Local hierarchical h -refinements in IgA based on generalized B-splines

Carla Manni Francesca Pelosi
Hendrik Speleers

Report TW 626, April 2013

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Local hierarchical h -refinements in IgA based on generalized B-splines

Carla Manni Francesca Pelosi
Hendrik Speleers

Report TW 626, April 2013

Department of Computer Science, K.U.Leuven

Abstract

In this paper we construct multilevel representations in terms of a hierarchy of tensor-product generalized B-splines. These representations combine the positive properties of a non-rational model with the possibility of dealing with local refinements. We discuss their use in the context of isogeometric analysis.

Keywords : Isogeometric analysis, generalized B-splines, hierarchical bases, multilevel representations.

Local Hierarchical h -Refinements in IgA Based on Generalized B-Splines

Carla Manni¹, Francesca Pelosi¹, and Hendrik Speleers²

¹ Dipartimento di Matematica, Università di Roma “Tor Vergata”,
Via della Ricerca Scientifica, 00133 Roma, Italy

² Departement Computerwetenschappen, Katholieke Universiteit Leuven,
Celestijnenlaan 200A, B-3001 Leuven, Belgium

Abstract. In this paper we construct multilevel representations in terms of a hierarchy of tensor-product generalized B-splines. These representations combine the positive properties of a non-rational model with the possibility of dealing with local refinements. We discuss their use in the context of isogeometric analysis.

Keywords: Isogeometric analysis; Generalized B-splines; Hierarchical bases; Multilevel representations

1 Introduction

Isogeometric Analysis (IgA) is a recent, but well established and successful, paradigm for the analysis of problems governed by partial differential equations [9, 17]. Its main goal is to improve the connection between numerical simulation and Computer Aided Design (CAD) systems.

In its original formulation, the main idea in IgA is to use directly the geometry provided by CAD systems – which usually is expressed in terms of tensor-product Non-Uniform Rational B-Splines (NURBS) – and to approximate the unknown solutions of differential equations by the same type of functions. This results in three principal advantages of IgA with respect to classical FEM (Finite Element Methods). First, complicated geometries are represented more accurately and some common profiles as conic sections are exactly described. This exact or accurate description of the geometry has a beneficial influence on the numerical solution of the addressed differential problem. Second, the description of the geometry is incorporated exactly at the coarsest mesh level and mesh refinement does not modify the geometry. This greatly simplifies the refinement process because it eliminates the necessity of interacting with the CAD system when mesh refinement is carried out. Finally, B-spline and NURBS representations allow an easy treatment and refinement of spaces with high approximation order and an inherent higher smoothness than those in classical FEM. This has been proved to be superior in various applications, see [9, 17], and references therein.

On the other hand, the three main advantages summarized above are not a distinguishing property of NURBS. Actually, NURBS suffer from some relevant geometric drawbacks. For instance, they lack an exact description of transcendental curves of interest in applications, and their parametrization of conic

sections does not correspond to natural arc length. In addition, NURBS poorly behave with respect to differentiation and integration which are crucial operators in analysis. On this concern, it is sufficient to have a look at the complicated structure of the derivative of a NURBS curve of a given order.

In the CAGD (Computer Aided Geometric Design) community several alternatives to the rational model have been proposed to overcome the above mentioned drawbacks. The so-called generalized B-splines are of relevant interest in this context, see [6, 21, 22, 28, 34] and references therein. They are piecewise functions with sections in more general spaces than algebraic polynomial spaces (like classical B-splines). Suitable selections of such spaces – typically including trigonometric or hyperbolic functions – allow an exact representation of polynomial curves, conic sections, helices and other profiles of salient interest in applications. Moreover, generalized B-splines possess all fundamental properties of algebraic B-splines (recurrence relation, compact minimum support, local linear independence, ...) which are shared by NURBS as well. Finally, contrarily to NURBS, they behave completely similar to B-splines with respect to differentiation and integration. Therefore, tensor-product generalized B-splines can offer an interesting alternative to NURBS in IgA as investigated in [8, 23, 24].

Adaptive local refinement is a crucial ingredient for obtaining, in an efficient way, an accurate solution of partial differential equations. However, NURBS rely on a tensor-product structure so they do not allow adequate local refinements. This motivates the interest in alternative structures for IgA that permit local refinements, such as T-splines [11], hierarchical splines [33], LR (Locally Refined) splines [10], or B-splines over triangulations [32].

Hierarchical B-splines were introduced in [12] as an accumulation of tensor-product B-splines with nested knot vectors, and they were further investigated in [19] and [13, 14, 33]. Of course, the concept of hierarchical bases can be considered for more general spaces than tensor-product B-splines, see also [14]. In particular, a hierarchical structure can be built for tensor-product generalized B-splines, with suitable section spaces, as they suffer from the same drawbacks of NURBS with respect to local refinements.

Besides the classical hierarchical bases, in the literature there also exist so-called quasi-hierarchical bases and truncated hierarchical bases. They are both alternative bases for a hierarchical space build on a sequence of nested linear spaces. The notion of quasi-hierarchical bases for hierarchical spaces has been introduced in [16, 20] and also used in [29, 30]. The concept of truncated bases has been described in [13, 14].

In this paper we define a multilevel representation in terms of a hierarchy of tensor-product generalized B-splines, and we discuss its use in the context of IgA. In this way, we can combine the positive properties of a non-rational model with the possibility of dealing with local refinements. The proposed hierarchical construction extends the classical one, because it does not necessarily require nested sequences of spaces.

The remainder of the paper consists of four sections. Section 2 briefly summarizes the definition and main properties of generalized B-splines, while in

Section 3 we define multilevel representations for them. Section 4 illustrates the performances of the generalized multilevel representations as a tool supporting local refinements in IgA in two benchmark problems taken from the literature. We end in Section 5 with some final comments.

2 Generalized B-Splines

To make the paper self contained, we devote this section to summarize the definition and basic properties of generalized B-splines. Further details can be found in the cited references and in [23, Section 3]. Assuming a sequence of knots is given,

$$\Xi := \{\xi_1 \leq \xi_2 \leq \dots \leq \xi_{n+p+1}\}, \quad n \in \mathbb{N}, \quad (1)$$

classical B-splines of degree p defined over (1) are a basis for piecewise polynomial functions with a suitable smoothness, i.e. functions with sections in the space of algebraic polynomials of degree p ,

$$\mathbb{P}_p := \langle 1, t, \dots, t^{p-2}, t^{p-1}, t^p \rangle.$$

Functions with a given smoothness and belonging piecewisely to more general spaces as

$$\mathbb{P}_p^{u_i, v_i} := \langle 1, t, \dots, t^{p-2}, u_i(t), v_i(t) \rangle, \quad t \in [\xi_i, \xi_{i+1}), \quad i = 1, \dots, n+p, \quad (2)$$

can be considered as well, see [6] and references therein. In the section spaces (2) the functions u_i, v_i can be selected such that salient profiles of interest are exactly represented and/or special characteristics are obtained. Popular choices for (2) are:

