

But since V_n is a nilpotent submodule this would imply that $v = 0$, which gives a contradiction.

Hence we can conclude that $V_n = 0$ and $\mathfrak{g}^\infty = V_0$. This implies that

$$H^i(\mathfrak{n}, \mathfrak{g}^\infty) = 0$$

for all $i \geq 0$. For $i = 2$ this means that the above extension splits. □

For the proof of our main theorem we need the following result, which can be found in [20] (Corollary 5.2):

Proposition 4.3.2. *Assume that* $\mathfrak{g} = \mathfrak{a} \rtimes_\varphi \mathfrak{b}$ *is a semidirect product of an abelian Lie algebra* $\mathfrak{a}$ *and a Lie algebra* $\mathfrak{b}$ *by a representation* $\varphi\colon \mathfrak{b} \to \mathrm{End}(\mathfrak{a}) = \mathrm{Der}(\mathfrak{a})$*, i.e., we have a split exact sequence*

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0.$$

If $\mathfrak{b}$ *admits an LR-structure* $(x,y)\mapsto x\cdot y$ *such that* $\varphi(x\cdot y)=0$ *for all* $x,y\in\mathfrak{b}$*, then also* $\mathfrak{g}$ *admits an LR-structure, given by*

$$(a,x)\star(b,y)=(\varphi(x)b,x\cdot y).$$

We are now ready to prove the main theorem of this chapter:

Theorem 4.3.3. *Let* $\mathfrak{g}$ *be a Lie algebra over a field* k *of characteristic 0. If* $\mathfrak{g}$ *admits an LR-structure, then* $\mathfrak{g}$ *also admits a complete LR-structure.*

Proof. As $\mathfrak{g}$ admits an LR-structure $x\cdot y$, we know that $\mathfrak{g}$ is 2-step solvable, see proposition 4.1.5. By the lemma above, we have that $\mathfrak{g}=\mathfrak{g}^\infty\rtimes\mathfrak{n}$, where $\mathfrak{n}=\mathfrak{g}/\mathfrak{g}^\infty$ is a nilpotent Lie algebra.

Since all terms of the lower central series of $\mathfrak{g}$ are 2-sided ideals for the LR-product, see lemma 4.1.7, $\mathfrak{g}^\infty$ is a 2-sided ideal and hence the LR-structure on $\mathfrak{g}$ induces an LR-structure on $\mathfrak{n}=\mathfrak{g}/\mathfrak{g}^\infty$.

Let us denote this induced product on $\mathfrak{n}$ by $x\bullet y$. By proposition 4.2.2 and remark 4.2.9 we know that $\mathfrak{n}$ also admits a complete LR-structure $x\circ y$ with $\mathfrak{n}\circ\mathfrak{n}\subseteq\mathfrak{n}\bullet\mathfrak{n}$. When we now view $\mathfrak{n}$ as a subalgebra of $\mathfrak{g}=\mathfrak{g}^\infty\rtimes\mathfrak{n}$, we get that $\mathfrak{n}\circ\mathfrak{n}\subseteq\mathfrak{n}\cdot\mathfrak{n}+\mathfrak{g}^\infty$.

As above, let $\varphi:\mathfrak{n}\to\mathrm{Der}(\mathfrak{g}^\infty)$ denote the homomorphism induced by the split-extension, i.e. $\varphi(n)(x)=[n,x]$ for all $n\in\mathfrak{n}$ and $x\in\mathfrak{g}^\infty$.

We claim that $\varphi(\mathfrak{n}\circ\mathfrak{n})=0$. Indeed, since $\mathfrak{g}^\infty$ is abelian, and $\mathfrak{g}^\infty\subseteq[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{g}\cdot\mathfrak{g}$, we have

$$[\mathfrak{n}\circ\mathfrak{n},\mathfrak{g}^\infty]\subseteq[\mathfrak{n}\cdot\mathfrak{n}+\mathfrak{g}^\infty,\mathfrak{g}^\infty]\subseteq[\mathfrak{n}\cdot\mathfrak{n},\mathfrak{g}^\infty]\subseteq[\mathfrak{g}\cdot\mathfrak{g},\mathfrak{g}\cdot\mathfrak{g}]=0,$$

where the last equality follows from the fact that in any LR-algebra A, we have the identity $(x\cdot y)\cdot(u\cdot v)-(u\cdot v)\cdot(x\cdot y)=0$ for all $x,y,u,v\in A$, see lemma 4.1.4.

It now follows from proposition 4.3.2, that $\mathfrak{g}$ admits an LR-structure $\star$, which is given by the following formula:

$$\forall(a,x),(b,y)\in\mathfrak{g}=\mathfrak{g}^\infty\rtimes\mathfrak{n}:\ (a,x)\star(b,y)=(\varphi(x)b,x\circ y).$$

Note that this structure is complete, since the structure on $\mathfrak{n}$ is complete and we have for all $k\geq 1$

$$R_\star^k(b,y)(a,x)=(\varphi(R_\circ^{k-1}(y)(x))b,R_\circ^k(y)(x)),\tag{4.17}$$

where $R_\star$ denotes the right multiplication for the $\star$-product in $\mathfrak{g}$ and $R_\circ$ denotes the right multiplication for the $\circ$-product in $\mathfrak{n}$.

This can be proven by induction on k. The case $k = 1$ is clear. Suppose we have the above identity for some k, then

$$\begin{aligned} R_\star^{k+1}(b,y)(a,x) &= R_\star(b,y)R_\star^k(b,y)(a,x) \\ &= R_\star(b,y)(\varphi(R_\circ^{k-1}(y)(x))b, R_\circ^k(y)(x)) \\ &= (\varphi(R_\circ^{k-1}(y)(x))b, R_\circ^k(y)(x)) \star (b,y) \\ &= (\varphi(R_\circ^k(y)(x))b, R_\circ^k(y)(x) \circ y) \\ &= (\varphi(R_\circ^k(y)(x))b, R_\circ^{k+1}(y)(x)). \end{aligned}$$

□

We already saw in the above proof that if a solvable Lie algebra $\mathfrak{g}$ admits an LR-structure, then also $\mathfrak{n} = \mathfrak{g}/\mathfrak{g}^\infty$ admits an LR-structure. The proof above also suggests a partial converse of this:

Theorem 4.3.4. *Let $\mathfrak{g}$ be a 2-step solvable Lie algebra and assume that $\mathfrak{n} = \mathfrak{g}/\mathfrak{g}^\infty$ admits an LR-structure satisfying*

$$\mathfrak{n} \cdot \mathfrak{n} \subseteq [\mathfrak{n}, \mathfrak{n}]. \tag{4.18}$$

Then, this LR-structure can be lifted to a complete LR-structure on $\mathfrak{g}$.

Proof. As before, $\mathfrak{g} = \mathfrak{g}^\infty \rtimes \mathfrak{n}$, where the action of $\mathfrak{n}$ on $\mathfrak{g}^\infty$ is given by $\varphi\colon \mathfrak{n} \to \operatorname{End}(\mathfrak{g}^\infty)$.

We have by lemma 4.1.4 that $[[\mathfrak{n},\mathfrak{n}],\mathfrak{g}^\infty] \subseteq [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 0$, so that $\varphi(\mathfrak{n}\cdot\mathfrak{n}) = 0$ by (4.18). Now the same formula $(a,x)\star(b,y) = (\varphi(x)b, x\cdot y)$ as before defines an LR-structure on $\mathfrak{g}$.

It follows easily that this structure on $\mathfrak{g}$ is complete. Indeed, note that also in this case we have (4.17), so we only need to prove that the product on $\mathfrak{n}$ is complete.

Let $R(x)$ denote the right multiplication by $x \in \mathfrak{n}$ in $\mathfrak{n}$. Now $R^k(x)(\mathfrak{n}) \subseteq \gamma_{k+1}(\mathfrak{n})$ for each $k \geq 1$.
Indeed, for $k = 1$ this is just (4.18). Suppose we know this for some $k \geq 1$, then by lemma 4.1.9, which holds here because of (4.18), it also holds for $k+1$ since

$$R^{k+1}(x)(\mathfrak{n}) = R(x)R^k(x)(\mathfrak{n}) \subseteq \gamma_{k+1}(\mathfrak{n}) \cdot \gamma_1(\mathfrak{n}) \subseteq \gamma_{k+2}(\mathfrak{n}).$$

Since $\mathfrak{n}$ is nilpotent, this implies that all right multiplications on $\mathfrak{n}$ are nilpotent and the structure is complete. □

We have the following corollary:

Corollary 4.3.5. *Let $\mathfrak{g}$ be a 2-step solvable Lie algebra with $\mathfrak{g}^\infty = \gamma_3(\mathfrak{g})$. Then $\mathfrak{g}$ admits an LR-structure.*

Proof. By assumption $\mathfrak{n} = \mathfrak{g}/\mathfrak{g}^\infty = \mathfrak{g}/\mathfrak{g}^3$ is 2-step nilpotent. By proposition 4.3 of [20], $x \cdot y = \frac{1}{2}[x, y]$ defines an LR-structure on $\mathfrak{n}$. It obviously satisfies $\mathfrak{n} \cdot \mathfrak{n} \subseteq [\mathfrak{n}, \mathfrak{n}]$, so the claim follows from the above theorem. □

Remark 4.3.6. *There are 2-step solvable Lie algebras with $\mathfrak{g}^\infty = \gamma_4(\mathfrak{g})$ without any LR-structure, see proposition 4.7 of [20], which gives an explicit example. In this sense the corollary cannot be improved.*

4.4 LR-structures on 2-step solvable 2-generated Lie algebras

Among the 2-step solvable Lie algebras with 2 generators are the filiform nilpotent Lie algebras of solvability class 2. For this class of Lie algebras an explicit LR-structure was constructed in [20]. We want to generalize this construction here to all 2-generated, 2-step solvable Lie algebras. We need two lemmas.

Lemma 4.4.1. *Let $\mathfrak{g}$ be a 2-step solvable Lie algebra, $x_i, y \in \mathfrak{g}$ for $1 \le i \le n$ and σ a permutation in $\mathcal{S}_n$. Then*

$$\operatorname{ad}(x_1) \cdots \operatorname{ad}(x_n) \operatorname{ad}(y) = \operatorname{ad}(x_{\sigma(1)}) \cdots \operatorname{ad}(x_{\sigma(n)}) \operatorname{ad}(y).$$

Proof. We use induction on $n \ge 1$, where the case $n = 1$ is clear. For $n \ge 2$ we first consider $\sigma \in \mathcal{S}_n$ with $\sigma(1) = 1$. Then, by the induction hypothesis, we have

$$\operatorname{ad}(x_1) \operatorname{ad}(x_{\sigma(2)}) \cdots \operatorname{ad}(x_{\sigma(n)}) \operatorname{ad}(y) = \operatorname{ad}(x_1) \operatorname{ad}(x_2) \cdots \operatorname{ad}(x_n) \operatorname{ad}(y). \quad (4.19)$$

For $z \in \mathfrak{g}$ the Jacobi identity and the solvability class 2 imply

$$\begin{aligned} \operatorname{ad}(x_1) \operatorname{ad}(x_2) \operatorname{ad}(y)(z) &= [x_1, [x_2, [y, z]]] \\ &= -[x_2, [[y, z], x_1]] - [[y, z], [x_1, x_2]] \\ &= [x_2, [x_1, [y, z]]] \\ &= \operatorname{ad}(x_2) \operatorname{ad}(x_1) \operatorname{ad}(y)(z). \end{aligned}$$

In the same way we obtain

$$\operatorname{ad}(x_1)\operatorname{ad}(x_2)\cdots\operatorname{ad}(x_n)\operatorname{ad}(y)(z) = \operatorname{ad}(x_2)\operatorname{ad}(x_1)\cdots\operatorname{ad}(x_n)\operatorname{ad}(y)(z). \quad (4.20)$$

The general case follows from (4.19) and (4.20). □

Lemma 4.4.2. *Let $\mathfrak{g}$ be a 2-step solvable Lie algebra and $x, y \in \mathfrak{g}$. Then the subspace spanned by x and*

$$\{\operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y \mid k, l \geq 0\}$$

is a Lie subalgebra of $\mathfrak{g}$.

Proof. We prove that the bracket of any two generators is again in this subspace. If $\ell \geq 1$ then lemma 4.4.1 implies

$$\begin{aligned} [x, \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y] &= \operatorname{ad}(x)\operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y \\ &= \operatorname{ad}(y)^k \operatorname{ad}(x)^{\ell+1} y. \end{aligned}$$

For $\ell, n \geq 1$ we have

$$[\operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y, \operatorname{ad}(y)^m \operatorname{ad}(x)^n y] = 0,$$

because $\mathfrak{g}$ is 2-step solvable. The other cases are trivial. □

Now we can show the following result:

Theorem 4.4.3. *Let $\mathfrak{g}$ be a 2-generated, 2-step solvable Lie algebra. Then $\mathfrak{g}$ admits a (complete) LR-structure.*

Proof. Let $\mathfrak{g}$ be generated as a Lie algebra by x and y. Then $\mathfrak{g}$ is spanned by x and all vectors $\operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ with $k, \ell \geq 0$.

Let us fix a basis consisting of x, y and a subset of the above vectors satisfying $\ell \geq 1$.

Using this basis we define a k-bilinear product on $\mathfrak{g}$ as follows

$$\begin{aligned} L(x) &= 0, \\ L(\operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y) &= \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y). \end{aligned}$$

In particular, $L(y) = \operatorname{ad}(y)$. We have to verify that this defines an LR-structure on $\mathfrak{g}$.

First of all we have $a \cdot b - b \cdot a = [a, b]$ for all basis vectors $a, b \in \mathfrak{g}$. For $a = x$ and $b = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ this means

$$\begin{aligned} a \cdot b - b \cdot a &= L(a)b - L(b)a \\ &= 0 - \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y)x \\ &= \operatorname{ad}(y)^k \operatorname{ad}(x)^{\ell+1} y \\ &= \operatorname{ad}(x)b \\ &= [a, b] \end{aligned}$$

by lemma 4.4.1.
For $a = y$ and $b = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ with $\ell \geq 1$ this means

$$\begin{aligned} a \cdot b - b \cdot a &= \operatorname{ad}(y) \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y - \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y)y \\ &= [y, \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y] - 0 \\ &= [a, b]. \end{aligned}$$

For $a = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ and $b = \operatorname{ad}(y)^m \operatorname{ad}(x)^n y$ with $\ell, n \geq 1$ this means

$$\begin{aligned} a \cdot b - b \cdot a &= \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y) \operatorname{ad}(y)^m \operatorname{ad}(x)^n y \\ &\quad - \operatorname{ad}(y)^m \operatorname{ad}(x)^n \operatorname{ad}(y) \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y \\ &= \operatorname{ad}(y)^{k+m+1} \operatorname{ad}(x)^{\ell+n} y - \operatorname{ad}(y)^{m+k+1} \operatorname{ad}(x)^{\ell+n} y \\ &= 0 \\ &= [a, b], \end{aligned}$$

since $\mathfrak{g}$ is 2-step solvable.

Next we verify that $[L(a), L(b)] = 0$ for all basis vectors $a, b \in \mathfrak{g}$. For $a = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ and $b = \operatorname{ad}(y)^m \operatorname{ad}(x)^n y$ this means

$$\begin{aligned} L(a)L(b) &= \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y) \operatorname{ad}(y)^m \operatorname{ad}(x)^n \operatorname{ad}(y) \\ &= \operatorname{ad}(y)^{k+m+1} \operatorname{ad}(x)^{\ell+n} \operatorname{ad}(y) \\ &= \operatorname{ad}(y)^m \operatorname{ad}(x)^n \operatorname{ad}(y) \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y) \\ &= L(b)L(a), \end{aligned}$$

where we have used lemma 4.4.1. The other cases are trivial.

Finally we have to show that $[R(a), R(b)] = 0$ for all basis vectors $a, b \in \mathfrak{g}$. We have $R(a) = L(a) - \operatorname{ad}(a)$, so that

$$R(x) = -\operatorname{ad}(x),$$

$$R(y) = 0.$$

It is not difficult to see that, for $a = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ with $\ell \geq 1$ we have

$$R(a) = \operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell.$$

Indeed, using lemma 4.4.1 and the Jacobi identity, we have for any $z \in \mathfrak{g}$

$$\begin{aligned} \operatorname{ad}(a)z &= -\operatorname{ad}(z)a = -\operatorname{ad}(z)\operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y \\ &= -\operatorname{ad}(y)^k \operatorname{ad}(x)^{\ell-1} \operatorname{ad}(z)\operatorname{ad}(x)y \\ &= \operatorname{ad}(y)^k \operatorname{ad}(x)^{\ell-1} \operatorname{ad}(x)\operatorname{ad}(y)z + \operatorname{ad}(y)^k \operatorname{ad}(x)^{\ell-1} \operatorname{ad}(y)\operatorname{ad}(z)x \\ &= \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell \operatorname{ad}(y)z - \operatorname{ad}(y)^k \operatorname{ad}(x)^{\ell-1} \operatorname{ad}(y)\operatorname{ad}(x)z \\ &= L(a)z - \operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell z. \end{aligned}$$

This gives directly that $R(a) = \operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell$.

For $a = x$ and $b = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ with $\ell \geq 1$ we have by lemma 4.4.1

$$\begin{aligned} R(a)R(b) &= -\operatorname{ad}(x)\operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell \\ &= -\operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell \operatorname{ad}(x) \\ &= R(b)R(a). \end{aligned}$$

For $a = \operatorname{ad}(y)^k \operatorname{ad}(x)^\ell y$ and $b = \operatorname{ad}(y)^m \operatorname{ad}(x)^n y$ with $\ell, n \geq 1$ we have by lemma 4.4.1

$$\begin{aligned} R(a)R(b) &= \operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell \operatorname{ad}(y)^{m+1} \operatorname{ad}(x)^n \\ &= \operatorname{ad}(y)^{m+1} \operatorname{ad}(x)^n \operatorname{ad}(y)^{k+1} \operatorname{ad}(x)^\ell \\ &= R(b)R(a). \end{aligned}$$

The other cases are trivial.

Note that this LR-structure is complete. Indeed, sort the basis elements by the number of Lie brackets they are build up from. Then for each basis element the right multiplication has a block matrix where only the blocks strictly under the diagonal can be nonzero. Hence it is nilpotent and this implies that the structure is complete. □

4.5 LR-structures on free nilpotent 2-step solvable Lie algebras

In this section we will study LR-structures on free p-step nilpotent 2-step solvable Lie algebras.

It appears that the situation for LR-structures is different than that for Novikov structures.

We already know from [16] and [20] that all free 2-step nilpotent Lie algebras admit Novikov and LR-structures.

In the previous chapter we proved that all free 3-step nilpotent Lie algebras admit a Novikov structure but for higher nilpotency class no Novikov structures exist.

For LR-structures such a restriction does not hold. In particular we already know from section 4.4 that the free p-step nilpotent 2-step solvable Lie algebra on 2 generators admits an LR-structure, and this for every nilpotency class p.

In this section we will prove that all free p-step nilpotent 2-step solvable Lie algebras on n generators admit an LR-structure, by explicitly constructing such a structure. In particular the structure will be complete.

To end this section we prove that all LR-structures on these Lie algebras are complete if the nilpotency class is at least 3.

