Hopf C*-algebras

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April 10, 2002

Proceedings of the London Mathematical Society (3) **82** (2001), 337–384.

Abstract

In this paper we define and study Hopf C*-algebras. Roughly speaking, a Hopf C*-algebra is a C*-algebra A with a comultiplication $\phi : A \rightarrow M(A \otimes A)$ such that the maps $a \otimes b \mapsto \phi(a)(1 \otimes b)$ and $a \otimes b \mapsto (a \otimes 1)\phi(b)$ have their range in $A \otimes A$ and are injective after being extended to a larger natural domain, the Haagerup tensor product $A \otimes_h A$. In a purely algebraic setting, these conditions on ϕ are closely related with the existence of a counit and antipode. In this topological context, things turn out to be much more subtle, but nevertheless one can show the existence of a suitable counit and antipode under these conditions.

The basic example is the C*-algebra $C_0(G)$ of continuous complex functions tending to zero at infinity on a locally compact group where the comultiplication is obtained by dualizing the group multiplication. But also the reduced group C*-algebra $C_r^*(G)$ of a locally compact group with the well known comultiplication falls in this category. In fact all locally compact quantum groups in the sense of [13] and [14] (such as the compact and discrete ones) as well as most of the known examples are included.

This theory differs from other similar approaches in that there is no Haar measure assumed.

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[†]Supported by the F.W.O. Research Network WO.011.96N

Introduction

Let G be a locally compact space. Consider the C^* -algebra $C_0(G)$ of all complex continuous functions on G tending to 0 at infinity. Any abelian C^* -algebra has this form. Moreover, the topological structure of G is completely determined by the C^* -algebra structure of $C_0(G)$. This is why an arbitrary C^* -algebra is sometimes thought of as ‘the continuous functions tending to 0 at infinity on a pseudo space’ (a quantum space or a non-commutative space). In this point of view, the study of C^* -algebras is non-commutative topology.

Now, one can consider extra structure on such a locally compact space, translate this structure to the C^* -algebra and then formulate it on the abstract level, for any C^* -algebra. This is the quantization procedure. This paper deals with quantizing the group structure of a locally compact group.

Before we can explain this, we need to recall some results in C^* -algebra theory. When A is a C^* -algebra, we denote by $M(A)$ the multiplier algebra of A . A $*$ -homomorphism $\gamma : A \rightarrow M(B)$ for C^* -algebras A and B is called *non-degenerate* if an approximate identity in A is mapped to a net strictly converging to 1 in $M(B)$. This is equivalent with the requirement that $\gamma(A)B$ is dense in B . In Woronowicz’ terminology one would call γ a morphism from A to B . An important property of such a non-degenerate $*$ -homomorphism is that it has a unique extension to a unital $*$ -homomorphism from $M(A)$ to $M(B)$. This extension is still denoted by γ and it is defined by

$$\gamma(x)\gamma(a)b = \gamma(xa)b$$

whenever $x \in M(A)$, $a \in A$ and $b \in B$. The extension is strictly continuous on bounded subsets of $M(A)$.

Now, let G be a locally compact group. First, we will explain how the group structure is translated to the C^* -algebra $C_0(G)$. Take $f \in C_0(G)$ and dualize the product in G to get a function $\phi(f)$ of two variables defined by $\phi(f)(p, q) = f(pq)$ for any $p, q \in G$. This function is bounded and continuous. We clearly get a $*$ -homomorphism ϕ from $C_0(G)$ to $C_b(G \times G)$, the C^* -algebra of bounded, continuous complex functions on $G \times G$. Now, there is a natural identification of $C_b(G \times G)$ with the multiplier algebra $M(C_0(G \times G))$ and further of $C_0(G \times G)$ with $C_0(G) \otimes C_0(G)$, the (minimal) C^* -tensor product of $C_0(G)$ with itself. Therefore, we may consider $\phi : C_0(G) \rightarrow M(C_0(G) \otimes C_0(G))$. This $*$ -homomorphism is non-degenerate. The associativity of the product in G gives rise to the coassociativity law $(\phi \otimes \iota)\phi = (\iota \otimes \phi)\phi$ of ϕ . Here ι denotes the identity map. And $\phi \otimes \iota$ and $\iota \otimes \phi$ are the unique extensions to $M(C_0(G) \otimes C_0(G))$ of the obvious maps on $C_0(G) \otimes C_0(G)$.

In definition 2.1, we will generalize this notion and define a comultiplication on a C^* -algebra A as a non-degenerate $*$ -homomorphism $\phi : A \rightarrow M(A \otimes A)$ satisfying coassociativity: $(\phi \otimes \iota)\phi = (\iota \otimes \phi)\phi$.

It is important to observe that, conversely, any comultiplication on an abelian C^* -algebra A comes from a continuous associative multiplication of the underlying locally compact space. Therefore we will think of a pair (A, ϕ) of a C^* -algebra A with a comultiplication ϕ as a *locally compact quantum semi-group*.

Next, consider the identity e of the group. This gives rise to a $*$ -homomorphism $\varepsilon : C_0(G) \rightarrow \mathbb{C}$ given by $\varepsilon(f) = f(e)$. The property $pe = ep = p$ for all $p \in G$ is expressed in terms of ε by

$$(\varepsilon \otimes \iota)\phi(f) = (\iota \otimes \varepsilon)\phi(f) = f$$

for all $f \in C_0(G)$. This naturally leads us to the notion of a counit for a general pair (A, ϕ) . A possible definition of a counit is a $*$ -homomorphism $\varepsilon : A \rightarrow \mathbb{C}$ satisfying

$$(\varepsilon \otimes \iota)\phi(x) = (\iota \otimes \varepsilon)\phi(x) = x$$

for all $x \in A$. Such a notion makes sense but unfortunately turns out to be too restrictive. The typical example to illustrate this comes from a locally compact non-amenable group G . Take for A the reduced group C^* -algebra $C_r^*(G)$. This carries a natural comultiplication ϕ characterized by $\phi(\lambda_p) = \lambda_p \otimes \lambda_p$ where λ_p is left translation by p on $L^2(G)$, considered as a multiplier of $C_r^*(G)$. If ε is a counit as above, it should verify $\varepsilon(\lambda_p) = 1$ for all p . However, as G is not amenable, the trivial representation of G is not weakly contained in the regular representation and so there is no such bounded $*$ -homomorphism on $C_r^*(G)$ (see e.g. [18], Theorem 8.9). This problem could be overcome by taking the full group C^* -algebra $C^*(G)$. This will create a new problem : the natural candidate for the Haar measure will no longer be faithful. This turns out to be more serious.

Therefore, we will work with another notion of a counit (see proposition 3.9), possibly not everywhere defined (and unbounded). A $*$ -homomorphism $\varepsilon : A \rightarrow \mathbb{C}$ satisfying $(\varepsilon \otimes \iota)\phi(x) = (\iota \otimes \varepsilon)\phi(x) = x$ for all $x \in A$ will be called a *bounded* counit.

Finally, consider the inverse in the group. We will see that this is even more problematic. Define the map $S : C_0(G) \rightarrow C_0(G)$ by $(Sf)(p) = f(p^{-1})$ for all $p \in G$: this map is called the antipode. Here it is a $*$ -homomorphism. The properties $pp^{-1} = e$ and $p^{-1}p = e$ for all $p \in G$ are translated in terms of the comultiplication ϕ , the counit ε and the antipode S on the C^* -algebra level into the formulas

$$\begin{aligned} m(\iota \otimes S)\phi(f) &= \varepsilon(f)1 \\ m(S \otimes \iota)\phi(f) &= \varepsilon(f)1 \end{aligned}$$

for all $f \in C_0(G)$. Here m denotes the multiplication map, defined from $C_b(G \times G)$ to $C_b(G)$ by $m(h)(p) = h(p, p)$ when $h \in C_b(G \times G)$. Observe that $m(h) = fg$ when $h = f \otimes g$ as then

$$m(h)(p) = h(p, p) = f(p)g(p) = (fg)(p)$$

for all $p \in G$.

If we want to formulate all of this for a general C^* -algebra A with a comultiplication ϕ , we run into all sorts of problems. First, we need a counit ε and this was already a problem. Secondly, as we know from general Hopf algebra theory, as soon as we work with non-abelian algebras, the antipode S should be an anti-homomorphism, rather than a homomorphism. Furthermore, in the interesting examples, the candidate for the antipode will not be a bounded everywhere defined map. This looks similar to the counit being unbounded, but it is a different problem. In the example $C_r^*(G)$ with G non-amenable ε is unbounded

while S is bounded. The unboundedness of S is related with the property that S also need not be a $*$ -map. Now, we are using $S \otimes \iota$ and $\iota \otimes S$ for characterizing the antipode and it is not clear how to give a meaning to these maps when S is unbounded. But even when S were a $*$ -anti-homomorphism, it is not obvious how to define this on the C^* -tensor product $A \otimes A$ and so also not on $M(A \otimes A)$. Finally the last problem that we are facing here is the multiplication map m . It is in general defined on the algebraic tensor product $A \odot A$ in the obvious way. But generally it will not extend to $A \otimes A$ and so not to $M(A \otimes A)$. That this can be done is typical for the abelian case.

