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Time-optimal path planning for flat systems with application to a wheeled mobile robot*

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Abstract—The time-optimal path planning problem aims at bringing a system from an initial to terminal state in minimal time while obeying both geometric and dynamic constraints. Path planning problems are often decoupled into a high-level path planning stage where a feasible geometric path is determined and a low level stage where system dynamics are taken into account. The paper combines the geometric and dynamic approach into a single optimization problem for so-called differentially flat systems. The geometric path is represented as a convex combination of two or more feasible paths. Relying on differential flatness, the dynamics of the system are projected onto the path which leads to a single input system. The resulting optimization problem is transformed into a fixed end-time optimal control problem that can be initialized easily. An application to a wheeled mobile robot, a challenging non-linear system, will illustrate the proposed approach throughout the paper.

I. INTRODUCTION

The time-optimal path planning problem considers bringing a system from an initial state to a terminal state, while obeying the dynamic and geometric constraints. Solving this problem efficiently can be of great interest for maximizing productivity of robot systems, minimizing lap time in car racing and many other applications.

Due to the complexity of the problem, a decoupled approach is often followed, which solves the motion planning problem in two stages [1]. First a high-level planner determines a geometric path, while accounting for obstacle avoidance and other geometric constraints but ignoring low-level, dynamic constraints. Subsequently, in the path following stage, a time-optimal trajectory along the geometric path is determined taking into account dynamic constraints [2], [1], [3]. Although this approach requires lower computational effort, it often results in sub-optimal solutions.

In this paper, we consider the motion planning for differentially flat systems. For such systems, there exists a particular set of outputs that characterize all states and inputs of the system. Therefore, solving the motion planning problem through flatness avoids the integration of the (often highly nonlinear) differential equations. For differentially

flat systems, an efficient formulation for the path following problem was recently derived in [4], [5]. Inspired by the problem formulation and the aforementioned two-level approach, a combined path planning–path following approach is proposed in this paper. The geometric path is parametrized as a convex combination of two or more outer paths, which are a function of a scalar path coordinate s with an unknown time dependency. Using a transformation of variables, the optimal control problem is formulated as a fixed end-time problem with s as independent variable that can be initialized easily.

The proposed approach is illustrated by means of a wheeled mobile robot. Other approaches for the time-optimal control of mobile robots have been proposed in [6], [7]. Both contributions follow an approach similar to the decoupled approach. First a geometric path is determined based on kinematic and geometric constraints. Afterwards, using an iterative procedure based on the path’s curvature and the dynamic constraints, the highest allowable velocity profile along the path is searched for. Although [6] also optimizes the control points of a spline that parametrizes the geometric path, no details on the corresponding optimization problem are provided. Furthermore, objects or walls in the environment are not taken into account, whereas in our approach this is easily accomplished without complicating the optimization problem.

Section II of the paper describes the basic concept of differential flatness and is followed by a general statement of the optimization problem in Section III. Then, in Section IV, an efficient formulation of the path following problem is reviewed [3], [4], [5], which is subsequently extended by our proposed combined path following and path planning approach. Finally, Section V illustrates the developed methodology on a wheeled mobile robot.

In the following, we will use $\partial_\tau^k g(\tau)$ to denote the k -th derivative of $g(\tau)$ with respect to τ . For small k , we will use the shorthand notation $\dot{g}$ to denote the derivative with respect to time.

II. DIFFERENTIAL FLATNESS

Before addressing the path following problem, let us first recall the notion of differential flatness, introduced in [8].

Definition (Differentially flat system). The system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (1)$$

with state $\mathbf{x} \in \mathbb{R}^n$ and input $\mathbf{u} \in \mathbb{R}^m$, $m \leq n$, is differentially flat if there exists a variable $\mathbf{z} \in \mathbb{R}^m$, the so-called flat output,

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such that

$$\mathbf{z}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), \dots, \partial_t^\alpha \mathbf{u}(t))$$

and

$$\begin{aligned} \mathbf{x}(t) &= \Phi(\mathbf{z}(t), \dots, \partial_t^{\beta-1} \mathbf{z}(t)), \\ \mathbf{u}(t) &= \Psi(\mathbf{z}(t), \dots, \partial_t^\beta \mathbf{z}(t)), \end{aligned} \quad (2)$$

where Φ and Ψ are smooth functions and $\alpha, \beta \in \mathbb{N}$.

