Interval fields to represent uncertainty on the output side of a static FE analysis

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Abstract

Intervals have been used extensively for the representation of parametric uncertainty in Finite Element Models when the available information is insufficient to build representative probabilistic models. Although the complexity of the interval concept is rather limited, the numerical problem arising at the core of the Interval Finite Element Method was proven to be non-trivial. Many research activities have focussed on this problem over the last decade. Still, one of the principal shortcomings of the current interval finite element procedures is that they are intrinsically not capable of representing dependencies among uncertain input and/or output quantities. This poses important limitations on the applicability of the interval concept in non-deterministic finite element analysis. These limits arise in the model definition phase as well as in the post-processing phase. In order to cope with this problem, this paper introduces the concept of interval fields. The focus in this work is on the static analysis of mechanical structures with interval uncertainties in the context of the Finite Element Method. However, the presented methodology is generic, and can be readily extended to any kind of finite element problem including interval uncertainties. In the first part of the paper, the theoretical background of the concept is presented, and it is shown how explicit and implicit interval fields can be used to represent dependent uncertain parameters in the model definition phase and in the post-processing phase. Furthermore, the applicability and accuracy of both types of interval fields are studied by means of a numerical validation example.

Keywords: non-deterministic FE analysis, interval analysis, dependency problem, interval fields

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Nomenclature

$\mathbf{y}^s$	=	a set of output vectors $\mathbf{y}$
$\mathbf{x}^I$	=	interval vector for input uncertain parameters
$f(\mathbf{x})$	=	function representing input-output relationship
$\mathbf{y}^I$	=	interval vector as hypercubic approximation of $\mathbf{y}^s$
$\mathbf{y}_{exp}^F$	=	explicit interval field approximation of $\mathbf{y}^s$
$\mathbf{y}_{imp}^F$	=	implicit interval field approximation of $\mathbf{y}^s$
$\psi_{.,i}$	=	basis vector with index i
$\eta_{.,i}^I$	=	interval factor with index i
$\Pi()$	=	a post-processing procedure
$\phi_{.,i}(\boldsymbol{\eta}_y)$	=	mapping from interval factor space $\boldsymbol{\eta}_y^I$ to component i value
$[\mathbf{K}]$	=	stiffness matrix
$[\mathbf{K}(\mathbf{x})]$	=	stiffness matrix influenced by uncertain input parameters $\mathbf{x}$
$\mathbf{u}$	=	vector of displacements and rotations
$\mathbf{u}(\mathbf{x})$	=	vector of displacements and rotations influenced by uncertain input parameters $\mathbf{x}$
$\mathbf{q}$	=	vector of forces and moments
$\mathbf{q}(\mathbf{x})$	=	vector of forces and moments influenced by uncertain input parameters $\mathbf{x}$
$\mathbf{u}^s$	=	a set of vectors $\mathbf{u}$
$\mathbf{u}^I$	=	interval vector as hypercubic approximation of $\mathbf{u}^s$
$\mathbf{u}^F$	=	interval field approximation of $\mathbf{u}^s$
$\phi_\eta(\mathbf{x})$	=	interval factors as a function of the input uncertainties
$\hat{\phi}_\eta(\mathbf{x})$	=	approximation of $\phi_\eta(\mathbf{x})$
$\phi_u(\mathbf{x})$	=	implicit function for $\mathbf{u}(\mathbf{x})$ as a function of the input uncertainties
$\hat{\phi}_u(\mathbf{x})$	=	approximation of $\phi_u(\mathbf{x})$

1. Introduction

In recent years many research activities have focussed on the development of non-deterministic Finite Element Methods (FEM). This has resulted in a variety of methods, ranging from the probabilistic approaches to the non-probabilistic Fuzzy and Interval Finite Element Methods (IFEM) [1], [2], [3], [4]. Two general difficulties arise in both families of methods: the problem of dependency and the problem of high calculation costs.

In the probabilistic methods the concept of random fields [5] is commonly used to take dependencies between uncertain parameters into account. The use of random fields requires the covariance function of the studied quantity to be known. Sufficient reliable information may not be available [6], especially in early design stages where no data on the studied non-deterministic quantities is available to derive the correct covariance function. The use of the probabilistic concepts requires good knowledge on the variability before realistic statements can be made on the probability of occurrence of an output quantity. Therefore, non-probabilistic approaches are gaining interest in design problems, where the knowledge of the non-determinism in general is rather limited. The objective of this paper is to study the applicability of the field concept for introducing dependencies in the context of interval analysis.

A second major aspect of non-deterministic approaches covered in literature is the high computational cost. The non-deterministic approaches (probabilistic and non-probabilistic) are computationally expensive due to the fact that at their core they have a deterministic problem that has to be solved multiple times to obtain the non-deterministic results. Generally speaking, this can be remedied in two ways: a reduction of the required number of deterministic analyses or a reduction of the calculation cost of one deterministic analysis [7], [8]. This paper presents a non-probabilistic interval field concept for taking into account the dependencies, while at the same time reducing the calculation cost of the deterministic analysis at the core of the problem. This is done by explicitly or implicitly separating dependency and uncertainty based on classic static reduction techniques.

A general introduction to non-deterministic fields is given in the next section. The focus is on the non-probabilistic static FEM. The third and fourth section present the interval field description of the Finite Element (FE) output. The fifth section gives room to a discussion and comparison of all the four implementations presented in the fourth section. In the sixth section a numerical case study illustrates the different interval field representations of static FE output for an industrial relevant case. The conclusions mark the end of the paper.

2. Non-deterministic fields in the Interval Finite Element Method

This section deals with non-deterministic fields in the IFEM. Firstly, the concept of the IFEM is briefly reviewed. Secondly, field representations of non-deterministic quantities are introduced.

2.1. Interval finite element analysis

The most common non-probabilistic concept is the interval representation.

An interval scalar $x^I = [x_{min} \ x_{max}] = [\underline{x} \ \overline{x}]$ is defined as a single continuous subset of $\mathbb{R}$. The range of an interval x^I is by definition bounded by a lower bound x_{min} or $\underline{x}$ and an upper bound x_{max} or $\overline{x}$. An interval vector $\mathbf{x}^I$ or interval matrix $[\mathbf{x}^I]$ has interval scalars as one or more of its components. The nature of such interval vectors or matrices doesn't allow for dependencies between their components, every component is an independent interval.

Generally an Interval Finite Element (IFE) problem can be represented by [9]:

$$\mathbf{y}^s = \{\mathbf{y} \mid (\mathbf{x} \in \mathbf{x}^I)(\mathbf{y} = \mathbf{f}(\mathbf{x}))\} \quad (1)$$

with $\mathbf{x}^I$ the interval vector representing the bounds on the input uncertain parameters and $\mathbf{f}(\mathbf{x})$ the function representing the input-output relationship. The solution is a set $\mathbf{y}^s$. For an overview of IFE methods see [2], [10] and [11]. The calculation of the set $\mathbf{y}^s$ is computationally intensive and thus $\mathbf{y}^s$ is often

approximated with an interval vector $\mathbf{y}^I$:

$$\mathbf{y}^I = \begin{Bmatrix} y_1^I \\ y_2^I \\ \vdots \\ y_n^I \end{Bmatrix}, \quad (2)$$

The individual components y_i^I of $\mathbf{y}^I$ are found by optimisation:

$$\underline{y}_i = \min_{\mathbf{x} \in \mathbf{x}^I} f_i(\mathbf{x}), \quad (3)$$

$$\overline{y}_i = \max_{\mathbf{x} \in \mathbf{x}^I} f_i(\mathbf{x}), \quad (4)$$

with $\mathbf{x}^I$ the vector defining the input interval space as in equation (1) and $f_i(\mathbf{x})$ is the i th output (degree of freedom i) of the deterministic analysis. To reduce the calculation cost of the deterministic analysis, it is often replaced by a response surface or so called surrogate model [12] (in the remainder of the paper a surrogate model is denoted by $\tilde{\cdot}$). Note that the optimisation approach finds the interval vector $\mathbf{y}^I$ circumscribing the set $\mathbf{y}^s$ by optimising each component independently, this is called a hypercubic approximation.

2.2. Non-deterministic fields

A field describes the spatial coherence and distribution of a parameter. In this way a field represents the variation of a physical property inside one realisation of a design. In an FE analysis, fields arise on the input side (e.g. pressure distribution) as well as on the output side (e.g. deformation). Every component (local value) of such a field is not independent of the other components because in reality, physical properties have a continuous behaviour through space. Such fields are useful tools to represent dependencies in uncertain parameters.