We start by proving the existence of LR-structures:

Proposition 4.5.1. *Let $\mathfrak{g}$ be the free p-step nilpotent 2-step solvable Lie algebra on n generators $x_1, \ldots, x_n$. Then $\mathfrak{g}$ admits a (complete) LR-structure.*

Proof. As a vector space $\mathfrak{g}$ has a basis

$$x_1, \ldots, x_n,$$

$$[\ldots[[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_k}]$$

$$\text{for all } n \geq i_1 > i_2 \leq i_3 \leq \ldots \leq i_k \leq n \text{ and } 2 \leq k \leq p.$$

Construct a bilinear product on $\mathfrak{g}$ by defining it on the basis elements as follows:

$$x_i \cdot x_j := [x_i, x_j] \text{ if } i > j \text{ and } 0 \text{ otherwise},$$

$$x_i \cdot \Big[\ldots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big]$$

$$:= \Big[\ldots\big[[x_i \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_i \cdot x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big],$$

$$\Big[\ldots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big] \cdot x_i$$

$$:= \Big[\ldots\big[[x_{j_1} \cdot x_i, x_{j_2}] + [x_{j_1}, x_{j_2} \cdot x_i], x_{j_3}\big], \ldots, x_{j_m}\Big],$$

$$\Big[\ldots\big[[x_{i_1}, x_{i_2}], x_{i_3}\big], \ldots, x_{i_k}\Big] \cdot \Big[\ldots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big]$$

$$:= \Big[\ldots\big[[\ldots[a, x_{j_3}], \ldots, x_{j_m}], x_{i_3}\big], \ldots, x_{i_k}\Big]$$

$$\text{with } a = \big[[x_{i_1} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{i_1} \cdot x_{j_2}], x_{i_2}\big]$$

$$+ \big[x_{i_1}, [x_{i_2} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{i_2} \cdot x_{j_2}]\big].$$

This product has the following properties:

Lemma 4.5.2. *We have that*

$$x_i \cdot \Big[\ldots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big] = \Big[\ldots\big[x_i \cdot [x_{j_1}, x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big],$$

$$\Big[\ldots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \ldots, x_{j_m}\Big] \cdot x_i = \Big[\ldots\big[[x_{j_1}, x_{j_2}] \cdot x_i, x_{j_3}\big], \ldots, x_{j_m}\Big].$$

Proof. We use the definition twice to get

$$\begin{aligned} x_i \cdot \Big[\dots[[x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}\Big] \\ &= \Big[\dots[[x_i \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_i \cdot x_{j_2}], x_{j_3}], \dots, x_{j_m}\Big] \\ &= \Big[\dots[x_i \cdot [x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}\Big]. \end{aligned}$$

Analogously we get the second result. □

Lemma 4.5.3. *The term*

$$\begin{aligned} a = \big[[x_{i_1} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{i_1} \cdot x_{j_2}], x_{i_2}\big] \\ + \big[x_{i_1}, [x_{i_2} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{i_2} \cdot x_{j_2}]\big], \end{aligned}$$

which appears in the definition of the product is equal to each of the following identities:

1. $\big[x_{i_1} \cdot [x_{j_1}, x_{j_2}], x_{i_2}\big] + \big[x_{i_1}, x_{i_2} \cdot [x_{j_1}, x_{j_2}]\big]$,
2. $[x_{i_1}, x_{i_2}] \cdot [x_{j_1}, x_{j_2}]$,
3. $\big[[x_{i_1}, x_{i_2}] \cdot x_{j_1}, x_{j_2}\big] + \big[x_{j_1}, [x_{i_1}, x_{i_2}] \cdot x_{j_2}\big]$.

Proof. Using the definition we find

$$\begin{aligned} \big[x_{i_1} \cdot [x_{j_1}, x_{j_2}], x_{i_2}\big] + \big[x_{i_1}, x_{i_2} \cdot [x_{j_1}, x_{j_2}]\big] = a, \\ [x_{i_1}, x_{i_2}] \cdot [x_{j_1}, x_{j_2}] = a, \end{aligned}$$

and also

$$\begin{aligned} a &= \big[[x_{i_1} \cdot x_{j_1}, x_{j_2}], x_{i_2}\big] + \big[[x_{j_1}, x_{i_1} \cdot x_{j_2}], x_{i_2}\big] \\ &\quad + \big[x_{i_1}, [x_{i_2} \cdot x_{j_1}, x_{j_2}]\big] + \big[x_{i_1}, [x_{j_1}, x_{i_2} \cdot x_{j_2}]\big] \\ &= \big[[x_{i_1} \cdot x_{j_1}, x_{i_2}], x_{j_2}\big] + \big[[x_{i_1}, x_{i_2} \cdot x_{j_1}], x_{j_2}\big] \\ &\quad + \big[x_{j_1}, [x_{i_1} \cdot x_{j_2}, x_{i_2}]\big] + \big[x_{j_1}, [x_{i_1}, x_{i_2} \cdot x_{j_2}]\big] \\ &= \big[[x_{i_1} \cdot x_{j_1}, x_{i_2}] + [x_{i_1}, x_{i_2} \cdot x_{j_1}], x_{j_2}\big] \\ &\quad + \big[x_{j_1}, [x_{i_1} \cdot x_{j_2}, x_{i_2}] + [x_{i_1}, x_{i_2} \cdot x_{j_2}]\big] \\ &= \big[[x_{i_1}, x_{i_2}] \cdot x_{j_1}, x_{j_2}\big] + \big[x_{j_1}, [x_{i_1}, x_{i_2}] \cdot x_{j_2}\big], \end{aligned}$$

where we have used that $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$ and lemma 4.4.1. □

We can now prove that the product is an LR-structure on $\mathfrak{g}$.

First of all we check that $[X, Y] = X \cdot Y - Y \cdot X$ for all basis elements X and Y. To start, let $X = x_i$ and $Y = x_j$, we may suppose that $i < j$. Then we have

$$X \cdot Y - Y \cdot X = x_i \cdot x_j - x_j \cdot x_i = 0 - [x_j, x_i] = [x_i, x_j] = [X, Y].$$

To proceed, we prove the following lemma:

Lemma 4.5.4. *We have*

$$x_i \cdot [x_{j_1}, x_{j_2}] - [x_{j_1}, x_{j_2}] \cdot x_i = [x_i, [x_{j_1}, x_{j_2}]],$$
$$[x_{i_1}, x_{i_2}] \cdot [x_{j_1}, x_{j_2}] - [x_{j_1}, x_{j_2}] \cdot [x_{i_1}, x_{i_2}] = 0.$$

Proof. We use the definition, the previous result and the Jacobi identity to get

$$\begin{aligned}
&x_i \cdot [x_{j_1}, x_{j_2}] - [x_{j_1}, x_{j_2}] \cdot x_i \\
&\quad = [x_i \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_i \cdot x_{j_2}] - [x_{j_1} \cdot x_i, x_{j_2}] - [x_{j_1}, x_{j_2} \cdot x_i] \\
&\quad = [x_i \cdot x_{j_1} - x_{j_1} \cdot x_i, x_{j_2}] + [x_{j_1}, x_i \cdot x_{j_2} - x_{j_2} \cdot x_i] \\
&\quad = \big[[x_i, x_{j_1}], x_{j_2}\big] + \big[x_{j_1}, [x_i, x_{j_2}]\big] \\
&\quad = \big[x_i, [x_{j_1}, x_{j_2}]\big].
\end{aligned}$$

For the second equality we have

$$\begin{aligned}
&[x_{i_1}, x_{i_2}] \cdot [x_{j_1}, x_{j_2}] \\
&\quad = \big[[x_{i_1} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{i_1} \cdot x_{j_2}], x_{i_2}\big] \\
&\qquad + \big[x_{i_1}, [x_{i_2} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{i_2} \cdot x_{j_2}]\big] \\
&\quad = \Big[\big[[x_{i_1}, x_{j_1}] + x_{j_1} \cdot x_{i_1}, x_{j_2}\big] + \big[x_{j_1}, [x_{i_1}, x_{j_2}] + x_{j_2} \cdot x_{i_1}\big], x_{i_2}\Big] \\
&\qquad + \Big[x_{i_1}, \big[[x_{i_2}, x_{j_1}] + x_{j_1} \cdot x_{i_2}, x_{j_2}\big] + \big[x_{j_1}, [x_{i_2}, x_{j_2}] + x_{j_2} \cdot x_{i_2}\big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\big[[x_{i_1}, x_{j_1}], x_{j_2}\big] + \big[x_{j_1} \cdot x_{i_1}, x_{j_2}\big] \\
&\qquad\qquad\qquad + \big[x_{j_1}, [x_{i_1}, x_{j_2}]\big] + \big[x_{j_1}, x_{j_2} \cdot x_{i_1}\big], x_{i_2}\Big] \\
&\quad + \Big[x_{i_1}, \big[[x_{i_2}, x_{j_1}], x_{j_2}\big] + \big[x_{j_1} \cdot x_{i_2}, x_{j_2}\big] \\
&\qquad\qquad\qquad + \big[x_{j_1}, [x_{i_2}, x_{j_2}]\big] + \big[x_{j_1}, x_{j_2} \cdot x_{i_2}\big]\Big] \\
&= \Big[[x_{j_1} \cdot x_{i_1}, x_{i_2}] + [x_{i_1}, x_{j_1} \cdot x_{i_2}], x_{j_2}\Big] \\
&\quad + \Big[x_{j_1}, [x_{j_2} \cdot x_{i_1}, x_{i_2}] + [x_{i_1}, x_{j_2} \cdot x_{i_2}]\Big] \\
&\quad - \Big[\big[[x_{j_1}, x_{j_2}], x_{i_1}\big], x_{i_2}\Big] - \Big[x_{i_1}, \big[[x_{j_1}, x_{j_2}], x_{i_2}\big]\Big] \\
&= [x_{j_1}, x_{j_2}] \cdot [x_{i_1}, x_{i_2}],
\end{aligned}
$$

where we have used the fact that $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$, lemma 4.4.1 and the Jacobi identity. □

Now suppose $X = x_i$ and $Y = [\dots[[x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}]$. By using lemmas 4.5.2 and 4.5.4 we get, using lemma 4.4.1,

$$
\begin{aligned}
&X \cdot Y - Y \cdot X \\
&\quad = \Big[\dots\big[x_i \cdot [x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\qquad - \Big[\dots\big[[x_{j_1}, x_{j_2}] \cdot x_i, x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = \Big[\dots\big[x_i \cdot [x_{j_1}, x_{j_2}] - [x_{j_1}, x_{j_2}] \cdot x_i, x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = \Big[\dots\big[[x_i, [x_{j_1}, x_{j_2}]\big], x_{j_3}], \dots, x_{j_m}\Big] \\
&\quad = \Big[x_i, \big[\dots[[x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}\big]\Big] \\
&\quad = [X, Y].
\end{aligned}
$$

The last case we have to consider is $X = \Big[\dots\big[[x_{i_1}, x_{i_2}], x_{i_3}\big], \dots, x_{i_k}\Big]$ and $Y = \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]$. Since $\mathfrak{g}$ is 2-step solvable, it is enough to prove that $X \cdot Y - Y \cdot X = 0$. We have by lemma 4.5.3

$$\begin{aligned} &X \cdot Y - Y \cdot X \\ &\quad = \Big[\dots\big[[\dots[[x_{i_1}, x_{i_2}] \cdot [x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}], x_{i_3}\big], \dots, x_{i_k}\Big] \\ &\qquad - \Big[\dots\big[[\dots[[x_{j_1}, x_{j_2}] \cdot [x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_k}], x_{j_3}\big], \dots, x_{j_m}\Big]. \end{aligned}$$

This is zero since $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$ and the use of lemma 4.4.1 and 4.5.4.

Secondly we have to check that $X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = 0$ for all basis elements X, Y, Z.

First let $X = x_i$, $Y = x_j$ and $Z = x_k$, we may suppose that $i < j$. We consider the following cases:

$$\begin{aligned} i < j \leq k: \quad & x_i \cdot (x_j \cdot x_k) - x_j \cdot (x_i \cdot x_k) = 0, \\ i \leq k < j: \quad & x_i \cdot (x_j \cdot x_k) - x_j \cdot (x_i \cdot x_k) = x_i \cdot [x_j, x_k] \\ & \quad = [x_i \cdot x_j, x_k] + [x_j, x_i \cdot x_k] = 0, \qquad (4.21) \\ k < i < j: \quad & x_i \cdot (x_j \cdot x_k) - x_j \cdot (x_i \cdot x_k) = x_i \cdot [x_j, x_k] - x_j \cdot [x_i, x_k] \\ & \quad = [x_i \cdot x_j, x_k] + [x_j, x_i \cdot x_k] - [x_j \cdot x_i, x_k] - [x_i, x_j \cdot x_k] \\ & \quad = [x_j, [x_i, x_k]] - [[x_j, x_i], x_k] - [x_i, [x_j, x_k]] = 0. \end{aligned}$$

Let $X = x_i$, $Y = x_j$ and $Z = \Big[\dots\big[[x_{r_1}, x_{r_2}], x_{r_3}\big], \dots, x_{r_p}\Big]$. We have

$$\begin{aligned} X \cdot (Y \cdot Z) &= x_i \cdot \Big[\dots\big[[x_j \cdot x_{r_1}, x_{r_2}] + [x_{r_1}, x_j \cdot x_{r_2}], x_{r_3}\big], \dots, x_{r_p}\Big] \\ &= \Big[\dots\big[[x_i \cdot (x_j \cdot x_{r_1}), x_{r_2}] + [x_{r_1}, x_i \cdot (x_j \cdot x_{r_2})], x_{r_3}\big], \dots, x_{r_p}\Big], \end{aligned}$$

where we have used that the product of two generators is either 0 or the Lie bracket of those generators and lemma 4.5.2. Note that for $Y \cdot (X \cdot Z)$ we find exactly the same result by using the previous result (4.21). Hence $X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = 0$.

Now let $X = x_i$, $Y = \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]$ and $Z = x_k$, we have

$$\begin{aligned}
&X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) \\
&\quad = x_i \cdot \Big(\Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \cdot x_k\Big) \\
&\qquad - \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \cdot (x_i \cdot x_k) \\
&\quad = x_i \cdot \Big[\dots\big[[x_{j_1} \cdot x_k, x_{j_2}] + [x_{j_1}, x_{j_2} \cdot x_k], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\qquad - \Big[\dots\big[[x_{j_1}, x_{j_2}] \cdot (x_i \cdot x_k), x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = \Big[\dots\big[[x_i \cdot (x_{j_1} \cdot x_k), x_{j_2}] + [x_{j_1}, x_i \cdot (x_{j_2} \cdot x_k)], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\qquad - \Big[\dots\big[[x_{j_1} \cdot (x_i \cdot x_k), x_{j_2}] + [x_{j_1}, x_{j_2} \cdot (x_i \cdot x_k)], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = 0.
\end{aligned}$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators, lemma 4.5.2, lemma 4.5.3 and (4.21).

Let $X = x_i$, $Y = \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]$ and
$Z = \Big[\dots\big[[x_{r_1}, x_{r_2}], x_{r_3}\big], \dots, x_{r_p}\Big]$, we have

$$\begin{aligned}
&X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) \\
&\quad = x_i \cdot \Big[\dots\big[[\dots[a, x_{r_3}], \dots, x_{r_p}], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\qquad - \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \cdot \\
&\qquad\qquad\qquad \Big[\dots\big[[x_i \cdot x_{r_1}, x_{r_2}] + [x_{r_1}, x_i \cdot x_{r_2}], x_{r_3}\big], \dots, x_{r_p}\Big] \\
&\qquad \text{with } a = [[x_{j_1} \cdot x_{r_1}, x_{r_2}] + [x_{r_1}, x_{j_1} \cdot x_{r_2}], x_{j_2}] \\
&\qquad\qquad + [x_{j_1}, [x_{j_2} \cdot x_{r_1}, x_{r_2}] + [x_{r_1}, x_{j_2} \cdot x_{r_2}]]
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\dots [[\dots [a, x_{r_3}], \dots, x_{r_p}], x_{j_3}], \dots, x_{j_m} \Big] \\
&\quad - \Big[\dots [[\dots [[[x_{j_1}, x_{j_2}] \cdot (x_i \cdot x_{r_1}), x_{r_2}] + [x_{r_1}, [x_{j_1}, x_{j_2}] \cdot (x_i \cdot x_{r_2})], \\
&\qquad\qquad x_{r_3}], \dots, x_{r_p}], x_{j_3}], \dots, x_{j_m} \Big] \\
&\quad \text{with } a = [[x_i \cdot (x_{j_1} \cdot x_{r_1}), x_{r_2}] + [x_{r_1}, x_i \cdot (x_{j_1} \cdot x_{r_2})], x_{j_2}] \\
&\qquad\qquad + [x_{j_1}, [x_i \cdot (x_{j_2} \cdot x_{r_1}), x_{r_2}] + [x_{r_1}, x_i \cdot (x_{j_2} \cdot x_{r_2})]] \\
&= \Big[\dots [[\dots [a, x_{r_3}], \dots, x_{r_p}], x_{j_3}], \dots, x_{j_m} \Big] \\
&\quad \text{with } a = [[x_{j_1} \cdot (x_i \cdot x_{r_1}), x_{r_2}] + [x_{r_1}, x_{j_1} \cdot (x_i \cdot x_{r_2})], x_{j_2}] \\
&\qquad\qquad + [x_{j_1}, [x_{j_2} \cdot (x_i \cdot x_{r_1}), x_{r_2}] + [x_{r_1}, x_{j_2} \cdot (x_i \cdot x_{r_2})]] \\
&\qquad\qquad - [[x_{j_1}, x_{j_2}] \cdot (x_i \cdot x_{r_1}), x_{r_2}] - [x_{r_1}, [x_{j_1}, x_{j_2}] \cdot (x_i \cdot x_{r_2})].
\end{aligned}
$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators, lemma 4.5.2, lemma 4.5.3 and (4.21). Now we have for a that

$$
\begin{aligned}
a &= [[x_{j_1} \cdot (x_i \cdot x_{r_1}), x_{r_2}], x_{j_2}] + [[x_{r_1}, x_{j_1} \cdot (x_i \cdot x_{r_2})], x_{j_2}] \\
&\quad + [x_{j_1}, [x_{j_2} \cdot (x_i \cdot x_{r_1}), x_{r_2}]] + [x_{j_1}, [x_{r_1}, x_{j_2} \cdot (x_i \cdot x_{r_2})]] \\
&\quad - [[x_{j_1} \cdot (x_i \cdot x_{r_1}), x_{j_2}], x_{r_2}] - [[x_{j_1}, x_{j_2} \cdot (x_i \cdot x_{r_1})], x_{r_2}] \\
&\quad - [x_{r_1}, [x_{j_1} \cdot (x_i \cdot x_{r_2}), x_{j_2}]] - [x_{r_1}, [x_{j_1}, x_{j_2} \cdot (x_i \cdot x_{r_2})]] \\
&= 0.
\end{aligned}
$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators, lemma 4.5.3, $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$ and lemma 4.4.1.

Let $X = \Big[\dots\big[[x_{i_1}, x_{i_2}], x_{i_3}\big], \dots, x_{i_k}\Big]$,
$Y = \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]$ and $Z = x_p$, we have

$$
\begin{aligned}
&X \cdot (Y \cdot Z)\\
&= \Big[\dots\big[[x_{i_1}, x_{i_2}], x_{i_3}\big], \dots, x_{i_k}\Big] \cdot\\
&\qquad \Big[\dots\big[[x_{j_1} \cdot x_p, x_{j_2}] + [x_{j_1}, x_{j_2} \cdot x_p], x_{j_3}\big], \dots, x_{j_m}\Big]\\
&= \Big[\dots\big[[\dots[[[x_{i_1} \cdot (x_{j_1} \cdot x_p), x_{i_2}] + [x_{i_1}, x_{i_2} \cdot (x_{j_1} \cdot x_p)], x_{j_2}],\\
&\qquad x_{j_3}], \dots, x_{j_m}], x_{i_3}\big], \dots, x_{i_k}\Big]\\
&\quad + \Big[\dots\big[[\dots[[x_{j_1}, [x_{i_1} \cdot (x_{j_2} \cdot x_p), x_{i_2}] + [x_{i_1}, x_{i_2} \cdot (x_{j_2} \cdot x_p)]],\\
&\qquad x_{j_3}], \dots, x_{j_m}], x_{i_3}\big], \dots, x_{i_k}\Big].
\end{aligned}
$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators and lemma 4.5.3. Note that for $Y \cdot (X \cdot Z)$ we find exactly the same result by using that $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$, lemma 4.4.1 and (4.21). Hence $X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = 0$.