The flat output characterizes all the state motions and corresponding input history. It is a powerful concept in motion planning as it avoids integration of the differential equations. In the following, we consider systems of the form (1) with output $\mathbf{y} \in \mathbb{R}^m$. We assume that the system is differentially flat with flat output $\mathbf{z} = \mathbf{y}$.

Example (A wheeled mobile robot [9]). The wheeled mobile robot is shown in figure 1. v and ω represent the forward and angular velocity of the robot with respect to an inertial reference frame, (x, y) the center position and θ the orientation with respect to the X -axis. The system is controlled by the torques, τ_r and τ_l , applied to the robot's right and left wheel respectively. Assuming a non-slip condition, the equations of motion are [9]

$$\begin{aligned} \dot{x} &= v \cos \theta \\ \dot{y} &= v \sin \theta \\ \dot{\theta} &= \omega \\ \dot{\omega} &= \alpha \\ \dot{v} &= a \\ \tau_r &= \frac{1}{2} \left(rma + \frac{rI}{b} \alpha \right) \\ \tau_l &= \frac{1}{2} \left(rma - \frac{rI}{b} \alpha \right). \end{aligned} \quad (3)$$

Here, m is the mass of the robot, r is the radius of the wheels, I is the moment of inertia of the robot about its center of mass, b is half the distance between the two wheels. This system is differentially flat with the coordinate $\mathbf{y} = (x, y)^T$ as flat output. Indeed, from the first two equations it follows that

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} \text{ and } \theta = \arctan \frac{\dot{y}}{\dot{x}}.$$

By differentiating both expressions with respect to t , we find

$$a = \frac{1}{2} \frac{\ddot{x} + \ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \text{ and } \omega = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$

Hence, the inputs τ_r and τ_l as well as the wheel velocities

$$\omega_r = \frac{1}{r}(v + b\omega) \text{ and } \omega_l = \frac{1}{r}(v - b\omega)$$

can be written as a function of x , y and their time derivatives.

III. OPTIMIZATION PROBLEM

Consider a geometric path $\mathbf{y}_r(s) \in \mathcal{C}^\beta$, a regular, i.e. $\partial_s \mathbf{y}_r(s) \neq 0, \forall s \in [0, 1]$, parametrized curve as a function of a scalar path coordinate s . The time dependency follows from the relation $s(t)$. Without loss of generality it is assumed that the trajectory starts at $t = 0$, ends at $t = T$

Fig. 1. A two-wheeled mobile robot described by the states x, y and θ

and that $s(0) = 0 \leq s(t) \leq s(T) = 1$. Furthermore, it is assumed that the velocity along the path is non-negative, i.e. $\dot{s}(t) \geq 0$ and that the boundary conditions $\mathbf{x}_0, \mathbf{x}_T$ are consistent with $\mathbf{y}_r(s)$, such that $\mathbf{x}_0 = \Phi(\mathbf{y}_r(0), \dots, \mathbf{y}_r^{(\beta-1)}(0))$ and $\mathbf{x}_T = \Phi(\mathbf{y}_r(1), \dots, \mathbf{y}_r^{(\beta-1)}(1))$.

In this paper we want to find an input signal $\mathbf{u}(t)$ and the a geometric path $\mathbf{y}_r(s)$ that allows for the fastest execution time such that (i) the geometric path is followed exactly by the system output $\mathbf{y}(t) = \mathbf{z}(t)$:

$$\mathbf{z}(t) = \mathbf{y}_r(s(t)),$$

(ii) without violating constraints on states and inputs

$$\mathbf{u}(t) \in \mathcal{U}, \mathbf{x}(t) \in \mathcal{X},$$

(iii) without violating geometric constraints

$$\mathbf{y}_r(s) \in \mathcal{Y},$$

and (iv) in minimal time T .