When non-determinism influences the model description, the common interval technique is to solve the interval problem for every component of the output separately. This is achieved by solving equations (3) and (4) for every component of the vector $\mathbf{y}$. As the individual response quantities and their bounds have been obtained after separate analysis runs that are not related to each other, and as the interval response at one position is necessarily considered to be unrelated to the response at another position, the response pattern of the entire structure can only be reconstructed using the bounds at adjacent positions. Inadvertent combination of bounds may lead to response patterns that are unrealistic. In figure 1 a cantilever beam with interval bending stiffness is shown. The blue dots show the independent bounds on every component of the interval vector $\mathbf{y}^I$ which represents the displacements of the nodes of the beam. The red (long dashed) line shows an unrealistic deformation of the cantilever beam when using $\mathbf{y}^I$ as the representation of the actual solution set $\mathbf{y}^s$. Formally, this deformation $\mathbf{y} \in \mathbf{y}^I$ and $\mathbf{y} \notin \mathbf{y}^s$. Especially when derivative results are required, conservatism is excessive, and the practical value of the results is limited.

Figure 1: An interval vector represents independent results for this beam with uncertain bending stiffness

Muhanna and Mullen [13] present an element-by-element approach that predicts sharp bounds on primary and derivative results [14]. However to achieve this, a much larger system of equations needs to be solved. The larger system of equations arises from the fact that all the elements are disassembled and for every degree of freedom belonging to more than one element an additional compatibility equation is needed.

The objective of this paper is to develop a formalism that intrinsically captures the output dependencies in an interval field that is much closer to the real, physical output set $\mathbf{y}^s$ than an interval vector representation. At the same time the formalism should maintain the capability to represent intervals on the response field. The theoretical basis for the interval field representation of field uncertainty is presented in [15]. Two general ways of solving the dependency problem are presented: explicit interval fields and implicit interval fields (see the remainder of this section for a summary). The extensions needed to make the interval field concept applicable to represent output fields of static finite element analysis are presented in the next section.

The quality of an interval field representation needs to be assessed on two levels: the accuracy and the conservatism. Firstly, for deterministic values of the uncertainty the interval field should be able to generate a deterministic realisation of the field sufficiently close to the exact deterministic realisation. Secondly, the interval field representation should correctly predict the bounds on all of the field components (local values) separately and most importantly it must not contain artefact realisations (inadvertent combinations of local values) of the field. This second level is referred to as the conservatism of the interval field.

2.2.1. Explicit interval field

To describe spatially dependent variation, numerical modelling approaches often use some type of shape functions. There are many examples of shape functions in FE methods: element shape functions interpolate the description of the geometry between nodes, or the field variable of the problem. Another example is an eigenmode that is used as basis vector in the modal superposition of the dynamic response of a structure. The actual solution is a linear combination of these shape functions.

Accordingly, the explicit interval field $\mathbf{y}_{exp}^F$ is defined as a superposition of n_b basis vectors $\boldsymbol{\psi}_i$ using interval factors η_i^I :

$$\mathbf{y}_{exp}^F = \sum_{i=1}^{n_b} \eta_{y,i}^I \boldsymbol{\psi}_{y,i} \quad (5)$$

The basis vectors represent a limited set of reference patterns, each of which is scaled by an independent interval factor. The components of the interval fields themselves are coupled through these basis vectors. Once the basis vectors are chosen, the definition of the interval field requires the specification of the interval factors that define the field on $\mathbf{y}$, which can be assembled in a classical (hypercubic) interval vector $\boldsymbol{\eta}_y^I$. In matrix notation, the interval field is denoted as:

$$\mathbf{y}_{exp}^F = [\boldsymbol{\psi}_y] \boldsymbol{\eta}_y^I \quad (6)$$

Note that the number of interval factors is limited to the number of basis vectors. This is an important reduction compared to the interval vector representation $\mathbf{y}^I$ with intervals for every component. However, the interval factors are still combined in a classical (hypercubic) interval vector $\boldsymbol{\eta}_y^I$, which cannot account for possible dependence between the interval factors.

Since expert knowledge about the modelled system dominates the definition of the uncertain parameters at the *input* side of an analysis, the freedom in choosing the basis vectors in the application of explicit interval fields is ideal to reflect this knowledge. For example, let the thickness of a rolled plate be uncertain due to the excentricity of the rolls. This uncertainty can be described by specifying a lower and upper bound for the thickness. However, such a description will not exclude discontinuous steps in the plate's thickness. A more informative description of the uncertainty is found by a combination of one basis vector and an interval factor. Figure 2 illustrates the sinusoidal (= basis vector) deviation of the thickness with uncertain amplitude (= interval factor).

The first step in the application of explicit interval fields at the *output* side of an FE analysis is in the selection of an appropriate set of basis vectors. The principle of describing an output field by a linear combination of basis vectors is based on Ritz vectors. Once these basis vectors are chosen, interval factors, which have to be predicted by the uncertain analysis run, are assigned to each of them. See sections 4.1 and 4.2 for further discussion. In section 4.3 an extension of this explicit interval field is presented. This extension eliminates the artificial independence of the interval factors which is introduced by combining them in the classical (hypercubic) interval vector $\boldsymbol{\eta}_y^I$.

With respect to post-processing of the explicit interval field result, the following procedure is suggested: let the function $\Pi()$ represent the post-processing procedure in a deterministic case, the final interval results after the post-processing phase $\mathbf{p}^I$ are calculated using an optimisation strategy, in which the interval factor hypercube $\boldsymbol{\eta}_y^I$ defines the search domain:

$$\underline{p}_i = \min_{\mathbf{y} \in \mathbf{y}_{exp}^F} \Pi_i(\mathbf{y}) = \min_{\boldsymbol{\eta}_y \in \boldsymbol{\eta}_y^I} \Pi_i([\boldsymbol{\psi}_y] \boldsymbol{\eta}_y), \quad (7)$$

Figure 2: Explicit interval field at input side of an analysis, example of thickness variation of a rolled plate. The plate is rolled from left to right, due to an excentricity of the rolls the thickness is uncertain and represented by an interval field. The figure shows different amplitudes (interval factor values) and the spatial coherence (basis vector) of the thickness uncertainty.

$$\bar{p}_i = \max_{\mathbf{y} \in \mathbf{y}_{exp}^F} \Pi_i(\mathbf{y}) = \max_{\boldsymbol{\eta}_y \in \boldsymbol{\eta}_y^I} \Pi_i([\boldsymbol{\psi}_y] \boldsymbol{\eta}_y) \quad (8)$$

The post-processing of an explicit interval field $\mathbf{y}_{exp}^F$ involving only spatial derivatives can be calculated easily. The spatial derivative of the basis vectors $[\nabla \boldsymbol{\psi}_y] = [\nabla \boldsymbol{\psi}_{y,1} \quad \nabla \boldsymbol{\psi}_{y,2} \quad \dots]$ needs to be taken and the same interval factors $\boldsymbol{\eta}_y^I$ can be used.

2.2.2. Implicit interval field

The implicit interval field is an alternative mathematical formalism to express dependency between vector components at the output side of an analysis. It is based on implicit functions $\phi_{y,i}(\boldsymbol{\eta}_y)$ of a set of predefined interval factors $\boldsymbol{\eta}_y$, describing the range of each individual component of the output vector:

$$\mathbf{y}_{imp}^F = \left\{ \left\{ \begin{array}{c} \phi_{y,1}(\boldsymbol{\eta}_y) \\ \vdots \\ \phi_{y,n}(\boldsymbol{\eta}_y) \end{array} \right\}, \boldsymbol{\eta}_y \in \boldsymbol{\eta}_y^I \right\} = \{ \phi_y(\boldsymbol{\eta}_y), \boldsymbol{\eta}_y \in \boldsymbol{\eta}_y^I \} \quad (9)$$

The implicit functions arise in a natural way from the deterministic solution of the FE problem, they are mappings from the interval factor space to the component value space for all components of the result vector. These functions provide the coupling between the components of the result vector through the common interval factors. The components of the interval output field are thus parametrised using the interval factors.

The main advantage of this approach is that it no longer assumes a linear relationship between the interval factors and the output field. Furthermore, the set of interval factors is chosen first, after which the implicit relationship with the output vector components is determined through the numerical analysis. As such, a set of independent interval factors can always be guaranteed. In many cases, the actual set of interval uncertain parameters present at the input side of the problem is a very good candidate set, as they often are known (or considered) to be independent. Other options for the choice of interval factors are a smaller set of the interval uncertain parameters at the input, for example by neglecting some parameters or (linearly or non-linearly) combining several input parameters. Section 4.4 describes the application of the implicit interval field to describe the output of a static FE analysis. Section 4.5 introduces an extension to the implicit interval field to increase its accuracy at the cost of increased conservatism.