So, it should be clear from the above discussion that there are serious problems with the concept of the antipode and (although less serious) with the counit for a general pair of a C^* -algebra A with a comultiplication ϕ .

This is probably the reason why, up to now, there was not a good notion of a Hopf C^* -algebra. In the literature, there existed essentially two directions. Some authors already used the name Hopf C^* -algebra for a pair of a C^* -algebra with a comultiplication, see e.g. [2], [22] and [16]. This is not very appropriate because the antipode is necessary to distinguish a Hopf algebra from a bialgebra. In the other approach, the existence of an antipode is assumed. In the older literature (e.g. when defining C^* -Kac algebras, see [11] and [22]), the map S is taken to be a $*$ -anti-automorphism satisfying $\phi(S(a)) = \sigma(S \otimes S)\phi(a)$ where σ is the flip. The latter is a property of the antipode in a Hopf algebra but much weaker than the usual requirement. This shortcoming was dissolved by imposing an extra condition involving the Haar measure. Moreover, from recent examples, we know that it is also too strong to require S to be everywhere defined and to satisfy $S(a^*) = S(a)^*$. This last difficulty can be overcome by using some kind of polar decomposition of the antipode. But still, the basic axiom for S is formulated in connection with the Haar measure (see e.g. [15]).

In this paper we propose a solution to this problem. We will start with a pair of a C^* -algebra A and a comultiplication ϕ on A . And we will impose extra, but natural conditions on ϕ so that it is possible to construct a counit ε and an antipode S and to give a meaning to the usual formulas

$$\begin{aligned} (\varepsilon \otimes \iota)\phi(a) &= (\iota \otimes \varepsilon)\phi(a) = a \\ m(S \otimes \iota)\phi(a) &= m(\iota \otimes S)\phi(a) = \varepsilon(a)1 \end{aligned}$$

for all a in a suitable subspace of A .

The approach will heavily rely on the Haagerup tensor product $A \otimes_h A$. We will recall in section 1 the definition and the basic properties. The main reason for using the Haagerup tensor product is that it is a natural domain for the multiplication map. Indeed, the multiplication $m : A \odot A \rightarrow A$ defined on the algebraic tensor product, is a contraction for the Haagerup norm and hence extends by continuity to a contraction from the completion $A \otimes_h A$. In fact, we will need to work with some larger space, denoted by $A \otimes_{mh} A$. Elements x in this space have the property that $(a \otimes 1)x$ and $x(1 \otimes a)$ belong to $A \otimes_h A$ for all $a \in A$. This will be true essentially by definition. This larger space will also be introduced in section 1, where we will give its properties needed further in the paper. The basic point is that multiplication is now a contraction from this large space $A \otimes_{mh} A$ to $M(A)$.

The conditions on ϕ are formulated in terms of the maps T_1 and T_2 defined by

$$\begin{aligned} T_1(a \otimes b) &= \phi(a)(1 \otimes b) \\ T_2(a \otimes b) &= (a \otimes 1)\phi(b). \end{aligned}$$

These maps are first defined on the algebraic tensor product $A \odot A$ and have range in $A \otimes A$ (by one of the assumptions on ϕ). Now, it should be no surprise that these maps are continuous for the Haagerup norm on $A \odot A$ and hence extend to contractions from $A \otimes_h A$ to $A \otimes A$. The extra requirement is that these extensions are injective.

Recall that in the case of a Hopf algebra (or more generally, a multiplier Hopf algebra), these maps T_1 and T_2 are bijective from the algebraic tensor product of the underlying algebra to itself. In fact, this is essentially the property characterizing Hopf algebras among bialgebras. We obviously have a topological analogue of this condition here. And we will see further in the paper why it is precisely the Haagerup tensor product that is natural to use.

The conditions that we will impose will obviously be fulfilled in our motivating example coming from a locally compact group. It will also be possible to verify them in many, if not all of the existing examples (see section 5).

Much of this paper is inspired by the work on multiplier Hopf algebras. In fact, what we do here is, to a great extent, consider the different steps in the original development of multiplier Hopf algebras (see [23]) and try to translate this to the C^* -framework. Therefore, many formulas in this paper are inspired by their analogues in multiplier Hopf algebras. We will sometimes refer to these and use them as a motivation or to illustrate the new formulas here.

However, it is not required to know the work on multiplier Hopf algebras. It is even not necessary to know Hopf algebras (although this will of course help in understanding the paper). What we do require is basic C^* -algebra theory. In subsection 5.3 we study a class of examples of Hopf C^* -algebras, namely the locally compact quantum groups. In order to understand this, one should know something about weights on C^* -algebras.

The paper is organized as follows. In section 1 we start by recalling the notion of the Haagerup tensor product and its basic properties. We also define a larger space and prove the necessary properties needed for the rest of the paper. In section 2 we recall the notion of a comultiplication ϕ on a C^* -algebra A and we explain the extra condition we will need to develop the theory of Hopf C^* -algebras. In section 3 we prove the existence of the counit and the antipode under the given condition. It should be said however that we can obtain this from two directions, left and right. In section 4 we treat the construction of the counit and the antipode in a ‘two-sided’ way. We will essentially get a domain on which the left and right counits and antipodes will coincide. Finally, in section 5 we will treat many examples and thus justify our axioms and our definition of Hopf C^* -algebras. Because we will be working with the Haagerup tensor product, the theory of operator spaces will be in the background throughout the paper although we never require more than elementary results from that theory.

The paper is a slight modification of the first author’s masters thesis [21]. The main difference lies in the use of another type of generalized Haagerup tensor product. We will say more

about this in section 2. In principle, the approach here is more general (although we still are not sure about this). It is somewhat more natural and in some sense simpler.

Let us now finish this introduction and fix some basic notations. We will use $(H, \Delta, \varepsilon, S)$ for a Hopf algebra (or a multiplier Hopf algebra) and in that context, all the tensor products will be algebraic tensor products. We will however use $\odot$ for the algebraic tensor product when we are dealing with topological spaces (C*-algebras most of the time) and $\otimes$ for completed tensor products. In particular $A \otimes A$ will be the minimal C*-tensor product of the C*-algebra A with itself. When we write $\otimes_h$, we will also mean the completed Haagerup tensor product (see section 1). We will use ι to denote the identity maps. We will identify $A \otimes \mathbb{C}$ and $\mathbb{C} \otimes A$ with A and so slice maps $\iota \otimes \omega$ and $\omega \otimes \iota$ will go e.g. from $A \otimes A$ to A (for any $\omega \in A^*$, the norm dual of A). As we mentioned already before, $M(A)$ will denote the multiplier algebra of the C*-algebra A and it will be considered both with its norm topology and the strict topology making the maps $x \mapsto ax$ and $x \mapsto xa$ from $M(A)$ to A continuous for all $a \in A$. We have already mentioned the extension property for non-degenerate *-homomorphisms. We will also need to extend functionals in A^* . It is important for this to notice that elements $\omega \in A^*$ are always of the form $\omega = \omega_1(\cdot c) = \omega_2(d \cdot)$ for some $c, d \in A$ and $\omega_1, \omega_2 \in A^*$. It follows that elements in A^* have unique strictly continuous extensions to $M(A)$. We will sometimes denote $\omega_1(\cdot c)$ by $c\omega_1$ and $\omega_2(d \cdot)$ by $\omega_2 d$. We will use the symbol ϕ to denote a comultiplication on a C*-algebra A (see def. 2.1). This will be a non-degenerate *-homomorphism from A to $M(A \otimes A)$ (hence a morphism from A to $A \otimes A$ in Woronowicz' terminology) satisfying $(\phi \otimes \iota)\phi = (\iota \otimes \phi)\phi$. Then we use the notation $\phi^{(2)} = (\phi \otimes \iota)\phi = (\iota \otimes \phi)\phi$. We will use sometimes the leg-numbering notation, as well for *-homomorphisms as for operators. So ϕ_{12} will simply denote $\phi \otimes \iota$ while ϕ_{13} equals ϕ_{12} composed with the flip map between the second and the third leg. Analogously one introduces u_{12} and u_{13} when u is a bounded operator or an element in some tensor product.