The problem can be seen as a combined path planning and path following approach. For differentially flat systems, the path following problem can be solved by projecting the system dynamics along the path onto a linear single-input system that is trivial to integrate [3], [4], [5], which avoids the integration of a high-dimensional non-linear state space model.

IV. SOLUTION STRATEGY

This section first reviews the optimal path following problem [3], [4], [5] in which the geometric path is assumed to be given. In a second step, the geometric path is taken into the optimization problem by defining the path as an unknown convex combination of two or more feasible paths.

A. Optimal path following

Let us first fix the geometric reference $\mathbf{y}_r$ and formulate the time-optimal path following problem. In a first step, the system dynamics are projected along the path by applying the chainrule [4].

$$\dot{\mathbf{y}}_r(s) = \partial_s \mathbf{y}_r(s) \dot{s},$$

$$\ddot{\mathbf{y}}_r(s) = \partial_s^2 \mathbf{y}_r(s) \dot{s}^2 + \partial_s \mathbf{y}_r(s) \ddot{s},$$

and so on, which allows us to rewrite states and input in terms of the path coordinate s and its time derivatives:

$$\begin{aligned}\mathbf{x} &= \Phi(s, \dot{s}, \dots, \partial_t^{\beta-1} s), \\ \mathbf{u} &= \Psi(s, \dot{s}, \dots, \partial_t^\beta s).\end{aligned}$$

Now, inspired by [3], instead of using the time t as independent variable, we rewrite states and inputs such that the path coordinate s is the independent variable. This is accomplished by the transformation of variables

$$\begin{aligned}\dot{s}^2 &= b(s), \\ \ddot{s} &= \frac{\partial_s b(s)}{2}, \\ \partial_t^3 s &= \frac{\partial_s^2 b(s) \dot{s}}{2} = \frac{\partial_s^2 b(s) \sqrt{b(s)}}{2},\end{aligned}\tag{4}$$

and so on, such that

$$\begin{aligned}\mathbf{x} &= \Phi(\mathbf{y}_r(s), b(s), \dots, \partial_s^{\beta-2} b(s)) = \Phi(s), \\ \mathbf{u} &= \Psi(\mathbf{y}_r(s), b(s), \dots, \partial_s^{\beta-1} b(s)) = \Psi(s).\end{aligned}$$

Also, by using $dt = \frac{ds}{\dot{s}}$, the terminal time

$$T = \int_0^1 \frac{1}{\sqrt{b(s)}} ds.\tag{5}$$

With the above transformation the optimization problem is formulated as the optimal control problem with pseudo-time s , differential states $b(s)$ up to $\partial_s^{\beta-2} b(s)$ and control $\partial_s^{\beta-1} b(s)$

$$\begin{aligned}\text{minimize}_{b(\cdot)} & \int_0^1 \frac{1}{\sqrt{b(s)}} ds, \\ \text{subject to} & \partial_s^i b(0) = b_{0_i}, \partial_s^i b(1) = b_{T_i}, i = 0, \dots, \beta - 2, \\ & b(s) \geq 0, \forall s \in [0, 1], \\ & \Phi(s) \in \mathcal{X}, \forall s \in [0, 1], \\ & \Psi(s) \in \mathcal{U}, \forall s \in [0, 1],\end{aligned}\tag{6}$$

where b_{0_i} and b_{T_i} are computed such that $\mathbf{x}_0 = \Phi(\mathbf{y}_r(0), b_{0_0}, \dots, b_{0_{\beta-2}})$ and $\mathbf{x}_T = \Phi(\mathbf{y}_r(1), b_{T_0}, \dots, b_{T_{\beta-2}})$. Note that we have obtained a fixed end-time optimal control problem, which, in general, is easier to solve than a free end-time problem. Indeed, the solution of free end-time problems solution can change strongly non-linearly with varying end-times.

In a second stage we also consider the geometric reference path as an unknown. How to deal with the geometric path and constraints is treated in the following section.