$$\mathbb{E}_{p, \alpha_i} := \langle 1, t, \dots, t^{p-2}, \exp(\alpha_i t), \exp(-\alpha_i t) \rangle, \quad 0 < \alpha_i \in \mathbb{R}, \quad (3)$$

$$\mathbb{T}_{p, \alpha_i} := \langle 1, t, \dots, t^{p-2}, \cos(\alpha_i t), \sin(\alpha_i t) \rangle, \quad 0 < \alpha_i(\xi_{i+1} - \xi_i) < \pi, \quad (4)$$

which lead to exponential and trigonometric splines respectively. They allow an exact representation of conic sections as well as of some transcendental curves (helix, cycloid, ...), see [4]. Exponential splines are often referred to as hyperbolic splines because the space (3) coincides with the space

$$\langle 1, t, \dots, t^{p-2}, \cosh(\alpha_i t), \sinh(\alpha_i t) \rangle.$$

This alternative formulation shows more clearly the connection between spaces (3) and (4). Other interesting section spaces are

$$\mathbb{V}\mathbb{D}_{p, \alpha_i} := \left\langle 1, t, \dots, t^{p-2}, \left(\frac{\xi_{i+1} - t}{\xi_{i+1} - \xi_i} \right)^{\alpha_i}, \left(\frac{t - \xi_i}{\xi_{i+1} - \xi_i} \right)^{\alpha_i} \right\rangle, \quad p \leq \alpha_i \in \mathbb{R}, \quad (5)$$

which lead to the so-called variable degree splines. Exponential and variable degree splines are a powerful tool in shape-preserving approximation and/or interpolation.

Results on the approximation power of the spaces (2) can be found in [6, Section 3] (see also [5, 25, 28]). In particular, for fixed values of the involved parameters, the spaces (3) and (4) have the same approximation power of $\mathbb{P}_p$.

Fig. 1. Generalized B-spline $B_{i,\Xi}^{(1)}$ with knot sequence $\Xi = \{0, 1, 2\}$. Left: the classical polynomial case. Right: the exponential case with $\alpha_i = 5$.

Remark 1. The section spaces in (2) may be different on each interval. Thus, generalized B-splines allow an exact representation of profiles composed by a sequence of curve segments of different kind: arcs of ellipses, hyperbolas, polynomial curves, etc.

It is well known that it is possible to construct B-spline-like functions with sections in spaces (2), see [5, 18, 21, 26–28, 34] and references therein. The so-called *generalized B-splines of degree p* , defined over the knot sequence (1), will be denoted by $B_{i,\Xi}^{(p)}$. To simplify the notation we omit the reference to the section spaces (2), even though this would be more appropriate. The specific section spaces will be clear from the context.

More precisely, we assume that³

$$u_i, v_i \in C^{p-1}[\xi_i, \xi_{i+1}],$$

and that $\{u_i^{(p-1)}, v_i^{(p-1)}\}$ is a Chebyshev system in $[\xi_i, \xi_{i+1}]$ and an Extended Chebyshev system in (ξ_i, ξ_{i+1}) . Thus, without loss of generality we can assume

$$u_i^{(p-1)}(\xi_i) > 0, \quad u_i^{(p-1)}(\xi_{i+1}) = 0, \quad v_i^{(p-1)}(\xi_i) = 0, \quad v_i^{(p-1)}(\xi_{i+1}) > 0.$$

According to [21] (see also [26, 34] and references therein), generalized B-splines can be defined with the following recurrence relation formula:

$$B_{i,\Xi}^{(1)}(t) := \begin{cases} \frac{v_i^{(p-1)}(t)}{v_i^{(p-1)}(\xi_{i+1})}, & \text{if } t \in [\xi_i, \xi_{i+1}), \\ \frac{u_{i+1}^{(p-1)}(t)}{u_{i+1}^{(p-1)}(\xi_{i+1})}, & \text{if } t \in [\xi_{i+1}, \xi_{i+2}), \\ 0, & \text{elsewhere;} \end{cases}$$

$$B_{i,\Xi}^{(p)}(t) := \delta_{i,\Xi}^{(p-1)} \int_{-\infty}^t B_{i,\Xi}^{(p-1)}(s) ds - \delta_{i+1,\Xi}^{(p-1)} \int_{-\infty}^t B_{i+1,\Xi}^{(p-1)}(s) ds, \quad p \geq 2,$$

where $\delta_{i,\Xi}^{(p)} := [\int_{-\infty}^{+\infty} B_{i,\Xi}^{(p)}(s) ds]^{-1}$ and fractions with zero denominators are considered to be zero. The knot sequence (1) allows us to define n generalized

³ For more general constructions with less restrictive hypotheses we refer to [26, 27].

Fig. 2. Examples of generalized B-splines of degree 3 defined on the knot sequence $\Xi = \{0, 0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1, 1\}$.

B-splines of degree p , namely $B_{1,\Xi}^{(p)}, \dots, B_{n,\Xi}^{(p)}$. Two generalized B-splines of degree 1 are depicted in Figure 1, and some sets of cubic generalized B-splines are illustrated in Figure 2.

Generalized B-splines possess all desirable properties of classical polynomial B-splines [3, 5, 21].

Proposition 1 *Let $B_{i,\Xi}^{(p)}$, $i = 1, \dots, n$, be generalized B-splines of degree $p \geq 2$ associated to the knot sequence (1). Then the following properties hold:*

- *piecewise structure:* $B_{i,\Xi}^{(p)}(t) \in \mathbb{P}_p^{u_j, v_j}$, $t \in [\xi_j, \xi_{j+1})$,
- *positivity:* $B_{i,\Xi}^{(p)}(t) \geq 0$,
- *partition of unity:* $\sum_{i=1}^n B_{i,\Xi}^{(p)}(t) \equiv 1$, $t \in [\xi_{p+1}, \xi_{n+1})$,
- *compact support:* $B_{i,\Xi}^{(p)}(t) = 0$, $t \notin [\xi_i, \xi_{i+p+1}]$,

- *smoothness*: $B_{i,\Xi}^{(p)}(t)$ is $p - \rho_j$ times continuously differentiable at ξ_j being ρ_j the multiplicity of ξ_j in the knot sequence,
- *local linear independence*: $B_{i-p,\Xi}^{(p)}(t), \dots, B_{i-1,\Xi}^{(p)}(t), B_{i,\Xi}^{(p)}(t)$ are linearly independent on $[\xi_i, \xi_{i+1})$.