Using the implicit interval field approach, post-processing of an output interval field is reformulated to an optimisation problem similar to the one derived for the explicit interval field in equations (7) and (8):

$$\underline{p}_i = \min_{\mathbf{y} \in \mathbf{y}_{imp}^F} \Pi_i(\mathbf{y}) = \min_{\boldsymbol{\eta}_y \in \boldsymbol{\eta}_y^I} \Pi_i(\phi_y(\boldsymbol{\eta}_y)), \quad (10)$$

$$\bar{p}_i = \max_{\mathbf{y} \in \mathbf{y}_{imp}^F} \Pi_i(\mathbf{y}) = \max_{\boldsymbol{\eta}_y \in \boldsymbol{\eta}_y^I} \Pi_i(\phi_y(\boldsymbol{\eta}_y)) \quad (11)$$

3. From the deterministic static FE procedure to the interval field FE procedure

Before describing in detail the different interval field implementations in the next section, this section gives a general overview of how the deterministic static FE procedure is adapted to deal with interval fields. First, an overview of the typical steps in using the deterministic static FE analysis are presented. Next, the changes made to these steps for uncertainty analysis with scalar interval uncertainty are presented. Finally, the adaptations to the deterministic steps needed to perform static FE analysis with interval fields are given in general terms.

3.1. Deterministic static FE analysis

The steps in a deterministic static FE analysis are:

1. Definition of the mesh
2. Definition of the elements' properties
3. Construction of the element stiffness matrices
4. Construction of the global stiffness matrix
5. Application of boundary conditions and loads with subsequent partitioning of the global stiffness matrix
6. Solving of the set of equations to find displacements and rotations followed by reaction forces
7. Post-processing to find secondary results e.g., stress

3.2. Static FE analysis with scalar interval uncertainty

The procedure to find bounds on FE output given scalar interval input uncertainty comes down to a maximisation and minimisation problem with the FE analysis at its core as described by Equations (3) and (4). The optimization procedure calls the FE analysis to get a deterministic output for sampled values of the scalar interval input uncertainty. Essentially this procedure requires the steps 2 through 7 to be carried out for every call of the optimisation procedure. The sampled value from the scalar interval input uncertainty requires the property in some elements (step 2) or the loads (step 5) to be changed to the sampled value. The rest of the steps are carried out again. Procedures to speed up this re-analysis can be found in [16]. These procedures take advantage of the fact that a large part of the global stiffness matrix is not modified when some local property is affected by uncertainty.

3.3. Static FE analysis with interval fields

To obtain interval field output from static FE analysis with uncertain (possibly interval field) input requires more adaptations to the deterministic procedure. The starting point is again interval uncertainty on the level of element properties or loads. The realisation of the input interval field determines a sample value for the uncertain property in all the elements or an altering of the loads. The influence on the rest of the FE-procedure depends on whether the explicit or implicit interval field implementation is adopted.

3.3.1. Explicit interval field

All the deterministic steps are kept till the partitioning (at the end of step 5). After the partitioning, a projection of the stiffness matrix and load vector on the basis vectors $[\psi_y]$ is carried out. The solution of this reduced set of equations gives a deterministic value for the interval factors η_y^I . The optimisation now looks to find the minimum and maximum value of these interval factors given the input uncertainty, by calling this adopted FE procedure. More details are given in sections 4.2 and 4.3.

3.3.2. Implicit interval field

The FE procedure remains exactly the same as in the case of scalar interval uncertainty. The important difference is found in keeping track of the functional relationship between the input uncertainty and the output interval field. When one has this relationship the realisation of the output interval field is uniquely defined by the sample values of the input uncertainty. In other words, the realisation of the output interval field is always consistent and does not combine different values from the input interval uncertainty. More details are given in sections 4.4 and 4.5.

4. Interval fields for static FE output

In this section the application of the explicit and implicit interval field concepts to static FE output is presented. A general static FE problem as constructed after step 5 of the procedure given in 3.1 is described by:

$$[\mathbf{K}]\mathbf{u} = \mathbf{q} \quad (12)$$

with $[\mathbf{K}]$ the stiffness matrix, $\mathbf{u}$ displacements in the non-constrained degrees of freedom (translations and rotations) and $\mathbf{q}$ the loads in the non-constrained degrees of freedom.

The uncertain model input parameters ($\mathbf{x} \in \mathbf{x}^I$) affect the stiffness matrix $[\mathbf{K}(\mathbf{x})]$ and/or the load vector $\mathbf{q}(\mathbf{x})$. This results in a set of displacement vectors $\mathbf{u}^s$. Instead of representing $\mathbf{u}^s$ by an interval vector $\mathbf{u}^I$, that forces a direct and independent influence of the uncertain parameters on each individual component, an interval field representation $\mathbf{u}^F$ is implemented to provide appropriate dependency between the displacement components. As such the three different representations of the uncertain displacements relate to each other in the following way:

$$\{\mathbf{u} \mid \mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}), \mathbf{x} \in \mathbf{x}^I\} = \mathbf{u}^s \subseteq \mathbf{u}^F \subseteq \mathbf{u}^I \quad (13)$$

This relation reflects the objective of the interval field description where $\mathbf{u}^F$ is very close to $\mathbf{u}^s$, while always remaining a conservative approximation. The interval field implementations require extensions of the general explicit and implicit interval field concept. In total four different implementations of the interval field concept are presented. The static reduction techniques provide an

important background for these implementations as the representative displacement patterns resulting from these techniques are used in all the interval field concepts. First a description of these displacement patterns is given. Next, a detailed definition of each of the interval field implementations and their properties is presented.

Academic example. The following academic example is used throughout the subsequent sections to illustrate how to obtain the interval field descriptions in practice and to assess their quality. A beam clamped at its left side and with an uncertain boundary condition on its right side is modelled using n nodes, $n_{dof} = 2n$ degrees of freedom and $n - 1$ elements (see figure 3). The $1m$ long beam has a circular cross-section with a diameter equal to $0.02m$ and a Young's modulus of $210GPa$. The beam is loaded in its mid-node with a vertical force of $1kN$. The boundary condition at the right end of the beam is uncertain and is modelled by an interval on the clamping stiffness $k = 10^x Nm/rad$ with $x \in [3 \ 5]$. The lower bound $1000Nm/rad$ is close to a simply supported boundary condition, and the upper bound $100000Nm/rad$ is close to a rigidly clamped condition.

The quality of an interval field representation will be evaluated on two levels. The first level consists in the assessment of the accuracy for a deterministic value of the uncertainty. The second level consists in checking if the interval field accurately captures the bounds on the components and if it avoids artefact realisations of the field. To measure the accuracy of the interval field representations, the error with respect to the exact deterministic solution is calculated for 17 equidistantly sampled values in the uncertain parameter interval x^I , $k = [10^3, 10^{3.125}, 10^{3.25}, \dots, 10^{4.875}, 10^5]$. For every sample (i.e. a deterministic value of the clamping stiffness k), the error is calculated using:

$$error = \frac{\sum_{i=1}^{n_{dof}} |u_i - \tilde{u}_i|}{n_{dof} u_{\frac{n_{dof}}{2}}} \quad (14)$$

with u_i the exact deterministic value of the i^{th} dof (this is a rotation or displacement), $\tilde{u}_i$ the value of the considered i^{th} dof in the interval field realisation:

$$\tilde{u}_i = [\psi_u]_i \boldsymbol{\eta}_u^* \quad (15)$$

for the explicit variant of the interval field, or

$$\tilde{u}_i = \phi_{u,i}(\boldsymbol{\eta}_u^*) \quad (16)$$

for the implicit variant of the interval field, with $\boldsymbol{\eta}_u^*$ the deterministic value of the interval factors obtained for the sampled value of the uncertain parameter x . To have a relative error measure (14) a division by $u_{\frac{n_{dof}}{2}}$, the exact deterministic value of the translation dof of the mid node for the considered deterministic value of the uncertain parameter, is done. To have a mean error measure for the whole structure, a summation over all the n_{dof} dofs is made and the sum is subsequently divided by n_{dof} . The level of conservatism associated with the different implementations is illustrated by plotting realisations of the different interval fields.

Figure 3: The dofs of the illustration beam with an uncertain clamping stiffness at the right side.

set-name	description	dofs for beam in figure 3
c-set	Known, constrained dofs	translation node 1 and n_{dof} , rotation node 1
l-set	Dofs that are loaded	translation node $\frac{n_{dof}}{2}$
o-set	Remaining dofs	Remaining dofs

Table 1: The three subsets of the degrees of freedom (dofs) for the illustration

4.1. Displacement patterns for static analysis

For static analysis, only a limited number of displacement patterns is commonly used. This section briefly presents two different displacement patterns that are used in the Component Mode Synthesis (CMS) method [17]: the ordinary and force displacement patterns.

Let the degrees of freedom (dofs) of a structure be divided in three sets. The c-set contains all the dofs of which the value is known. The l-set contains the dofs that are loaded. The remaining dofs are members of the o-set. Table 1 shows the division of the dofs of the beam into the three sets.