The basic references for C*-algebras are [17] and [20]. For Hopf algebras we refer to [1] and for multiplier Hopf algebras to [23].

Acknowledgment. The second author likes to thank his colleagues at the University of Fukuoka (where this work was started). We are also grateful for the hospitality we both experienced at the University of Trondheim (where this work was completed). Moreover we enjoy the pleasant working atmosphere in Leuven (where most work on this paper was done).

1 The multiplier Haagerup tensor product

In this preliminary section we will first recall the definition of the Haagerup tensor product and its properties we will need further in the paper. For details, we refer to e.g. [4], [5] and [9].

Let A be a C*-algebra. Denote by $A \odot A$ the algebraic tensor product of A with itself.

Definition 1.1. *The Haagerup tensor product on $A \odot A$ is defined by*

$$\|x\|_h = \inf \left\{ \left\| \sum_{i=1}^n p_i p_i^* \right\|^{1/2} \left\| \sum_{i=1}^n q_i^* q_i \right\|^{1/2} \mid n \in \mathbb{N}, x = \sum_{i=1}^n p_i \otimes q_i \right\}.$$

So, the infimum is taken over all the possible representations of x in $A \odot A$. One can prove that this is indeed a norm. The completion of $A \odot A$ with respect to this norm will be denoted here by $A \otimes_h A$.

Let now I be an index set and suppose that $(p_i)_{i \in I}$ and $(q_i)_{i \in I}$ are families of elements of A such that the nets $\sum_{i \in I} p_i p_i^*$ and $\sum_{i \in I} q_i^* q_i$ are norm convergent in A . Then the net $\sum_{i \in I} p_i \otimes q_i$ will be norm convergent in $A \otimes_h A$. This is quite easy to see, because for $I_0 \subseteq I$ and I_0 finite we have

$$\left\| \sum_{i \in I_0} p_i \otimes q_i \right\|_h^2 \leq \left\| \sum_{i \in I_0} p_i p_i^* \right\| \left\| \sum_{i \in I_0} q_i^* q_i \right\|.$$

It is clear that

$$\left\| \sum_{i \in I} p_i \otimes q_i \right\|_h^2 \leq \left\| \sum_{i \in I} p_i p_i^* \right\| \left\| \sum_{i \in I} q_i^* q_i \right\|.$$

It is also possible to prove a converse of this statement: every element of $A \otimes_h A$ can be written as such an infinite sum. We do not go into this because we will not need this result.

Quite often we will extend maps defined on the algebraic tensor product $A \odot A$ to $A \otimes_h A$ by showing their continuity. We will use the following obvious lemma (see also lemma 4.2).

Lemma 1.2. *Let B be a C^* -algebra and $p_i, q_i \in B$ for $i = 1, \dots, n$. Then*

$$\left\| \sum_{i=1}^n p_i q_i \right\|^2 \leq \left\| \sum_{i=1}^n p_i p_i^* \right\| \left\| \sum_{i=1}^n q_i^* q_i \right\|.$$

Now we prove the following crucial result.

Proposition 1.3. *The inclusion of $A \odot A$ in $A \otimes A$ is a contraction for the Haagerup norm on $A \odot A$ and can be extended uniquely to a contraction from $A \otimes_h A$ to $A \otimes A$. This extension is injective.*

Proof. When $p_i, q_i \in A$ we have by the previous lemma

$$\left\| \sum_{i=1}^n p_i \otimes q_i \right\|^2 = \left\| \sum_{i=1}^n (p_i \otimes 1)(1 \otimes q_i) \right\|^2 \leq \left\| \sum_{i=1}^n p_i p_i^* \right\| \left\| \sum_{i=1}^n q_i^* q_i \right\|.$$

So we can extend the inclusion of $A \odot A$ in $A \otimes A$ to a contraction j from $A \otimes_h A$ to $A \otimes A$. Now we have to prove the injectivity of j . Suppose $j(x) = 0$ for $x \in A \otimes_h A$. Then we have $(\omega \otimes \mu)(x) = (\omega \otimes \mu)j(x) = 0$ for all $\omega, \mu \in A^*$. Because the inclusion of $A \otimes_h A$ in $(A^* \otimes_h A^*)^*$ is completely isometric (see e.g. [9]), we may conclude that $x = 0$.

From now on we will identify $A \otimes_h A$ with a dense subspace of $A \otimes A$.

From lemma 1.2 follows immediately the following result.

Proposition 1.4. *The multiplication map $m : p \otimes q \mapsto pq$ can be uniquely extended to a contraction from the Haagerup tensor product $A \otimes_h A$ to A .*

It is easy to see that the maps

$$x \mapsto x(1 \otimes a) \quad \text{and} \quad x \mapsto (a \otimes 1)x$$

are bounded from $A \otimes_h A$ to itself with norms majorized by $\|a\|$. This, together with the following lemma will be the starting point for the introduction of a larger space, the so-called multiplier Haagerup tensor product.

Lemma 1.5. *If (e_λ) is a bounded approximate identity in A , then $x(1 \otimes e_\lambda) \rightarrow x$ and $(e_\lambda \otimes 1)x \rightarrow x$ in $A \otimes_h A$ for all $x \in A \otimes_h A$.*

Proof. It is sufficient to do this for elements in $A \odot A$. If $x = \sum_{i=1}^n p_i \otimes q_i$ then $x - x(1 \otimes e_\lambda) = \sum_{i=1}^n p_i \otimes q_i(1 - e_\lambda)$ and so

$$\|x - x(1 \otimes e_\lambda)\|_h \leq \left\| \sum_{i=1}^n p_i p_i^* \right\|^{1/2} \left\| \sum_{i=1}^n (1 - e_\lambda^*) q_i^* q_i (1 - e_\lambda) \right\|^{1/2}$$

and this last expression will tend to 0.

We will now introduce the multiplier Haagerup tensor product $A \otimes_{mh} A$. The way we will do this here is different from [21]. In principle we get a larger space, although we are not completely sure about that. In any case, this new approach seems to be somehow more natural. We define $A \otimes_{mh} A$ in a way which is very similar to the definition of the multiplier algebra $M(A \otimes A)$ of $A \otimes A$. We will see later in the next section why this is natural. In fact, this should also become clear from certain properties that we will prove already here.

Definition 1.6. *Denote by $A \otimes_{mh} A$ the space of pairs of maps (ρ_1, ρ_2) from A to $A \otimes_h A$ satisfying $(a \otimes 1)\rho_1(b) = \rho_2(a)(1 \otimes b)$ for all $a, b \in A$.*

We clearly have $A \otimes_h A$ sitting in $A \otimes_{mh} A$ by putting $\rho_1(a) = x(1 \otimes a)$ and $\rho_2(a) = (a \otimes 1)x$ where $x \in A \otimes_h A$ and $a \in A$. We also have an imbedding as $x(1 \otimes a) = 0$ for all a will imply $x = 0$ by the previous lemma.

We will now show that $A \otimes_{mh} A$ carries a norm such that this space is complete and contains $A \otimes_h A$ isometrically.

Proposition 1.7. *Let $x = (\rho_1, \rho_2) \in A \otimes_{mh} A$. Then ρ_1 and ρ_2 are bounded and $\|\rho_1\| = \|\rho_2\|$. Further $\|x\| := \|\rho_1\| = \|\rho_2\|$ defines a norm on $A \otimes_{mh} A$ for which this space is complete. Moreover, the imbedding of $A \otimes_h A$ in $A \otimes_{mh} A$ described above is isometric.*

Proof. We will first show that the maps ρ_1 and ρ_2 are closed. Take $a_n \in A$ and assume that $a_n \rightarrow a$ in A and $\rho_1(a_n) \rightarrow x$ in $A \otimes_h A$. Then for any $b \in A$ we have

$$\begin{aligned} (b \otimes 1)x &= \lim_n (b \otimes 1)\rho_1(a_n) \\ &= \lim_n \rho_2(b)(1 \otimes a_n) \\ &= \rho_2(b)(1 \otimes a) \\ &= (b \otimes 1)\rho_1(a). \end{aligned}$$

Then $x = \rho_1(a)$. By the closed graph theorem we have that ρ_1 is bounded. Similarly ρ_2 is bounded. Now, because of lemma 1.5 we have

$$\begin{aligned}\|\rho_1\| &= \sup\{\|\rho_1(a)\|_h \mid \|a\| \leq 1\} \\ &= \sup\{\|(b \otimes 1)\rho_1(a)\|_h \mid \|a\| \leq 1, \|b\| \leq 1\} \\ &= \sup\{\|\rho_2(b)(1 \otimes a)\|_h \mid \|a\| \leq 1, \|b\| \leq 1\} \\ &= \|\rho_2\|.\end{aligned}$$

As for $x \in A \otimes_h A$ we have $\|x\|_h = \sup\{\|x(1 \otimes a)\|_h \mid \|a\| \leq 1\}$ we get that the imbedding of $A \otimes_h A$ in $A \otimes_{mh} A$ is isometric.