For a given degree p and a fixed knot sequence Ξ , generalized B-splines with section spaces as in (3), (4) and (5) will be referred to as *exponential*, *trigonometric* and *variable degree B-splines of degree p* , respectively. We will denote by

$$\mathbb{E}\mathbb{S}_{\Xi,\alpha}^p, \mathbb{T}\mathbb{S}_{\Xi,\alpha}^p, \mathbb{V}\mathbb{D}\mathbb{S}_{\Xi,\alpha}^p, \mathbb{S}_{\Xi}^p$$

the spaces spanned by exponential, trigonometric, variable degree and classical B-splines of degree p , respectively. Here $\alpha = \{\dots, \alpha_i, \dots\}$ stands for the set of real parameters in (3), (4) and (5). The spaces $\mathbb{E}\mathbb{S}_{\Xi,\alpha}^p, \mathbb{T}\mathbb{S}_{\Xi,\alpha}^p, \mathbb{V}\mathbb{D}\mathbb{S}_{\Xi,\alpha}^p$ are called *exponential*, *trigonometric* and *variable degree spline spaces*, respectively.

Stable evaluation algorithms are crucial for applications. It is well known that a stable evaluation of exponential functions is a difficult task. On the other hand, variable degree splines possess graphical properties analogous to exponential splines as the parameters α_i increase, see Figure 2 (b)–(c), but they profit of an efficient evaluation. Indeed, variable degree splines can be obtained by a *geometric construction* consisting of simple corner cutting schemes which are applied to the (generalized) de Boor control polygon and produce the (generalized) Bézier control polygons associated with the Bernstein-like representation of the section spaces (5), see [5, 7]. Thanks to this geometric construction, the evaluation of the spline results in a stable computation.

Tensor-product exponential, trigonometric and variable degree splines have been profitably used in the context of IgA, including the treatment of advection-diffusion problems, see [8, 23, 24]. However, as for classical B-splines and NURBS, the tensor-product structure prevents the possibility of local refinements. On this concern, in the next section we address the problem of constructing multilevel bases in terms of a hierarchy of generalized B-splines.

3 Generalized Multilevel Bases

In this section we construct multilevel bases in terms of a hierarchy of generalized B-splines. Hierarchical tensor-product B-splines have been introduced in [12] as an accumulation of tensor-product B-splines with nested knot vectors, in order to allow local editing of tensor-product spline surfaces. Nevertheless, the definition of hierarchical spaces and bases can be more general, and it is not confined to tensor-product algebraic B-splines, see [14].

The classical definition of hierarchical bases and spaces assumes to deal with a sequence of nested linear spaces. Here, we extend the usual construction to sequences of not necessarily nested spaces. We prove that, under the hypothesis of local linear independence for the basis functions considered in each space, the resulting set of functions still possesses the fundamental property of linear independence.

Let Ω_0 be the closure of a bounded domain in $\mathbb{R}^d$. We consider a sequence of finite-dimensional linear spaces of functions defined on Ω_0 ,

$$V^0, V^1, \dots, V^\ell, \dots \quad (6)$$

and let Ω_0^ℓ be the closure of domains in $\mathbb{R}^d$ such that

$$\Omega_0 \supset \Omega_0^1 \supset \dots \supset \Omega_0^\ell \supset \dots \quad (7)$$

Setting $n_\ell := \dim(V^\ell)$, we assume

$$\mathcal{B}^\ell := \{b_{1,\ell}, \dots, b_{n_\ell,\ell}\}$$

is a basis of V^ℓ . A multilevel basis can be obtained by considering first all the basis elements in $\mathcal{B}^0$ whose support overlaps Ω_0 , and by considering then an iterative procedure which selects at each level ℓ all the basis functions of the previous level whose support is not entirely contained in Ω_0^ℓ and all the basis functions in $\mathcal{B}^\ell$ whose support is entirely contained in Ω_0^ℓ , see Figure 3.

More precisely, the multilevel space associated to a hierarchy of depth L is the space spanned by the set of functions constructed according to the following definition, see [13, 14, 33]. We denote by $\text{supp}(b_{i,\ell})$ the intersection of the support of $b_{i,\ell}$ with Ω_0 .

Definition 1. *The multilevel set of basis functions $\mathcal{H}$ associated to a hierarchy of domains (7) of depth L is recursively constructed as follows:*

- i) $\mathcal{H}^0 := \{b_{i,0} : \text{supp}(b_{i,0}) \neq \emptyset\};$
- ii) for $\ell = 0, \dots, L-2$:

$$\mathcal{H}^{\ell+1} := \mathcal{H}_C^{\ell+1} \cup \mathcal{H}_F^{\ell+1},$$

where

$$\begin{aligned} \mathcal{H}_C^{\ell+1} &:= \{b_{i,j} \in \mathcal{H}^\ell : \text{supp}(b_{i,j}) \not\subset \Omega_0^{\ell+1}\}, \\ \mathcal{H}_F^{\ell+1} &:= \{b_{i,\ell+1} \in \mathcal{B}^{\ell+1} : \text{supp}(b_{i,\ell+1}) \subset \Omega_0^{\ell+1}\}; \end{aligned}$$

- iii) $\mathcal{H} := \mathcal{H}^{L-1}.$

We are focusing on (local) mesh refinements in the context of IgA – the so-called h -refinements. Therefore, in the following we assume that each space V^ℓ in (6) is a space of generalized B-splines with fixed degree p defined over a given knot sequence. For $\ell = 0, \dots, L-1$, we assume that the knot vectors

$$\Xi_\ell := \{\xi_{1,\ell} \leq \xi_{2,\ell} \leq \dots \leq \xi_{n_\ell+p+1,\ell}\}, \quad n_\ell \in \mathbb{N}, \quad (8)$$

and the section spaces

$$\mathbb{P}_{p,\ell}^{u_i,v_i} := \langle 1, t, \dots, t^{p-2}, u_{i,\ell}(t), v_{i,\ell}(t) \rangle, \quad t \in [\xi_{i,\ell}, \xi_{i+1,\ell}), \quad i = 1, \dots, n_\ell + p, \quad (9)$$

are given. Moreover, we assume that the domains Ω_0^ℓ in (7) are selected as the union of a number of knot intervals defined by Ξ_ℓ in (8).

We will refer to the elements of $\mathcal{H}$ as *generalized multilevel B-splines* and to the spaces they span as *generalized multilevel spline spaces*. This multilevel construction generalizes the classical definition of hierarchical B-splines (see [33]) in the following ways:

Fig. 3. Illustration of the construction of a multilevel representation of cubic exponential B-splines with $\alpha_{i,\ell} = 50$. The light blue functions are removed in the construction of the multilevel basis.

- the multilevel basis is constructed as a hierarchy of generalized B-splines;
- different section spaces may be chosen not only on the different knot intervals (see Remark 1) but also on the different levels;
- the different spaces V^ℓ are not necessarily nested.

Despite this flexibility, the generalized multilevel B-splines preserve many properties of the classical hierarchical B-splines. Thanks to the local linear independence of generalized B-splines, the construction in Definition 1 ensures that the obtained set $\mathcal{H}$ consists of linear independent functions, as proved in the following theorem. Therefore, we will refer to them as *generalized multilevel bases*.