Let $X = \Big[\dots\big[[x_{i_1}, x_{i_2}], x_{i_3}\big], \dots, x_{i_k}\Big]$, $Y = \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]$ and $Z = \Big[\dots\big[[x_{r_1}, x_{r_2}], x_{r_3}\big], \dots, x_{r_p}\Big]$, we have

$$
\begin{aligned}
&X \cdot (Y \cdot Z)\\
&= \Big[\dots\big[[x_{i_1}, x_{i_2}], x_{i_3}\big], \dots, x_{i_k}\Big]\\
&\qquad \cdot \Big[\dots\big[[\dots[a, x_{r_3}], \dots, x_{r_p}], x_{j_3}\big], \dots, x_{j_m}\Big]\\
&\quad \text{with } a = \big[[x_{j_1} \cdot x_{r_1}, x_{r_2}] + [x_{r_1}, x_{j_1} \cdot x_{r_2}], x_{j_2}\big]\\
&\qquad + \big[x_{j_1}, [x_{j_2} \cdot x_{r_1}, x_{r_2}] + [x_{r_1}, x_{j_2} \cdot x_{r_2}]\big]
\end{aligned}
$$

$$= \Big[\dots [[\dots [[\dots [a, x_{r_3}], \dots, x_{r_p}], x_{j_3}], \dots, x_{j_m}], x_{i_3}], \dots, x_{i_k} \Big]$$

$$\begin{aligned} \text{with } a = {} & [[[x_{i_1}, x_{i_2}] \cdot (x_{j_1} \cdot x_{r_1}), x_{r_2}], x_{j_2}] \\ & + [[x_{r_1}, [x_{i_1}, x_{i_2}] \cdot (x_{j_1} \cdot x_{r_2})], x_{j_2}] \\ & + [x_{j_1}, [[x_{i_1}, x_{i_2}] \cdot (x_{j_2} \cdot x_{r_1}), x_{r_2}]] \\ & + [x_{j_1}, [x_{r_1}, [x_{i_1}, x_{i_2}] \cdot (x_{j_2} \cdot x_{r_2})]]. \end{aligned}$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators and lemma 4.5.3. Using this again, we get

$$\begin{aligned} a = {} & [[[x_{i_1} \cdot (x_{j_1} \cdot x_{r_1}), x_{i_2}], x_{r_2}], x_{j_2}] + [[[x_{i_1}, x_{i_2} \cdot (x_{j_1} \cdot x_{r_1})], x_{r_2}], x_{j_2}] \\ & + [[x_{r_1}, [x_{i_1} \cdot (x_{j_1} \cdot x_{r_2}), x_{i_2}]], x_{j_2}] + [[x_{r_1}, [x_{i_1}, x_{i_2} \cdot (x_{j_1} \cdot x_{r_2})]], x_{j_2}] \\ & + [x_{j_1}, [[x_{i_1} \cdot (x_{j_2} \cdot x_{r_1}), x_{i_2}], x_{r_2}]] + [x_{j_1}, [[x_{i_1}, x_{i_2} \cdot (x_{j_2} \cdot x_{r_1})], x_{r_2}]] \\ & + [x_{j_1}, [x_{r_1}, [x_{i_1} \cdot (x_{j_2} \cdot x_{r_2}), x_{i_2}]]] + [x_{j_1}, [x_{r_1}, [x_{i_1}, x_{i_2} \cdot (x_{j_2} \cdot x_{r_2})]]]. \end{aligned}$$

Note that for $Y \cdot (X \cdot Z)$ we find exactly the same result by using that $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$, lemma 4.4.1 and (4.21). Hence $X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = 0$.

Finally we check that $(Z \cdot Y) \cdot X - (Z \cdot X) \cdot Y = 0$ for all basis elements X, Y, Z.

First let $X = x_i$, $Y = x_j$ and $Z = x_k$, we may suppose that $i < j$. We consider the following cases:

$$\begin{aligned} i < j < k: \quad & (x_k \cdot x_j) \cdot x_i - (x_k \cdot x_i) \cdot x_j = [x_k, x_j] \cdot x_i - [x_k, x_i] \cdot x_j \\ & = [x_k \cdot x_i, x_j] + [x_k, x_j \cdot x_i] - [x_k \cdot x_j, x_i] - [x_k, x_i \cdot x_j] \\ & = [[x_k, x_i], x_j] + [x_k, [x_j, x_i]] - [[x_k, x_j], x_i] = 0, \\ i < k \leq j: \quad & (x_k \cdot x_j) \cdot x_i - (x_k \cdot x_i) \cdot x_j = -[x_k, x_i] \cdot x_j \\ & = -[x_k \cdot x_j, x_i] - [x_k, x_i \cdot x_j] = 0, \\ k \leq i < j: \quad & (x_k \cdot x_j) \cdot x_i - (x_k \cdot x_i) \cdot x_j = 0. \end{aligned} \tag{4.22}$$

Let $X = x_i$, $Y = x_j$ and $Z = \Big[\dots\big[[x_{r_1}, x_{r_2}], x_{r_3}\big], \dots, x_{r_p}\Big]$. We have

$$\begin{aligned}
&(Z \cdot Y) \cdot X \\
&\quad = \Big[\dots\big[[x_{r_1} \cdot x_j, x_{r_2}] + [x_{r_1}, x_{r_2} \cdot x_j], x_{r_3}\big], \dots, x_{r_p}\Big] \cdot x_i \\
&\quad = \Big[\dots\big[[(x_{r_1} \cdot x_j) \cdot x_i, x_{r_2}] + [x_{r_1}, (x_{r_2} \cdot x_j) \cdot x_i], x_{r_3}\big], \dots, x_{r_p}\Big],
\end{aligned}$$

where we have used that the product of two generators is either 0 or the Lie bracket of those generators and lemma 4.5.2. Note that for $(Z \cdot X) \cdot Y$ we find exactly the same result by using (4.22). Hence $(Z \cdot Y) \cdot X - (Z \cdot X) \cdot Y = 0$.

Now let $X = x_i$, $Y = \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]$ and $Z = x_k$, we have

$$\begin{aligned}
&(Z \cdot Y) \cdot X - (Z \cdot X) \cdot Y \\
&\quad = \Big(x_k \cdot \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big]\Big) \cdot x_i \\
&\qquad - (x_k \cdot x_i) \cdot \Big[\dots\big[[x_{j_1}, x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = \Big[\dots\big[[x_k \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_k \cdot x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \cdot x_i \\
&\qquad - \Big[\dots\big[[(x_k \cdot x_i) \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, (x_k \cdot x_i) \cdot x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = \Big[\dots\big[[(x_k \cdot x_{j_1}) \cdot x_i, x_{j_2}] + [x_{j_1}, (x_k \cdot x_{j_2}) \cdot x_i], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\qquad - \Big[\dots\big[[(x_k \cdot x_i) \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, (x_k \cdot x_i) \cdot x_{j_2}], x_{j_3}\big], \dots, x_{j_m}\Big] \\
&\quad = 0.
\end{aligned}$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators, lemma 4.5.2, lemma 4.5.3 and (4.22).

Let $X = x_i$, $Y = \Big[\dots [[x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}\Big]$ and $Z = \Big[\dots [[x_{r_1}, x_{r_2}], x_{r_3}], \dots, x_{r_p}\Big]$, we have

$$
\begin{aligned}
&(Z \cdot Y) \cdot X - (Z \cdot X) \cdot Y \\
&\quad = \Big[\dots [[\dots [a, x_{j_3}], \dots, x_{j_m}], x_{r_3}], \dots, x_{r_p}\Big] \cdot x_i \\
&\qquad - \Big[\dots [[x_{r_1} \cdot x_i, x_{r_2}] + [x_{r_1}, x_{r_2} \cdot x_i], x_{r_3}], \dots, x_{r_p}\Big] \\
&\qquad\qquad \cdot \Big[\dots [[x_{j_1}, x_{j_2}], x_{j_3}], \dots, x_{j_m}\Big] \\
&\qquad \text{with } a = \big[[x_{r_1} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{r_1} \cdot x_{j_2}], x_{r_2}\big] \\
&\qquad\qquad + \big[x_{r_1}, [x_{r_2} \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, x_{r_2} \cdot x_{j_2}]\big] \\
&\quad = \Big[\dots [[\dots [a, x_{j_3}], \dots, x_{j_m}], x_{r_3}], \dots, x_{r_p}\Big] \\
&\qquad \text{with } a = \big[[(x_{r_1} \cdot x_{j_1}) \cdot x_i, x_{j_2}] + [x_{j_1}, (x_{r_1} \cdot x_{j_2}) \cdot x_i], x_{r_2}\big] \\
&\qquad\qquad + \big[x_{r_1}, [(x_{r_2} \cdot x_{j_1}) \cdot x_i, x_{j_2}] + [x_{j_1}, (x_{r_2} \cdot x_{j_2}) \cdot x_i]\big] \\
&\qquad\qquad - [[(x_{r_1} \cdot x_i) \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, (x_{r_1} \cdot x_i) \cdot x_{j_2}], x_{r_2}] \\
&\qquad\qquad - [x_{r_1}, [(x_{r_2} \cdot x_i) \cdot x_{j_1}, x_{j_2}] + [x_{j_1}, (x_{r_2} \cdot x_i) \cdot x_{j_2}]] \\
&\qquad\quad = 0.
\end{aligned}
$$

We have used that the product of two generators is either 0 or the Lie bracket of those generators, lemma 4.5.2, lemma 4.5.3 and (4.22).

Suppose $X, Y \in \gamma_2(\mathfrak{g})$, then, since $\mathfrak{g} \cdot \mathfrak{g} \subseteq \gamma_2(\mathfrak{g})$, $\mathfrak{g}$ is 2-step solvable and left multiplications commute, we have

$$
\begin{aligned}
(Z \cdot Y) \cdot X - (Z \cdot X) \cdot Y &= X \cdot (Z \cdot Y) - Y \cdot (Z \cdot X) \\
&= Z \cdot (X \cdot Y) - Z \cdot (Y \cdot X) \\
&= Z \cdot [X, Y] \\
&= 0.
\end{aligned}
$$

Note that this LR-structure is complete. Indeed, sort the basis elements by the number of Lie brackets they are build up from. Then for each basis element X, the left multiplication $L(X)$ has a block matrix where only the blocks strictly under the diagonal can be nonzero. Hence $L(X)$ is nilpotent and since $\mathfrak{g}$ is nilpotent, this implies by proposition 4.1.11 that the structure is complete. $\square$

To end this section, we will prove that all LR-structures on the free p-step nilpotent 2-step solvable Lie algebras are complete if $p \geq 3$:

Proposition 4.5.5. *Let $\mathfrak{g}$ be the free p-step nilpotent 2-step solvable Lie algebra on n generators $x_1, x_2, \ldots, x_n$ ($p \geq 3$). Then any LR-structure on $\mathfrak{g}$ is complete.*

Proof. This proposition will be proved in several parts. First of all we prove it for nilpotency class 3 on 2 generators, then for nilpotency class 3 on more than 2 generators. From these cases we can easily deduce the proposition for nilpotency class bigger than 3.

Proof for nilpotency class 3 and 2 generators

This case will we treated separately and will be proved by explicitly computing the LR-structures on the free 3-step nilpotent (and hence 2-step solvable) Lie algebra $\mathfrak{g}$ on 2 generators.

Take the following basis of $\mathfrak{g}$:

$$x_1, x_2, \qquad z_{2,1,1} = [[x_2, x_1], x_1],$$

$$y_{2,1} = [x_2, x_1], \qquad z_{2,1,2} = [[x_2, x_1], x_2].$$

Suppose we have some LR-structure on $\mathfrak{g}$. We have to prove that this structure is complete.

We will prove that all left multiplications of the basis elements above have only eigenvalue zero. Since all these left multiplications commute, they are simultaneously triangularizable. So we can find some other basis in which the left multiplications of the basis elements above all have upper triangular form with on the diagonal their only eigenvalue 0.

Hence, the left multiplication of a general element will also have this form in this new basis, so is nilpotent. This proves that the LR-structure is complete by proposition 4.1.11.

From lemma 4.1.8 it follows that all left multiplications in our chosen basis have a block matrix with all blocks above the diagonal 0.

For $x \in \gamma_2(\mathfrak{g})$ also all diagonal blocks are 0 and hence $L(x)$ has only eigenvalue 0.

Now, we only have to show that $L(x_1)$ and $L(x_2)$ have only eigenvalue 0. Denote their matrices by

$$L(x_1) = \begin{pmatrix} a_{1,1,1} & a_{1,1,2} & & & \\ a_{1,2,1} & a_{1,2,2} & & 0 & \\ & & a_{1,3,3} & & \\ & * & & a_{1,4,4} & a_{1,4,5} \\ & & & a_{1,5,4} & a_{1,5,5} \end{pmatrix},$$

$$L(x_2) = \begin{pmatrix} a_{2,1,1} & a_{2,1,2} & & & \\ a_{2,2,1} & a_{2,2,2} & & 0 & \\ & & a_{2,3,3} & & \\ & * & & a_{2,4,4} & a_{2,4,5} \\ & & & a_{2,5,4} & a_{2,5,5} \end{pmatrix}.$$

We have the following identities:

$$\begin{aligned} y_{2,1} &= x_2 \cdot x_1 - x_1 \cdot x_2, \\ a_{i,3,3} y_{2,1} + A &= x_i \cdot y_{2,1} = x_i \cdot [x_2, x_1] \\ &= [x_i \cdot x_2, x_1] + [x_2, x_i \cdot x_1] \\ &= (a_{i,2,2} + a_{i,1,1}) y_{2,1} + B, \\ a_{i,4,j+3} z_{2,1,1} + a_{i,5,j+3} z_{2,1,2} &= x_i \cdot z_{2,1,j} = x_i \cdot [y_{2,1}, x_j] \\ &= [x_i \cdot y_{2,1}, x_j] + [y_{2,1}, x_i \cdot x_j] \\ &= a_{i,3,3} z_{2,1,j} + a_{i,1,j} z_{2,1,1} + a_{i,2,j} z_{2,1,2}, \end{aligned}$$

where A and B are elements of $\gamma_3(\mathfrak{g})$.

From these identities we can successively derive that

$$\begin{aligned} a_{2,1,1} &= a_{1,1,2}, \\ a_{2,2,1} &= a_{1,2,2}, \\ \\ a_{1,3,3} &= a_{1,2,2} + a_{1,1,1}, \\ a_{2,3,3} &= a_{2,2,2} + a_{2,1,1} = a_{2,2,2} + a_{1,1,2}, \end{aligned}$$

$$a_{1,4,4} = a_{1,3,3} + a_{1,1,1} = a_{1,2,2} + 2a_{1,1,1},$$
$$a_{1,5,4} = a_{1,2,1},$$
$$a_{1,4,5} = a_{1,1,2},$$
$$a_{1,5,5} = a_{1,3,3} + a_{1,2,2} = 2a_{1,2,2} + a_{1,1,1},$$

$$a_{2,4,4} = a_{2,3,3} + a_{2,1,1} = a_{2,2,2} + 2a_{1,1,2},$$
$$a_{2,5,4} = a_{2,2,1} = a_{1,2,2},$$
$$a_{2,4,5} = a_{2,1,2},$$
$$a_{2,5,5} = a_{2,3,3} + a_{2,2,2} = 2a_{2,2,2} + a_{1,1,2}.$$

Hence we have for the left multiplications:

$$L(x_1) = \begin{pmatrix} a_{1,1,1} & a_{1,1,2} & & & \\ a_{1,2,1} & a_{1,2,2} & & 0 & \\ & & a_{1,2,2} + a_{1,1,1} & & \\ & * & & a_{1,2,2} + 2a_{1,1,1} & a_{1,1,2} \\ & & & a_{1,2,1} & 2a_{1,2,2} + a_{1,1,1} \end{pmatrix},$$

$$L(x_2) = \begin{pmatrix} a_{1,1,2} & a_{2,1,2} & & & \\ a_{1,2,2} & a_{2,2,2} & & 0 & \\ & & a_{2,2,2} + a_{1,1,2} & & \\ & * & & a_{2,2,2} + 2a_{1,1,2} & a_{2,1,2} \\ & & & a_{1,2,2} & 2a_{2,2,2} + a_{1,1,2} \end{pmatrix}.$$

This is actually the general form of a derivation of $\mathfrak{g}$, see [41].

We proceed by distinguishing two cases and repeatedly use lemma 4.1.8.

<u>*Case 1:*</u> $a_{2,2,2} + a_{1,1,2} = 0$.

We consider the coefficient of $z_{2,1,1}$ in the following identity:

$$\begin{aligned} 0 &= x_1 \cdot (z_{2,1,1} \cdot x_1) - z_{2,1,1} \cdot (x_1 \cdot x_1) \\ &= x_1 \cdot (x_1 \cdot z_{2,1,1}) - (x_1 \cdot x_1) \cdot z_{2,1,1} \\ &= x_1 \cdot ((a_{1,2,2} + 2a_{1,1,1})z_{2,1,1} + a_{1,2,1}z_{2,1,2}) - (a_{1,1,1}x_1 + a_{1,2,1}x_2) \cdot z_{2,1,1} \\ &= (a_{1,2,2} + a_{1,1,1})x_1 \cdot z_{2,1,1} + a_{1,2,1}x_1 \cdot z_{2,1,2} - a_{1,2,1}x_2 \cdot z_{2,1,1}. \end{aligned}$$

This coefficient is given by

$$
\begin{aligned}
0 &= (a_{1,2,2} + a_{1,1,1})(a_{1,2,2} + 2a_{1,1,1}) - a_{1,2,1}a_{2,2,2} + a_{1,2,1}a_{2,2,2} \\
&= (a_{1,2,2} + a_{1,1,1})(a_{1,2,2} + 2a_{1,1,1}).
\end{aligned} \tag{4.23}
$$

Furthermore, consider the coefficient of $z_{2,1,2}$ in the following identity:

$$
\begin{aligned}
0 &= x_1 \cdot (z_{2,1,1} \cdot x_2) - z_{2,1,1} \cdot (x_1 \cdot x_2) \\
&= x_1 \cdot (x_2 \cdot z_{2,1,1}) - (x_1 \cdot x_2) \cdot z_{2,1,1} \\
&= x_1 \cdot (-a_{2,2,2} z_{2,1,1} + a_{1,2,2} z_{2,1,2}) \\
&\quad - (-a_{2,2,2} x_1 + a_{1,2,2} x_2) \cdot z_{2,1,1} \\
&= a_{1,2,2} x_1 \cdot z_{2,1,2} - a_{1,2,2} x_2 \cdot z_{2,1,1}.
\end{aligned}
$$

This coefficient is given by

$$
0 = a_{1,2,2}(2a_{1,2,2} + a_{1,1,1}) - a_{1,2,2}^2 = a_{1,2,2}(a_{1,2,2} + a_{1,1,1}). \tag{4.24}
$$

Now (4.23) and (4.24) imply that $a_{1,2,2} + a_{1,1,1} = 0$. The left multiplications of x_1 and x_2 are now given by:

$$
L(x_1) = \begin{pmatrix} a_{1,1,1} & -a_{2,2,2} & & & \\ a_{1,2,1} & -a_{1,1,1} & & 0 & \\ & & 0 & & \\ & * & & a_{1,1,1} & -a_{2,2,2} \\ & & & a_{1,2,1} & -a_{1,1,1} \end{pmatrix},
$$

$$
L(x_2) = \begin{pmatrix} -a_{2,2,2} & a_{2,1,2} & & & \\ -a_{1,1,1} & a_{2,2,2} & & 0 & \\ & & 0 & & \\ & * & & -a_{2,2,2} & a_{2,1,2} \\ & & & -a_{1,1,1} & a_{2,2,2} \end{pmatrix}.
$$

Hence the eigenvalues of $L(x_1)$ are given by 0 and $\pm\sqrt{a_{1,1,1}^2 - a_{1,2,1}a_{2,2,2}}$ and the eigenvalues of $L(x_2)$ are given by 0 and $\pm\sqrt{a_{2,2,2}^2 - a_{1,1,1}a_{2,1,2}}$.