Finally we show that $A \otimes_{mh} A$ is complete. Take a Cauchy sequence (x_n) in $A \otimes_{mh} A$. Write $x_n = (\rho_{1n}, \rho_{2n})$. Then $(\rho_{1n}(a))_n$ and $(\rho_{2n}(a))_n$ are Cauchy sequences in $A \otimes_h A$. They have limits and we get maps (ρ_1, ρ_2) defined by the pointwise limit. Obviously they still satisfy $(a \otimes 1)\rho_1(b) = \rho_2(a)(1 \otimes b)$ for all $a, b \in A$. By a standard technique we get that $\rho_{1n} \rightarrow \rho_1$ and $\rho_{2n} \rightarrow \rho_2$ also in norm. This completes the proof.

Further we will write $x(1 \otimes a)$ for $\rho_1(a)$ and $(a \otimes 1)x$ for $\rho_2(a)$, when $x = (\rho_1, \rho_2)$ for any $x \in A \otimes_{mh} A$. This notation is compatible with the relation between ρ_1 and ρ_2 (now expressed as associativity), with the imbedding of $A \otimes_h A$ in $A \otimes_{mh} A$ and with the norm properties. Moreover, we can justify this notation even further because we have the following result.

Proposition 1.8. *The inclusion $j : A \otimes_h A \rightarrow A \otimes A$ can be uniquely extended to a contraction from $A \otimes_{mh} A$ to $M(A \otimes A)$, still denoted by j , such that*

$$j(x)(1 \otimes a) = j(x(1 \otimes a)) \quad \text{and} \quad (a \otimes 1)j(x) = j((a \otimes 1)x)$$

for all $x \in A \otimes_{mh} A$ and $a \in A$. Also this extension is injective.

Proof. Take $x \in A \otimes_{mh} A$. As $x(1 \otimes a) \in A \otimes_h A \subseteq A \otimes A$ we can clearly define a left multiplier from $A \odot A$ to $A \otimes A$ by $x(b \otimes a) = x(1 \otimes a)(b \otimes 1)$. We will show that this is bounded and therefore extends to a left multiplier from $A \otimes A$ to $A \otimes A$. To see this, take a bounded approximate unit (e_λ) of A and observe that

$$(e_\lambda \otimes 1) \left(x \left(\sum_{i=1}^n b_i \otimes a_i \right) \right) = ((e_\lambda \otimes 1)x) \left(\sum_{i=1}^n b_i \otimes a_i \right).$$

Now $\|(e_\lambda \otimes 1)x\| \leq \|(e_\lambda \otimes 1)x\|_h \leq \|x\|_h$ and so

$$\left\| x \left(\sum_{i=1}^n b_i \otimes a_i \right) \right\| \leq \|x\|_h \left\| \sum_{i=1}^n b_i \otimes a_i \right\|.$$

Similarly, x defines a bounded right multiplier of $A \otimes A$. And obviously $((a \otimes b)x)(c \otimes d) = (a \otimes b)(x(c \otimes d))$ so that indeed $x \in M(A \otimes A)$.

From now on we will identify $A \otimes_{mh} A$ with a subspace of $M(A \otimes A)$. So we have $A \otimes_{mh} A$ in $M(A \otimes A)$ and now $x(1 \otimes a)$ makes sense in $M(A \otimes A)$. In fact we get the following characterization.

Proposition 1.9. *If $x \in M(A \otimes A)$, then $x \in A \otimes_{mh} A$ if and only if $x(1 \otimes a) \in A \otimes_h A$ and $(a \otimes 1)x \in A \otimes_h A$ for all $a \in A$.*

It is clear that we could also have started with this property to define $A \otimes_{mh} A$ from the beginning as sitting in $M(A \otimes A)$.

Now, very often, we will need to extend maps from $A \otimes_h A$ to $A \otimes_{mh} A$. The first example we have in mind is multiplication :

Proposition 1.10. *The multiplication map m can be extended uniquely to a contraction from $A \otimes_{mh} A$ to $M(A)$ satisfying*

$$\begin{aligned} m(x)a &= m(x(1 \otimes a)) \\ am(x) &= m((a \otimes 1)x) \end{aligned}$$

when $x \in A \otimes_{mh} A$ and $a \in A$.

Proof. Now, if $x \in A \otimes_{mh} A$ we can define two maps from A to A by

$$\begin{aligned} a &\mapsto m(x(1 \otimes a)) \\ a &\mapsto m((a \otimes 1)x) \end{aligned}$$

where we use now m for the multiplication from $A \otimes_h A$ to A . Because

$$\begin{aligned} am(x(1 \otimes b)) &= m((a \otimes 1)x(1 \otimes b)) \\ &= m((a \otimes 1)x)b \end{aligned}$$

this pair of maps gives an element in $M(A)$.

This is the kind of technique we will be using, but it is good to know what is really behind. This is what we will explain now.

Definition 1.11. *By the strict topology on $A \otimes_{mh} A$ we mean the locally convex topology given by the semi-norms $x \mapsto \|x(1 \otimes a)\|_h$ and $x \mapsto \|(a \otimes 1)x\|_h$ for $a \in A$.*

We have the following density result.

Proposition 1.12. *When $x \in A \otimes_{mh} A$ and (e_λ) is a bounded approximate identity in A , then $x(1 \otimes e_\lambda) \rightarrow x$ and $(e_\lambda \otimes 1)x \rightarrow x$ strictly.*

Proof. Take $a \in A$ and observe that $(a \otimes 1)(x(1 \otimes e_\lambda)) = ((a \otimes 1)x)(1 \otimes e_\lambda) \rightarrow (a \otimes 1)x$ in $A \otimes_h A$ (lemma 1.5). On the other hand also $x(1 \otimes e_\lambda)(1 \otimes a) = x(1 \otimes e_\lambda a) \rightarrow x(1 \otimes a)$ as $e_\lambda a \rightarrow a$ in norm in A .

It follows that the unit ball of $A \otimes_h A$ is strictly dense in the unit ball of $A \otimes_{mh} A$. Therefore to extend maps one can show that they are continuous for the strict topology on bounded sets. Such is e.g. the case for the multiplication map. In fact, m as defined in 1.10 from $A \otimes_{mh} A$ to $M(A)$ is continuous when both spaces are taken with the strict topology. From the proof of proposition 1.7 also follows immediately that the inclusion of $A \otimes_{mh} A$ in $M(A \otimes A)$ is strictly continuous on bounded sets.

It is possible to define slice maps $\omega \otimes \iota$ from $A \otimes_h A$ to A when $\omega \in A^*$ (see e.g. the discussion after lemma 4.2). These slice maps can be extended to $A \otimes_{mh} A$ as follows.

Proposition 1.13. *Let $\omega \in A^*$. The maps $\omega \otimes \iota$ and $\iota \otimes \omega$ from $A \otimes_h A$ to A extend uniquely to strict-norm continuous maps from $A \otimes_{mh} A$ to A satisfying*

$$a(\omega \otimes \iota)(x) = (\omega \otimes \iota)((1 \otimes a)x) \quad \text{and} \quad (\omega \otimes \iota)(x)a = (\omega \otimes \iota)(x(1 \otimes a))$$

and satisfying analogous formulas for $\iota \otimes \omega$. The norms of these maps are majorized by $\|\omega\|$.

Remark that we mean by strict-norm continuity from $A \otimes_{mh} A$ to A continuity when $A \otimes_{mh} A$ is equipped with the strict topology and A with the norm topology.

Proof. This is quite easy to see. We look at the case of $\omega \otimes \iota$. Then the formulas in the proposition define an element in $M(A)$. Now write $\omega = \mu b$ with $\mu \in A^*$ and $b \in A$ and observe that $(\omega \otimes \iota)(x) = (\mu \otimes \iota)((b \otimes 1)x) \in A$. This gives the result.

There is one more result on this space we will need.

Proposition 1.14. *The map $\Gamma : p \otimes q \rightarrow q^* \otimes p^*$ extends to an isometry of $A \otimes_{mh} A$ satisfying*

$$\Gamma(x(1 \otimes a)) = (a^* \otimes 1)\Gamma(x) \quad \Gamma((a \otimes 1)x) = \Gamma(x)(1 \otimes a^*)$$

for $x \in A \otimes_{mh} A$ and $a \in A$. The map Γ is a homeomorphism for the strict topology on $A \otimes_{mh} A$.