B. Combined optimal path planning and path following

Due to the possible complex geometry of the environment, imposing $\mathbf{y}_r(s) \in \mathcal{Y}$ directly in the optimization problem can pose quite a challenge. To circumvent this difficulty, the geometric reference path $\mathbf{y}_r(s)$ is defined as a convex combination of feasible paths $\mathbf{y}_i(s) \in \mathcal{Y}, i = 1, \dots, k$:

$$\mathbf{y}_r(s) = \sum_{i=1}^k \mathbf{y}_i(s) p_i(s),$$

Fig. 2. Geometric reference $\mathbf{y}_r(s)$ as a convex combination of two paths $\mathbf{y}_1(s)$ and $\mathbf{y}_2(s)$ in a constrained environment

with

$$\sum_{i=1}^k p_i(s) = 1 \text{ and } p_i(s) \geq 0, \forall s \in [0, 1],$$

and $p_i(s), \mathbf{y}_i(s) \in \mathcal{C}^\beta$. Fig. 2 shows two functions $\mathbf{y}_1(s)$ and $\mathbf{y}_2(s)$, which stay clear of the geometric constraints in the hatched areas. The geometric reference $\mathbf{y}_r(s)$ is defined as the convex combination of both functions, here with $p_1(s) = 1 - \sqrt{s}$ and $p_2(s) = \sqrt{s}$.

Similarly, the system dynamics can be projected along the path onto a linear single-input system:

$$\begin{aligned}\dot{\mathbf{y}}_r(s) &= \sum_{i=1}^k (\partial_s \mathbf{y}_i(s) p_i(s) + \partial_s p_i(s) \mathbf{y}_i(s)) \sqrt{b(s)}, \\ \ddot{\mathbf{y}}_r(s) &= \sum_{i=1}^k (\partial_s \mathbf{y}_i(s) p_i(s) + \partial_s p_i(s) \mathbf{y}_i(s)) \partial_s b(s) / 2 \\ &+ \sum_{i=1}^k (\partial_s^2 \mathbf{y}_i(s) p_i(s) + 2 \partial_s p_i(s) \partial_s \mathbf{y}_i(s) + \partial_s^2 p_i(s) \mathbf{y}_i(s)) b(s),\end{aligned}\tag{7}$$

and so on, such that we can rewrite the states and inputs to the system as

$$\begin{aligned}\mathbf{x} &= \Phi(p_i, \partial_s p_i, \dots, \partial_s^{\beta-1} p_i, b(s), \dots, \partial_s^{\beta-2} b(s)) = \Phi(s), \\ \mathbf{u} &= \Psi(p_i, \partial_s p_i, \dots, \partial_s^\beta p_i, b(s), \dots, \partial_s^{\beta-1} b(s)) = \Psi(s).\end{aligned}$$

The path following problem (6) is extended with the control $\partial_s^\beta p_i(s), i = 1, \dots, k$ and states $p_i(s)$ up to $\partial_s^{\beta-1} p_i(s), i = 1, \dots, k$ to arrive at following optimal control problem

$$\begin{aligned}\text{minimize}_{b(\cdot), p_i(\cdot)} & \int_0^1 \frac{1}{\sqrt{b(s)}} ds, \\ \text{subject to} & \partial_s^i b(0) = b_{0_i}, \partial_s^i b(1) = b_{T_i}, i = 0, \dots, \beta - 2, \\ & \sum_{i=1}^k p_i(s) = 1, \forall s \in [0, 1], \\ & b(s) \geq 0, \forall s \in [0, 1], \\ & p_i(s) \geq 0, \forall s \in [0, 1], i = 1, \dots, k, \\ & \Phi(s) \in \mathcal{X}, \forall s \in [0, 1], \\ & \Psi(s) \in \mathcal{U}, \forall s \in [0, 1].\end{aligned}\tag{8}$$

Note that the optimal control problem does not require the integration of a possibly high-dimensional non-linear state

parameter	value	unit
m	18.2	kg
r	0.0825	m
I	0.39375	kg m ²
b	0.1624	m

TABLE I
NUMERICAL PARAMETERS FOR THE WHEELED MOBILE ROBOT

space model and that it has been reduced to a smaller dimension. Indeed only the scalar functions $b(s)$ and $p_i(s)$ have to be determined. Moreover, the problem is transformed into a fixed end-time problem, which can be initialized easily.