Now consider the following identities:

$$\begin{aligned}0 &= x_1 \cdot (x_2 \cdot x_1) - x_2 \cdot (x_1 \cdot x_1)\\ &= x_1 \cdot (-a_{2,2,2}x_1 - a_{1,1,1}x_2) - x_2 \cdot (a_{1,1,1}x_1 + a_{1,2,1}x_2) + A,\\ 0 &= x_1 \cdot (x_2 \cdot x_2) - x_2 \cdot (x_1 \cdot x_2)\\ &= x_1 \cdot (a_{2,1,2}x_1 + a_{2,2,2}x_2) - x_2 \cdot (-a_{2,2,2}x_1 - a_{1,1,1}x_2) + B,\end{aligned}$$

where $A, B \in \gamma_2(\mathfrak{g})$. The coefficient of x_2 in the first identity is

$$\begin{aligned}0 &= -a_{2,2,2}a_{1,2,1} + a_{1,1,1}^2 + a_{1,1,1}^2 - a_{1,2,1}a_{2,2,2}\\ &= 2(a_{1,1,1}^2 - a_{1,2,1}a_{2,2,2})\end{aligned}$$

and the coefficient of x_1 in the second identity is

$$\begin{aligned}0 &= a_{2,1,2}a_{1,1,1} - a_{2,2,2}^2 - a_{2,2,2}^2 + a_{1,1,1}a_{2,1,2}\\ &= -2(a_{2,2,2}^2 - a_{1,1,1}a_{2,1,2}).\end{aligned}$$

This implies that $L(x_1)$ and $L(x_2)$ have only eigenvalue 0, which was to be shown.

Case 2: $a_{2,2,2} + a_{1,1,2} \neq 0$.

Consider the following identity:

$$\begin{aligned}0 &= x_1 \cdot (z_{2,1,1} \cdot x_2) - z_{2,1,1} \cdot (x_1 \cdot x_2)\\ &= x_1 \cdot (x_2 \cdot z_{2,1,1}) - (x_1 \cdot x_2) \cdot z_{2,1,1}\\ &= x_1 \cdot \big((a_{2,2,2} + 2a_{1,1,2})z_{2,1,1} + a_{1,2,2}z_{2,1,2}\big)\\ &\quad - \big(a_{1,1,2}x_1 + a_{1,2,2}x_2 + A\big) \cdot z_{2,1,1}\\ &= (a_{2,2,2} + a_{1,1,2})x_1 \cdot z_{2,1,1} + a_{1,2,2}x_1 \cdot z_{2,1,2} - a_{1,2,2}x_2 \cdot z_{2,1,1},\end{aligned}$$

with $A \in \gamma_2(\mathfrak{g})$. The coefficient of $z_{2,1,1}$ is given by

$$\begin{aligned}0 &= (a_{2,2,2} + a_{1,1,2})(a_{1,2,2} + 2a_{1,1,1})\\ &\quad + a_{1,2,2}a_{1,1,2} - a_{1,2,2}(a_{2,2,2} + 2a_{1,1,2})\\ &= 2(a_{2,2,2} + a_{1,1,2})a_{1,1,1}.\end{aligned}$$

Using the assumption $a_{2,2,2} + a_{1,1,2} \neq 0$, it follows that $a_{1,1,1} = 0$.

Now consider the identity:

$$\begin{aligned}
0 &= x_2 \cdot (z_{2,1,2} \cdot x_2) - z_{2,1,2} \cdot (x_2 \cdot x_2) \\
&= x_2 \cdot (x_2 \cdot z_{2,1,2}) - (x_2 \cdot x_2) \cdot z_{2,1,2} \\
&= x_2 \cdot (a_{2,1,2} z_{2,1,1} + (2a_{2,2,2} + a_{1,1,2}) z_{2,1,2}) \\
&\quad - (a_{2,1,2} x_1 + a_{2,2,2} x_2 + A) \cdot z_{2,1,2} \\
&= a_{2,1,2} x_2 \cdot z_{2,1,1} + (a_{2,2,2} + a_{1,1,2}) x_2 \cdot z_{2,1,2} \\
&\quad - a_{2,1,2} x_1 \cdot z_{2,1,2},
\end{aligned} \tag{4.25}$$

with $A \in \gamma_2(\mathfrak{g})$. The coefficient of $z_{2,1,1}$ is given by

$$\begin{aligned}
0 &= a_{2,1,2}(a_{2,2,2} + 2a_{1,1,2}) + (a_{2,2,2} + a_{1,1,2}) a_{2,1,2} - a_{2,1,2} a_{1,1,2} \\
&= 2a_{2,1,2}(a_{2,2,2} + a_{1,1,2}).
\end{aligned}$$

This implies that $a_{2,1,2} = 0$.

Next, using that $a_{1,1,1} = 0$, we consider the identity:

$$\begin{aligned}
0 &= x_2 \cdot (z_{2,1,2} \cdot x_1) - z_{2,1,2} \cdot (x_2 \cdot x_1) \\
&= x_2 \cdot (x_1 \cdot z_{2,1,2}) - (x_2 \cdot x_1) \cdot z_{2,1,2} \\
&= x_2 \cdot (a_{1,1,2} z_{2,1,1} + 2a_{1,2,2} z_{2,1,2}) \\
&\quad - (a_{1,1,2} x_1 + a_{1,2,2} x_2 + A) \cdot z_{2,1,2} \\
&= a_{1,1,2} x_2 \cdot z_{2,1,1} + a_{1,2,2} x_2 \cdot z_{2,1,2} - a_{1,1,2} x_1 \cdot z_{2,1,2},
\end{aligned}$$

with $A \in \gamma_2(\mathfrak{g})$. Using that $a_{2,1,2} = 0$, the coefficient of $z_{2,1,1}$ is given by

$$\begin{aligned}
0 &= a_{1,1,2}(a_{2,2,2} + 2a_{1,1,2}) - a_{1,1,2} a_{1,1,2} \\
&= a_{1,1,2}(a_{2,2,2} + a_{1,1,2}).
\end{aligned}$$

This implies that $a_{1,1,2} = 0$ and hence $a_{2,2,2} \neq 0$.

To finish, we again consider identity (4.25) and use that $a_{2,1,2} = 0$ and that $a_{1,1,2} = 0$, we find

$$0 = a_{2,2,2} x_2 \cdot z_{2,1,2}.$$

The coefficient of $z_{2,1,2}$ is given by $0 = 2a_{2,2,2}^2$, which gives a contradiction.

Hence, this second case does not occur.

Proof for nilpotency class 3 and at least 3 generators

To prove the proposition in this case, we can use almost exactly the same proof as in the case of Novikov structures (proposition 3.3.4). Almost all results about Novikov structures that are used in that proof are also true for LR-structures. The only difference is the factor $\frac{1}{2}$ which appears in lemma 3.2.3 but not in lemma 4.1.6. However, this does not change anything about the development of the proof. We can, without problems, remove this factor everywhere in the proof and get the same result.

Proof for nilpotency class at least 4

Suppose $\mathfrak{g}$ is the free p-step nilpotent 2-step solvable Lie algebra on n generators, with $p \geq 4$, and suppose we have an LR-structure on $\mathfrak{g}$. To prove that this LR-structure is complete, it suffices, by proposition 4.1.11, to prove that all left multiplications are nilpotent.

From lemma 4.1.7 we know that $\gamma_4(\mathfrak{g})$ is a 2-sided ideal for the LR-product, hence we have an induced LR-structure on the quotient $\mathfrak{g}/\gamma_4(\mathfrak{g})$.

Remark that this quotient $\mathfrak{g}/\gamma_4(\mathfrak{g})$ is the free 3-step nilpotent Lie algebra on n generators and hence the previous part of our proof implies that the induced LR-structure should be complete.

If we denote by $L(x)_{(q)}$ the induced map of $L(x)$ on $\mathfrak{g}/\gamma_q(\mathfrak{g})$, we know that $L(x)_{(4)}$ is a nilpotent map on $\mathfrak{g}/\gamma_4(\mathfrak{g})$. We use the following lemma:

Lemma 4.5.6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let D be a derivation of $\mathfrak{g}$. Suppose there exists some $q \geq 2$ such that the induced map $D_{(q)}$ on $\mathfrak{g}/\gamma_q(\mathfrak{g})$ is nilpotent, then for all $r \geq q$ it holds that $D_{(r)}$ is nilpotent on $\mathfrak{g}/\gamma_r(\mathfrak{g})$.*

Proof. We prove this lemma by induction on r, where the case $r = q$ is clear.

Now suppose that for some $r > q$ it holds that $D_{(r-1)}$ is nilpotent on $\mathfrak{g}/\gamma_{r-1}(\mathfrak{g})$. So there exists some $k \in \mathbb{N}_0$ such that $D^k_{(r-1)} = 0$, or else:

$$D^k(x) \in \gamma_{r-1}(\mathfrak{g}) \text{ for all } x \in \mathfrak{g}. \tag{4.26}$$

Hence, to prove that $D_{(r)}$ is nilpotent on $\mathfrak{g}/\gamma_r(\mathfrak{g})$, it is enough to prove that there exists some $l \in \mathbb{N}_0$ such that $D^l(x) \in \gamma_r(\mathfrak{g})$ for all $x \in \gamma_{r-1}(\mathfrak{g})$.

Any element of $\gamma_{r-1}(\mathfrak{g})$ is a linear combination of Lie brackets $[x, y]$ with $x \in \mathfrak{g}$ and $y \in \gamma_{r-2}(\mathfrak{g})$.

By induction on $l \in \mathbb{N}_0$ we have that

$$D^l([x,y]) = \sum_{s=0}^{l} \binom{l}{s} [D^s(x), D^{l-s}(y)], \tag{4.27}$$

since D is a derivation.

Indeed, the case $l = 1$ is just the definition of a derivation. Suppose the identity is true for some $l \in \mathbb{N}_0$, then

$$\begin{aligned} D^{l+1}([x,y]) &= D\Big(\sum_{s=0}^{l} \binom{l}{s} [D^s(x), D^{l-s}(y)]\Big) \\ &= \sum_{s=0}^{l} \binom{l}{s} ([D^{s+1}(x), D^{l-s}(y)] + [D^s(x), D^{l-s+1}(y)]) \\ &= \sum_{s=1}^{l+1} \binom{l}{s-1} [D^s(x), D^{l+1-s}(y)] + \sum_{s=0}^{l} \binom{l}{s} [D^s(x), D^{l+1-s}(y)] \\ &= [D^{l+1}(x), y] + \sum_{s=1}^{l} \left(\binom{l}{s-1} + \binom{l}{s}\right) [D^s(x), D^{l+1-s}(y)] \\ &\quad + [x, D^{l+1}(y)] \\ &= \sum_{s=0}^{l+1} \binom{l+1}{s} [D^s(x), D^{l+1-s}(y)], \end{aligned}$$

which proves the identity for $l + 1$.

Now, take $l \geq 2k$ in (4.27). Then, by using (4.26), we have for $s < k$ that $D^{l-s}(y) \in \gamma_{r-1}(\mathfrak{g})$ and for $s \geq k$ we have $D^s(x) \in \gamma_{r-1}(\mathfrak{g})$.

Hence, $D^l([x,y]) \in \gamma_r(\mathfrak{g})$ and this implies $D^l(\gamma_{r-1}(\mathfrak{g})) \subset \gamma_r(\mathfrak{g})$, which was to be shown. □

Using this lemma and the fact that $L(x)_{(4)}$ is nilpotent on $\mathfrak{g}/\gamma_4(\mathfrak{g})$ for all $x \in \mathfrak{g}$, we find that $L(x)_{(p+1)}$ is nilpotent on $\mathfrak{g}/\gamma_{p+1}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. Since $\mathfrak{g}$ is p-step nilpotent, this just means that $L(x)$ is nilpotent on $\mathfrak{g}$. □

Remark that we can not include nilpotency class 2 in this proposition. This is shown in the following example for 2 generators. Also for more generators counterexamples can be found.

Example 4.5.7. *The proposition does not hold for the free 2-step nilpotent Lie algebras. The following LR-structure on the free 2-step nilpotent Lie algebra on 2 generators is not complete:*

$$x_1 \cdot x_1 = x_1, \qquad x_2 \cdot x_1 = -y_{1,2},$$
$$x_1 \cdot y_{1,2} = y_{1,2}, \qquad y_{1,2} \cdot x_1 = y_{1,2}.$$

4.6 The (non) existence of LR-structures on triangular matrix algebras

As in the study of Novikov structures we could ask which of the Lie algebras of (strictly) upper triangular matrices admit an LR-structure.

It turns out that we have the same result as for Novikov structures, although the proofs become much easier.

We first study the case of strictly upper triangular matrices:

Proposition 4.6.1. *The Lie algebra* $\mathfrak{n}(n,k)$ *admits an LR-structure if and only if* $n \leq 4$.

Proof. If $n \leq 4$, the Lie algebra $\mathfrak{n}(n,k)$ is abelian ($n = 2$), 2-step nilpotent ($n = 3$) or 3-step nilpotent and generated by 3 elements ($n = 4$). In any of these cases, we know that an LR-structure exists. Indeed, in [20] it is proved that any 2-step nilpotent Lie algebra and any 3-step nilpotent Lie algebra on 3 generators admits an LR-structure.

For $n > 4$ the Lie algebra $\mathfrak{n}(n,k)$ is not 2-step solvable and hence LR-structures cannot exist. □

Also the case of upper triangular matrices can be proved very easily:

Proposition 4.6.2. *The Lie algebra* $\mathfrak{t}(n,k)$ *admits an LR-structure if and only if* $n \leq 2$.

Proof. It is easy to construct an LR-structure on $\mathfrak{t}(1,k) \cong k$ and on $\mathfrak{t}(2,k)$. On $\mathfrak{t}(1,k)$ the zero product will be an LR-structure. For $\mathfrak{t}(2,k)$ the classification of LR-structures is given in [20].

For $n > 2$ the Lie algebra $\mathfrak{t}(n,k)$ is not 2-step solvable and hence LR-structures cannot exist. □

Chapter 5

Post-Lie algebras and post-Lie algebra structures

In this chapter we introduce the notion of a post-Lie algebra and a post-Lie algebra structure on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ defined on a fixed vector space V. We consider the existence question for post-Lie algebra structures, link them to NIL-affine actions on nilpotent Lie groups and consider special cases and examples.

In section 5.1 we give the definition of a post-Lie algebra and a post-Lie algebra structure and prove some easy properties.

In section 5.2 we have a look at some special cases of post-Lie algebra structures, in particular LR-structures and left-symmetric structures.

In section 5.3 we give some easy examples. We look at the case where $\mathfrak{n}$ is complete, and more generally at the case where $\mathfrak{n}$ has trivial center.

In section 5.4 we prove some interesting one-one correspondences between post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ on the one hand and specific embeddings of $\mathfrak{g}$ in $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ or specific subalgebras of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ on the other hand. We have a closer look at the special case of $\mathfrak{n}$ being semisimple where we have a one-one correspondence with specific subalgebras of $\mathfrak{n} \oplus \mathfrak{n}$.

In section 5.5 we show that post-Lie algebra structures arise naturally in the study of NIL-affine actions on nilpotent Lie groups. In fact, there is a relation between simply transitive NIL-affine actions of a real connected, simply connected nilpotent Lie group on another one and complete post-Lie algebra structures on Lie algebras.

In section 5.6 we classify all complex two-dimensional post-Lie algebra structures.

In section 5.7 we present several results on the existence of post-Lie algebra structures in terms of the algebraic structure of the two Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$. This also leads to a classification of the 3-dimensional Lie algebras $\mathfrak{g}$ for which the pair $(\mathfrak{g}, \mathfrak{sl}_2(\mathbb{C}))$ admits a post-Lie algebra structure.

All algebras we consider are assumed to be finite dimensional over a field k of characteristic 0.

Many results of this chapter are presented in [23].

5.1 Post-Lie algebras and post-Lie algebra structures

In chapter 2 we already stated that the study of simply transitive NIL-affine actions of a real connected, simply connected Lie group G on another such Lie group N, leads naturally to the study of post-Lie algebra structures on a pair of Lie algebras. We will prove this in section 5.5 of this chapter.

Let us first define the notion of a post-Lie algebra and a post-Lie algebra structure and prove some identities they satisfy.

Definition 5.1.1 (Post-Lie algebra). *A post-Lie algebra* $(V, \cdot, \{\,,\})$ *is a vector space* V *over a field* k *equipped with two* k*-bilinear operations* $x \cdot y$ *and* $\{x, y\}$*, such that* $(V, \{,\})$ *is a Lie algebra and*

$$\{x, y\} \cdot z = (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z), \tag{5.1}$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \tag{5.2}$$

for all $x, y, z \in V$.

We denote the Lie algebra $(V, \{,\})$ by $\mathfrak{n}$. If the bracket $\{x, y\}$ is zero, then a post-Lie algebra is just a left-symmetric algebra.

Condition (5.2) says that the left multiplication $L(x)$, defined by $L(x)y = x \cdot y$, is a derivation of the Lie algebra $\mathfrak{n} = (V, \{,\})$.

When working with post-Lie algebras, we can define another Lie bracket as follows:

Proposition 5.1.2. *A post-Lie algebra* $(V, \cdot, \{,\})$ *has another associated Lie bracket, defined by the formula*

$$[x, y] := x \cdot y - y \cdot x + \{x, y\}. \tag{5.3}$$

Proof. Of course we have $[x, y] = -[y, x]$, because of the anti-commutativity of $\{,\}$.
Furthermore, by the definition, we have for all $x, y, z \in V$

$$\begin{aligned}
[x, [y, z]] &= [x, y \cdot z - z \cdot y + \{y, z\}] \\
&= [x, y \cdot z] - [x, z \cdot y] + [x, \{y, z\}] \\
&= x \cdot (y \cdot z) - (y \cdot z) \cdot x + \{x, y \cdot z\} \\
&\quad - x \cdot (z \cdot y) + (z \cdot y) \cdot x - \{x, z \cdot y\} \\
&\quad + x \cdot \{y, z\} - \{y, z\} \cdot x + \{x, \{y, z\}\}.
\end{aligned}$$

It follows from the definition of a post-Lie algebra that

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= \{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} \\
&\quad + x \cdot \{y, z\} + \{z, x \cdot y\} - \{y, x \cdot z\} \\
&\quad + y \cdot \{z, x\} + \{x, y \cdot z\} - \{z, y \cdot x\} \\
&\quad + z \cdot \{x, y\} + \{y, z \cdot x\} - \{x, z \cdot y\} \\
&\quad + (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z) - \{x, y\} \cdot z \\
&\quad + (x \cdot z) \cdot y - x \cdot (z \cdot y) - (z \cdot x) \cdot y + z \cdot (x \cdot y) - \{z, x\} \cdot y \\
&\quad + (z \cdot y) \cdot x - z \cdot (y \cdot x) - (y \cdot z) \cdot x + y \cdot (z \cdot x) - \{y, z\} \cdot x \\
&= 0.
\end{aligned}$$

This shows that the Jacobi identity is satisfied. □

We denote the Lie algebra $(V, [\,,])$ by $\mathfrak{g}$. The Lie bracket of $\mathfrak{g}$ satisfies the following identity:

Proposition 5.1.3. *The second Lie bracket* (5.3) *associated to a post-Lie algebra* $(V, \cdot, \{\,,\})$ *satisfies the identity*

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z),$$

i.e., the post-Lie algebra is a left-module over the Lie algebra $\mathfrak{g}$.

Proof. For all $x, y, z \in V$ we have by (5.3) and (5.1)

$$\begin{aligned}[x, y] \cdot z &= (x \cdot y) \cdot z - (y \cdot x) \cdot z + \{x, y\} \cdot z \\ &= (x \cdot y) \cdot z - (y \cdot x) \cdot z + (y \cdot x) \cdot z \\ &\quad - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z) \\ &= x \cdot (y \cdot z) - y \cdot (x \cdot z).\end{aligned}$$

The other conditions for a left-module are satisfied since the product is bilinear. □

The above proposition says that the map $L\colon V \to \mathrm{End}(V)$, given by $x \mapsto L(x)$, is a linear representation of the Lie algebra $\mathfrak{g} = (V, [\,,])$.

Two post-Lie algebras $(V, \cdot, \{\,,\})$ and $(W, \cdot, \{\,,\})$ are called *isomorphic* if and only if there exists a bijective linear map $\varphi : V \to W$, which preserves both products:

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y),$$

$$\varphi(\{x, y\}) = \{\varphi(x), \varphi(y)\},$$

for all $x, y \in V$. It is obvious that isomorphic post-Lie algebras have isomorphic associated Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$.