Theorem 1. *Let $\mathcal{H}$ be given as in Definition 1 where V^ℓ are spaces of generalized B-splines defined on the knot sequences (8) with section spaces as in (9). Then,*

- *the elements of $\mathcal{H}$ are linearly independent;*
- *the elements of $\mathcal{H}$ are non-negative.*

Proof. Let us consider a linear combination of the elements of $\mathcal{H}$ which vanishes in Ω_0 . Denoting by $I_\ell \subset \{1, 2, \dots, n_\ell\}$ the set of indices of the elements of $\mathcal{B}^\ell$ belonging to $\mathcal{H}$, $\forall t \in \Omega_0$, we have

$$0 = \sum_{\ell=0}^{L-1} \sum_{k \in I_\ell} \beta_{k,\ell} b_{k,\ell}(t), \quad \beta_{k,\ell} \in \mathbb{R}.$$

In order to show the linear independence of the elements of $\mathcal{H}$, we have to prove that all the coefficients in this representation are zero. By (7) and by Definition 1, the only functions in $\mathcal{H}$ which are possibly not zero in $\Omega_0 \setminus \Omega_0^1$ belong to $\mathcal{B}^0$. Therefore, $\forall t \in \Omega_0 \setminus \Omega_0^1$, we have

$$0 = \sum_{\ell=0}^{L-1} \sum_{k \in I_\ell} \beta_{k,\ell} b_{k,\ell}(t) = \sum_{k \in I_0} \beta_{k,0} b_{k,0}(t).$$

From Proposition 1 we know that the elements of $\mathcal{B}^0$ are locally linearly independent, and so we obtain that $\beta_{k,0} = 0$, $k \in I_0$. We now proceed in a similar way for all levels $\ell = 1, \dots, L-2$ sequentially. By considering all $t \in \Omega_0^\ell \setminus \Omega_0^{\ell+1}$ for a given ℓ , we obtain that $\beta_{k,\ell} = 0$, $k \in I_\ell$. Finally, imposing that the linear combination must vanish for all $t \in \Omega_0^{L-1}$, we obtain that $\beta_{k,L-1} = 0$, $k \in I_{L-1}$.

In addition, from Proposition 1 it follows that any $b_{i,\ell}$ is non-negative. Hence, the functions in $\mathcal{H}$ are non-negative as well. $\square$

The procedure described in Definition 1 allows a great flexibility in the construction of generalized multilevel B-splines because spline spaces with different section spaces and different knot sequences are allowed at different levels. Nevertheless, when nested spaces are of interest, we can also consider nested spaces of generalized B-splines defined over nested knot vectors. More precisely, the following result can be easily proved.

Proposition 2 *Let $\mathcal{B}^\ell := \{B_{i,\Xi_\ell}^{(p)}, i = 1, \dots, n_\ell\}$. The spaces $V^\ell := \langle \mathcal{B}^\ell \rangle$ are nested, if the knot sequences are nested, i.e.,*

$$\Xi_\ell \subset \Xi_{\ell+1}, \quad (10)$$

and if

$$\langle 1, t, \dots, t^{p-2}, u_{i,\ell}(t), v_{i,\ell}(t) \rangle = \langle 1, t, \dots, t^{p-2}, u_{j,0}(t), v_{j,0}(t) \rangle, \quad (11)$$

for each $[\xi_{i,\ell}, \xi_{i+1,\ell}) \subset [\xi_{j,0}, \xi_{j+1,0})$.

The main advantage of nested spaces is the ensured reduction (not deterioration) of the error.⁴ Moreover, as a particular case of the results in [14] (see also [33]), we obtain that the intermediate spaces $\langle \mathcal{H}^\ell \rangle$ are nested in the construction of Definition 1.

Proposition 3 *Let $\mathcal{H}$ be constructed as in Definition 1, where V^ℓ are spaces of generalized B-splines defined on the knot sequences (8) with section spaces as in (9). If (10) and (11) hold, then $\langle \mathcal{H}^\ell \rangle \subseteq \langle \mathcal{H}^{\ell+1} \rangle$.*

We now consider two multilevel spaces $\langle \mathcal{H} \rangle$ and $\langle \hat{\mathcal{H}} \rangle$ constructed by using the same nested generalized spline spaces V^ℓ over two different domain hierarchies $\{\Omega_0^\ell\}_{\ell=0,\dots,L-1}$ and $\{\hat{\Omega}_0^\ell\}_{\ell=0,\dots,L-1}$. From [14] (see also [33]) we know that, if the second domain sequence enlarges the first one, then this property is inherited by the corresponding generalized multilevel spline spaces.

⁴ This is not always the case for non-nested spaces, but when particular features (as layers) have to be detected, non-nested spaces can be powerful as well, see e.g. [32].

Proposition 4 *Let $\{\Omega_0^\ell\}_{\ell=0,\dots,L-1}$ and $\{\hat{\Omega}_0^\ell\}_{\ell=0,\dots,L-1}$ be two domain hierarchies such that $\Omega^0 = \hat{\Omega}^0$ and $\Omega^\ell \subseteq \hat{\Omega}^\ell$ for $\ell = 1, \dots, L-1$. Let $\mathcal{H}^\ell$ and $\hat{\mathcal{H}}^\ell$ be constructed as in Definition 1 on both domain hierarchies, where V^ℓ are spaces of generalized B-splines defined on the knot sequences (8) with section spaces as in (9). If (10) and (11) hold, then $\langle \mathcal{H}^\ell \rangle \subseteq \langle \hat{\mathcal{H}}^\ell \rangle$.*

Remark 2. Exponential and trigonometric splines easily provide nested sequences of spaces. As far as variable degree splines are concerned, we note that, if $[\xi_{i,\ell}, \xi_{i+1,\ell})$ is a proper subset of $[\xi_{j,0}, \xi_{j+1,0})$, but $\alpha_{j,0} \neq p$, then the space

$$\left\langle 1, t, \dots, t^{p-2}, \left(\frac{\xi_{i+1,\ell} - t}{\xi_{i+1,\ell} - \xi_{i,\ell}} \right)^{\alpha_{j,0}}, \left(\frac{t - \xi_{i,\ell}}{\xi_{i+1,\ell} - \xi_{i,\ell}} \right)^{\alpha_{j,0}} \right\rangle$$

is not a subspace of

$$\left\langle 1, t, \dots, t^{p-2}, \left(\frac{\xi_{j+1,0} - t}{\xi_{j+1,0} - \xi_{j,0}} \right)^{\alpha_{j,0}}, \left(\frac{t - \xi_{j,0}}{\xi_{j+1,0} - \xi_{j,0}} \right)^{\alpha_{j,0}} \right\rangle.$$

In general, the elements of $\mathcal{H}$ do not sum up to one. When a sequence of nested spaces is taken in (6) such that V_0 contains constants, different strategies can be considered to obtain a normalized basis in $\langle \mathcal{H} \rangle$. For example, the so-called truncated hierarchical bases presented in [13, 14] are particularly interesting for their stability properties. For the sake of simplicity, we just consider the elements of $\mathcal{H}$ as basis elements and we are not concerned with normalization.