The division of the dofs into the three sets allows for the partitioning of equation (12) for the non-constrained dofs:

$$\begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{lo} \\ \mathbf{K}_{ol} & \mathbf{K}_{oo} \end{bmatrix} \begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_o \end{bmatrix} = \begin{bmatrix} \mathbf{q}_l \\ \mathbf{q}_o \end{bmatrix} \quad (17)$$

with a notation like $[\mathbf{K}_{ol}]$ to describe the part of $[\mathbf{K}]$ that shows the influence of the dofs from the l-set on the forces of the dofs from the o-set.

The *ordinary displacement patterns* $[\chi^C]$ are calculated one by one by forcing a single dof from the l-set to its unit value, while the other dofs in the l-set are fixed at zero. The calculation is done by

$$[\chi^C] = \begin{bmatrix} \mathbf{I}_{ll} \\ \chi_{ol}^C \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{ll} \\ -\mathbf{K}_{oo}^{-1} \mathbf{K}_{ol} \end{bmatrix} \quad (18)$$

Figure 4: Beam transverse displacement pattern χ^C for unit imposed displacement at the middle node, for 3 realisations (with minimum, nominal and maximum value of the end rotational stiffness)

Figure 4 shows the ordinary displacement pattern calculated using equation (18) and evaluated for the minimum, nominal ($\mathbf{x} = \mathbf{x}_{nom} = \frac{\mathbf{x}_{max} + \mathbf{x}_{min}}{2}$) and maximum value of the uncertain clamping stiffness.

The set of *force displacement patterns* $[\chi^A]$ is calculated by successively setting the load in every dof of the l-set to the unit value, while the other dofs in the l-set remain unloaded

$$[\chi^A] = \begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{lo} \\ \mathbf{K}_{ol} & \mathbf{K}_{oo} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I}_{ll} \\ \mathbf{0} \end{bmatrix} \quad (19)$$

Figure 5 shows the force displacement pattern calculated using equation (19) and evaluated for the minimum, nominal and maximum value of the uncertain clamping stiffness.

The only difference between the two types of patterns in linear elasticity is the scaling, since generally forcing a dof to the unit value does not result in the same magnitude of displacements as forcing the corresponding force to the unit value. This difference in magnitude is linked to the distinctive behaviour of these displacement patterns with respect to an uncertainty on a stiffness property. I.e., a uniformly stiffer structure (for example having a higher Young's modulus) has the same magnitude of displacements for its ordinary displacement patterns. Its force displacement patterns however have a lower magnitude with increasing uniform stiffness.

Figure 5: Beam transverse displacement pattern χ^A for unit imposed force at the middle node, for 3 realisations (with minimum, nominal and maximum value of the end rotational stiffness)

4.2. The explicit interval field for static FE output

Definition. The explicit interval field $\mathbf{u}_{exp}^F$ is constructed based on the explicit interval field concept as described in section 2.2.1. The uncertain displacements are described by an approximation based on fixed basis vectors and linear scaling interval factors

$$\mathbf{u}^s = \{ \mathbf{u} \mid (\mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x})), (\mathbf{x} \in \mathbf{x}^I) \} \simeq \mathbf{u}_{exp}^F = \{ [\psi_u] \boldsymbol{\eta}_u \mid \boldsymbol{\eta}_u \in \boldsymbol{\eta}_u^I \} \quad (20)$$

with

- $\mathbf{u}^s$ the exact set of displacement vectors $\mathbf{u}$
- $\mathbf{u}$ a deterministic realisation of the displacements
- $\mathbf{K}(\mathbf{x})$ the uncertain stiffness matrix
- $\mathbf{q}(\mathbf{x})$ the uncertain load vector
- $\mathbf{x}$ a deterministic realisation of the vector of uncertain model input parameters
- $\mathbf{x}^I$ the interval vector of uncertain model input parameters
- $\mathbf{u}_{exp}^F$ the explicit interval field representation
- ψ_u a basis vector
- $\boldsymbol{\eta}_u$ a deterministic realisation of the vector of interval factors
- $\boldsymbol{\eta}_u^I$ the vector of interval factors

Properties. Each basis vector $[\psi_u] = [\psi_{u,1} \ \psi_{u,2} \ \dots]$ represents a fixed dependency between the components of the displacement field. The displacement patterns $[\chi^C]$ or $[\chi^A]$ from section 4.1 are taken as basis vectors.

Calculation of unknowns. The unknowns in the explicit interval field are the bounds on the interval factors $\boldsymbol{\eta}_u^I$. Given the uncertain model input parameters $\mathbf{x}^I$, the bounds on $\boldsymbol{\eta}_u^I$ need to be calculated. The calculation of a deterministic value for $\boldsymbol{\eta}_u$ is done by substituting

$$\mathbf{u} = [\psi_u] \boldsymbol{\eta}_u \quad (21)$$

in

$$\mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}) \quad (22)$$

resulting in

$$[\mathbf{K}(\mathbf{x})][\psi_u] \boldsymbol{\eta}_u = \mathbf{q}(\mathbf{x}) \quad (23)$$

Next, both sides are premultiplied by $[\psi_u]^T$, so $\boldsymbol{\eta}_u$ can be calculated as

$$\boldsymbol{\eta}_u = [[\psi_u]^T [\mathbf{K}(\mathbf{x})] [\psi_u]]^{-1} [\psi_u]^T \mathbf{q}(\mathbf{x}) \quad (24)$$

The calculation of the bounds on $\boldsymbol{\eta}_u$ is done using equations (3) and (4) with:

$$\mathbf{f}(\mathbf{x}) = [[\boldsymbol{\psi}_u]^T [\mathbf{K}(\mathbf{x})] [\boldsymbol{\psi}_u]]^{-1} [\boldsymbol{\psi}_u]^T \mathbf{q}(\mathbf{x}) \quad (25)$$

the relation between the model input parameters $\mathbf{x}$ and a deterministic value for the interval factors $\boldsymbol{\eta}_u$. The bounds on $\boldsymbol{\eta}_u$ result from

$$\underline{\eta}_{u,i} = \min_{\mathbf{x} \in \mathbf{x}^I} f_i(\mathbf{x}), \quad (26)$$

$$\bar{\eta}_{u,i} = \max_{\mathbf{x} \in \mathbf{x}^I} f_i(\mathbf{x}), \quad (27)$$

Comments. The accuracy of this explicit interval field is determined both by the quantity and the quality of the set of basis vectors. If only one basis vector is used, then a displacement pattern calculated from the nominal configuration of the structure with each input uncertainty set at its nominal value is the obvious choice. The linear scaling of this single basis vector is only an exact representation of the uncertain displacements for two specific cases: the uncertainty influences the stiffness in a uniform manner or it scales the loading on the structure linearly. For other cases the single basis vector does not capture the influence of the uncertainty on the displacements well. Augmenting the set of basis vectors with displacement patterns calculated for additional values of the input uncertainties increases the accuracy of the interval field for deterministic realisations of the input uncertainties. Since the basis on which the actual static FE problem is projected is larger, more means to match the exact deterministic realisations are available.

On the other hand, when considering the full range of the uncertain model input parameters, the values taken by the interval factors $\boldsymbol{\eta}_u$ are not independent of each other. Yet they are represented with an interval vector $\boldsymbol{\eta}_u^I$, since their bounds are calculated one by one using equations (26) and (27). This results in a loss of sharpness on the bounds of the actual displacements in every component and/or the introduction of artefact realisations of the displacements. This problem will be addressed in the extended version of the explicit interval field in the next subsection.

A priori assessment of the number of basis vectors and corresponding interval factors needed, is impossible in the general case. The standard procedure used in numerical methods involving a reduced representation is to perform a convergence study. In such a study the basis vector set is gradually augmented with additional basis vectors and the change in the output is monitored. If this change drops below a preset threshold the result is considered to have converged. With the above described interval field implementation a convergence test can be easily performed.

Academic illustration. Several explicit interval field implementations are applied to the beam problem. The error of every implementation is calculated as in (14) and the conservatism is discussed.

Let a *single basis vector* be equal to the ordinary displacement pattern calculated for the nominal value of the clamping stiffness (cfr. the green solid curve

Figure 6: The error for the explicit interval field with the nominal ordinary or force displacement pattern as a basis vector.

in figure 4)

$$[\psi_{u,1}] = [\chi_{nom}^C] \quad (28)$$

The bounds on the interval factor are determined by the minimum and maximum values of equation (25):

$$\eta_u^I = [0.0018 \ 0.0047] \quad (29)$$

If the basis vector is chosen equal to the nominal value of the force displacement pattern (cfr. the green solid curve in figure 5),

$$[\psi_{u,1}] = [\chi_{nom}^A] \quad (30)$$

then following bounds are found for the interval factor:

$$\eta_u^I = [445.37 \ 1142.2] \quad (31)$$

Both interval field representations are identical with respect to the error and conservatism measure, since both nominal basis vectors are equal apart from a linear scaling factor. Figure 6 shows the error for all the samples as a function of the clamping stiffness. Figure 7 shows the bounds on the displacement of the beam for the exact set $\mathbf{u}^s$ and this interval field representation u_{exp}^F . It is clear that this explicit interval field with only a nominal displacement pattern is not an accurate representation of the true uncertain displacement of this beam.