Let $(\mathfrak{g}, [\,,])$ and $(\mathfrak{n}, \{\,,\})$ be two Lie algebras with the same underlying vector space V over a field k. We call $(\mathfrak{g}, \mathfrak{n})$ a *pair* of Lie algebras over k. So as sets or vector spaces, $\mathfrak{g} = \mathfrak{n} = V$. In the sequel, when talking about a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, we will always denote the Lie bracket in $\mathfrak{g}$ with square brackets $[x, y]$ and the one in $\mathfrak{n}$ with curly brackets $\{x, y\}$ and the underlying vector space of both $\mathfrak{g}$ and $\mathfrak{n}$ will be denoted by V. Furthermore, when we write $\mathrm{ad}(x)$, we will always mean the adjoint operator in the Lie algebra $\mathfrak{n}$.

The following structure arises in the study of NIL-affine actions of nilpotent Lie groups, see theorem 5.5.1:

Definition 5.1.4 (Post-Lie algebra structure)**.** *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras. A post-Lie algebra structure on the pair* $(\mathfrak{g}, \mathfrak{n})$ *is a* k*-bilinear product* $x \cdot y$ *on* V *satisfying the following identities:*

$$x \cdot y - y \cdot x = [x, y] - \{x, y\}, \tag{5.4}$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \tag{5.5}$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \tag{5.6}$$

for all $x, y, z \in V$*.*

Evidently this just means that $(V, \cdot, \{\,,\})$ is a post-Lie algebra with associated second Lie algebra $\mathfrak{g} = (V, [\,,])$.

We can derive some more consequences of the above identities:

Lemma 5.1.5. *The axioms* (5.4)*,* (5.5) *and* (5.6) *imply the following identities:*

$$\{x, y\} \cdot z = (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z), \tag{5.7}$$

$$z \cdot [x, y] = z \cdot (x \cdot y) - z \cdot (y \cdot x) + z \cdot \{x, y\}, \tag{5.8}$$

$$\begin{aligned} &[x \cdot y, z] + [y, x \cdot z] - x \cdot [y, z] \\ &\quad = (x \cdot y) \cdot z - (x \cdot z) \cdot y + y \cdot (x \cdot z) \\ &\qquad - x \cdot (y \cdot z) + x \cdot (z \cdot y) - z \cdot (x \cdot y), \end{aligned} \tag{5.9}$$

$$\begin{aligned} &x \cdot \{y, z\} + y \cdot \{z, x\} + z \cdot \{x, y\} \\ &\quad = \{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\{x, y\} \cdot z + \{y, z\} \cdot x + \{z, x\} \cdot y \\ &\quad = \{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\} \\ &\qquad + [\{x, y\}, z] + [\{y, z\}, x] + [\{z, x\}, y] \end{aligned} \tag{5.11}$$

for all $x, y, z \in V$*.*

Proof. Using (5.4) and (5.5) we obtain

$$\begin{aligned} \{x, y\} \cdot z &= ([x, y] - x \cdot y + y \cdot x) \cdot z \\ &= [x, y] \cdot z - (x \cdot y) \cdot z + (y \cdot x) \cdot z \\ &= (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z). \end{aligned}$$

This gives (5.7), which is just (5.1). Identity (5.8) follows directly from (5.4). Using (5.4) and (5.6) we obtain

$$\begin{aligned} x\cdot(y\cdot z)-x\cdot(z\cdot y) &= x\cdot([y,z]-\{y,z\}) \\ &= x\cdot[y,z]-x\cdot\{y,z\} \\ &= x\cdot[y,z]-\{x\cdot y,z\}-\{y,x\cdot z\} \\ &= x\cdot[y,z]-([x\cdot y,z]+z\cdot(x\cdot y)-(x\cdot y)\cdot z) \\ &\quad -([y,x\cdot z]+(x\cdot z)\cdot y-y\cdot(x\cdot z)). \end{aligned}$$

This gives (5.9). Using the Jacobi identity for $\{\,,\}$, (5.4) and (5.6) we have

$$\begin{aligned} 0 &= \{\{x,y\},z\}+\{\{y,z\},x\}+\{\{z,x\},y\} \\ &= \{[x,y]-x\cdot y+y\cdot x,z\}+\{[y,z]-y\cdot z+z\cdot y,x\} \\ &\quad +\{[z,x]-z\cdot x+x\cdot z,y\} \\ &= \{[x,y],z\}-\{x\cdot y,z\}+\{y\cdot x,z\}+\{[y,z],x\}-\{y\cdot z,x\} \\ &\quad +\{z\cdot y,x\}+\{[z,x],y\}-\{z\cdot x,y\}+\{x\cdot z,y\} \\ &= \{[x,y],z\}+\{[y,z],x\}+\{[z,x],y\} \\ &\quad -x\cdot\{y,z\}-y\cdot\{z,x\}-z\cdot\{x,y\}. \end{aligned}$$

This is (5.10). For (5.11) use (5.4) in the following way:

$$\begin{aligned} 0 &= \{\{x,y\},z\}+\{\{y,z\},x\}+\{\{z,x\},y\} \\ &= [\{x,y\},z]-\{x,y\}\cdot z+z\cdot\{x,y\}+[\{y,z\},x]-\{y,z\}\cdot x \\ &\quad +x\cdot\{y,z\}+[\{z,x\},y]-\{z,x\}\cdot y+y\cdot\{z,x\}. \end{aligned}$$

By applying (5.10) the identity (5.11) follows. □

5.2 Special cases

In this section we look at some special cases by taking the product or one of the Lie brackets to be zero. In particular, we are led back to the situation of left-symmetric structures and LR-structures.

The zero product

Suppose that the post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ is given by the zero product. Then $(\mathfrak{g}, [\,,]) = (\mathfrak{n}, \{\,,\})$.

Indeed, $x \cdot y = 0$ implies $[x, y] = \{x, y\}$ for all $x, y \in V$ because of the first axiom of a post-Lie algebra structure.

Conversely, the zero product is a post-Lie algebra structure on the pair $(\mathfrak{g}, \mathfrak{g})$ for any Lie algebra $\mathfrak{g}$ since all axioms are trivially satisfied.

The case where $\mathfrak{n}$ is abelian

If $\mathfrak{n}$ is abelian, then a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ corresponds to a left-symmetric structure on $\mathfrak{g}$.

If $\{x, y\} = 0$ for all $x, y \in V$, then the conditions reduce to

$$x \cdot y - y \cdot x = [x, y],$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z),$$

i.e., $x \cdot y$ is a left-symmetric structure on the Lie algebra $\mathfrak{g}$.

Conversely, a left-symmetric structure on a Lie algebra $\mathfrak{g}$ induces a post-Lie algebra structure on the pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is the abelian Lie algebra on the same underlying vector space as $\mathfrak{g}$.

The case where $\mathfrak{g}$ is abelian

If $\mathfrak{g}$ is abelian, then a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ corresponds to an LR-structure on $\mathfrak{n}$.

If $\mathfrak{g}$ is abelian, then the conditions reduce to

$$x \cdot y - y \cdot x = -\{x, y\},$$

$$x \cdot (y \cdot z) = y \cdot (x \cdot z),$$

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y.$$

The first and second condition follow trivially from the definition of a post-Lie algebra structure. If we now use (5.9) (which is equivalent to (5.6) using (5.4)) and apply the second condition and the fact that $\mathfrak{g}$ is abelian, we get the third condition. Hence $-x \cdot y$ is an LR-structure on the Lie algebra $\mathfrak{n}$.

Conversely, an LR-structure on a Lie algebra $\mathfrak{n}$ induces a post-Lie algebra structure on the pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is the abelian Lie algebra on the same underlying vector space as $\mathfrak{n}$.

5.3 Easy examples

Let us give some easy, but useful examples of post-Lie algebra structures.

Proposition 5.3.1. *Suppose* $(\mathfrak{g}, \mathfrak{n})$ *is a pair of Lie algebras and let* $\lambda \notin \{0, 1\}$*. Then* $x \cdot y = \lambda[x, y]$ *defines a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *if and only if* $\{x, y\} = (1 - 2\lambda)[x, y]$*, and both* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are nilpotent of class at most 2.*

Proof. Suppose that $x \cdot y = \lambda[x, y]$ defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then (5.4) implies $\{x, y\} = (1 - 2\lambda)[x, y]$. By (5.5) and the Jacobi identity for $\mathfrak{g}$ we obtain

$$\begin{aligned} \lambda[[x, y], z] &= [x, y] \cdot z \\ &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ &= \lambda^2[x, [y, z]] - \lambda^2[y, [x, z]] \\ &= \lambda^2[[x, y], z]. \end{aligned}$$

Because $\lambda \neq 0, 1$ this yields $[[x, y], z] = \{\{x, y\}, z\} = (x \cdot y) \cdot z = 0$, hence $\mathfrak{g}$ and $\mathfrak{n}$ are nilpotent of class at most 2.

Conversely, let $(\mathfrak{g}, \mathfrak{n})$ be a pair of nilpotent Lie algebras of class ≤ 2 with $\{x, y\} = (1 - 2\lambda)[x, y]$, and let $x \cdot y = \lambda[x, y]$, $\lambda \notin \{0, 1\}$.
Obviously, the identities (5.4) and (5.5) are satisfied. To show (5.6), we use that $\{x, y\} = \mu\, x \cdot y$ with $\mu = \frac{1-2\lambda}{\lambda}$, we have

$$\begin{aligned} x \cdot \{y, z\} &= \mu\, x \cdot (y \cdot z) \\ &= \mu\lambda^2[x, [y, z]] \\ &= \mu\lambda^2[[x, y], z] + \mu\lambda^2[y, [x, z]] \\ &= \mu\, (x \cdot y) \cdot z + \mu\, y \cdot (x \cdot z) \\ &= \{x \cdot y, z\} + \{y, x \cdot z\}. \end{aligned}$$

Hence the product defines a post-Lie algebra structure. □

Note that for $\lambda = \frac{1}{2}$ we have $x \cdot y = \frac{1}{2}[x,y]$ and $\{x,y\} = 0$. Hence $\mathfrak{n}$ is abelian, and the product defines a left-symmetric structure (even a Novikov structure) on $\mathfrak{g}$ (still assuming $\mathfrak{g}$ is nilpotent of class ≤ 2).

It is easy to discuss the cases $\lambda = 0, 1$ which we have excluded above. For $\lambda = 0$ we have the zero product $x \cdot y = 0$ with $[x,y] = \{x,y\}$. It gives the trivial post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{g})$ for any $\mathfrak{g}$.
For $\lambda = 1$ we have $x \cdot y = [x,y] = -\{x,y\}$. This defines a post-Lie algebra structure on $(\mathfrak{g}, -\mathfrak{g})$ for any $\mathfrak{g}$.

Remark 5.3.2. *We have an analogous result for post-Lie algebra structures defined by* $x \cdot y = \mu\{x,y\}$.

For $\mu = 0$ *of* $\mu = -1$ *we are respectively in the cases* $\lambda = 0$ *and* $\lambda = 1$.

For $\mu \notin \{0, -1\}$ *this means that* $[x,y] = (1+2\mu)\{x,y\}$, *and both* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are nilpotent of class at most 2. In fact, we obtain the same post-Lie algebra structures as above, except for the case* $\mu = -\frac{1}{2}$, *where* $\mathfrak{g}$ *is abelian, and* $-x \cdot y = \frac{1}{2}\{x,y\}$ *defines an LR-structure on a nilpotent Lie algebra* $\mathfrak{n}$ *of class* ≤ 2.

Another special case arises if $\{x,y\} = \rho[x,y]$ for some nonzero scalar ρ:

Example 5.3.3. *Let* $\rho \notin \{0,1\}$ *and* $\{x,y\} = \rho[x,y]$. *Then* $x \cdot y$ *defines a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *if and only if*

$$
\begin{aligned}
x \cdot y - y \cdot x &= (1-\rho)[x,y], \\
(1-\rho)(x \cdot (y \cdot z) - y \cdot (x \cdot z)) &= (x \cdot y) \cdot z - (y \cdot x) \cdot z, \\
x \cdot (y \cdot z) - y \cdot (x \cdot z) &= (x \cdot y) \cdot z - z \cdot (x \cdot y) \\
&\quad - (x \cdot z) \cdot y + x \cdot (z \cdot y).
\end{aligned}
$$

The first identity is just the first axiom of a post-Lie algebra structure in this setting. Using this identity in the other two axioms, we get the second and third identity above.

This says that $x \cdot y$ is a certain deformed left-symmetric structure on $\mathfrak{g}$. In general, it seems difficult to classify such products. For semisimple Lie algebras however it is possible, see [17].

There is also the interesting case $\rho = 1$, i.e., $\{x,y\} = [x,y]$:

Example 5.3.4. *Let* $\{x, y\} = [x, y]$*. Then* $x \cdot y$ *defines a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *if and only if*

$$x \cdot y = y \cdot x,$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z),$$

$$x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z].$$

Hence $x \cdot y$ is a commutative product on $\mathfrak{g}$ such that the operators $L(x)$ are derivations and $L([x, y]) = [L(x), L(y)]$.

For semisimple Lie algebras this can be classified, see [17]. In general however, this seems to be difficult. Already for the Heisenberg Lie algebra $\mathfrak{n}_3(\mathbb{C})$ there are many such structures: let (e_1, e_2, e_3) be a basis of $\mathbb{C}^3$ and define the nonzero Lie brackets of $\mathfrak{g}$ and $\mathfrak{n}$ by $[e_1, e_2] = e_3$, $\{e_1, e_2\} = e_3$.

Example 5.3.5. *Let* $\mathfrak{g} = \mathfrak{n} = \mathfrak{n}_3(\mathbb{C})$ *and* $\alpha, \beta, \gamma \in \mathbb{C}$*. Then*

$$e_1 \cdot e_1 = \alpha e_2 + \beta e_3,$$

$$e_1 \cdot e_2 = e_2 \cdot e_1 = \gamma e_3,$$

defines a commutative post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$*, where we did not write down the zero products between basis vectors.*

Post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ when $\mathfrak{n}$ has trivial center

If the Lie algebra $\mathfrak{n}$ is complete, then we can say more on post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$. Recall that a Lie algebra $\mathfrak{n}$ is called *complete*, if $\mathrm{Der}(\mathfrak{n}) = \mathrm{ad}(\mathfrak{n})$ and $Z(\mathfrak{n}) = 0$.

Lemma 5.3.6. *Suppose that* $x \cdot y$ *is a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{n}$ *is complete. Then there is a unique linear map* $\varphi \colon V \to V$ *such that* $x \cdot y = \{\varphi(x), y\}$*, i.e., satisfying* $L(x) = \mathrm{ad}(\varphi(x))$*.*

Proof. For any $x \in V$, we have $L(x) \in \mathrm{Der}(\mathfrak{n}) = \mathrm{ad}(\mathfrak{n})$. As $\mathfrak{n}$ has trivial center, there is a unique element $\varphi(x) \in \mathfrak{n}$ such that $L(x) = \mathrm{ad}(\varphi(x))$, which defines the map $\varphi \colon V \to V$.

For $x, x', y \in V$ we have

$$\begin{aligned}\{\varphi(x+x'), y\} &= (x+x')\cdot y \\ &= x\cdot y + x'\cdot y \\ &= \{\varphi(x)+\varphi(x'), y\}.\end{aligned}$$

It follows that $\varphi(x+x') = \varphi(x)+\varphi(x')$, because $\mathfrak{n}$ has trivial center. In the same way we obtain $\varphi(\lambda x) = \lambda\varphi(x)$, hence φ is linear. □

Inspired by the above, we now show the following result, which applies in particular for $\mathfrak{n}$ being semisimple:

Proposition 5.3.7. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras such that $\mathfrak{n}$ has trivial center. Let $\varphi \in \operatorname{End}(V)$. Then the product $x \cdot y = \{\varphi(x), y\}$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if*

$$\{\varphi(x), y\} + \{x, \varphi(y)\} = [x, y] - \{x, y\},$$

$$\varphi([x, y]) = \{\varphi(x), \varphi(y)\}$$

for all $x, y \in V$.

Proof. Assume that $x \cdot y = \{\varphi(x), y\}$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then the first identity follows immediately from (5.4). The second one follows from (5.5) and the Jacobi identity for $\mathfrak{n}$. For $x, y, z \in V$ we have

$$\begin{aligned}\{\varphi([x, y]), z\} &= [x, y]\cdot z \\ &= x\cdot(y\cdot z) - y\cdot(x\cdot z) \\ &= x\cdot\{\varphi(y), z\} - y\cdot\{\varphi(x), z\} \\ &= \{\varphi(x), \{\varphi(y), z\}\} - \{\varphi(y), \{\varphi(x), z\}\} \\ &= \{\{\varphi(x), \varphi(y)\}, z\}.\end{aligned}$$

Since $Z(\mathfrak{n}) = 0$ the claim follows, i.e., the map $\varphi\colon \mathfrak{g} \to \mathfrak{n}$ is a Lie algebra homomorphism.

Conversely, one can also show that when the two identities are satisfied, the product $x \cdot y = \{\varphi(x), y\}$ does indeed define a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.
Since φ is linear, the product is bilinear. The first identity directly gives (5.4).

To get (5.5) we use the second identity and the Jacobi identity for $\mathfrak{n}$. For $x, y, z \in V$ we have

$$\begin{aligned}
[x, y] \cdot z &= \{\varphi([x, y]), z\} \\
&= \{\{\varphi(x), \varphi(y)\}, z\} \\
&= \{\varphi(x), \{\varphi(y), z\}\} - \{\varphi(y), \{\varphi(x), z\}\} \\
&= x \cdot \{\varphi(y), z\} - y \cdot \{\varphi(x), z\} \\
&= x \cdot (y \cdot z) - y \cdot (x \cdot z).
\end{aligned}$$

We again use the Jacobi identity for $\mathfrak{n}$ to prove (5.6). For $x, y, z \in V$ we have

$$\begin{aligned}
x \cdot \{y, z\} &= \{\varphi(x), \{y, z\}\} \\
&= \{y, \{\varphi(x), z\}\} - \{z, \{\varphi(x), y\}\} \\
&= \{y, x \cdot z\} - \{z, x \cdot y\} \\
&= \{x \cdot y, z\} + \{y, x \cdot z\}.
\end{aligned}$$

□

5.4 One-one correspondences concerning post-Lie algebra structures

In this section we will prove some useful one-one correspondences concerning post-Lie algebra structures and specialize to the case where $\mathfrak{n}$ is semisimple.

The first result shows that we have a one-one correspondence between post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and embeddings $\mathfrak{g} \hookrightarrow \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$, where we recall that the Lie bracket on $\mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$ is given by

$$[(x, D), (x', D')] = (\{x, x'\} + D(x') - D'(x), [D, D']).$$

Proposition 5.4.1. *Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then the map*

$$\varphi \colon \mathfrak{g} \to \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) : \ x \mapsto (x, L(x))$$

is an injective homomorphism of Lie algebras. Conversely, any such embedding, with the identity map on the first factor, yields a post-Lie algebra structure onto $(\mathfrak{g}, \mathfrak{n})$.
This gives a one-one correspondence.

Proof. Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. First of all, (5.6) implies that φ is well-defined since $L(x) \in \mathrm{Der}(\mathfrak{n})$. Furthermore, we have

$$\begin{aligned} [\varphi(x), \varphi(y)] &= [(x, L(x)), (y, L(y))] \\ &= (\{x, y\} + x \cdot y - y \cdot x, [L(x), L(y)]) \\ &= ([x, y], L([x, y])) \\ &= \varphi([x, y]), \end{aligned}$$

where we have used (5.4) and (5.5). So φ is indeed a Lie algebra homomorphism.

Conversely, if we have a given embedding $\varphi(x) = (x, L(x))$ with a derivation $L(x)$, define $x \cdot y$ by $L(x)y$. Since φ is a homomorphism we have

$$\varphi([x, y]) = [\varphi(x), \varphi(y)].$$

Working out both sides of this equality gives

$$\begin{aligned} ([x, y], L([x, y])) &= [(x, L(x)), (y, L(y))] \\ &= (\{x, y\} + x \cdot y - y \cdot x, [L(x), L(y)]). \end{aligned}$$

This proves (5.4) and (5.5). Identity (5.6) follows from the fact that $L(x)$ is a derivation.