Proof. First we need to verify that Γ leaves the Haagerup norm invariant. This is more or less straightforward. Then Γ is extended to $A \otimes_h A$. Finally the formulas in the proposition will yield the extension of Γ to $A \otimes_{mh} A$ and also show the strict continuity.

Let us consider the following natural subspace of $A \otimes_{mh} A$.

Proposition 1.15. *Let $(p_i), (q_i)$ be families of elements of A such that the $\sum_{i \in I} p_i p_i^*$ and $\sum_{i \in I} q_i^* q_i$ converge strictly and remain bounded in $M(A)$, then the net $\sum_{i \in I} p_i \otimes q_i$ converges strictly to an element in $A \otimes_{mh} A$.*

Proof. Let $a \in A$. Then the net $\sum_{i \in I} a p_i p_i^* a^*$ is norm convergent, and the net $\sum_{i \in I} q_i^* q_i$ is bounded. So, just as in the remark after definition 1.1 you get that the net

$$\sum_{i \in I} a p_i \otimes q_i$$

is norm convergent in $A \otimes_h A$. We denote the limit by $\rho_2(a)$. Analogously we define

$$\rho_1(a) = \sum_{i \in I} p_i \otimes q_i a.$$

Then it is clear that $x = (\rho_1, \rho_2)$ belongs to $A \otimes_{mh} A$, and by definition the net $\sum_{i \in I} p_i \otimes q_i$ converges strictly to x .

In [21], the present theory has been developed using this subspace of $A \otimes_{mh} A$. We do not know whether this subspace is the whole of $A \otimes_{mh} A$ or not.

2 Hopf C*-algebras

As before, let A be a C*-algebra. We consider the minimal C*-tensor product $A \otimes A$ of A by itself. Let ϕ be a non-degenerate *-homomorphism of A into the multiplier algebra $M(A \otimes A)$ of $A \otimes A$. Denote by ι the identity map of A and consider the maps $\iota \otimes \phi$ and $\phi \otimes \iota$ from $A \otimes A$ to $M(A \otimes A \otimes A)$. They are again non-degenerate and therefore have unique extensions to *-homomorphisms from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$. These extensions will still be denoted by $\iota \otimes \phi$ and $\phi \otimes \iota$.

Then we are ready for the following definition.

Definition 2.1. *A comultiplication on a C*-algebra A is a non-degenerate *-homomorphism $\phi : A \rightarrow M(A \otimes A)$ such that $(\phi \otimes \iota)\phi = (\iota \otimes \phi)\phi$.*

The condition of coassociativity, namely $(\phi \otimes \iota)\phi = (\iota \otimes \phi)\phi$, is given a meaning by extending $\phi \otimes \iota$ and $\iota \otimes \phi$ to $M(A \otimes A)$ as above. For this it is necessary to have ϕ non-degenerate.

The above conditions are satisfied in the following basic example.

Proposition 2.2. *Let G be a locally compact semi-group. Let A be the C*-algebra $C_0(G)$ of continuous complex functions on G tending to 0 at infinity. Then ϕ defined by $\phi(f)(p, q) = f(pq)$ when $f \in C_0(G)$ and $p, q \in G$, is a comultiplication on A .*

Proof. First observe that $A \otimes A$ is naturally identified with $C_0(G \times G)$ and that $M(A \otimes A)$ is hence identified with $C_b(G \times G)$, the C*-algebra of bounded continuous complex functions on $G \times G$. Therefore, ϕ as defined in the formulation of the proposition is a map from A to $M(A \otimes A)$. It is clearly a *-homomorphism and it is not hard to show that ϕ is non-degenerate. The coassociativity comes from the associativity of the product in G .

It follows that we can think of a pair of a C*-algebra A and a comultiplication ϕ on A as a locally compact quantum semi-group. Now, as already explained in the introduction, it is not at all clear how to impose extra conditions in order to get a locally compact quantum group. One cannot simply require the existence of a counit and an antipode.

The most ambitious project would be to look for extra conditions that are easy to verify in special cases (such as the locally compact groups) and nice non-trivial examples and such

that a theory can be developed along the lines of locally compact groups. In such a frame, the existence of (the equivalent of) the Haar measure would be one of the main results. It seems however that such an objective is still far out of reach.

All the known existence proofs for the Haar measure in the classical case seem to fail in the non-commutative setting (except for special cases like the compact and the discrete quantum groups). Therefore, it is quite acceptable to develop a theory where the existence of the Haar measure is assumed. After all, just as in the classical case, also in the quantum case the Haar measures are simply there. From this point of view, the uniqueness property is at least as important.

Such a theory has now, very recently, been developed and the results are very satisfactory. In [13] and [14], a pair (A, ϕ) of a C^* -algebra with a comultiplication is called a locally compact quantum group if the spaces $\phi(A)(1 \otimes A)$ and $(A \otimes 1)\phi(A)$ are dense in $A \otimes A$ and if there exist 'good' left and right Haar measures. We remark that the abelian locally compact quantum groups are precisely those of the form $(C_0(G), \phi)$ with G a locally compact group and ϕ the comultiplication defined in proposition 2.2.

In case $A = C_0(G)$ with G a locally compact semi-group, the above mentioned density conditions have a special meaning. If the closed linear span of the elements $\phi(a)(1 \otimes b)$ contains the whole of $A \otimes A$, then G satisfies the right cancellation law: $xz = yz$ implies $x = y$. An analogous remark can be made for $(A \otimes 1)\phi(A)$. On the other hand, if $\phi(A)(1 \otimes A)$ is contained in $A \otimes A$, then G satisfies the following property: the set $\{x \in G \mid \text{there exists an element } y \in K \text{ such that } xy \in K'\}$ is compact for all compact subsets K and K' of G . This implies the first lemma in the existence proof of the Haar measure in Bourbaki [6, Chapitre 7, par. 1, no. 1, lemme 1].

Also observe that in case A has an identity, the density conditions are sufficient to prove the existence of a Haar measure. This gives the theory of compact quantum groups. But of course, this will not be enough in general (when A has no identity).

Nevertheless, the question remains what is a good notion for a counit and an antipode in this context. In fact, these notions do play an important role in the theory of locally compact quantum groups of [14]. Moreover, it is worthwhile to develop such notions independently of the existence of Haar measures. This is what we will do further in the paper. And to distinguish from the theory of locally compact quantum groups with Haar measure, we will call the objects we get here Hopf C^* -algebras. This name is appropriate for a C^* -algebra with a comultiplication, a counit and an antipode. Cfr. the discussion concerning this terminology in the introduction.

To see what extra conditions might be natural, let us think of the theory of Hopf algebras. If $(H, \Delta, \varepsilon, S)$ is a Hopf algebra, then the maps T_1, T_2 , defined on $H \otimes H$ by

$$\begin{aligned} T_1(a \otimes b) &= \Delta(a)(1 \otimes b) \\ T_2(a \otimes b) &= (a \otimes 1)\Delta(b) \end{aligned}$$

are bijective. The inverses are given in terms of the antipode by

$$\begin{aligned} T_1^{-1}(a \otimes b) &= ((\iota \otimes S)\Delta(a))(1 \otimes b) \\ T_2^{-1}(a \otimes b) &= (a \otimes 1)((S \otimes 1)\Delta(b)). \end{aligned}$$

In multiplier Hopf algebra theory, the bijectivity of these maps was the starting point from which the counit and the antipode were constructed (see [23]).

If we want to develop a C^* -algebra theory in this direction, it is more or less obvious what we have to do : we need to require that the maps T_1 and T_2 above have dense range in $A \otimes A$ and that proper extensions are (still) injective.

The abelian case here is very misleading. If $A = C_0(G)$ with G a locally compact group, then the maps T_1 and T_2 extend to $*$ -isomorphisms of $A \otimes A$ given by $(T_1 f)(p, q) = f(pq, q)$ and $(T_2 f)(p, q) = f(p, pq)$ when $f \in C_0(G \times G)$ and $p, q \in G$. This is not at all the case in general where e.g. T_1 and T_2 will not even be bounded (the reason being that multiplication is no longer bounded on $A \otimes A$ with the C^* -norm).

Fortunately, as we have seen in the previous section, multiplication is continuous for the Haagerup norm. And it is precisely that property of the Haagerup norm that allows the following result which is the starting point in the development of our notion of a Hopf C^* -algebra.