Figure 7: The bounds on the displacement of the beam for the exact set and the explicit interval field with only the nominal ordinary or force displacement pattern as a basis vector.

Now *two basis vectors* are chosen: the ordinary displacement pattern calculated for the minimal and maximal value of the clamping stiffness (cfr. the dotted red and dashed blue line in figure 4):

$$[\psi_{u,1} \quad \psi_{u,2}] = [\chi_{min}^C \quad \chi_{max}^C] \quad (32)$$

The bounds on both interval factors are determined by looking for the minimum and maximum value of each of them independently in the solution of equation (25) over all the 17 considered realisations:

$$\boldsymbol{\eta}_u^I = \begin{bmatrix} [0. 0.0052] \\ [0. 0.0033] \end{bmatrix} \quad (33)$$

If the basis vectors are chosen equal to the minimal and maximal evaluation of the force displacement patterns (cfr. the dotted red and dashed blue line in figure 5),

$$[\psi_{u,1} \quad \psi_{u,2}] = [\chi_{min}^A \quad \chi_{max}^A] \quad (34)$$

then following bounds are found for the interval factors:

$$\boldsymbol{\eta}_u^I = \begin{bmatrix} [0. 1000] \\ [0. 1000] \end{bmatrix} \quad (35)$$

Since the basis vectors for the interval fields based on the ordinary displacement patterns and the force displacement patterns only differ in scaling, their error

Figure 8: The bounds on the displacement of the beam for the exact case and the explicit interval field with the minimal and maximal ordinary or force displacement patterns as basis vectors.

and conservatism measures coincide. The error for all the deterministic samples is equal to zero, since a linear combination of the two displacement patterns can match the exact solution for every realisation of the clamping stiffness. Whatever value the clamping stiffness takes, the ratio between the dofs of the o-set remains the same, only the ratio of the displacement of the mid node with respect to the dofs in the o-set changes and as such only two basis vectors are needed to capture the latter ratio change. The conservatism is high, since the bounds on the two interval factors can be combined in a hypercubic way. This leads to large overestimation of the bounds on each dof (i.e., the realisation $[00]^T$ from equation (35) gives the dashed lower bound in figure 8) and to artefact realisations of the interval field. The red dotted line in figure 8 shows such an artefact realisation of the field that respects the exact bounds on all of the dofs, but still is not a possible displacement of this beam.

4.3. Extension of the explicit interval field for static FE output

Definition. In the extended explicit interval field $\mathbf{u}_{exp*}^F$ the uncertain displacements are described by an approximation based on fixed basis vectors and functions of the input uncertainty $\mathbf{x}^I$ for the interval factors

$$\mathbf{u}^s = \{ \mathbf{u} \mid (\mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x})), (\mathbf{x} \in \mathbf{x}^I) \} \simeq \mathbf{u}_{exp*}^F = \{ [\psi_u] \phi_\eta(\mathbf{x}) \mid \mathbf{x} \in \mathbf{x}^I \} \quad (36)$$

with

- $\mathbf{u}^s$ the exact set of displacement vectors $\mathbf{u}$

- $\mathbf{u}$ a deterministic realisation of the displacements
- $\mathbf{K}(\mathbf{x})$ the uncertain stiffness matrix
- $\mathbf{q}(\mathbf{x})$ the uncertain load vector
- $\mathbf{x}$ a deterministic realisation of the vector of uncertain model input parameters
- $\mathbf{x}^I$ the interval vector of uncertain model input parameters
- $\mathbf{u}_{exp*}^F$ the extended explicit interval field representation
- ψ_u a basis vector
- $\phi_\eta(\mathbf{x})$ a deterministic realisation of the vector of interval factors as a function of deterministic values for the uncertain model input parameters

Properties. Each basis vector $[\psi_u] = [\psi_{u,1} \ \psi_{u,2} \ \dots]$ represents a fixed dependency between the components of the displacement field. The displacement patterns $[\chi^C]$ or $[\chi^A]$ from section 4.1 are taken as basis vectors.

Calculation of unknowns. The unknowns in $\mathbf{u}_{exp*}^F$ are the functions $\phi_\eta(\mathbf{x})$. Given the uncertain model input parameters $\mathbf{x}^I$, the functions $\phi_\eta(\mathbf{x})$ need to be determined. Let us substitute

$$\mathbf{u} = [\psi_u] \phi_\eta(\mathbf{x}) \quad (37)$$

in

$$\mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}) \quad (38)$$

resulting in

$$[\mathbf{K}(\mathbf{x})][\psi_u] \phi_\eta(\mathbf{x}) = \mathbf{q}(\mathbf{x}) \quad (39)$$

Next, both sides are premultiplied by $[\psi_u]^T$, so $\phi_\eta(\mathbf{x})$ can be calculated as

$$\phi_\eta(\mathbf{x}) = [[\psi_u]^T [\mathbf{K}(\mathbf{x})] [\psi_u]]^{-1} [\psi_u]^T \mathbf{q}(\mathbf{x}) \quad (40)$$

with $\mathbf{x} \in \mathbf{x}^I$.

To have a computationally less intensive expression, approximations $\tilde{\phi}_\eta(\mathbf{x})$ over the uncertain model input parameter space are used. In practice, first some regression function on the model input parameters $\mathbf{x}^I$ is postulated, next the coefficients in the regression function are determined with a set of exact evaluations of equation (40) and as such the approximation $\tilde{\phi}_\eta(\mathbf{x})$ is built.

Comments. The extended explicit interval field eliminates the interval vector $\boldsymbol{\eta}_u^I$ in the representation of the uncertain displacement field. Instead a direct, but approximative, link to the input uncertainties $\mathbf{x}^I$ is established. The accuracy for deterministic sample values of the input uncertainties is determined by the size and quality of the basis vector set as well as the quality of the approximation $\tilde{\phi}_\eta(\mathbf{x})$ over the input uncertainty space $\mathbf{x}^I$. The latter can be improved by using more elaborated regression functions and a higher number of exact evaluations of equation (40). The computation cost for such evaluations is rather limited since the stiffness matrix and/or load vector only need to be rebuilt and their projected versions solved. The conservatism of the extended explicit interval field is greatly reduced thanks to the direct link with the input uncertainties. The hypercubic combination of the basis vectors is avoided and as such the number of artefact realisations allowed by the explicit interval field is reduced to zero.

Academic example. Let the fixed basis vectors be chosen equal to the ordinary displacement pattern evaluated for the minimal and maximal value of the uncertain clamping stiffness:

$$[\psi_{u,1} \quad \psi_{u,2}] = [\chi_{min}^C \quad \chi_{max}^C] \quad (41)$$

A linear regression function for each of the two components of $\phi_\eta(\mathbf{x})$ is postulated and the exact evaluation of equation (40) for the minimal and maximal value of the clamping stiffness is used to determine the coefficients. This results in:

$$\tilde{\phi}_\eta(\mathbf{x}) = \begin{bmatrix} 0.0026 - 0.0104\Delta x \\ 0.0017 + 0.0066\Delta x \end{bmatrix} \quad (42)$$

with $\Delta x = (x - x_{nom})/x_{nom}$. Note that this results in the same interval factors as in eq. (33), but now with a coupling between both interval factors through the single uncertain model input parameter: x . The black solid line in figure 9 shows the error measure over the sample values of the clamping stiffness uncertainty. If a quadratic regression function for each of the two interval factors is postulated and the exact evaluation of equation (25) for the nominal value of the clamping stiffness is added to the two exact evaluations already used, then following parameters are found:

$$\tilde{\phi}_\eta(\mathbf{x}) = \begin{bmatrix} 0.0022 - 0.0104\Delta x + 0.007(\Delta x)^2 \\ 0.0019 + 0.0066\Delta x - 0.0044(\Delta x)^2 \end{bmatrix} \quad (43)$$

The green dashed line in figure 9 shows the error measure for this regression function of the interval factors. The linear regression function as defined in equation (42) appears to perform better for lower values of the clamping stiffness. A closer look at for example the mid node displacement (figure 10) shows its behaviour through the uncertain clamping stiffness domain for the exact case (cyan solid line) and for the extended explicit interval field with the ordinary minimal and maximal displacement patterns as basis vectors and linear (black

Figure 9: The error for the extended explicit interval field with the ordinary minimal and maximal displacement patterns as basis vectors and linear (black solid line) or quadratic (green dashed line) functions for the corresponding interval factors.

solid line with x marks) or quadratic (green dashed line with o marks) functions for the corresponding interval factors. It is clear that the interval field with the linear functions for the interval factors captures not only the intended minimum and maximum value correctly, but also the displacement for a clamping stiffness of 3150 Nm/rad. It is very close to the exact displacement for values of the clamping stiffness in this region. The interval field with the quadratic functions for the interval factors captures exactly the minimum, nominal and maximum value of the displacement as required. It approximates the displacement for lower values of the clamping stiffness worse but does a better job for the higher values when compared to the linear regression function. The bounds on all the individual components of the displacement field are exactly satisfied since the extreme realisations of the field are used to determine the functions $\tilde{\phi}_\eta(\mathbf{x})$. No artefact realisations can be introduced because of the coupling of the interval factors through the functions $\tilde{\phi}_\eta(\mathbf{x})$. As such the conservatism is eliminated.