V. EXAMPLES

To illustrate the developed methodology, we apply our approach to a wheeled mobile robot. The numerical values of the system's parameters from (3) are listed in table I. The motor torques are constrained to $[-2, 2]$ N m and the angular velocity of the wheels is constrained to $[-12, 12]$ rad/s. These constraints can be chosen such that saturation is avoided and the no-slip assumption remains valid [6], [7]. The initial and terminal states are chosen such that the robot moves from (0,0) m to (1,1) m and $\theta(0) = \theta(T) = v(0) = v(T) = \omega(0) = \omega(T) = 0$.

The optimization problem is discretized using a direct transcription method for $b(s)$ as in [3] and $p_i(s)$ are parametrized as B -splines with 15 equidistant knots. The discretized problems are modeled in CasADi [10] and solved with Ipopt [11] on a 2.66 GHz pc with 4 GB of RAM using exact Hessians. The reported solver times only include the time spent in Ipopt. All examples are solved using automatic initialization. Therefore, the user is not required to supply an initial guess, whereas in many other techniques a good initial guess is required for the solver to converge.

A. Time-optimal path following

In a first example we consider steering the mobile robot in a constrained environment, as shown in Fig. 3, time-optimally along a predefined path $\mathbf{y}_r(s) = (x(s), y(s))^T$. $x(s)$ and $y(s)$ are degree 4 splines for which the control points are chosen such that the walls are avoided and $x(0) = y(0) = 0$, $x(1) = y(1) = 1$, $\partial_s x(0) = \partial_s x(1) = 0$ and $\partial_s^i y(0) = \partial_s^i y(1) = 0$ for $i = 1, 2, 3$, which assures the system starts and stops in the desired state. The geometric path is drawn in Fig. 3.

An optimal solution is calculated in 0.05 s. In Fig. 3 the location of the robot is shown for 20 equally spaced timesteps. Fig. 4 shows the associated wheel velocities and torques. Note that at each time instant at least one of the constraints is active due to time-optimality. The optimal time amounts 4.14 s.

B. Time-optimal point-to-point motion in a constrained environment

Now let us also optimize the path. To this end, similar to the previous example, the outer paths are defined as splines of degree 4 and are shown as the dotted lines in Fig. 5.

Fig. 3. Time-optimal path following for a wheeled mobile robot. The robot is drawn for 20 equally spaced time steps.

Fig. 4. The associated wheel velocities and torques for the time optimal movement of Fig. 3.

The calculation time of 0.88 s is significantly more than in previous example. However, the calculation time is still low considering the complexity of the problem. The optimal path is shown in Fig. 5 by the solid line and the robot is drawn for 20 equally spaced time steps. The solution is symmetrical due to the symmetry of the constraints and the neglect of friction in the model. Fig. 6 shows the associated wheel speeds and torques. The optimal time is 3.44 s, which is a significant improvement of 17 % compared to previous example. Note that the obstacles are only avoided by a suitable choice of the outer paths, which have to be determined before the optimization. Obviously, their choice influences directly the amount of freedom for the optimized path.

Fig. 5. Time-optimal movement of a wheeled mobile robot in a constrained environment. The robot is drawn for 20 equally spaced time steps.

CONCLUSIONS

The paper proposes a framework to solve the time-optimal path planning problem for differentially flat systems. Hereto we developed a combined path planning, path following approach. The geometric path is represented as a convex combination of two or more feasible outer paths, which avoids complex geometric constraints in the optimization problem. Subsequently, the system dynamics are projected onto the parametrized path, which leads to a single input system. The resulting optimization problem is transformed into a fixed end-time optimal control problem which can be solved efficiently by interior-point solvers, such as Ipopt [11]. The approach is illustrated for a wheeled mobile robot, showing the efficiency of the approach. In both examples the solver converges in under a second, without the need of an initial guess and static objects can easily be avoided by a suitable choice of the outer functions without complicating the optimization problem. By supplying a good initial guess and only performing a few iterations an online implementation of the algorithm may be possible.

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Fig. 6. The associated wheel velocities and torques for the time optimal movement of Fig. 5.

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