Proposition 2.3. *Let (A, ϕ) be a pair of a C^* -algebra and a comultiplication such that $\phi(A)(1 \otimes A)$ and $\phi(A)(A \otimes 1)$ are subspaces of $A \otimes A$. Then the maps T_1 and T_2 defined from $A \odot A$ to $A \otimes A$ by*

$$T_1(a \otimes b) = \phi(a)(1 \otimes b) \quad T_2(a \otimes b) = (a \otimes 1)\phi(b)$$

are contractive from $A \odot A$ with the Haagerup norm to $A \otimes A$ with the C^ -norm. We use the same symbols T_1 and T_2 to denote the extensions to $A \otimes_h A$.*

Proof. Take $x \in A \odot A$ and write $x = \sum_{i=1}^n p_i \otimes q_i$. By using lemma 1.2 we get

$$\begin{aligned} \|T_1(x)\| &= \left\| \sum_{i=1}^n \phi(p_i)(1 \otimes q_i) \right\| \\ &\leq \left\| \sum_{i=1}^n \phi(p_i p_i^*) \right\|^{1/2} \left\| \sum_{i=1}^n q_i^* q_i \right\|^{1/2} \\ &\leq \left\| \sum_{i=1}^n p_i p_i^* \right\|^{1/2} \left\| \sum_{i=1}^n q_i^* q_i \right\|^{1/2}. \end{aligned}$$

So we can conclude that $\|T_1(x)\| \leq \|x\|_h$. Analogously $\|T_2(x)\| \leq \|x\|_h$.

We see that we use essentially the same argument as in the proof of proposition 1.4 where we show that multiplication is contractive. This is no surprise as we can write $T_1 = (\iota \otimes m)(\phi \otimes \iota)$ and $T_2 = (m \otimes \iota)(\iota \otimes \phi)$. Remark that the order of a and b is important (the Haagerup norm is not symmetric, i.e. not invariant under the flip). This means that a map like $a \otimes b \mapsto (1 \otimes b)\phi(a)$ will in general not be continuous for the Haagerup norm, but a map like $a \otimes b \mapsto (1 \otimes a)\phi(b)$ will be. We will use this last map also in section 3.

Then we are ready for the main definition.

Definition 2.4. Let (A, ϕ) be a pair of a C^* -algebra A with a comultiplication ϕ such that $\phi(A)(1 \otimes A)$ and $\phi(A)(A \otimes 1)$ are subspaces of $A \otimes A$. We call this pair a Hopf C^* -algebra if the maps T_1 and T_2 as defined above, are injective on $A \otimes_h A$.

Observe that this definition implies immediately that the comultiplication ϕ is injective.

As we have said before, the injectivity of T_1 and T_2 on $A \otimes_h A$ is obvious in the case of $A = C_0(G)$ with G a locally compact group (where we have much stronger properties). But what is rather striking is that also conversely, all abelian Hopf C^* -algebras are of this form. We even have the following stronger result.

Proposition 2.5. Let (A, ϕ) be a pair of an abelian C^* -algebra A with a comultiplication ϕ such that $\phi(A)(1 \otimes A)$ and $\phi(A)(A \otimes 1)$ are subspaces of $A \otimes A$. Assume further that

$$\begin{aligned} \phi(a)(1 \otimes b) = 0 & \text{ implies } a \otimes b = 0 \\ \text{and } \phi(a)(b \otimes 1) = 0 & \text{ implies } a \otimes b = 0 \end{aligned}$$

for all $a, b \in A$. Then there exists a locally compact group G such that $A \cong C_0(G)$ and ϕ is given by $\phi(f)(p, q) = f(pq)$ for all $p, q \in C_0(G)$ under this isomorphism.

Proof. It is clear by the Gelfand-Naimark theorem that there exists a locally compact space G and a continuous associative binary operation on G such that $A \cong C_0(G)$ and such that ϕ is given by the formula above under this isomorphism. We will prove now that G is a group and that the inverse is continuous.

First choose a non-empty open subset V of G . We claim that the set $\{rs \mid r \in G, s \in V\}$ is dense in G . Indeed, suppose that it is not dense. Then there is an element $f \in C_0(G)$ such that $f \neq 0$ but $f(rs) = 0$ for all $r \in G$ and $s \in V$. Choose an element $g \in C_0(G)$ such that $g \neq 0$ and such that g has support in V . Then $f(rs)g(s) = 0$ for all $r, s \in G$. Hence $\phi(f)(1 \otimes g) = 0$ and by assumption we may conclude that $f = 0$ or $g = 0$. So we get a contradiction and this gives the claim.

Now let $s, t \in G$. For any pair $\alpha = (V, W)$ of open neighborhoods of s and t respectively, we have, by the property above, points $r_\alpha \in G$, $s_\alpha \in V$ and $t_\alpha \in W$ such that $r_\alpha s_\alpha = t_\alpha$. These pairs of neighborhoods give an index set I when ordered by opposite inclusions. By definition the net (s_α) converges to s and (t_α) to t .

For every $f, g \in C_0(G)$ we have

$$(\phi(f)(1 \otimes g))(r_\alpha, s_\alpha) = f(r_\alpha s_\alpha)g(s_\alpha) = f(t_\alpha)g(s_\alpha)$$

and this converges to $f(t)g(s)$. Now choose f and g such that $f(t)g(s) \neq 0$. Because $\phi(f)(1 \otimes g) \in C_0(G \times G)$ by assumption, we can take compact sets K_1 and K_2 such that $|\phi(f)(1 \otimes g)|$ is smaller than $\frac{1}{2}|f(t)g(s)|$ outside $K_1 \times K_2$. Then, for α large enough, we will have $|(\phi(f)(1 \otimes g))(r_\alpha, s_\alpha)| \geq \frac{1}{2}|f(t)g(s)|$ and so $r_\alpha \in K_1$. Then we can find a subnet of (r_α) that converges to a point r . Hence we find $r \in G$ such that $rs = t$.

Because $s, t \in G$ were chosen arbitrarily we get $Gs = G$ for all $s \in G$. By symmetry we also get $sG = G$ for all $s \in G$. Then it is an easy exercise to prove that G is a group. It now

follows from Bourbaki [7, Chapitre 10, par. 3, ex. 25] that G is a locally compact group. But we can also give an easy direct proof of the continuity of the inverse. Because G is a group we have the cancellation property. Therefore the linear span of $\{\phi(f)(1 \otimes g) \mid f, g \in C_0(G)\}$ is a $*$ -subalgebra of $C_0(G \times G)$ that separates points of $G \times G$ and hence is dense in $C_0(G \times G)$. Now for any $f, g \in C_0(G)$ the map

$$s \mapsto (\phi(f)(1 \otimes g))(s^{-1}, s) = f(s^{-1}s)g(s) = f(e)g(s)$$

is continuous. By density we get that the map $s \mapsto f(s^{-1})g(s)$ is continuous for all $f, g \in C_0(G)$. From this it follows that $s \mapsto s^{-1}$ is continuous and that concludes the proof.

This proposition is one reason for not assuming the density of the ranges of T_1 and T_2 in the definition of a Hopf C^* -algebra, which is a little surprising after the discussion we have had. Another reason is that on the one hand we do not need these density conditions for the development of our theory, while on the other hand we will formulate another density condition that will be stronger. We will come back to this in section 3.

A problem with the notion of a Hopf C^* -algebra is that injectivity on this completed space $A \otimes_h A$ may sometimes be hard to verify. We will however give techniques to do this in section 5, where we treat examples and special cases. There we will see that we do get Hopf C^* -algebras in most interesting cases.

Now we need also the extension of T_1 and T_2 to $A \otimes_{mh} A$. The proof that these extensions exist, is essentially similar to the fact that also multiplication can be extended to $A \otimes_{mh} A$.

Proposition 2.6. *The maps T_1 and T_2 can be uniquely extended to contractions from $A \otimes_{mh} A$ to $M(A \otimes A)$ satisfying*

$$\begin{aligned} T_1(x)(1 \otimes a) &= T_1(x(1 \otimes a)) & \text{and} & & \phi(a)T_1(x) &= T_1((a \otimes 1)x) \\ (a \otimes 1)T_2(x) &= T_2((a \otimes 1)x) & \text{and} & & T_2(x)\phi(a) &= T_2(x(1 \otimes a)) \end{aligned}$$

whenever $x \in A \otimes_{mh} A$ and $a \in A$. These extensions are still injective and they are strictly continuous on bounded sets.

Proof. The injectivity of the extensions is an immediate consequence of these defining formulas. If e.g. $T_1(x) = 0$ for $x \in A \otimes_{mh} A$, then $x(1 \otimes a) = 0$ for all $a \in A$ by the injectivity of T_1 on $A \otimes_h A$ and hence $x = 0$. So it essentially remains to show that the above formulas yield maps into $M(A \otimes A)$.