4.4. The implicit interval field for static FE output

Definition. The implicit interval field $\mathbf{u}_{imp}^F$ is constructed based on the implicit interval field concept as described in 2.2.2. The uncertain displacements are described by implicit functions of the uncertain model input parameters

$$\mathbf{u}^s = \{\mathbf{u} \mid \mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}), \mathbf{x} \in \mathbf{x}^I\} \simeq \mathbf{u}_{imp}^F = \{\phi_u(\mathbf{x}) \mid \mathbf{x} \in \mathbf{x}^I\} \quad (44)$$

with

Figure 10: The value of the mid node displacement for the exact case (cyan solid line) and the extended explicit interval field with the ordinary minimal and maximal displacement patterns as basis vectors and linear (black solid line with x marks) or quadratic (green dashed line with o marks) functions for the corresponding interval factors.

- $\mathbf{u}^s$ the exact set of displacement vectors $\mathbf{u}$
- $\mathbf{u}$ a deterministic realisation of the displacements
- $\mathbf{K}(\mathbf{x})$ the uncertain stiffness matrix
- $\mathbf{q}(\mathbf{x})$ the uncertain load vector
- $\mathbf{x}$ a deterministic realisation of the vector of uncertain model input parameters
- $\mathbf{x}^I$ the interval vector of uncertain model input parameters
- $\mathbf{u}_{imp}^F$ the implicit interval field representation
- $\phi_u(\mathbf{x})$ a deterministic realisation of the implicit functions describing the mapping from the uncertain model input parameters onto the displacements

Properties. The spatial dependency is captured through the implicit functions (one for every component of the field) which all have the same combination of input parameters. Referring to equation (9), the $\boldsymbol{\eta}_y^I$ are taken equal to the uncertain model input parameters $\mathbf{x}^I$.

Calculation of unknowns. The unknowns in $\mathbf{u}_{imp}^F$ are the functions $\phi_u(\mathbf{x})$. Let us substitute

$$\mathbf{u} = \phi_u(\mathbf{x}) \quad (45)$$

in

$$\mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}) \quad (46)$$

resulting in

$$\mathbf{K}(\mathbf{x})\phi_u(\mathbf{x}) = \mathbf{q}(\mathbf{x}) \quad (47)$$

then the implicit functions $\phi_u(\mathbf{x})$ can be found as

$$\phi_u(\mathbf{x}) = [\mathbf{K}(\mathbf{x})]^{-1}\mathbf{q}(\mathbf{x}) \quad (48)$$

Naturally, this solution of the static FE problem for any deterministic value of the uncertain model input parameters yields the actual $\phi_u(\mathbf{x})$.

To have a computationally less intensive expression, approximations $\tilde{\phi}_u(\mathbf{x})$ are obtained by using response surface methods (RSM). This allows a fast calculation of the output field while keeping the dependencies between the components over the full range of the uncertain parameters. Different types of response surfaces [18, 19] are possible, for a state of the art overview, see [20].

Comments. The accuracy of the implicit interval field is determined by the quality of the chosen RSM and the number of exact evaluations used. The conservatism of the implicit interval field with respect to the bounds on the components, depends on the quality of the response surface in the vicinity of the extrema of each component. The conservatism of the implicit interval field with respect to artefact realisations is avoided, since the dependencies between all the components in the interval field are preserved through the implicit functions. The implicit interval field has the advantage of allowing displacement patterns that are adapted to actual changes in the space of uncertain parameters. When the effect of the uncertainty is hard to predict, such a property may be advantageous over the explicit interval field concept that uses displacement patterns that are fixed beforehand.

Academic example. The use of the implicit interval field requires the construction of a response surface. Three exact evaluations (for the minimal, nominal and maximal value of the clamping stiffness) of the beam problem are used to construct a quadratic approximation

$$\phi(\mathbf{x}) \doteq \tilde{\phi}(\mathbf{x}) = \phi_{nom} + \phi_{linear} \Delta x + \phi_{quadratic} \Delta x^2 \quad (49)$$

with $\Delta x = (x - x_{nom})/x_{nom}$ and ϕ_{nom} , ϕ_{linear} and $\phi_{quadratic}$ the coefficients of the quadratic approximation calculated from the exact evaluations $\phi(\mathbf{x}_{min})$, $\phi(\mathbf{x}_{nom})$ and $\phi(\mathbf{x}_{max})$. This representation coincides with the above example of the extended explicit interval field where a quadratic approximation for the interval factors is used. The error is thus also shown in figure 9 by the green dashed line. Naturally, the error is zero for the exact evaluations. It reaches maxima of around 5% in between those exact evaluations. There is no conservatism in this interval field representation with respect to the bounds on its components. This is a result of the fact that the exact evaluations $\phi(\mathbf{x}_{min})$, $\phi(\mathbf{x}_{nom})$ and $\phi(\mathbf{x}_{max})$, used to construct the quadratic approximation (49) represent the actual bounds on the displacement for this particular problem.

4.5. Extension of the implicit interval field for static FE output

Definition. In the extended implicit interval field $\mathbf{u}_{imp*}^F$ the uncertain displacements are described by an approximation based on adaptive basis vectors and linear scaling interval factors

$$\mathbf{u}^s = \{\mathbf{u} \mid \mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}), \mathbf{x} \in \mathbf{x}^I\} \doteq \mathbf{u}_{imp*}^F = \{[\phi_u(\mathbf{x})] \boldsymbol{\eta}_u \mid \mathbf{x} \in \mathbf{x}^I, \boldsymbol{\eta}_u \in \boldsymbol{\eta}_u^I\} \quad (50)$$

with

- $\mathbf{u}^s$ the exact set of displacement vectors $\mathbf{u}$
- $\mathbf{u}$ a deterministic realisation of the displacements
- $\mathbf{K}(\mathbf{x})$ the uncertain stiffness matrix
- $\mathbf{q}(\mathbf{x})$ the uncertain load vector

- $\mathbf{x}$ a deterministic realisation of the vector of uncertain model input parameters
- $\mathbf{x}^I$ the interval vector of uncertain model input parameters
- $\mathbf{u}_{imp}^F$ the extended implicit interval field representation
- $\phi_u(\mathbf{x})$ an adaptive basis vector which is function of $\mathbf{x}$
- $\boldsymbol{\eta}_u$ a deterministic realisation of the vector of interval factors
- $\boldsymbol{\eta}_u^I$ the vector of interval factors

Properties. The spatial dependency is captured through the adaptive $\phi_u(\mathbf{x})$ which represent changing basis vectors through the input uncertainty space.

Calculation of unknowns. First, the adaptive basis vectors $\phi_u(\mathbf{x})$ need to be calculated. They result from equation (18) (ordinary displacement patterns) or equation (19) (force displacement patterns) evaluated for the considered value of the uncertain model input parameters $\mathbf{x}$.

To have a computationally less intensive expression for the adaptive basis vectors, approximations $\tilde{\phi}_u(\mathbf{x})$ are obtained by using response surface methods (RSM). In practice, first some regression functions on the model input parameters $\mathbf{x}^I$ are postulated, next the coefficients in the regression function are determined with a set of exact evaluations of equation (18) (ordinary displacement patterns) or equation (19) (force displacement patterns).