The results and the proofs of the previous theorem and propositions immediately extend to the multidimensional tensor-product setting.

4 Numerical Examples

For the sake of completeness, we first briefly summarize the formulation of the isogeometric analysis approach for numerical approximation of PDE solutions. For a more comprehensive presentation we refer to [9], see also [8, 11, 17] for a short summary. For the sake of simplicity, we explain the paradigm for linear stationary problems in two dimensions with Dirichlet boundary conditions.

Let $\mathcal{L}$ be a second order (elliptic) operator on the domain Ω with Lipschitz boundary $\partial\Omega$. Let us consider the problem

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (12)$$

for the unknown $u : \Omega \rightarrow \mathbb{R}$. Assuming without loss of generality homogeneous Dirichlet boundary conditions (i.e. $g = 0$), the weak formulation of (12) is formulated as follows:

$$\text{find } u \in \mathbb{V}_D, \quad \text{such that } a(u, v) = F(v), \quad \forall v \in \mathbb{V}_D, \quad (13)$$

where

$$\begin{aligned} a : \mathbb{V} \times \mathbb{V} &\rightarrow \mathbb{R} \text{ is a bilinear form depending on } \mathcal{L}, \\ F : \mathbb{V} &\rightarrow \mathbb{R} \text{ is a linear form depending on } f, \\ \mathbb{V} &:= H^1(\Omega), \quad \mathbb{V}_D := H_0^1(\Omega). \end{aligned}$$

It is well known that the Galerkin approach to approximate the solution of (13) consists in selecting a suitable subspace

$$\mathbb{V}_h := \langle \phi_1, \phi_2, \dots, \phi_{n_h}, \phi_{n_h+1}, \dots, \phi_{n_h+n_h^b} \rangle \subset \mathbb{V},$$

and in looking for $u_h \in \mathbb{V}_{D_h} \subset \mathbb{V}_D$, such that

$$a(u_h, v_h) = F(v_h), \quad \forall v_h \in \mathbb{V}_{D_h}. \quad (14)$$

Here, we assume that an approximation (which is possibly exact) Ω_h of the physical domain Ω is given, and

$$\mathbb{V}_{D_h} := \{v_h \in \mathbb{V}_h : v_h|_{\partial\Omega_h} = 0\}.$$

Assuming $\mathbb{V}_{D_h} = \langle \phi_1, \phi_2, \dots, \phi_{n_h} \rangle$ and setting $u_h := \sum_{i=1}^{n_h} q_i \phi_i$, condition (14) gives rise to a linear system $A\mathbf{q} = \mathbf{F}$, where $A \in \mathbb{R}^{n_h \times n_h}$ is the stiffness matrix $A_{i,j} := a(\phi_j, \phi_i)$ and $\mathbf{F} \in \mathbb{R}^{n_h}$ is the data vector defined by $F_i := F(\phi_i)$, $i = 1, \dots, n_h$. Different methods correspond to different choices of $\mathbb{V}_h$.

In IgA, assuming the parametric domain $\Omega_0 := [0, 1] \times [0, 1]$ is given, the physical domain, Ω , is represented by a *global geometry function*

$$\mathbf{G} : \Omega_0 \rightarrow \overline{\Omega}, \quad \mathbf{G}(\boldsymbol{\omega}) = \sum_{i=1}^{n_h+n_h^b} \mathcal{N}_i(\boldsymbol{\omega}) \mathbf{P}_i, \quad \mathbf{P}_i \in \mathbb{R}^2, \quad \boldsymbol{\omega} \in \Omega_0, \quad (15)$$

where the basis functions

$$\{\mathcal{N}_1, \dots, \mathcal{N}_{n_h+n_h^b}\} \quad (16)$$

have to be selected to produce an exact representation of the geometry. The space $\mathbb{V}_h$ is then defined by

$$\phi_i(\mathbf{x}) := \mathcal{N}_i \circ \mathbf{G}^{-1}(\mathbf{x}) = \mathcal{N}_i(\boldsymbol{\omega}), \quad i = 1, \dots, n_h + n_h^b, \quad \mathbf{x} = \mathbf{G}(\boldsymbol{\omega}).$$

In IgA based on NURBS, the functions in (16) are (tensor-product) NURBS. Here we select the basis (16) as generalized multilevel B-spline bases constructed from tensor-product exponential, trigonometric and/or variable degree generalized B-splines.

In the next subsections we consider two examples illustrating the potential of these generalized multilevel spaces in IgA. For the sake of simplicity, we do not use advanced error estimators and corresponding automatic refinement strategies. Hence, the presented hierarchical meshes are constructed manually. For automatic refinement strategies we refer to the literature, see e.g. [1, 11, 33].

Fig. 4. Example 1. The domain with the boundary conditions.

4.1 Example 1: Reaction-Diffusion Problem

We consider the following problem, see [2]:

$$\begin{cases} -\kappa \Delta u + u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (17)$$

with $\kappa = 10^{-3}$ and Dirichlet boundary conditions as depicted in Figure 4.

We consider the approximated solutions in the bivariate spaces spanned by the tensor-product of three different univariate bases, all with sections in four-dimensional spaces: cubic exponential B-splines (EXP₃ BSP), cubic variable degree B-splines (VDP₃ BSP) and classical cubic B-splines. Moreover, we also consider the solutions in the bivariate spaces spanned by generalized multilevel bases built from the same tensor-product spaces mentioned before, defined on the hierarchical meshes shown in Figure 5.

The discontinuous Dirichlet boundary conditions have been approximated by a continuous function: a Schoenberg-like quasi-interpolant obtained by using the values of the boundary data at the Greville abscissas of the cubic B-splines as depicted in Figure 5 (right). The “true” Greville abscissas could have been used (dealing with cubic-like spaces) but for the sake of simplicity the standard ones have been considered.

We compute in all cases the condition number of the corresponding stiffness matrices in the 2-norm and the minimum value of the solution evaluated on a regular 400×400 grid. We observe that all the solutions have 1.000 as maximum value on the same evaluation grid.

Table 1 shows the results obtained on uniform refinements of the initial rectangular 5×5 mesh, and the parameters of the spaces of generalized B-splines are fixed for all refinement levels: $\alpha_{i,\ell} = 50$ for all sections i and all levels ℓ in case of the exponential B-splines and $\alpha_{i,\ell} = 5$ for all i and all ℓ in case of the variable degree B-splines. On the other hand, the results in Table 2 are obtained by varying the values of the tension parameters for different refinement levels. Tables 3

Fig. 5. Example 1. Hierarchical meshes (left) and Greville abscissas of cubic tensor-product B-splines (right) for different refinement levels.