It is obvious that this leads to a one-one correspondence. □

The previous result can be rephrased in terms of subalgebras, this leads to the following proposition:

Proposition 5.4.2. *There is a one-one correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and the subalgebras $\mathfrak{h}$ of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ for which the projection $p_1 \colon \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \to \mathfrak{n}$ onto the first factor induces a Lie algebra isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.*

Note that, as vector spaces $\mathfrak{n} = V = \mathfrak{g}$, so that p_1 can indeed be seen as a map onto $\mathfrak{g}$.

Proof. Assume that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, and denote by φ the corresponding embedding as above. Then $\mathfrak{h} = \operatorname{im} \varphi = \{(x, L(x)) \mid x \in \mathfrak{g}\}$ is the Lie subalgebra corresponding to $\mathfrak{g}$.
It is obviously a subalgebra of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ and φ induces an isomorphism of $\mathfrak{g}$ onto $\mathfrak{h}$. It is clear that the restriction of p_1 to $\mathfrak{h}$ is the inverse of this isomorphism, and so is itself an isomorphism.

Conversely, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$, for which $p_1|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. Then the inverse map

$$\varphi = (p_1|_{\mathfrak{h}})^{-1} : \mathfrak{g} \to \mathfrak{h} \leq \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$$

is an embedding inducing the identity on the first factor. Hence, by proposition 5.4.1, φ determines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.

In the above, we showed how to assign a subalgebra $\mathfrak{h}$ to a post-Lie algebra structure and vice versa. It is obvious that these two operations are each others inverse. □

We can use the previous result to form post-Lie algebra structures, with a given Lie algebra $\mathfrak{n}$ as second Lie algebra of the pair $(\mathfrak{g}, \mathfrak{n})$:

Remark 5.4.3. *Given a Lie algebra $\mathfrak{n}$, let $\mathfrak{h}$ be any subalgebra of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ for which the projection p_1 onto the first factor is a bijection. Then, for any $x \in \mathfrak{n}$, there is exactly one $L(x) \in \mathrm{Der}(\mathfrak{n})$ such that $(x, L(x)) \in \mathfrak{h}$. We can define a new Lie bracket on $\mathfrak{n}$ by*

$$[x, y] := p_1([(x, L(x)), (y, L(y))])$$

and denote the corresponding Lie algebra by $\mathfrak{g}$.

Now, $\varphi : \mathfrak{g} \to \mathfrak{h} : \ x \mapsto (x, L(x))$ is an isomorphism of Lie algebras, and $x \cdot y := L(x)y$ is the post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ corresponding to $\mathfrak{h}$.

Let us show that the bracket $[x, y]$ is indeed a well-defined Lie bracket.

First of all note that $L(\sum_i \alpha_i x_i) = \sum_i \alpha_i L(x_i)$.
Indeed, since $(x_i, L(x_i)) \in \mathfrak{h}$, also $\sum_i \alpha_i (x_i, L(x_i)) \in \mathfrak{h}$, hence

$$\Big(\sum_i \alpha_i x_i, \sum_i \alpha_i L(x_i)\Big) \in \mathfrak{h}.$$

On the other hand $(\sum_i \alpha_i x_i, L(\sum_i \alpha_i x_i)) \in \mathfrak{h}$, because of the definition of L. Since the first factors of these elements are equal, so should the second factors be, since the projection on the first factor is a bijection. Hence L is a linear map.

Now let us prove the Jacobi identity. Note that $[(x, L(x)), (y, L(y))] \in \mathfrak{h}$ and take

$$a = p_1([(x, L(x)), (y, L(y))]) = [x, y] \in \mathfrak{n}.$$

There is exactly one element $L(a) \in \mathrm{Der}(\mathfrak{n})$ such that $(a, L(a)) \in \mathfrak{h}$. Since $a = p_1([(x, L(x)), (y, L(y))])$, we have that

$$(a, L(a)) = [(x, L(x)), (y, L(y))]$$

since they have the same first factor. Hence

$$([x,y], L([x,y])) = [(x, L(x)), (y, L(y))],$$

so we have

$$\begin{aligned}[][x,y],z] &= p_1([([x,y], L([x,y])), (z, L(z))]) \\ &= p_1([[(x, L(x)), (y, L(y))], (z, L(z))]).\end{aligned}$$

Because of the Jacobi identity for the semidirect product and the linearity of p_1 we get the Jacobi identity for the new bracket.

One-one correspondence in the semisimple case

In the special case where $\mathfrak{n}$ is semisimple we can say more on the above one-one correspondence.

Now $\mathrm{Der}(\mathfrak{n}) = \mathrm{ad}(\mathfrak{n}) = \mathfrak{n}$, and the Lie algebra $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) = \mathfrak{n} \rtimes \mathfrak{n}$ is isomorphic to the direct sum $\mathfrak{n} \oplus \mathfrak{n}$. Indeed, the map $\psi\colon \mathfrak{n} \rtimes \mathfrak{n} \to \mathfrak{n} \oplus \mathfrak{n}\colon (x,y) \mapsto (x+y, y)$ is a Lie algebra isomorphism:

Lemma 5.4.4. *Let $\mathfrak{n}$ be a Lie algebra. The map*

$$\psi : \mathfrak{n} \rtimes \mathfrak{n} \to \mathfrak{n} \oplus \mathfrak{n} : (x,y) \mapsto (x+y, y)$$

is an isomorphism of Lie algebras.

Proof. Clearly ψ is a bijective linear map. Furthermore

$$\begin{aligned}[][\psi(x,y), \psi(x',y')] &= [(x+y, y), (x'+y', y')] \\ &= (\{x+y, x'+y'\}, \{y, y'\}) \\ &= (\{x,x'\} + \{y,y'\} + \{x,y'\} + \{y,x'\}, \{y,y'\}) \\ &= \psi(\{x,x'\} + \{x,y'\} + \{y,x'\}, \{y,y'\}) \\ &= \psi(\{x,x'\} + \{y,x'\} - \{y',x\}, \{y,y'\}) \\ &= \psi(\{x,x'\} + y \cdot x' - y' \cdot x, \{y,y'\}) \\ &= \psi([(x,y),(x',y')]),\end{aligned}$$

where the action of $\mathfrak{n}$ on itself is given by the adjoint representation $\mathrm{ad} : \mathfrak{n} \to \mathfrak{gl}(\mathfrak{n})$. □

Now we get the following result:

Proposition 5.4.5. *Let $\mathfrak{n}$ be a semisimple Lie algebra. Then there is a one-one correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and the subalgebras $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ for which the map $p_1 - p_2 \colon \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n} : (x, y) \mapsto x - y$ induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$. Here $p_i : \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n}$ denotes projection onto the i-th factor ($i = 1, 2$).*

Proof. This follows from proposition 5.4.2 by noting that a subalgebra $\mathfrak{h}$ of $\mathfrak{n} \rtimes \mathfrak{n}$ for which p_1 induces an isomorphism of $\mathfrak{h}$ on $\mathfrak{g}$ corresponds, via ψ, to a subalgebra $\mathfrak{h}' = \psi(\mathfrak{h})$ of $\mathfrak{n} \oplus \mathfrak{n}$ such that $p_1 - p_2 \colon \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n}$: $(x, y) \mapsto x - y$ induces an isomorphism of $\mathfrak{h}'$ on $\mathfrak{g}$. This is visualized by the following diagram:

$$\begin{array}{ccc} \mathfrak{h} \leq \mathfrak{n} \rtimes \mathfrak{n} & \xrightarrow{p_1} & \mathfrak{n} \\ {\scriptstyle \psi^{-1}} \uparrow \; \downarrow {\scriptstyle \psi} & \nearrow {\scriptstyle p_1 - p_2} & \\ \mathfrak{h}' \leq \mathfrak{n} \oplus \mathfrak{n}. & & \end{array}$$

□

Let us make some remarks about the subalgebras of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$, $\mathfrak{n} \rtimes \mathfrak{n}$ and $\mathfrak{n} \oplus \mathfrak{n}$ corresponding to post-Lie algebra structures.

Remark 5.4.6. *For subalgebras of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ the elements are exactly those elements (x, D) with $x \in \mathfrak{n}$ and $D \in \mathrm{Der}(\mathfrak{n})$ such that $L(x) = D$.*

Now suppose we are in the case where $\mathfrak{n}$ is semisimple. For every $x \in \mathfrak{n}$ there exists some $y \in \mathfrak{n}$ such that $L(x) = \mathrm{ad}(y)$. The subalgebras of $\mathfrak{n} \rtimes \mathfrak{n}$ corresponding to a post-Lie algebra structure now exist exactly of those elements (x, y) with $x, y \in \mathfrak{n}$ such that $L(x) = \mathrm{ad}(y)$.

By the isomorphism ψ an element (x, y) is mapped to $(x + y, y)$. Hence the subalgebras of $\mathfrak{n} \oplus \mathfrak{n}$ corresponding to post-Lie algebra structures exist exactly of those elements (x, y) with $x, y \in \mathfrak{n}$ for which $L(x) - L(y) = \mathrm{ad}(y)$.

5.5 From simply transitive NIL-affine actions to post-Lie algebra structures

In this section we show how post-Lie algebra structures arise naturally in the study of NIL-affine actions on nilpotent Lie groups. We say here that a post-Lie algebra structure is *complete* if all left multiplications are nilpotent.

Theorem 5.5.1. *Let G and N be real connected, simply connected nilpotent Lie groups with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$. Then there exists a simply transitive NIL-affine action of G on N if and only if there is a Lie algebra $\mathfrak{g}' \simeq \mathfrak{g}$, with the same underlying vector space as $\mathfrak{n}$, such that the pair of Lie algebras $(\mathfrak{g}', \mathfrak{n})$ admits a complete post-Lie algebra structure.*

To prove this theorem we will use the one-one correspondence between simply transitive NIL-affine actions and complete NIL-affine structures, as stated in chapter 2 theorem 2.3.6.

Proof. Let G and N be real connected, simply connected nilpotent Lie groups with corresponding Lie algebras respectively $\mathfrak{g}$ and $\mathfrak{n}$. Let $\rho : G \to \mathrm{Aff}(N)$ be a representation with corresponding differential $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : x \mapsto (t(x), D(x))$. Recall that $\mathfrak{aff}(\mathfrak{n}) = \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is the Lie algebra of the Lie group $\mathrm{Aff}(N)$.

Then ρ induces a simply transitive NIL-affine action of G on N if and only if $d\rho$ is a complete NIL-affine structure on $\mathfrak{g}$ (see theorem 2.3.6). This means that $d\rho$ is a Lie algebra homomorphism such that $t : \mathfrak{g} \to \mathfrak{n} : x \mapsto t(x)$ is bijective and such that $D(x)$ is nilpotent for all $x \in \mathfrak{g}$.

Now suppose we have a complete post-Lie algebra structure on a pair of Lie algebras $(\mathfrak{g}', \mathfrak{n})$ where $\mathfrak{g}'$ is isomorphic to $\mathfrak{g}$, say via $\psi : \mathfrak{g} \to \mathfrak{g}'$. Hence, by proposition 5.4.1, we have that

$$\varphi\colon \mathfrak{g}' \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) :\ x \mapsto (x, L(x))$$

is an injective Lie algebra homomorphism such that all $L(x)$ are nilpotent. The composition $\varphi \circ \psi : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is then clearly a complete NIL-affine structure on $\mathfrak{g}$.

For the converse statement, suppose that $d\rho : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) : x \mapsto (t(x), D(x))$ is a complete NIL-affine structure on $\mathfrak{g}$. Then $\mathfrak{h} = d\rho(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ for which the projection on the first factor induces a bijection of $\mathfrak{h}$ on $\mathfrak{n}$ since t is bijective. By remark 5.4.3 this gives rise to a post-Lie algebra structure on $(\mathfrak{g}', \mathfrak{n})$, where the Lie bracket on $\mathfrak{g}'$ is given by

$$(x, y) = p_1([(x, D(t^{-1}(x))), (y, D(t^{-1}(y)))]),$$

and p_1 is the projection on the first factor. The left multiplications are given by $L(x) = D(t^{-1}(x))$, so these are all nilpotent and hence the post-Lie algebra structure is complete. Note that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ which is in its turn isomorphic to $\mathfrak{g}'$. □

This theorem shows that a deeper study of (complete) post-Lie algebra structures is valuable to obtain a good understanding of simply transitive NIL-affine actions on nilpotent Lie groups.

5.6 Classification of complex two-dimensional post-Lie algebras

In the following we want to classify all complex two-dimensional post-Lie algebras.

If $(V, \cdot, \{\,,\})$ is a two-dimensional complex post-Lie algebra, then the associated Lie algebras are either $\mathbb{C}^2$, or $\mathfrak{r}_2(\mathbb{C})$, the non-abelian Lie algebra of dimension 2. In our classification, we distinguish between four cases, depending on the isomorphism types of these associated Lie algebras.

Case 1: $(\mathfrak{g}, [\,,])$ and $(\mathfrak{n}, \{\,,\})$ are abelian.

There is a basis (e_1, e_2) of V such that $[e_1, e_2] = \{e_1, e_2\} = 0$. Then (5.4) says that $x \cdot y = y \cdot x$ for all $x, y \in V$, (5.5) says that $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for all $x, y, z \in V$, and (5.6) says $0 = 0$. This implies

$$x \cdot (z \cdot y) = x \cdot (y \cdot z) = y \cdot (x \cdot z) = (x \cdot z) \cdot y,$$

so that post-Lie algebra structures on $(\mathbb{C}^2, \mathbb{C}^2)$ correspond to 2-dimensional commutative and associative algebras and visa versa. The classification is well known, see for example [15]:

V	Products	$[\,,]$	$\{\,,\}$
V_1	$-$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_2	$e_1 \cdot e_1 = e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_3	$e_1 \cdot e_1 = e_1,\ e_2 \cdot e_2 = e_2$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_4	$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_2$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_5	$e_2 \cdot e_2 = e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$

Case 2: $(\mathfrak{g}, [\,,])$ is abelian, and $(\mathfrak{n}, \{\,,\})$ is not abelian.

We may choose a basis (e_1, e_2) of V such that $[e_1, e_2] = 0$ and $\{e_1, e_2\} = -e_1$. Then post-Lie algebra structures on $(\mathbb{C}^2, \mathfrak{r}_2(\mathbb{C}))$ are just LR-structures on $\mathfrak{n}$, which were classified in [20]:

V	Products	$[\,,]$	$\{\,,\}$
V_6	$e_1 \cdot e_1 = e_1,\ e_2 \cdot e_1 = -e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = -e_1$
V_7	$e_1 \cdot e_2 = e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = -e_1$
V_8	$e_2 \cdot e_1 = -e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = -e_1$

Note that $(V_8, \cdot)$ is also an LSA.

<u>*Case 3:*</u> $(\mathfrak{g}, [\,,])$ is not abelian, and $(\mathfrak{n}, \{\,,\})$ is abelian.

We may choose a basis (e_1, e_2) of V such that $[e_1, e_2] = e_1$ and $\{e_1, e_2\} = 0$. Then post-Lie algebra structures on $(\mathfrak{r}_2(\mathbb{C}), \mathbb{C}^2)$ are just LSA-structures on $\mathfrak{g}$, which were classified in [15]:

V	Products	$[\,,]$	$\{\,,\}$
$V_9(\alpha)$	$e_2 \cdot e_1 = -e_1,\ e_2 \cdot e_2 = \alpha e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
$V_{10}(\beta)$ $\beta \neq 0$	$e_1 \cdot e_2 = \beta e_1,\ e_2 \cdot e_1 = (\beta - 1)e_1,$ $e_2 \cdot e_2 = \beta e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
V_{11}	$e_2 \cdot e_1 = -e_1,\ e_2 \cdot e_2 = e_1 - e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
V_{12}	$e_1 \cdot e_1 = e_2,\ e_2 \cdot e_1 = -e_1$ $e_2 \cdot e_2 = -2e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
V_{13}	$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_2 = e_1 + e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$

Note that $(V_9(0), \cdot)$ is also a complete LR-algebra.

<u>*Case 4:*</u> $(\mathfrak{g}, [\,,])$ and $(\mathfrak{n}, \{\,,\})$ are not abelian.

We may choose a basis (e_1, e_2) of V such that $[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2$ with $(\alpha_1, \alpha_2) \neq (0, 0)$, and $\{e_1, e_2\} = e_1$. Note that we cannot make further assumptions on α_1 or α_2.

On the other hand, the conditions (5.4), (5.5) and (5.6) become very restrictive. They immediately imply that $\alpha_2 = 0$, and hence $\alpha_1 \neq 0$. Indeed, the conditions are given by

$$e_1 \cdot e_2 - e_2 \cdot e_1 = [e_1, e_2] - \{e_1, e_2\} = (\alpha_1 - 1)e_1 + \alpha_2 e_2,$$

$$\alpha_1 e_1 \cdot e_1 + \alpha_2 e_2 \cdot e_1 = [e_1, e_2] \cdot e_1 = e_1 \cdot (e_2 \cdot e_1) - e_2 \cdot (e_1 \cdot e_1),$$

$$\alpha_1 e_1 \cdot e_2 + \alpha_2 e_2 \cdot e_2 = [e_1, e_2] \cdot e_2 = e_1 \cdot (e_2 \cdot e_2) - e_2 \cdot (e_1 \cdot e_2),$$

$$e_1 \cdot e_1 = e_1 \cdot \{e_1, e_2\} = \{e_1 \cdot e_1, e_2\} + \{e_1, e_1 \cdot e_2\},$$

$$e_2 \cdot e_1 = e_2 \cdot \{e_1, e_2\} = \{e_2 \cdot e_1, e_2\} + \{e_1, e_2 \cdot e_2\}.$$

The last two identities imply that multiplying with e_1 on the right can only give multiples of e_1.
Suppose $e_1 \cdot e_1 = \alpha e_1$ and $e_1 \cdot e_2 = \beta e_1 + \beta' e_2$ then the 4th identity becomes

$$\alpha e_1 = \{\alpha e_1, e_2\} + \{e_1, \beta e_1 + \beta' e_2\} = \alpha e_1 + \beta' e_1.$$

Hence $\beta' = 0$ and $e_1 \cdot e_2 = \beta e_1$. Now, in the first identity the left hand side gives a multiple of e_1, so α_2 should be zero. This also implies that α_1 cannot be zero.

We can assume that $e_1 \cdot e_1 = \alpha e_1, e_1 \cdot e_2 = \beta e_1, e_2 \cdot e_1 = \gamma e_1$ and $e_2 \cdot e_2 = \delta e_1 + \epsilon e_2$. Filling this in in the identities gives

$$\beta e_1 - \gamma e_1 = (\alpha_1 - 1) e_1,$$

$$\alpha_1 \alpha e_1 = e_1 \cdot (\gamma e_1) - e_2 \cdot (\alpha e_1) = \gamma \alpha e_1 - \alpha \gamma e_1 = 0,$$

$$\alpha_1 \beta e_1 = e_1 \cdot (\delta e_1 + \epsilon e_2) - e_2 \cdot (\beta e_1) = \delta \alpha e_1 + \epsilon \beta e_1 - \beta \gamma e_1,$$

$$\alpha e_1 = \{\alpha e_1, e_2\} + \{e_1, \beta e_1\} = \alpha e_1,$$

$$\gamma e_1 = \{\gamma e_1, e_2\} + \{e_1, \delta e_1 + \epsilon e_2\} = \gamma e_1 + \epsilon e_1.$$

Immediately, we see that $\alpha = 0$ and $\epsilon = 0$. The two remaining identities can now be given by

$$\beta - \gamma = \alpha_1 - 1,$$

$$\alpha_1 \beta = -\beta \gamma.$$

Hence we have two cases. In the first case $\beta = 0$ and $\gamma = 1 - \alpha_1$. If $\beta \neq 0$ then $\gamma = -\alpha_1$ and $\beta = -\alpha_1 + \alpha_1 - 1 = -1$.

We can easily list *all* possible products, regardless of post-Lie algebra isomorphisms. We obtain two families of algebras, the first one given by

$$e_2 \cdot e_1 = (1 - \alpha_1) e_1, \; e_2 \cdot e_2 = \delta e_1, \tag{5.12}$$

where δ is an arbitrary complex number, and the second one given by

$$e_1 \cdot e_2 = -e_1, \; e_2 \cdot e_1 = -\alpha_1 e_1, \; e_2 \cdot e_2 = \delta e_1, \tag{5.13}$$

where δ is an arbitrary complex number.