The other unknowns are the bounds on $\boldsymbol{\eta}_u^I$. They are calculated by substituting

$$\mathbf{u} = [\tilde{\phi}_u(\mathbf{x})]\boldsymbol{\eta}_u^I \quad (51)$$

in

$$\mathbf{K}(\mathbf{x})\mathbf{u} = \mathbf{q}(\mathbf{x}) \quad (52)$$

resulting in

$$\mathbf{K}(\mathbf{x})[\tilde{\phi}_u(\mathbf{x})]\boldsymbol{\eta}_u^I = \mathbf{q}(\mathbf{x}) \quad (53)$$

Next, both sides are premultiplied by $[\tilde{\phi}_u(\mathbf{x})]^T$, so $\boldsymbol{\eta}_u$ can be calculated as

$$\boldsymbol{\eta}_u = \left[[\tilde{\phi}_u(\mathbf{x})]^T [\mathbf{K}(\mathbf{x})] [\tilde{\phi}_u(\mathbf{x})] \right]^{-1} [\tilde{\phi}_u(\mathbf{x})]^T \mathbf{q}(\mathbf{x}) \quad (54)$$

The calculation of the bounds on $\boldsymbol{\eta}_u$ is done using equations (3) and (4) with:

$$\mathbf{f}(\mathbf{x}) = \left[[\tilde{\phi}_u(\mathbf{x})]^T [\mathbf{K}(\mathbf{x})] [\tilde{\phi}_u(\mathbf{x})] \right]^{-1} [\tilde{\phi}_u(\mathbf{x})]^T \mathbf{q}(\mathbf{x}) \quad (55)$$

the relation between a deterministic value for the model input parameters $\mathbf{x}$ and a deterministic value for the interval factors $\boldsymbol{\eta}_u$. The bounds on $\boldsymbol{\eta}_u$ result from

$$\underline{\eta}_{u,i} = \min_{\mathbf{x} \in \mathbf{x}^I} f_i(\mathbf{x}), \quad (56)$$

$$\bar{\eta}_{u,i} = \max_{\mathbf{x} \in \mathbf{x}^I} f_i(\mathbf{x}), \quad (57)$$

Comments. A high accuracy with respect to deterministic realisations of the field can be achieved because the basis vectors are changing with the input uncertainty and the scaling interval factors can adjust for mismatches between the approximation for the basis vectors and the exact realisation of the displacement field. With respect to the conservatism, the coupling between changing basis vectors $\tilde{\phi}_u(\mathbf{x})$ and interval factors η_u^I is not captured in this representation. This fact allows artefact realisations of the field through combination of the interval factors η_u^I and any shape of the basis vector $\tilde{\phi}_u(\mathbf{x})$ over the input uncertainty space.

Academic example. The extended implicit interval field requires the construction of adaptive basis vectors through the uncertainty space and the calculation of the bounds on the interval factors. For the adaptive basis vectors and their approximation two options are presented: one based on the ordinary displacement patterns and one based on the force displacement patterns. A *single adaptive basis vector* approximation is constructed using the *ordinary* displacement patterns $\chi_{nom}^C, \chi_{min}^C$ and χ_{max}^C :

$$\left[\tilde{\phi}(x)\right] = \left[\chi_{nom}^C + \phi_{linear}\Delta x + \phi_{quadratic}\Delta x^2\right] \quad (58)$$

with $\Delta x = (x - x_{nom})/x_{nom}$ and ϕ_{linear} and $\phi_{quadratic}$, the coefficients in the approximation which are determined based on the exact evaluations χ_{min}^C and χ_{max}^C . This adaptive basis vector results in an interval factor:

$$\eta_u^I = [0.0033 \ 0.0052] \quad (59)$$

The black solid line in figure 11 shows the error. It reaches maxima around 4% in between the exact evaluations of the basis vectors. In figure 12 two possible realisations of the interval field obtained by hypercubic combinations of the basis vector and the interval factor are plotted on a background of the exact bounds on the displacements. The realisations illustrate that local bounds on the displacement are not satisfied and artefact realisations occur in this extended version of the implicit interval field.

A *single adaptive basis vector* approximation is constructed using the *force* displacement patterns $\chi_{nom}^A, \chi_{min}^A$ and χ_{max}^A :

$$\left[\tilde{\phi}(x)\right] = \left[\chi_{nom}^A + \phi_{linear}\Delta x + \phi_{quadratic}\Delta x^2\right] \quad (60)$$

with $\Delta x = (x - x_{nom})/x_{nom}$ and ϕ_{linear} and $\phi_{quadratic}$, the coefficients in the approximation which are determined based on the exact evaluations χ_{min}^A and χ_{max}^A . This adaptive basis vector results in an interval factor:

$$\eta_u^I = [969 \ 1029] \quad (61)$$

The error is shown in figure 11 by the green dashed line and is slightly below the error of the extended implicit interval field constructed using the ordinary displacement patterns. The problem of conservatism is significantly reduced as

Figure 11: The error for the extended implicit interval field with a quadratic approximation based on the ordinary and force displacement patterns.

Figure 12: The bounds on the displacement of the beam for the exact case and two realisations of the extended implicit interval field representation with a quadratic approximation of the basis vector based on the ordinary displacement patterns.

Figure 13: The bounds on the displacement of the beam for the exact case and two realisations of the extended implicit interval field representation with a quadratic approximation of the basis vector based on the force displacement patterns.

shown in figure 13, where the hypercubic combination of the basis vector and the interval factor is plotted against a background of the exact bounds on the displacement. The bounds on the local values are overestimated but artefact realisations are not present in this interval field. The latter is a result of the nature of the force displacement patterns, where a unit force is imposed for all the exact evaluations of the clamping stiffness. In this way, only a slight scaling mismatch can occur, but not the artefact realisations seen in figure 12.

4.6. Post-processing

The post-processing of the interval fields is carried out based on equations (7), (8) and (10), (11) for the explicit and implicit implementations respectively. The quality of the interval field post-processing depends on both the accuracy and conservatism of this primary interval field. It is clear that both calculating correct bounds and excluding artefact realisations in the displacement interval field are important to compute correct bounds on for example stress levels.

5. Discussion and comparison

In this section the four interval field implementations are reviewed and their accuracy and conservatism properties are compared. The *accuracy* of an interval field implementation is determined by its ability to represent a deterministic realisation accurately. The *conservatism* of an interval field is determined on two levels. The first level deals with the ability of the interval field to represent

the exact bounds on all its components individually. The second level deals with the ability to exclude artefact realisations, i.e. realisations of the field that do not occur in the exact set $\mathbf{u}^s$, within these bounds.

The four interval field implementations are all approximations of the exact set $\mathbf{u}^s$:

$$\mathbf{u}_{exp}^F = \{[\psi_u] \boldsymbol{\eta}_u \mid \boldsymbol{\eta}_u \in \boldsymbol{\eta}_u^I\} \quad (62)$$

$$\mathbf{u}_{exp*}^F = \{[\psi_u] \phi_\eta(\mathbf{x}) \mid \mathbf{x} \in \mathbf{x}^I\} \quad (63)$$

$$\mathbf{u}_{imp}^F = \{\phi_u(\mathbf{x}) \mid \mathbf{x} \in \mathbf{x}^I\} \quad (64)$$

$$\mathbf{u}_{imp*}^F = \{[\phi_u(\mathbf{x})] \boldsymbol{\eta}_u \mid \mathbf{x} \in \mathbf{x}^I, \boldsymbol{\eta}_u \in \boldsymbol{\eta}_u^I\} \quad (65)$$

The *explicit interval field* implementations $\mathbf{u}_{exp}^F$ and $\mathbf{u}_{exp*}^F$ adopt fixed basis vectors and interval factors to approximate the set $\mathbf{u}^s$. The accuracy of these explicit interval fields is determined by the quantity and quality of the basis vectors. A higher number of basis vectors increases the dimension of the vector space on which the uncertain static FE problem is projected in equation (24) and equation (40) and as such allows for increased accuracy. The similarity between basis vectors and possible realisations of the uncertain displacement determines the quality of the basis vectors. The conservatism with respect to the exact component bounds of the explicit interval field $\mathbf{u}_{exp}^F$ and the extended explicit interval field $\mathbf{u}_{exp*}^F$ is equal, provided in both cases the extreme realisations are part of the set of basis vectors. The conservatism with respect to artefact realisations of $\mathbf{u}_{exp}^F$ and $\mathbf{u}_{exp*}^F$ is drastically different. In $\mathbf{u}_{exp}^F$ the interval factors are represented by the interval vector $\boldsymbol{\eta}_u^I$. This interval vector cannot capture the dependencies between the different interval factors and as such allows artefact realisations by inadvertent combinations of deterministic values from these interval factors. In $\mathbf{u}_{exp*}^F$ on the other hand, the interval factors are actually functions $\phi_\eta(\mathbf{x})$ which depend on the same set of uncertain model input parameters, thus providing a coupling between the multiple interval factors and excluding artefact realisations.

The *implicit interval field* implementation $\mathbf{u}_{imp}^F$ adopts implicit functions of the uncertain model input parameters to describe the uncertain displacements. The most straightforward implicit functions are found as the solution of the uncertain static FE problem itself in equation (48). A computationally less intensive expression is found using approximations $\tilde{\phi}_u(\mathbf{x})$ based on RSM. The quality of these response surfaces determines directly the accuracy of the interval field. The conservatism of $\mathbf{u}_{imp}^F$ with respect to the bounds on the components, depends on the quality of the response surface in the vicinity of the extrema of each component. The conservatism with respect to artefact realisations is avoided, since the dependencies between all the components is preserved through the functions $\tilde{\phi}_u(\mathbf{x})$.