So, let $x \in A \otimes_{mh} A$. We define a left multiplier from $A \odot A$ to $A \otimes A$ by $T_1(x)(b \otimes a) = T_1(x(1 \otimes a))(b \otimes 1)$. When (e_λ) is an approximate identity in A , we have

$$\begin{aligned} \phi(e_\lambda)T_1(x)\left(\sum_{i=1}^n b_i \otimes a_i\right) &= \phi(e_\lambda)\left(\sum_{i=1}^n T_1(x(1 \otimes a_i))(b_i \otimes 1)\right) \\ &= \sum_{i=1}^n T_1((e_\lambda \otimes 1)x(1 \otimes a_i))(b_i \otimes 1) \\ &= T_1((e_\lambda \otimes 1)x)\left(\sum_{i=1}^n b_i \otimes a_i\right). \end{aligned}$$

This implies that

$$\begin{aligned} \left\| \phi(e_\lambda) T_1(x) \left(\sum_{i=1}^n b_i \otimes a_i \right) \right\| &\leq \| (e_\lambda \otimes 1)x \|_h \left\| \sum_{i=1}^n b_i \otimes a_i \right\| \\ &\leq \|x\|_h \left\| \sum_{i=1}^n b_i \otimes a_i \right\|. \end{aligned}$$

Taking the limit we get

$$\left\| T_1(x) \left(\sum_{i=1}^n b_i \otimes a_i \right) \right\| \leq \|x\|_h \left\| \sum_{i=1}^n b_i \otimes a_i \right\|.$$

This shows that we have a left multiplier of $A \otimes A$ with norm less than $\|x\|_h$. Similarly, we define $T_1(x)$ as a right multiplier by

$$\left(\sum_{i=1}^n (a_i \otimes b_i) \phi(c_i) \right) T_1(x) = \sum_{i=1}^n (a_i \otimes b_i) T_1((c_i \otimes 1)x).$$

A similar argument as above, by looking at $x(1 \otimes e_\lambda)$, together with the non-degeneracy of ϕ , will give that $T_1(x)$ is also a bounded right multiplier of $A \otimes A$.

Hence $T_1(x) \in M(A \otimes A)$ and $\|T_1(x)\| \leq \|x\|_h$. From the formulas above it follows immediately that T_1 extended to $A \otimes_{mh} A$ will be strictly continuous on bounded sets. We can give a similar argument for T_2 .

3 The left counit and antipode

Now let A be a C^* -algebra with a comultiplication ϕ satisfying the assumptions of the previous section, making (A, ϕ) a Hopf C^* -algebra. So, the maps T_1 and T_2 defined from $A \odot A$ to $A \otimes A$ by $T_1(a \otimes b) = \phi(a)(1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1)\phi(b)$ have ranges in $A \otimes A$ and their continuous extensions to $A \otimes_h A$ are injective. Again as in the previous section, we consider their extensions to $A \otimes_{mh} A$, still using the same symbols. These extensions are still injective. Now the range is in $M(A \otimes A)$. Recall that the maps T_1, T_2 from $A \otimes_{mh} A$ to $M(A \otimes A)$ are strictly continuous on bounded sets.

We will now first define a candidate for the domain of the counit and the antipode.

Notation 3.1. *Let A_0 be the space of elements $a \in A$ such that there is an element $x \in A \otimes_{mh} A$ satisfying $a \otimes 1 = T_1 x$*

In case $A = C_0(G)$ with G a locally compact group, it is possible to show that A_0 is dense. Also for compact and discrete and more generally locally compact quantum groups this is the case. We will prove these things in section 5. However, without further assumptions, it seems that nothing can be said about the size of A_0 . Nevertheless it is immediately clear

that the density of A_0 implies that $\phi(A)(1 \otimes A)$ is dense in $A \otimes A$, so that we will have to assume at least that to get A_0 dense in A .

We will not get more elements if we would allow $a \in M(A)$. Indeed, if $a \in M(A)$ and $x \in A \otimes_{mh} A$ satisfy $a \otimes 1 = T_1 x$, then $a \otimes b = (T_1 x)(1 \otimes b) = T_1(x(1 \otimes b))$ which belongs to $A \otimes A$ for all $b \in A$ and this will imply $a \in A$.

It is obvious that A_0 is a subspace of A . But we have more :

Proposition 3.2. *A_0 is a subalgebra of A .*

Proof. We first make some introductory remarks. Let $x = \sum_{j=1}^n r_j \otimes s_j$ and $y = \sum_{i=1}^m p_i \otimes q_i$ be elements of $A \odot A$. Define

$$M(x, y) = \sum_{j,i=1}^{n,m} r_j p_i \otimes q_i s_j.$$

Then we have

$$\begin{aligned} \|M(x, y)\|_h^2 &\leq \left\| \sum_{i,j} r_j p_i p_i^* r_j^* \right\| \left\| \sum_{i,j} s_j^* q_i^* q_i s_j \right\| \\ &\leq \left\| \sum_i p_i p_i^* \right\| \left\| \sum_j r_j r_j^* \right\| \left\| \sum_i q_i^* q_i \right\| \left\| \sum_j s_j^* s_j \right\|. \end{aligned}$$

From this we can conclude that $\|M(x, y)\|_h \leq \|x\|_h \|y\|_h$, and we extend M to a contraction from $(A \otimes_h A) \times (A \otimes_h A)$ to $A \otimes_h A$ by continuity. If now $y \in A \otimes_{mh} A$ and $x = \sum_{j=1}^n r_j \otimes s_j \in A \odot A$, we define, consistently with the previous notation if $y \in A \otimes_h A$

$$M(x, y) = \sum_{j=1}^n (r_j \otimes 1) y (1 \otimes s_j) \in A \otimes_h A.$$

Then we have, with (e_λ) an approximate identity for A

$$\|M(x, y)\|_h = \lim_{\lambda} \left\| \sum_{j=1}^n (r_j \otimes 1) y (1 \otimes e_\lambda s_j) \right\|_h = \lim_{\lambda} \|M(x, y(1 \otimes e_\lambda))\|_h \leq \|x\|_h \|y\|_h.$$

So again we can extend M to a contraction from $(A \otimes_h A) \times (A \otimes_{mh} A)$ to $A \otimes_h A$. If finally $x, y \in A \otimes_{mh} A$ we define $z \in A \otimes_{mh} A$ by

$$z(1 \otimes c) = M(x(1 \otimes c), y) \quad (c \otimes 1)z = M((c \otimes 1)x, y)$$

for all $c \in A$. Suppose now $a, b \in A_0$. Take $x, y \in A \otimes_{mh} A$ satisfying $T_1(x) = a \otimes 1$ and $T_1(y) = b \otimes 1$. Put $z = M(x, y)$. Then we have $z \in A \otimes_{mh} A$ and we compute $T_1(z)$. Let $c \in A$. Then

$$T_1(z)(1 \otimes c) = T_1(z(1 \otimes c)) = T_1(M(x(1 \otimes c), y)).$$

If $p, q \in A$ we have

$$\begin{aligned} T_1(M(p \otimes q, y)) &= T_1((p \otimes 1)y(1 \otimes q)) \\ &= \phi(p)T_1(y)(1 \otimes q) = \phi(p)(1 \otimes q)(b \otimes 1) = T_1(p \otimes q)(b \otimes 1). \end{aligned}$$

By continuity we may conclude from this and the previous formula that

$$T_1(z)(1 \otimes c) = T_1(x(1 \otimes c))(b \otimes 1) = (a \otimes c)(b \otimes 1) = ab \otimes c.$$

So we get $T_1(z) = ab \otimes 1$ and hence $ab \in A_0$.

Next, we need to work with the map

$$\psi : p \otimes q \mapsto \phi_{13}(p)\phi_{23}(q)$$

from $A \odot A$ to $M(A \otimes A \otimes A)$ where we use the common leg numbering notation : $\phi_{13}(p)$ stands for $\phi(p)$ sitting in the first and third factor, while $\phi_{23}(q)$ is simply $1 \otimes \phi(p)$. We first extend this map to $A \otimes_h A$:

Proposition 3.3. *The map ψ is continuous for the Haagerup norm and extends to a contraction from $A \otimes_h A$ to $M(A \otimes A \otimes A)$.*

Proof. This is analogous to the proof of proposition 1.4. Just remark that by lemma 1.2

$$\begin{aligned} \left\| \psi \left(\sum_{i=1}^n p_i \otimes q_i \right) \right\|^2 &= \left\| \sum_{i=1}^n \phi_{13}(p_i) \phi_{23}(q_i) \right\|^2 \\ &\leq \left\| \sum_{i=1}^n \phi_{13}(p_i p_i^*) \right\| \left\| \sum_{i=1}^n \phi_{23}(q_i^* q_i) \right\| \\ &\leq \left\| \sum_{i=1}^n p_i p_i^* \right\| \left\| \sum_{i=1}^n q_i^* q_i \right\|. \end{aligned}$$

So we get $\|\psi(x)\| \leq \|x\|_h$ for all $x \in A \odot A$ and that proves the result.