Example 1, uniform mesh refinements

h	dof	EXP ₃ BSP, $\alpha_{i,\ell} = 50$		VDP ₃ BSP, $\alpha_{i,\ell} = 5$		Cubic BSP	
		min	cond	min	cond	min	cond
0.2	36	-4.982 10 ⁻³	13.04	-1.048 10 ⁻²	30.99	-8.352 10 ⁻²	250.77
0.1	121	-6.452 10 ⁻⁴	14.21	-6.648 10 ⁻⁴	13.84	-1.603 10 ⁻³	121.99
0.05	441	-9.312 10 ⁻⁵	14.81	-1.017 10 ⁻⁴	5.50	-1.061 10 ⁻⁴	40.76
0.025	1681	-3.743 10 ⁻⁸	21.34	0.000 10 ⁰	4.95	-1.306 10 ⁻⁷	29.67

Table 1. The results for Example 1 using tensor-products of cubic exponential B-splines (EXP₃ BSP) with $\alpha_{i,\ell} = 50$ for all sections i and all levels ℓ , cubic variable degree B-splines (VDP₃ BSP) with $\alpha_{i,\ell} = 5$ for all i and all ℓ , and cubic B-splines. They are all defined on uniformly refined meshes with h -length intervals per side.

Example 1, uniform mesh refinements

h	dof	EXP ₃ BSP			VDP ₃ BSP		
		$\alpha_{i,\ell}$	min	cond	$\alpha_{i,\ell}$	min	cond
0.2	36	35	-3.102 10 ⁻³	20.30	6	-3.296 10 ⁻³	22.02
0.1	121	35	-1.304 10 ⁻⁴	25.95	4	-1.171 10 ⁻⁴	26.41
0.05	441	60	-9.187 10 ⁻⁵	10.96	4	-9.283 10 ⁻⁵	9.14
0.025	1681	100	-9.080 10 ⁻¹⁸	11.06	4	0.000 10 ⁰	7.81

Table 2. The results for Example 1 using tensor-products of cubic exponential B-splines (EXP₃ BSP) and cubic variable degree B-splines (VDP₃ BSP) with different values of the tension parameters $\alpha_{i,\ell}$ for each refinement level ℓ in Table 1.

Example 1, hierarchical mesh refinements

$h_{\min}$	dof	EXP ₃ BSP, $\alpha_{i,\ell} = 50$		VDP ₃ BSP, $\alpha_{i,\ell} = 5$		Cubic BSP	
		min	cond	min	cond	min	cond
0.2	36	-4.982 10 ⁻³	13.04	-1.048 10 ⁻²	30.99	-8.352 10 ⁻²	250.77
0.1	56	-6.576 10 ⁻⁴	489.46	-6.648 10 ⁻⁴	172.44	-1.423 10 ⁻³	484.56
0.05	92	-1.062 10 ⁻⁴	957.39	-1.017 10 ⁻⁴	352.45	-4.782 10 ⁻⁴	1092.37
0.025	144	-2.382 10 ⁻⁶	1784.63	-6.789 10 ⁻⁶	520.76	-1.215 10 ⁻⁵	1434.21

Table 3. The results for Example 1 using tensor-products of cubic exponential B-splines (EXP₃ BSP), cubic variable degree B-splines (VDP₃ BSP) and cubic B-splines, defined on the hierarchical meshes shown in Figure 5 and consisting of quadrilaterals with $h_{\min}$ as minimum side length.

Example 1, hierarchical mesh refinements

$h_{\min}$	dof	EXP ₃ BSP			VDP ₃ BSP		
		$\alpha_{i,\ell}$	min	cond	$\alpha_{i,\ell}$	min	cond
0.2	36	35	-3.102 10 ⁻³	20.30	6	-3.296 10 ⁻³	22.02
0.1	56	35	-4.431 10 ⁻⁴	556.82	4	-5.171 10 ⁻⁴	172.71
0.05	92	60	-8.703 10 ⁻⁵	1351.18	4	-9.581 10 ⁻⁵	352.78
0.025	144	100	-9.881 10 ⁻⁷	2085.02	4	-1.063 10 ⁻⁶	522.41

Table 4. The results for Example 1 using tensor-products of cubic exponential B-splines (EXP₃ BSP) and cubic variable degree B-splines (VDP₃ BSP) with different values of the tension parameters $\alpha_{i,\ell}$ for each refinement level ℓ and defined on the same four hierarchical meshes as in Table 3.

Fig. 6. Example 1. Generalized multilevel cubic exponential B-spline solutions on the first and last mesh shown in Figure 5 with tension parameters $\alpha_{i,\ell} = 35, 35, 60, 100$ at the levels $\ell = 0, 1, 2, 3$ for all i .

and 4 report the corresponding results obtained in the considered locally refined multilevel space setting.

Figures 6 and 7 show the computed approximate solutions and their contour plots using cubic exponential B-splines and cubic variable degree B-splines respectively, defined on the first and last hierarchical mesh given in Figure 5 and with the parameters given in Table 4. Figure 8 shows the reduction of undershoots versus the number of degrees of freedom (dof), in the cases of uniform and local hierarchical refinements with the different considered generalized spline spaces.

As illustrated by the numerical and graphical results, exponential and variable degree B-splines have a similar behavior, and they provide a good identification of the boundary layer with less undershoots than in the case of classical B-splines. Considering their similar performances, variable degree B-splines appear to be particularly attractive because of their reduced computational cost and the available stable evaluation algorithms.

Finally, we remark that the increase in the condition number of the stiffness matrices (see Tables 3 and 4) is a common behavior when dealing with local

Fig. 7. Example 1. Generalized multilevel cubic variable degree B-spline solutions on the first and last mesh shown in Figure 5 with tension parameters $\alpha_{i,0} = 6$ at level 0 and $\alpha_{i,\ell} = 4$ at the other levels $\ell = 1, 2, 3$ for all i .

Fig. 8. Example 1. Min values (as absolute value) of the solution using cubic exponential B-splines (red), cubic variable degree B-splines (black) and classical cubic B-splines (blue) versus the number of degrees of freedom, defined on locally refined hierarchical meshes (solid) and uniformly refined meshes (dashed). Left: the tension parameters are fixed as in Tables 1 and 3. Right: see Tables 2 and 4 for the tension parameters used at each level.

Fig. 9. Example 2. Elastic plate with a circular hole: problem setting.

refinements. Other results with alternative spaces and local refinement strategies for this problem can be found in [2, 32].