Let $\varphi = (\varphi_{ij}) \in \mathrm{End}(V)$. It is an automorphism of $\mathfrak{n}$ if and only if $\varphi_{21} = 0$, $\varphi_{22} = 1$ and $\det(\varphi) = \varphi_{11} \neq 0$. Applying these automorphisms we obtain the classification of the above products as post-Lie algebras (assuming that $\alpha_1 \neq 0$):

V	Products	$[\,,]$	$\{\,,\}$
V_{14,α_1}	$e_2 \cdot e_1 = (1-\alpha_1)e_1$	$[e_1, e_2] = \alpha_1 e_1$	$\{e_1, e_2\} = e_1$
V_{15}	$e_2 \cdot e_2 = e_1$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = e_1$
V_{16,α_1}	$e_1 \cdot e_2 = -e_1,\ e_2 \cdot e_1 = -\alpha_1 e_1$	$[e_1, e_2] = \alpha_1 e_1$	$\{e_1, e_2\} = e_1$
V_{17}	$e_1 \cdot e_2 = -e_1,\ e_2 \cdot e_1 = e_1$ $e_2 \cdot e_2 = e_1$	$[e_1, e_2] = -e_1$	$\{e_1, e_2\} = e_1$

Note first that all four cases are examples of the above two products. V_{14} is the first product with $\delta = 0$, V_{15} is the first product with $\alpha_1 = 1$ and $\delta = 1$, V_{16} is the second product with $\delta = 0$ and V_{17} is the second product with $\alpha_1 = -1$ and $\delta = 1$.

We will now prove that those four cases include all post-Lie algebras. Suppose we have a product as in (5.12) or (5.13), for some $\alpha_1 \neq 0$ and δ. For $\varphi_{11} \neq 0$, we look at the following change of basis:

$$E_1 = \varphi_{11} e_1,$$

$$E_2 = \varphi_{12} e_1 + e_2,$$

where the Lie brackets are now given by

$$[E_1, E_2] = \varphi_{11}\alpha_1 e_1 = \alpha_1 E_1,$$

$$\{E_1, E_2\} = \varphi_{11} e_1 = E_1.$$

For the first type of product we get as nonzero products

$$E_2 \cdot E_1 = \varphi_{11}(1-\alpha_1)e_1 = (1-\alpha_1)E_1,$$

$$E_2 \cdot E_2 = \varphi_{12}(1-\alpha_1)e_1 + \delta e_1 = \frac{\varphi_{12}(1-\alpha_1)+\delta}{\varphi_{11}} E_1.$$

For the second type of product we get as nonzero products

$$E_1 \cdot E_2 = -\varphi_{11} e_1 = -E_1,$$

$$E_2 \cdot E_1 = \varphi_{11}(-\alpha_1)e_1 = -\alpha_1 E_1,$$

$$E_2 \cdot E_2 = -\varphi_{12} e_1 + \varphi_{12}(-\alpha_1)e_1 + \delta e_1 = \frac{-\varphi_{12}(1+\alpha_1)+\delta}{\varphi_{11}} E_1.$$

Consider the first type of product. If $\alpha_1 \neq 1$, set $\varphi_{12} = \frac{\delta}{\alpha_1 - 1}$, then we get an automorphism with V_{14,α_1}. If $\alpha_1 = 1$ and $\delta = 0$, then we have $V_{14,1}$. If on the other hand $\delta \neq 0$, set $\varphi_{11} = \delta$, then we get an automorphism with V_{15}.

Now consider the second type of product. If $\alpha_1 \neq -1$, set $\varphi_{12} = \frac{\delta}{1+\alpha_1}$, then we get an automorphism with V_{16,α_1}. If $\alpha_1 = -1$ and $\delta = 0$, then we have $V_{16,-1}$. If on the other hand $\delta \neq 0$, set $\varphi_{11} = \delta$, then we get an automorphism with V_{17}.

We still have to prove that the above four cases do not contain isomorphic algebras. Suppose we take an isomorphism φ of one of the above algebras V such that the isomorphic algebra $\varphi(V)$ is again one of our four cases.

First of all, if V is V_{14,α_1} or V_{15}, then $E_1 \cdot E_2 = 0$ so $\varphi(V)$ is again V_{14,α_1} or V_{15}. Similarly, if V is V_{16,α_1} or V_{17}, then $E_1 \cdot E_2 = -E_1$ so $\varphi(V)$ is again V_{16,α_1} or V_{17}.

Suppose $V = V_{14,\alpha_1}$, when $\alpha_1 \neq 1$, then $E_2 \cdot E_1 = (1 - \alpha_1)E_1$, so we need that $\varphi(V) = V_{14,\alpha_1}$. When $\alpha_1 = 1$ then $E_2 \cdot E_1 = E_2 \cdot E_2 = 0$, so $\varphi(V) = V_{14,1}$.
For $V = V_{15}$ we have that $E_2 \cdot E_2 = \frac{1}{\varphi_{11}} E_1$, so it can not be anything else then that $\varphi(V) = V_{15}$.
For $V = V_{16,\alpha_1}$ with $\alpha_1 \neq -1$, we have $E_2 \cdot E_1 = -\alpha_1 E_1 \neq E_1$ so we have that $\varphi(V) = V_{16,\alpha_1}$. If $\alpha_1 = -1$, then $E_2 \cdot E_2 = 0$ and $E_2 \cdot E_1 = E_1$, so $\varphi(V) = V_{16,-1}$.
If $V = V_{17}$, then $E_2 \cdot E_2 = \frac{1}{\varphi_{11}} E_1$, so $\varphi(V)$ has to be V_{17}.

Remark that the algebras $(V_{14,\alpha_1}, \cdot)$ and $(V_{15}, \cdot)$ are LR and LSA, but the algebras $(V_{16,\alpha_1}, \cdot)$ and $(V_{17}, \cdot)$ are not.

5.7 Structure results for $\mathfrak{g}$ and $\mathfrak{n}$

The existence of post-Lie algebra structures on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ imposes certain algebraic conditions on $\mathfrak{g}$ and $\mathfrak{n}$. In particular, the algebraic structures of $\mathfrak{g}$ and $\mathfrak{n}$ depend on each other in a certain way.

We will show, for example, that if $\mathfrak{g}$ is nilpotent and $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure, then $\mathfrak{n}$ must be solvable. But first we study the situation in which $\mathfrak{n}$ is 2–step nilpotent.

Lemma 5.7.1. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra and $\mathfrak{m}$ be the abelian Lie algebra with the same underlying vector space as $\mathfrak{n}$. Then*

$$\psi\colon \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \to \mathfrak{m} \rtimes \mathrm{Der}(\mathfrak{m}) = \mathfrak{m} \rtimes \mathfrak{gl}(\mathfrak{m}) :$$

$$(x, D) \mapsto \big(x, \frac{1}{2}\,\mathrm{ad}(x) + D\big)$$

is an embedding of Lie algebras.

Proof. The map is obviously an injective linear map. It remains to show that it is a Lie algebra homomorphism. We have

$$\begin{aligned}\psi([(x_1, D_1), (x_2, D_2)]) &= \psi\Big(\{x_1, x_2\} + D_1(x_2) - D_2(x_1), [D_1, D_2]\Big)\\ &= \Big(\{x_1, x_2\} + D_1(x_2) - D_2(x_1),\\ &\qquad \frac{1}{2}\,\mathrm{ad}(D_1(x_2)) - \frac{1}{2}\,\mathrm{ad}(D_2(x_1)) + [D_1, D_2]\Big),\end{aligned}$$

since we have $\mathrm{ad}(\{x_1, x_2\}) = 0$ because $\mathfrak{n}$ is 2-step nilpotent.

For a derivation D and $x, y \in \mathfrak{n}$ we have

$$[D, \mathrm{ad}(x)](y) = D(\{x, y\}) - \{x, D(y)\} = \{D(x), y\} = \mathrm{ad}(D(x))(y).$$

Using this identity $[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x))$ and the fact that $\mathfrak{m}$ is abelian and $\mathfrak{n}$ is 2-step nilpotent we obtain

$$\begin{aligned}&[\psi(x_1, D_1), \psi(x_2, D_2)]\\ &= \Big[\big(x_1, \frac{1}{2}\,\mathrm{ad}(x_1) + D_1\big), \big(x_2, \frac{1}{2}\,\mathrm{ad}(x_2) + D_2\big)\Big]\\ &= \Big(\frac{1}{2}\,\mathrm{ad}(x_1)(x_2) - \frac{1}{2}\,\mathrm{ad}(x_2)(x_1) + D_1(x_2) - D_2(x_1),\\ &\qquad \Big[\frac{1}{2}\,\mathrm{ad}(x_1) + D_1, \frac{1}{2}\,\mathrm{ad}(x_2) + D_2\Big]\Big)\\ &= \Big(\{x_1, x_2\} + D_1(x_2) - D_2(x_1),\\ &\qquad \frac{1}{2}[\mathrm{ad}(x_1), D_2] + \frac{1}{2}[D_1, \mathrm{ad}(x_2)] + [D_1, D_2]\Big)\end{aligned}$$

$$= \Big(\{x_1, x_2\} + D_1(x_2) - D_2(x_1),$$

$$\frac{1}{2}\operatorname{ad}(D_1(x_2)) - \frac{1}{2}\operatorname{ad}(D_2(x_1)) + [D_1, D_2] \Big).$$

We can conclude that ψ is indeed an injective Lie algebra homomorphism. □

We can now prove the following proposition:

Proposition 5.7.2. *Suppose that there exists a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{n}$ *is* 2*-step nilpotent. Then* $\mathfrak{g}$ *admits a left-symmetric structure. In particular,* $\mathfrak{g}$ *is not semisimple.*

Proof. Proposition 5.4.1 and the above lemma imply that also $(\mathfrak{g}, \mathfrak{m})$ admits a post-Lie algebra structure, with $\mathfrak{m}$ the abelian Lie algebra with the same underlying vector space as $\mathfrak{n}$. Since $\mathfrak{m}$ is abelian, $\mathfrak{g}$ admits a left-symmetric structure.

A complex semisimple Lie algebra does not admit a left-symmetric structure (see [14] and [17]). Hence $\mathfrak{g}$ can not be semisimple. □

If we assume that $\mathfrak{g}$ is nilpotent, we obtain the following result:

Proposition 5.7.3. *Suppose that there exists a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{g}$ *is nilpotent. Then* $\mathfrak{n}$ *is solvable.*

Proof. Consider the map

$$\varphi\colon \mathfrak{g} \to \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) : \; x \mapsto (x, L(x))$$

induced by the post-Lie algebra structure. Then $\mathfrak{h} = L(\mathfrak{g})$ is a nilpotent Lie algebra.

We claim that $\mathfrak{n} \rtimes \mathfrak{h} = \varphi(\mathfrak{g}) \oplus \mathfrak{h}$ as vector space. Indeed, for $(x, y) \in \mathfrak{n} \rtimes \mathfrak{h}$ we have

$$(x, y) - \varphi(x) = (x, y) - (x, L(x)) = (0, y - L(x)).$$

Hence $(x, y) = \varphi(x) + (0, y - L(x)) \in \varphi(\mathfrak{g}) \oplus \mathfrak{h}$.

Conversely, for $(x, y) \in \varphi(\mathfrak{g}) \oplus \mathfrak{h}$ there exist $a, b \in \mathfrak{g}$ such that

$$(x, y) = \varphi(a) + (0, L(b)) = (a, L(a) + L(b)) = (a, L(a + b)) \in \mathfrak{n} \rtimes \mathfrak{h}.$$

It follows that $\mathfrak{n} \rtimes \mathfrak{h}$ is the vector space sum of two nilpotent Lie algebras. Goto [32] has shown that the sum of two nilpotent Lie algebras is solvable. Hence $\mathfrak{n} \rtimes \mathfrak{h}$ is solvable, and so $\mathfrak{n}$ itself is solvable. □

In the case where $\mathfrak{n}$ is solvable and non-nilpotent we obtain the following proposition:

Proposition 5.7.4. *Suppose that there exists a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{n}$ *is solvable and non-nilpotent. Then* $\mathfrak{g}$ *is not perfect.*

Proof. By assumption the nilradical $\mathrm{nil}(\mathfrak{n})$ of $\mathfrak{n}$ is different from $\mathfrak{n}$. We have to show that $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$.

For all $x \in \mathfrak{n}$ the left multiplication $L(x)$ is a derivation of $\mathfrak{n}$. Any derivation D satisfies $D(\mathrm{rad}(\mathfrak{n})) \subseteq \mathrm{nil}(\mathfrak{n})$. Since $\mathfrak{n}$ is solvable we have $D(\mathfrak{n}) \subseteq \mathrm{nil}(\mathfrak{n})$. In particular we have $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathrm{nil}(\mathfrak{n})$.

It follows that for any $x, y \in \mathfrak{g}$ we have

$$[x, y] = x \cdot y - y \cdot x + \{x, y\} \in \mathrm{nil}(\mathfrak{n}).$$

This implies $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{nil}(\mathfrak{n}) \subsetneq \mathfrak{n} = \mathfrak{g}$ as vector spaces, so that $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$. □

We now have the following result, in case $\mathfrak{g}$ is $\mathfrak{sl}_2(\mathbb{C})$:

Corollary 5.7.5. *Suppose that there exists a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$*. Then* $\mathfrak{n}$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*.*

Proof. Suppose that $\mathfrak{n}$ is nilpotent. Since $\dim(\mathfrak{n}) = 3$, it has to be 2-step nilpotent. This gives a contradiction to proposition 5.7.2.

On the other hand, $\mathfrak{n}$ cannot be solvable, non-nilpotent by proposition 5.7.4. It follows that $\mathfrak{n}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. □

On a pair of simple Lie algebras $(\mathfrak{g}, \mathfrak{n})$ only trivial post-Lie algebra structures are possible, this is proven in the following proposition:

Proposition 5.7.6. *The post-Lie algebra structures* $x \cdot y$ *on* $(\mathfrak{g}, \mathfrak{n})$*, where both* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are simple, are given by* $x \cdot y = 0$ *for all* x *and* y *and* $[x, y] = \{x, y\}$*, or* $x \cdot y = [x, y] = -\{x, y\}$*.*

Proof. By proposition 5.4.5, any such post-Lie algebra structure corresponds to a subalgebra $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ for which the map $p_1 - p_2 \colon \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n}$ induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$. So since $\mathfrak{g}$ is simple, $\mathfrak{h}$ has to be simple too.

Both projection maps p_1 and p_2 are Lie algebra homomorphisms. Hence the kernels $\ker(p_1|_{\mathfrak{h}})$ and $\ker(p_2|_{\mathfrak{h}})$ are ideals in $\mathfrak{h}$, and so must be either 0 or $\mathfrak{h}$. This yields three possible cases.

Note that the elements of $\mathfrak{h}$ are of the form (x, y) with $x, y \in \mathfrak{n}$ and $L(x)-L(y) = \operatorname{ad}(y)$ (see remark 5.4.6).

Case 1: $p_2(\mathfrak{h}) = 0$
Then we have $\mathfrak{h} = \{(x, 0) \mid x \in \mathfrak{n}\}$. Because $L(x) = \operatorname{ad}(0) = 0$ for all $x \in \mathfrak{n}$ we have $x \cdot y = 0$ and $[x, y] = \{x, y\}$ for all $x, y \in \mathfrak{n}$. In particular, $\mathfrak{g} = \mathfrak{n}$.

Case 2: $p_1(\mathfrak{h}) = 0$
Then we have $\mathfrak{h} = \{(0, x) \mid x \in \mathfrak{n}\}$, and $L(x) = -\operatorname{ad}(x)$. It follows that $[x, y] = -\{x, y\}$ for all $x, y \in \mathfrak{n}$. Hence $\mathfrak{g} = -\mathfrak{n}$.

Case 3: $p_1(\mathfrak{h}) \neq 0$ *and* $p_2(\mathfrak{h}) \neq 0$
Then we have $\ker(p_{1|\mathfrak{h}}) = \ker(p_{2|\mathfrak{h}}) = 0$. Hence p_1 and p_2 are both bijective when restricted to $\mathfrak{h}$, since $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.

This implies that there is a bijective linear map $\varphi : \mathfrak{n} \to \mathfrak{n}$ such that $\mathfrak{h} = \{(x, \varphi(x)) \mid x \in \mathfrak{n}\}$. Indeed, define $\varphi : \mathfrak{n} \to \mathfrak{n} : x \mapsto p_2(p_1^{-1}(x))$.

As $\mathfrak{h}$ is a subalgebra of $\mathfrak{n} \oplus \mathfrak{n}$, we know that $[(x, \varphi(x)), (y, \varphi(y))] \in \mathfrak{h}$ for all $x, y \in \mathfrak{n}$. So $(\{x, y\}, \{\varphi(x), \varphi(y)\}) = (z, \varphi(z))$ for some $z \in \mathfrak{n}$. It follows that $z = \{x, y\}$ and hence

$$\varphi(\{x, y\}) = \varphi(z) = \{\varphi(x), \varphi(y)\}.$$

This shows that $\varphi \in \operatorname{Aut}(\mathfrak{n})$. By a result of Jacobson (see [35]), $\lambda = 1$ must be an eigenvalue of φ. But then $p_1 - p_2 : \mathfrak{h} \to \mathfrak{g} : (x, \varphi(x)) \mapsto x - \varphi(x)$ cannot be an isomorphism. This is a contradiction. Hence this third case cannot occur. □

Remark 5.7.7. *As we have seen in corollary* 5.7.5*, a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ *can only exist if* $\mathfrak{n}$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*. In that case we just have the two post-Lie algebra structures* $x \cdot y = 0$*, or* $x \cdot y = [x, y]$*, see proposition* 5.7.6*.*

In [17] it is shown that if there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{g}$ *semisimple, then* $\mathfrak{n}$ *cannot be solvable. For* $\dim(\mathfrak{g}) = 3$ *this again yields corollary* 5.7.5*.*

In the case where $\mathfrak{g}$ is simple and $\mathfrak{n}$ is semisimple, we get the following result:

Proposition 5.7.8. *Let* $x \cdot y$ *be a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{g}$ *is simple and* $\mathfrak{n}$ *is semisimple. Then* $\mathfrak{n}$ *is also simple and either* $x \cdot y = 0$ *and* $[x, y] = \{x, y\}$*, or* $x \cdot y = [x, y] = -\{x, y\}$*.*

Proof. We know that the map $L\colon \mathfrak{g} \to \operatorname{Der}(\mathfrak{n}) = \operatorname{ad}(\mathfrak{n}) \simeq \mathfrak{n}\colon x \mapsto L(x)$ is a Lie algebra homomorphism. Its kernel is an ideal in $\mathfrak{g}$.

If L is the zero map, then $x \cdot y = 0$ for all $x, y \in \mathfrak{g}$. Hence $[x, y] = \{x, y\}$ and $\mathfrak{n}$ is also simple.

Otherwise, L is a monomorphism and $\mathfrak{g}$ embeds into $\mathfrak{n}$, so that $\mathfrak{n} \simeq \mathfrak{g}$ is also simple. The claim follows from proposition 5.7.6. $\square$

The next result classifies the possible 3-dimensional Lie algebras $\mathfrak{g}$ for which the pair $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$ admits a post-Lie algebra structure.

We denote by $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ the series of solvable, non-nilpotent Lie algebras with basis (e_1, e_2, e_3) and Lie brackets $[e_1, e_2] = e_2$, $[e_1, e_3] = \lambda e_3$. Here $\lambda \in \mathbb{C}$ is a parameter. For $\lambda = 0$ we obtain the decomposable Lie algebra $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$.

Proposition 5.7.9. *Suppose that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is $\mathfrak{sl}_2(\mathbb{C})$. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ or to one of the Lie algebras $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ for $\lambda \neq -1$. Moreover, all these possibilities do occur.*

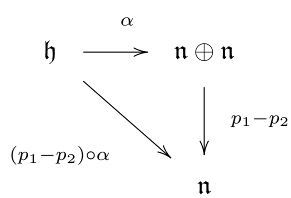
Proof. Assume first that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ for $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$. Then $\mathfrak{g}$ cannot be nilpotent by proposition 5.7.3.