The *extended implicit interval field* implementation $\mathbf{u}_{imp*}^F$ is constructed with adaptive basis vectors and interval factors. The accuracy of this implementation depends on the quantity and quality of the adaptive basis vectors. In general the adaptive nature of the basis vectors together with the interval

Notation	Min [mm]	Nominal [mm]	Max [mm]
t_1	3	4	5
t_2	4	5	6
t_3	4	6	8
t_4	4	6	8
t_5	4	7	10

Table 2: The uncertainties on the thicknesses

factors allows for an accurate representation of deterministic realisations of the field. The problem of artefact realisations is present since deterministic interval factor values can be combined without taking into account their dependencies.

To summarize, high accuracy can be achieved with $\mathbf{u}_{exp}^F$ and $\mathbf{u}_{imp*}^F$. While little conservatism is found using $\mathbf{u}_{exp*}^F$ and $\mathbf{u}_{imp}^F$. Depending on the application a trade off between those two groups of interval field implementations is needed. This trade off can be made by the number of basis vectors used in $\mathbf{u}_{exp}^F$, $\mathbf{u}_{exp*}^F$ and $\mathbf{u}_{imp*}^F$ or the sophistication of the applied approximation schemes in $\mathbf{u}_{exp*}^F$, $\mathbf{u}_{imp}^F$ and $\mathbf{u}_{imp*}^F$.

6. Example

In this section the applicability of the four different interval field implementations is checked in the context of a realistic IFE analysis. The IFE analysis is presented first. Secondly, the accuracy of the four implementations is checked by computing the difference between the interval field representation of the displacement field and the true solution over the whole uncertainty input space. This difference is representative for the error on a requested interval output when such an interval result is calculated using an optimisation approach over the input uncertainty space.

6.1. Numerical case study

The static analysis of a conical structure (see figure 14) is done using the different interval field representations and the classic (exact) FE solution method. The difference between both (i.e. the error of the interval field representation) is presented over the input uncertainty space.

The FE model consists of 6726 dofs. The structure is subject to a vertical and transverse nodal force, each with a magnitude of $10kN$, and it is clamped at the lower side. The effect of 5 different uncertain sizing parameters (thicknesses, see table 2) and one uncertain YZ force component on the static response of the whole structure is investigated. The sizing uncertainties can be represented by an explicit interval field where each of the five basis vectors has unity components in the corresponding thickness region (i.e., the thickness is constant in a ring) and zero values for the other regions. The uncertain YZ force is implemented as a vector with fixed magnitude ($10kN$), but uncertain direction. The angle θ (measured counter clockwise when looking in the positive X-direction) between this YZ-component and the Z-axis is bounded by the interval $[20\ 240]^\circ$.

Figure 14: The FE model of the conical structure, showing the five different shell properties, the load and boundary conditions

6.2. Accuracy of the proposed interval field approaches

The error of the interval field representation with respect to the exact FE solution for the displacements is calculated for $5^6 = 15625$ samples in the uncertain parameter space (each of the five thickness intervals and the force angle is sampled for five equidistant values and then all possible combinations are made). For every sample (i.e. a deterministic value of the thicknesses and angle), the error is calculated using:

$$error = \frac{\sum_{n=1}^{n_{dof}} |u_n - \tilde{u}_n|}{n_{dof} u_{max}} \quad (66)$$

with u_n the exact value of the n^{th} dof (this is a rotation or displacement), $\tilde{u}_n$ the interval field value of the considered n^{th} dof. To have a relative error measure a division by u_{max} , the maximum value of all the dofs in the structure for the considered deterministic value of the uncertain parameters, is done. To have a mean error measure for the whole structure, a summation over all the n_{dof} dofs is made and the sum is subsequently divided by n_{dof} .

Table 3 gives a summary of the different investigated representations. Indices a and b are given for alternatives of the same interval field concept.

The key properties of every interval field representation are given together with its measure for accuracy, expressed as the mean and standard deviation of the error compared to the exact solution. In the column for the basis vectors the dimension of the basis vector space is given. For example, $[\chi_{nom} [\chi_{min} \chi_{max}].:6]$

denotes a basis vector space consisting of 13 basis vectors: the nominal displacement pattern and the displacement pattern evaluated for the minimal and maximal value of each uncertainty, while the other uncertain parameters are kept at their nominal value. The conservatism-measure which is a priori determined by the choice of the interval field representation is given for reference purpose. The label *high* is given when no dependency between the interval factors can be taken into account, the label *low* is given when only one interval factor is involved or the interval factors are coupled through the input uncertain parameters.

For the extended explicit interval field $\mathbf{u}_{exp*}^F$, two implementations are presented. In the first $\mathbf{u}_{exp*a}^F$, the interval factor for the nominal basis vector is a quadratic function in every uncertain thickness t_i . The effect of plate thicknesses is approximated by a quadratic polynomial function:

$$\begin{aligned} \phi_{nom}(\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, \Delta t_5) = & a + b\Delta t_1 + c\Delta t_1^2 + d\Delta t_2 + e\Delta t_2^2 + \dots \\ & f\Delta t_3 + g\Delta t_3^2 + h\Delta t_4 + i\Delta t_4^2 + j\Delta t_5 + k\Delta t_5^2 + l\Delta\theta + m\Delta\theta^2 \end{aligned} \quad (67)$$

the interval factor for a minimal or maximal basis vector is a quadratic function of the corresponding uncertainty (i.e. one plate thickness t_i or the angle θ):

$$\phi_{\{min,max\}}(\Delta t_i) = p + q\Delta t_i + r\Delta t_i^2 \quad (68)$$

In the second extended explicit interval field implementation $\mathbf{u}_{exp*b}^F$, the interval factor for the nominal basis vector is a 4th order polynomial in every uncertain thickness t_i , the interval factor for a minimal or maximal basis vector is a 4th order polynomial of the corresponding uncertainty (i.e. one plate thickness or the angle θ), based on solving equation (40) for values of the uncertain parameter half way in between minimal and nominal and nominal and maximal values.

The best approximation is the explicit interval field $\mathbf{u}_{exp}^F$ using a basis vector space with dimension 11. This leads to a good accuracy since the basis vector space is the biggest and the values of the interval factors are determined independently each time a particular value for the uncertain parameters is chosen. Since no dependency between the interval factors can be taken into account it yields high conservatism. The best trade-off between accuracy and conservatism is found for the explicit interval field $\mathbf{u}_{exp*}^F$ with a 4th order approximation of the interval factors.

7. Conclusion

This paper presents the concept of interval fields for the representation of dependent uncertainty in numerical analysis. The application of interval fields in the static FEM is discussed in detail, leading to a range of interval field descriptions from the explicit to the implicit concept. These alternative descriptions have been applied to a numerical case study involving a static IFE analysis of a conical structure. For the test case that is considered, it was shown that the best trade-off between accuracy and conservatism is found when using the extended explicit interval field. In this representation the basis vectors

Description	basis vectors	Interval factors
$\mathbf{u}_{exp}^F \text{ }^a$	$[\psi_{nom}]$	η^I
$\mathbf{u}_{exp}^F \text{ }^b$	$[\psi_{nom} \ [\psi_{min} \ \psi_{max}]_{:,6}]$	η^I
$\mathbf{u}_{exp*}^F \text{ }^a$	$[\psi_{nom} \ [\psi_{min} \ \psi_{max}]_{:,6}]$	$\phi(\mathbf{x}^I)$, 2 nd order
$\mathbf{u}_{exp*}^F \text{ }^b$	$[\psi_{nom} \ [\psi_{min} \ \psi_{max}]_{:,6}]$	$\phi(\mathbf{x}^I)$, 4 th order
$\mathbf{u}_{imp}^F$	$[\tilde{\phi}(\mathbf{x})]$, 2 nd order	none
$\mathbf{u}_{imp*}^F$	$[\phi_{nom} \ [\phi_{dev}(\mathbf{x})]_{:,6}]$	η^I

Description	Error mean (%)	Error std (%)	conservatism
$\mathbf{u}_{exp}^F \text{ }^a$	6.88	3.51	low
$\mathbf{u}_{exp}^F \text{ }^b$	0.91	0.68	high
$\mathbf{u}_{exp*}^F \text{ }^a$	1.62	0.99	low
$\mathbf{u}_{exp*}^F \text{ }^b$	0.95	0.73	low
$\mathbf{u}_{imp}^F$	1.62	0.99	low
$\mathbf{u}_{imp*}^F$	1.04	0.72	high

Table 3: The accuracy of the different interval field representations. With ^a and ^b indicating alternative implementations of the same interval field concept.

are constant and the interval factors depend on the input uncertain parameters using response surfaces. Furthermore this interval field representation allows for realistic post-processing of primary interval results, which needs continued research.

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