4.2 Example 2: a Problem in Solid Mechanics

This example aims to illustrate the performances of local refinements based on multilevel trigonometric B-splines by solving a classical problem in solid mechanics [11, 17, 23, 31]. We consider an infinite plate with a circular hole of radius r , subject to an in-plane uniform tension T_x in x -direction, see Figure 9 (left). For a homogeneous and isotropic material this problem features an exact solution which can be found in [15, Section 7.6]. The infinite plate is modeled by a finite circular domain with radius R . Due to the symmetry, the computational domain Ω is restricted to a quarter, see Figure 9 (right). We study the linear elastic behavior of the displacement field $\mathbf{w} : \Omega \rightarrow \mathbb{R}^2$ described by

$$\operatorname{div} \sigma(\mathbf{w}) = 0 \quad \text{in } \Omega.$$

The exact solution is applied as a Neumann boundary condition, see Figure 9 (right). For the sake of completeness we recall that $\sigma(\mathbf{w}) := \{\sigma_{ij}(\mathbf{w})\}_{i,j=1,2}$ with

$$\sigma_{ij}(\mathbf{w}) := \lambda \operatorname{div}(\mathbf{w}) \delta_{ij} + 2\mu \epsilon_{ij}(\mathbf{w}), \quad \epsilon_{ij}(\mathbf{w}) := \frac{1}{2} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right), \quad i, j = 1, 2,$$

$$\mathbf{w} := (w_1, w_2), \quad (x, y) := (x_1, x_2), \quad \lambda := \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu := \frac{E}{2(1+\nu)},$$

where E denotes the Young's modulus and ν the Poisson's ratio. In our computed example we have taken

$$r = 1, \quad R = 4, \quad E = 10^5, \quad \nu = 0.3, \quad T_x = 10.$$

Without the hole, the stress would be uniform

$$\sigma_{1,1} = T_x, \quad \sigma_{1,2} = \sigma_{2,2} = 0.$$

k	$\mathbf{P}_{k,1}$	$\mathbf{P}_{k,2}$	$\mathbf{P}_{k,3}$
1	(-1,0)	(-2.5,0)	(-4,0)
2	(-1,1)	(-2.5,2.5)	(-4,4)
3	(0,1)	(0,2.5)	(0,4)

Table 5. Example 2. Control points of the geometry function.

Example 2, uniform mesh refinements						
dof	TRG ₂ BSP			Quadratic BSP		
	error of $\mathbf{w}$	$\sigma_{1,1}(0,1)$	cond	error of $\mathbf{w}$	$\sigma_{1,1}(0,1)$	cond
96	4.421 10^{-5}	26.722	25.02	8.564 10^{-5}	26.931	23.43
300	8.881 10^{-6}	28.921	33.24	1.893 10^{-5}	28.243	33.02
1044	2.612 10^{-6}	29.502	151.81	5.563 10^{-6}	28.905	151.35
3876	9.341 10^{-8}	29.996	693.45	1.371 10^{-7}	29.802	691.34

Table 6. The results for Example 2 using tensor-product quadratic trigonometric B-splines and tensor-product quadratic B-splines defined on uniformly refined meshes. The error is computed for the displacement $\mathbf{w}$ of the exact solution in the L_2 -norm, and the approximated value of $\sigma_{1,1}$ is given at the upper side of the hole.

Example 2, hierarchical mesh refinements						
dof	TRG ₂ BSP			Quadratic BSP		
	error of $\mathbf{w}$	$\sigma_{1,1}(0,1)$	cond	error of $\mathbf{w}$	$\sigma_{1,1}(0,1)$	cond
96	4.421 10^{-5}	26.722	25.02	8.564 10^{-5}	26.931	23.43
230	5.025 10^{-6}	28.993	31.42	1.423 10^{-5}	28.762	29.34
574	6.703 10^{-7}	29.745	101.04	1.758 10^{-6}	29.243	96.13
1122	8.052 10^{-8}	30.012	245.25	2.132 10^{-7}	29.962	231.81

Table 7. The results for Example 2 using quadratic trigonometric B-splines and quadratic B-splines defined on the hierarchical meshes shown in Figure 10. The error is computed for the displacement $\mathbf{w}$ of the exact solution in the L_2 -norm, and the approximated value of $\sigma_{1,1}$ is given at the upper side of the hole.

This distribution will alter only in the vicinity on the hole. More precisely, we get a peak stress concentration at the upper side of the hole: the stress component $\sigma_{1,1}$ takes the value 30 at the point (0, 1).

To exactly represent the geometry we construct a global geometry function as in (15) considering the tensor-product space $\mathbb{TS}_{\Xi, \alpha}^2 \otimes \mathbb{S}_{\Xi}^2$ with $\alpha_i = \frac{\pi}{2}$. The control points $\mathbf{P}_i = \mathbf{P}_{k,j}$, $k, j = 1, 2, 3$, are depicted in Table 5 for a coarse grid consisting of one interval per edge. Then we approximate both components of the displacement in the bivariate tensor-product spaces $\mathbb{TS}_{\Xi, \alpha}^2 \otimes \mathbb{S}_{\Xi}^2$ (TRG₂ BSP) with $\alpha_i = \frac{\pi}{2}$. Classical quadratic B-splines $\mathbb{S}_{\Xi}^2 \otimes \mathbb{S}_{\Xi}^2$ (quadratic BSP) have also been considered for the sake of comparison. Next, we also consider bivariate spaces spanned by generalized multilevel bases built from the same tensor-product spaces mentioned before, defined on the hierarchical meshes shown in Figure 10. Note that all the considered spaces are nested. We compute in all cases the condition number of the corresponding stiffness matrices in the 2-norm and the error with respect to the exact displacement in the L_2 -norm.

Fig. 10. Example 2. Hierarchical meshes (left) and Greville abscissas for quadratic tensor-product B-splines (right) for four refinement levels.

Fig. 11. Example 2. Left: control net of the geometry function. Right: exact $\sigma_{1,1}$.

Fig. 12. Example 2. Multilevel quadratic trigonometric B-spline approximations of $\sigma_{1,1}$ with $\alpha_{i,\ell} = \pi/2$ on the first mesh (left) and last mesh (right) shown in Figure 10.

Fig. 13. Example 2. L_2 -norm of the error for the displacement using quadratic trigonometric B-splines (red line) and classical quadratic B-splines (blue line) versus the number of degrees of freedom, computed on locally refined hierarchical meshes (solid) and uniformly refined meshes (dashed).

Table 6 shows the results obtained on uniform refinements of the initial rectangular mesh, see Figure 10 (top). Table 7 reports the corresponding results obtained in the locally refined multilevel space setting using the meshes given in Figure 10. Figure 13 shows the reduction of the computed error versus the number of degrees of freedom, in the cases of uniform and local hierarchical refinements with the different considered spaces. Finally, Figure 12 shows the computed stress component $\sigma_{1,1}$ on the hierarchical meshes as in Figure 10.

Other results with different spaces and local refinement strategies for this problem can be found in [11, 31, 33].

5 Closure

Generalized B-splines have been used as an alternative to standard NURBS as an efficient problem-oriented tool in isogeometric analysis. However, they share with NURBS a tensor-product structure. This prevents the possibility of local refinements, which are a crucial ingredient for obtaining, in an efficient way, an accurate solution of partial differential equations.

Multilevel bases – originally introduced for tensor-product B-splines – offer a natural structure to obtain local refinements and can be extended to tensor-product generalized B-splines with suitable section spaces.

Moreover, multilevel bases can be constructed from not necessarily nested sequences of spaces. Combining this with the flexibility of generalized B-splines results in a variety of generalized multilevel spaces, whose characteristics can be tuned according to the considered problem.