As we have seen by proposition 5.7.6, the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is possible.

It remains to consider the 3-dimensional solvable, non-nilpotent Lie algebras. They are given by the Lie algebras $\mathfrak{r}_{3,\lambda}(\mathbb{C})$, and the Lie algebra $\mathfrak{r}_3(\mathbb{C})$, with Lie brackets $[e_1, e_2] = e_2$ and $[e_1, e_3] = e_2 + e_3$.

By proposition 5.4.5 a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ corresponds to a subalgebra $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ for which the map $p_1 - p_2 : \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n} : (x, y) \mapsto x - y$ induces a bijection when restricted to $\mathfrak{h}$. Note that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ in this case.

In other words, we want to classify the 3-dimensional solvable, non-nilpotent Lie algebras $\mathfrak{h}$ for which there exists an injective Lie algebra homomorphism α such that $(p_1 - p_2) \circ \alpha$ is bijective:



Since both $p_1 \circ \alpha$ and $p_2 \circ \alpha$ are Lie algebra homomorphisms, their kernels are ideals of $\mathfrak{h}$. If one of them equals $\mathfrak{h}$, then the post-Lie algebra product is either zero, or it is given by $x \cdot y = [x, y]$, this is proven in the same way as the cases 1 and 2 in the proof of proposition 5.7.6. In both cases $\mathfrak{h}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, so not solvable, non-nilpotent. Hence the kernels can not be $\mathfrak{h}$.

Suppose that one of these kernels equals zero. Then this map is an injective Lie algebra homomorphism, so that $\mathfrak{h} \simeq \mathfrak{n}$ is simple. Hence, since $\mathfrak{h}$ should be solvable, non-nilpotent, the kernels $\ker(p_1 \circ \alpha)$ and $\ker(p_2 \circ \alpha)$ are both non-trivial ideals of $\mathfrak{h}$.

We may assume that $\mathfrak{h}$ is one of the Lie algebras $\mathfrak{r}_3(\mathbb{C})$ or $\mathfrak{r}_{3,\lambda}(\mathbb{C})$.

The non-trivial ideals of $\mathfrak{r}_3(\mathbb{C})$ are represented by $\langle e_2\rangle$ and $\langle e_2, e_3\rangle$. This means that $p_1(\alpha(e_2)) = p_2(\alpha(e_2)) = 0$, so that $((p_1 - p_2)\circ\alpha)(e_2) = 0$. Hence $(p_1 - p_2)\circ\alpha$ is not bijective when restricted to $\mathfrak{h} = \mathfrak{r}_3(\mathbb{C})$. Hence there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g} \simeq \mathfrak{r}_3(\mathbb{C})$.

There is also no post-Lie algebra structure for the unimodular Lie algebra $\mathfrak{r}_{3,-1}(\mathbb{C})$. This follows from a general result in [17].

On the other hand it is easy to find a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ for $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g} \simeq \mathfrak{r}_{3,\lambda}(\mathbb{C})$ for all $\lambda \neq -1$ by direct calculation. Let $\alpha \neq \beta$ be two complex parameters. If the brackets of $\mathfrak{n}$ are given by

$$\{e_1, e_2\} = e_3,\ \{e_1, e_3\} = -2e_1,\ \{e_2, e_3\} = 2e_2,$$

then the following product

$$\begin{aligned} e_2 \cdot e_1 &= -\alpha e_1 + e_3, & e_3 \cdot e_1 &= \tfrac{2\beta}{\alpha-\beta} e_1, \\ e_2 \cdot e_2 &= \alpha e_2 + \tfrac{\alpha^2-\beta^2}{4} e_3, & e_3 \cdot e_2 &= -\tfrac{2\beta}{\alpha-\beta} e_2 - \beta e_3, \\ e_2 \cdot e_3 &= \tfrac{\beta^2-\alpha^2}{2} e_1 - 2e_2, & e_3 \cdot e_3 &= 2\beta e_1, \end{aligned}$$

defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is given by the Lie brackets

$$\begin{aligned} [e_1, e_2] &= \alpha e_1, \\ [e_1, e_3] &= -\frac{2\alpha}{\alpha-\beta} e_1, \\ [e_2, e_3] &= \frac{\beta^2-\alpha^2}{2} e_1 + \frac{2\beta}{\alpha-\beta} e_2 + \beta e_3. \end{aligned}$$

For $\beta \neq 0$ and $\lambda = -\alpha/\beta \neq -1$ it is isomorphic to $\mathfrak{r}_{3,\lambda}(\mathbb{C})$. Indeed, if $\alpha = 0$, the following change of basis gives the desired isomorphism

$$\begin{aligned} E_1 &= \frac{e_3}{2}, \\ E_2 &= -\frac{\beta^2}{4}e_1 + e_2 - \frac{\beta}{2}e_3, \\ E_3 &= e_1. \end{aligned}$$

If $\alpha \neq 0$, the following change of basis gives the desired isomorphism

$$\begin{aligned} E_1 &= \frac{1}{\beta}e_2, \\ E_2 &= \frac{\beta - \alpha}{2}e_1 + \frac{2}{\alpha - \beta}e_2 + e_3, \\ E_3 &= e_1. \end{aligned}$$

□

Chapter 6

Concluding questions and future work

To end this thesis, we will formulate some open questions and interesting future perspectives concerning our research.

6.1 The existence of complete Novikov structures

For nilpotent Lie algebras, Mizuhara showed in [46] that any complex nilpotent Lie algebra admitting a left-symmetric structure, also admits a complete left-symmetric structure. For LR-structures we were able to prove an analogue in chapter 4. Also for Novikov structures we have a lot of hope that a similar result is true:

> **Conjecture:** Any nilpotent Lie algebra over an algebraically closed field of characteristic 0 admitting a Novikov structure, also admits a complete Novikov structure.

As Novikov structures are left-symmetric structures, it follows from Mizuharas result that any complex nilpotent Lie algebra admitting a Novikov structure, also admits a complete left-symmetric structure. However, this complete left-symmetric structure, as constructed in the proof of Mizuhara, is not a Novikov structure in general.

Suppose $\mathfrak{g}$ is a nilpotent Lie algebra over an algebraically closed field of characteristic 0 and assume there exists a Novikov structure on $\mathfrak{g}$. Using proposition 4.2.4 for the representation $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by the left multiplication operator, we find that, as vector space, $\mathfrak{g}$ is the direct sum of the weight spaces.

We were able to prove that this direct sum is also a direct sum as Lie algebras (this is not true in general for left-symmetric structures or LR-structures). Hence, to prove our conjecture, it suffices to construct a complete Novikov structure on each of these weight spaces separately.

However, up till now, we weren't able to construct a complete Novikov structure in the case of only one, nonzero, weight.

We did find some structural properties about the given non-complete structure (in the case of one weight), saying that there exists a right identity for this product and that the product and Lie bracket of the basis elements, as taken in the above proposition, belong to specific subspaces.

We examined a lot of examples where for each nilpotent Lie algebra admitting a Novikov structure, also a complete Novikov structure exists, but no general proof was found yet.

6.2 More algebraic questions concerning post-Lie algebra structures

In chapter 5 we studied the relation between the algebraic structure of the Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$, in case the pair $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Other results in this context were found by Dekimpe and Burde in [17].

Together, this gives us the following results when a post-Lie algebra structure exists on the pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$:

- If $\mathfrak{n}$ is 2-step nilpotent, then $\mathfrak{g}$ is not semisimple.
- If $\mathfrak{n}$ is solvable, then $\mathfrak{g}$ is not semisimple.
- If $\mathfrak{n}$ is solvable, non-nilpotent, then $\mathfrak{g}$ is not perfect.
- If $\mathfrak{n}$ is not solvable, then $\mathfrak{g}$ is not nilpotent.
- If $\mathfrak{n}$ is semisimple, then $\mathfrak{g}$ can not be solvable, unimodular.
- If $\mathfrak{g}$ and $\mathfrak{n}$ are both simple, then they are isomorphic.

Also the following results about the existence of a post-Lie algebra structure hold:

- If $\mathfrak{n}$ is semisimple, then there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ for some solvable non-nilpotent Lie algebra $\mathfrak{g}$.
- If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then $\mathfrak{n}$ is isomorphic to $\mathfrak{g}$.
- If $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ or one of the Lie algebras $\tau_{3,\lambda}(\mathbb{C})$ for $\lambda \neq -1$ and all these possibilities occur.

It would be interesting to continue this study and find more relations between the algebraic structures of the Lie algebras involved or to refine the existing results.

For example we could ask what happens in the following cases:

- $\mathfrak{n}$ nilpotent, or more specifically p-step nilpotent.
- $\mathfrak{n}$ not solvable, not semisimple.
- Does the nilpotency class of $\mathfrak{g}$ imply anything about the solvability class of $\mathfrak{n}$?
- $\mathfrak{g}$ solvable, non-nilpotent.
- $\mathfrak{g}$ not solvable, not semisimple.

6.3 Geometric meaning of Novikov structures

We pointed out in chapter 2 how left-symmetric structures arise naturally in the study of simply transitive affine actions of Lie groups. In particular the question whether a given real nilpotent (or more general solvable) Lie algebra admits a complete left-symmetric structure corresponds to the question whether its Lie group admits a simply transitive affine action.

Since Novikov structures form a subclass of the left-symmetric structures, we could ask what the geometric meaning of a Novikov structure is, if it has any geometric meaning at all.

6.4 Geometric meaning of LR- and post-Lie algebra structures in the solvable case

In chapter 2 and 5 we saw how left-symmetric structures, LR-structures and post-Lie algebra structures arise in the study of simply transitive NIL-affine actions of nilpotent Lie groups.

For left-symmetric structures we pointed out in chapter 2 that this geometric meaning could be extended to the general (solvable) case, saying that a (solvable) Lie group admits a simply transitive affine action if and only if its Lie algebra admits a complete left-symmetric structure.

We could now ask if we can also extend this geometric meaning of LR- and post-Lie algebra structures to the solvable case:

> **Open question:** Let G be a (necessarily 2–step) solvable and simply connected Lie group such that the associated Lie algebra $\mathfrak{g}$ admits a complete LR–structure. Is it true that G admits an abelian simply transitive affine action $\rho : \mathbb{R}^n \to \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$?

> **Open question:** Let G be a connected, simply connected solvable Lie group and let N be a connected, simply connected nilpotent Lie group. Is there a relation between the simply transitive NIL-affine actions of G on N and a particular class of post-Lie algebra structures on $(\mathfrak{g}', \mathfrak{n})$ where $\mathfrak{g}'$ is a Lie algebra isomorphic to $\mathfrak{g}$, the Lie algebra corresponding to the Lie group G.

In case of an affirmative answer, this particular class of post-Lie algebra structures would be called complete. In case of a negative answer, we could ask to what notion these actions do correspond on the Lie algebra level.

In both cases, the study on the Lie algebra level can learn us a lot about these simply transitive actions.

Bibliography

[1] Abels, H. *Properly Discontinuous Groups of Affine Transformations: A survey.* Geometriae Dedicata, 2001, 87 pp. 309–333. pages 18

[2] Abels, H., Margulis, G., and Soifer, G. *Semigroups containing proximal linear maps.* Israel J. Math., 1995, 91 pp. 1–30. pages 18

[3] Abels, H., Margulis, G., and Soifer, G. *Properly Discontinuous Groups of Affine Transformations with Orthogonal Linear Part.* C. R. Acad. Sci. Paris Sér. I Math., 1997, 324 I pp. 253–258. pages 18

[4] Abels, H., Margulis, G. A., and Soifer, G. A. *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations.* J. Differential Geom., 2002, 60 2, 315–344. pages 18

[5] Auslander, L. *The structure of complete locally affine manifolds.* Topology, 1964, 3 Suppl. 1., pp. 131–139. pages 15

[6] Auslander, L. *Simply Transitive Groups of Affine Motions.* Amer. J. Math., 1977, 99 (4), pp. 809–826. pages 11, 23

[7] Auslander, L. and Markus, L. *Holonomy of Flat Affinely Connected Manifolds.* Ann. of Math., 1955, 62 (1), pp. 139–151. pages 15

[8] Bai, C., Guo, L., and X, N. *Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras.* Comm. Math. Phys., 2010, 297 2, pp. 553–596. pages 31

[9] Balinskiĭ, A. A. and Novikov, S. P. *Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras.* Sov. Math. Dokl., 1985, 32 pp. 228–231. pages 22, 35

[10] Baues, O. *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups.* Topology, 2004, 43 4, pp. 903–924. pages 25

[11] Benoist, Y. *Une nilvariété non affine.* C. R. Acad. Sci. Paris Sér. I Math., 1992, 315 pp. 983–986. pages 11, 17, 35, 65

[12] Burde, D. *Affine structures on nilmanifolds.* Internat. J. Math, 1996, 7 5, pp. 599–616. pages 17

[13] Burde, D. *Classical r-matrices and Novikov algebras.* Geom. Dedicata, 2006, 122 1, pp. 145–157. pages 35

[14] Burde, D. *Left-symmetric algebras, or pre-Lie algebras in geometry and physics.* Central European Journal of Mathematics, 2006, 4 3, pp. 323–357. pages 136

[15] Burde, D. and Beneš, T. *Degenerations of pre-Lie algebras.* J. Math. Phys., 2009, 50 11, 112102. pages 130, 131

[16] Burde, D. and Dekimpe, K. *Novikov structures on solvable Lie algebras.* J. Geom. Phys., 2006, 56 9, pp. 1837–1855. pages 35, 40, 44, 61, 65, 76, 92

[17] Burde, D. and Dekimpe, K. *Post-Lie algebra structures and generalized derivations of semisimple Lie algebras.* Moscow Math. J., 2013, 13 1, pp. 1–18. pages 121, 122, 136, 138, 140, 144

[18] Burde, D., Dekimpe, K., and Deschamps, S. *The Auslander conjecture for NIL-affine crystallographic groups.* Math. Ann., 2005, 332 1, pp. 161–176. pages 25

[19] Burde, D., Dekimpe, K., and Deschamps, S. *Affine Actions on Nilpotent Lie Groups.* Forum Math, 2009, 21 5, pp. 921–934. pages 27, 28

[20] Burde, D., Dekimpe, K., and Deschamps, S. *LR-algebras.* Contemporary Mathematics, 2009, 491 pp. 125–140. pages vi, 28, 71, 85, 88, 92, 112, 131

[21] Burde, D., Dekimpe, K., and Vercammen, K. *Novikov Algebras and Novikov Structures on Lie algebras.* Linear Algebra and its Applications, 2008, 429 1, pp. 31–41. pages 34

[22] Burde, D., Dekimpe, K., and Vercammen, K. *Complete LR–structures on solvable Lie algebras.* Journal of Group Theory, 2010, 13 5, pp. 703–719. pages 70

[23] Burde, D., Dekimpe, K., and Vercammen, K. *Affine actions on Lie groups and post-Lie algebra structures.* Linear Algebra and its Applications, 2012, 437 5, pp. 1250–1263. pages 114

[24] Burde, D. and Grunewald, F. *Modules for certain Lie algebras of maximal class.* J. Pure Appl. Algebra, 1995, 99 3, pp. 239–254. pages 11, 17

[25] Dekimpe, K. *Any virtually polycyclic group admits a NIL-affine crystallographic action.* Topology, 2003, 42 4, pp. 821–832. pages 12, 25

[26] Dekimpe, K. and Petrosyan, N. *Crystallographic actions on contractible algebraic manifolds.* Preprint (22 pages), 2010. pages 25

[27] Frenkel, I., Y.Z., H., and L., L. *On axiomatic approaches to vertex operator algebras and modules.* Mem. Am. Math. Soc., 1993, 494 pp. 1–64. pages 22

[28] Fried, D., Goldman, W., and Hirsch, M. *Affine manifolds with nilpotent holonomy.* Comment. Math. Helv., 1981, 56 pp. 487–523. pages 11, 18, 20

[29] Fried, D. and Goldman, W. M. *Three–Dimensional Affine Crystallographic Groups.* Adv. in Math., 1983, 47 1, pp. 1–49. pages 11

[30] Gel′fand, I. M. and Dorfman, I. J. *Hamiltonian operators and algebraic structures related to them.* Funct. Anal. Appl., 1980, 13 pp. 248–262. pages 22

[31] Gorbacevič, V. V. *Discrete subgroups of solvable Lie groups of type (E).* Math. USSR Sb., 1971, 14 2, pp. 233–251. pages 10

[32] Goto, M. *Note on a characterization of solvable Lie algebras.* J. Sci. Hiroshima Univ. Ser. A-I Math., 1962, 26 1, pp. 1–2. pages 136

[33] Goto, M. and Grosshans, F. D. *Semisimple Lie algebras*, volume 38 of *Lecture Notes in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1978. pages 77

[34] Jacobson, N. *Lie Algebras*, volume 10 of *Interscience Tracts in Pure and Applied Mathematics.* Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1962. pages 77

[35] Jacobson, N. *A note on automorphisms of Lie algebras.* Pacific J. Math., 1962, 12 1, pp. 303–315. pages 138

[36] Kamber, F. and Tondeur, P. *Flat manifolds with parallel torsion.* J. Differential Geom., 1968, 2, pp. 358–389. pages 24

[37] Kim, H. *Complete left-invariant affine structures on nilpotent Lie groups.* J. Differential Geom., 1986, 24 pp. 373–394. pages 11, 23, 35

[38] Knapp, A. *Lie groups, Lie algebras and cohomology*, volume 34 of *Mathematical Notes.* Princeton University Press, Princetopn, 1988. pages 6

[39] Kobayashi, S. and Nomizu, K. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963. pages 14

[40] Loday, J.-L. *Generalized bialgbras and triples of operads.* Asterisque, 2008, 320 116 pp. pages 30

[41] Magnin, L. *Adjoint and Trivial Cohomology Tables for Indecomposable Nilpotent Lie Algebras of Dimension* ≤ 7 *over* $\mathbb{C}$. Electronic Book, 1995. pages 106

[42] Mal'cev, A. I. *On a class of homogeneous spaces.* Amer. Math. Soc. Translations, 1951, 39, pp. 1–33. pages 9

[43] Margulis, G. A. *Free properly discontinuous groups of affine transformations.* Dokl. Akad. Nauk SSSR, 1983, 272, pp. 937–940. pages 15

[44] Margulis, G. A. *Complete affine locally flat manifolds with a free fundamental group.* J. Soviet Math., 1987, 134, pp. 129–134. pages 15

[45] Milnor, J. *On fundamental groups of complete affinely flat manifolds.* Adv. Math., 1977, 25 pp. 178–187. pages 11, 14, 15, 22

[46] Mizuhara, A. *On a complete left symmetric algebra over a nilpotent Lie algebra.* Tensor (N.S.), 1983, 40 2, pp. 144–148. pages vi, 30, 143

[47] Raghunathan, M. S. *Discrete Subgroups of Lie Groups*, volume 68 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, 1972. pages 16

[48] Scheuneman, J. *Affine structures on three-step nilpotent Lie algebras.* Proc. A.M.S., 1974, 46 (3), pp. 451–454. pages 40

[49] Scheuneman, J. *Translations in certain groups of affine motions.* Proc. Amer. Math. Soc., 1975, 47 (1), pp. 223–228. pages 20

[50] Tits, J. *Free subgroups of linear groups.* J. Algebra, 1972, 20, pp. 250–270. pages 16

[51] Valette, B. *Homology of generalized partition posets.* J. Pure and Applied Algebra, 2007, 208 2, pp. 699–725. pages 30

[52] Warner, F. W. *Foundations of differentiable manifolds and Lie groups.* Scott, Foresman and Company, Glenview, Illinois, 1971. pages 14

[53] Wolf, J. A. *Spaces of constant curvature.* Publish or Perish, Inc. Berkeley, 1977. pages 14, 15

[54] Zelmanov, E. *On a class of local translation invariant Lie algebras*. Soviet Math. Dokl., 1987, 35 pp. 216–218. pages 35

Arenberg Doctoral School of Science, Engineering & Technology
Faculty of Sciences
Department of Mathematics
Research group Algebraic Topology and Group Theory
Etienne Sabbelaan 53, 8500 Kortrijk