In particular, generalized multilevel spaces based on variable degree B-splines present interesting performances because they couple the ability of modelling sharp features of exponential B-splines, with the low computational cost and stable evaluation algorithms of classical algebraic B-splines.

Of course, if non-nested spaces are considered to build the hierarchy, finer levels do not result in a nested sequence of the (multilevel) discretization spaces. Nested spaces are not a necessary ingredient in IgA, see for example [31, 32]. Nevertheless, nested hierarchical structures can be obtained with generalized B-splines as well, whenever preferred. In this case, at each level of the hierarchy, generalized B-spline spaces defined over nested sequences of knots and with sections in nested spaces have to be considered.

Finally, we have focused on local mesh refinements (h -refinements), but as for classical B-splines hierarchical p - and k -refinements can be considered as well. Since the section spaces (2) support degree-raising, nested hierarchical structures can be obtained also in this case if preferred.

References

1. Bank, R.E., Smith, R.K.: A posteriori error estimates based on hierarchical bases *SIAM J. Numer. Anal.* **30** (1993) 921–935

2. Bazilevs, Y., Calo, V.M., Cottrell, J.A., Evans, J.A., Hughes, T.J.R., Lipton, S., Scott, M.A., Sederberg, T.W.: Isogeometric analysis using T-splines. *Comput. Methods Appl. Mech. Engrg.* **199** (2010) 229–263
3. de Boor, C.: A Practical Guide to Splines, Revised edition. Springer (2001)
4. Carnicer, J.M., Mainar, E., Peña, J.M.: Critical length for design purposes and Extended Chebyshev spaces. *Constr. Approx.* **20** (2004) 55–71
5. Costantini, P.: Curve and surface construction using variable degree polynomial splines. *Comput. Aided Geom. Design* **17** (2000) 419–446
6. Costantini, P., Lyche, T., Manni, C.: On a class of weak Tchebycheff systems. *Numer. Math.* **101** (2005) 333–354
7. Costantini, P., Manni, C.: Geometric construction of generalized cubic splines. *Rend. Matem. Appl.* **26** (2006) 327–338
8. Costantini, P., Manni, C., Pelosi, F., Sampoli, M.L.: Quasi-interpolation in isogeometric analysis based on generalized B-splines. *Comput. Aided Geom. Design* **27** (2010) 656–668
9. Cottrell, J.A., Hughes, T.J.R., Bazilevs, Y.: Isogeometric Analysis: Toward Integration of CAD and FEA. John Wiley & Sons (2009)
10. Dokken, T., Lyche, T., Pettersen, K.F.: Polynomial splines over locally refined box-partitions. *Comput. Aided Geom. Design* **30** (2013) 331–356
11. Dörfler, M., Jüttler, B., Simeon, B.: Adaptive isogeometric analysis by local h -refinement with T-splines. *Comput. Methods Appl. Mech. Engrg.* **199** (2010) 264–275
12. Forsey, D.R., Bartels, R.H.: Hierarchical B-spline refinement. *Comput. Graph.* **22** (1988) 205–212
13. Giannelli, C., Jüttler, B., Speleers, H.: THB-splines: the truncated basis for hierarchical splines. *Comput. Aided Geom. Design* **29** (2012) 485–498
14. Giannelli, C., Jüttler, B., Speleers, H.: Strongly stable bases for adaptively refined multilevel spline spaces. Preprint (2012)
15. Gould, P.L.: Introduction to Linear Elasticity. Springer Verlag, Berlin (1999)
16. Grinspun, E., Krysl, P., Schröder, P.: CHARMS: a simple framework for adaptive simulation. *ACM Trans. Graphics* **21** (2002) 281–290
17. Hughes, T.J.R., Cottrell, J.A., Bazilevs, Y.: Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.* **194** (2005) 4135–4195
18. Koch, P.E., Lyche, T.: Interpolation with exponential B-splines in tension. In: Farin, G., Hagen, H., Noltemeier, H., Knödel, W. (eds.), Geometric modelling, pp. 173–190. Springer-Verlag (1993)
19. Kraft, R.: Adaptive and linearly independent multilevel B-splines. In: Le Méhauté, A., Rabut, C., Schumaker, L.L. (eds.), Surface Fitting and Multiresolution Methods, pp. 209–218. Vanderbilt University Press, Nashville (1997)
20. Krysl, P., Grinspun, E., Schröder, P.: Natural hierarchical refinement for finite element methods. *Int. J. Numer. Meth. Eng.* **56** (2003) 1109–1124
21. Kvasov, B.I., Sattayatham, P.: GB-splines of arbitrary order. *J. Comput. Appl. Math.* **104** (1999) 63–88
22. Mainar, E., Peña, J.M., Sánchez-Reyes, J.: Shape preserving alternatives to the rational Bézier model. *Comput. Aided Geom. Design* **18** (2001) 37–60
23. Manni, C., Pelosi, F., Sampoli, M.L.: Generalized B-splines as a tool in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.* **200** (2011) 867–881
24. Manni, C., Pelosi, F., Sampoli, M.L.: Isogeometric analysis in advection-diffusion problems: tension splines approximation. *J. Comput. Appl. Math.* **236** (2011) 511–528

25. Marušić, M., Rogina, M.: Sharp error bounds for interpolating splines in tension. *J. Comput. Appl. Math.* **61** (1995) 205–223
26. Mazure, M.L.: Chebyshev-Bernstein bases. *Comput. Aided Geom. Design* **16** (1999) 649–669
27. Mazure, M.L.: How to build all Chebyshevian spline spaces good for geometric design? *Numer. Math.* **119** (2011) 517–556
28. Schumaker, L.L.: Spline Functions: Basic Theory, Third edition. Cambridge U.P. (2007)
29. Speleers, H., Dierckx, P., Vandewalle, S.: Quasi-hierarchical Powell-Sabin B-splines. *Comput. Aided Geom. Design* **26** (2009) 174–191
30. Speleers, H., Dierckx, P., Vandewalle, S.: On the local approximation power of quasi-hierarchical Powell-Sabin splines. In: Dæhlen, M., Floater, M.S., Lyche, T., Merrien, J.L., Mørken, K., Schumaker, L.L. (eds.), *Mathematical Methods for Curves and Surfaces*, Lecture Notes in Computer Science 5862, pp. 419–433 (2010)
31. Speleers, H., Manni, C., Pelosi, F.: From NURBS to NURPS geometries. *Comput. Methods Appl. Mech. Engrg.* **255** (2013) 238–254
32. Speleers, H., Manni, C., Pelosi, F., Sampoli, M.L.: Isogeometric analysis with Powell-Sabin splines for advection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.* **221–222** (2012) 132–148
33. Vuong, A.-V., Giannelli, C., Jüttler, B., Simeon, B.: A hierarchical approach to adaptive local refinement in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.* **200** (2011) 3554–3567
34. Wang, G., Fang, M.: Unified and extended form of three types of splines. *J. Comput. Appl. Math.* **216** (2008) 498–508