We can say more about this map, but before we give the proof of the next proposition we need the following lemma.

Lemma 3.4. *Let $x \in M(A \otimes A)$ such that $(\iota \otimes \omega)(x) \in A$ and $(\iota \otimes \omega)(xx^*) \in A$ for all $\omega \in A^*$. Let A act on a Hilbert space H , $\xi \in H$ and let $(\eta_i)_{i \in I}$ be an orthonormal basis for H . Then the increasing net in A^+*

$$\sum_{i \in I} (\iota \otimes \omega_{\eta_i, \xi})(x) (\iota \otimes \omega_{\eta_i, \xi})(x)^* \tag{1}$$

is norm convergent in A with limit $(\iota \otimes \omega_{\xi, \xi})(xx^)$.*

Remark that we use the notation $\omega_{\xi, \eta}$ for the vector functional given by $\omega_{\xi, \eta}(a) = \langle a\xi, \eta \rangle$.

Proof. Choose $\omega \in A_+^*$. Let π be the universal representation of A on the Hilbert space K . Take $\rho \in K$ such that $\omega(x) = \omega_{\rho, \rho}(\pi(x))$ for all $x \in A$. Let $(\rho_j)_{j \in J}$ be an orthonormal basis for K . Let I_0 be a finite subset of I . Then we have

$$\begin{aligned} \omega\left(\sum_{i \in I_0} (\iota \otimes \omega_{\eta_i, \xi})(x)(\iota \otimes \omega_{\eta_i, \xi})(x)^*\right) &= \sum_{i \in I_0} \|(\iota \otimes \omega_{\xi, \eta_i})(\pi \otimes \iota)(x^*)\rho\|^2 \\ &= \sum_{i \in I_0, j \in J} |\langle (\pi \otimes \iota)(x^*)(\rho \otimes \xi), \rho_j \otimes \eta_i \rangle|^2. \end{aligned}$$

So the increasing net of positive numbers

$$\left(\omega\left(\sum_{i \in I_0} (\iota \otimes \omega_{\eta_i, \xi})(x)(\iota \otimes \omega_{\eta_i, \xi})(x)^*\right)\right)_{I_0 \subseteq I}$$

converges to

$$\|(\pi \otimes \iota)(x^*)(\rho \otimes \xi)\|^2 = \omega((\iota \otimes \omega_{\xi, \xi})(xx^*)).$$

Now we can apply Dini's theorem to the compact Hausdorff space $\{\omega \in A_+^* \mid \|\omega\| \leq 1\}$, and the increasing pointwise converging net of positive continuous functions on it given by formula 1. So we get uniform convergence of this net and hence the lemma follows.

Proposition 3.5. *For all $\omega \in A^*$ and $x \in A \otimes_h A$ we have that $(\iota \otimes \iota \otimes \omega)\psi(x) \in A \otimes_h A$ and*

$$\|(\iota \otimes \iota \otimes \omega)\psi(x)\|_h \leq \|\omega\| \|x\|_h.$$

Proof. By continuity it is enough to verify this for $x \in A \odot A$. So let $x = \sum_{i=1}^n p_i \otimes q_i$ and choose $\omega \in A^*$. Let A act on its universal Hilbert space H and take vectors ξ and η in H such that $\omega = \omega_{\xi, \eta}$ and $\|\omega\| = \|\xi\| \|\eta\|$. Let $(\gamma_j)_{j \in J}$ be an orthonormal base for H . By the previous lemma we get that for every $i = 1, \dots, n$ the net

$$\sum_{j \in J} (\iota \otimes \omega_{\gamma_j, \eta})\phi(p_i)(\iota \otimes \omega_{\gamma_j, \eta})\phi(p_i)^*$$

is norm convergent in A with limit $(\iota \otimes \omega_{\eta, \eta})\phi(p_i p_i^*)$. Using the remark after definition 1.1 we may conclude from this and an analogous statement for the second leg that

$$\sum_{i, j} (\iota \otimes \omega_{\gamma_j, \eta})\phi(p_i) \otimes (\iota \otimes \omega_{\xi, \gamma_j})\phi(q_i) \in A \otimes_h A \quad (2)$$

with norm convergence. The norm of this element is majorized by

$$\left\|\sum_{i=1}^n p_i p_i^*\right\|^{1/2} \left\|\sum_{i=1}^n q_i^* q_i\right\|^{1/2} \|\xi\| \|\eta\|.$$

Because the net in 2 is also norm convergent in $A \otimes A$ with limit

$$\sum_{i=1}^n (\iota \otimes \iota \otimes \omega_{\xi, \eta})(\phi_{13}(p_i)\phi_{23}(q_i)) = (\iota \otimes \iota \otimes \omega)\psi(x)$$

the result follows.

Now we extend ψ to $A \otimes_{mh} A$ by strict continuity:

Proposition 3.6. *The map ψ can be extended uniquely to a contraction from $A \otimes_{mh} A$ to $M(A \otimes A \otimes A)$, still denoted by ψ , such that for all $x \in A \otimes_{mh} A$ and $a \in A$*

$$\psi(x)\phi_{23}(a) = \psi(x(1 \otimes a)) \quad \text{and} \quad \phi_{13}(a)\psi(x) = \psi((a \otimes 1)x).$$

This extension is strictly continuous on bounded sets. For every $\omega \in A^$ the map*

$$x \mapsto (\iota \otimes \iota \otimes \omega)\psi(x)$$

maps $A \otimes_{mh} A$ into $A \otimes_{mh} A$, is bounded with norm majorized by $\|\omega\|$ and is strictly continuous on bounded sets.

Proof. The extension of ψ to $A \otimes_{mh} A$ is completely analogous to the extension of the maps T_1 and T_2 . If $x \in A \otimes_{mh} A$ we can obtain $\psi(x)$ as a right multiplier by putting $(a \otimes b \otimes c)\phi_{13}(d)\psi(x)$ equal to $(a \otimes b \otimes c)\psi((d \otimes 1)x)$. One shows just as in the proof of proposition 2.6 that we obtain a bounded right multiplier of $A \otimes A \otimes A$ this way. In a similar way we define a bounded left multiplier. This gives us the extension of ψ .

Now write for every $\omega \in A^*$, $x \in A \otimes_h A$ and $u \in A \otimes A$

$$R(\omega, x, u) = (\iota \otimes \iota \otimes \omega)(u_{13}\psi(x)).$$

At the moment we only have $R(\omega, x, u) \in M(A \otimes A)$. We claim that $R(\omega, x, u) \in A \otimes_h A$ and

$$\|R(\omega, x, u)\|_h \leq \|\omega\| \|x\|_h \|u\|. \quad (3)$$

When $x = \sum_{i=1}^n p_i \otimes q_i$, when A acts on a Hilbert space H with $\omega = \omega_{\xi, \eta}$, $\|\omega\| = \|\xi\| \|\eta\|$ and when $(\gamma_j)_{j \in J}$ is an orthonormal basis for H , then we have as in the proof of the previous proposition

$$R(\omega, x, u) = \sum_{i,j} (\iota \otimes \omega_{\gamma_j, \eta})(u\phi(p_i)) \otimes (\iota \otimes \omega_{\xi, \gamma_j})(\phi(q_i))$$

with norm convergence in $A \otimes_h A$. Further we get

$$\|R(\omega, x, u)\|_h \leq \|\eta\| \|u\| \left\| \sum_{i=1}^n p_i p_i^* \right\|^{1/2} \|\xi\| \left\| \sum_{i=1}^n q_i^* q_i \right\|^{1/2}.$$

This proves our claim that $R(\omega, x, u) \in A \otimes_h A$ and also proves inequality 3.

Now let $\omega \in A^*$, $x \in A \otimes_{mh} A$, $u \in A \otimes A$ and $a \in A$. Then

$$(\iota \otimes \iota \otimes \omega)((u\phi(a))_{13}\psi(x)) = (\iota \otimes \iota \otimes \omega)(u_{13}\psi((a \otimes 1)x)) \in A \otimes_h A$$

and

$$\begin{aligned} \|(\iota \otimes \iota \otimes \omega)((u\phi(a))_{13}\psi(x))\|_h &= \|R(\omega, (a \otimes 1)x, u)\|_h \\ &= \lim_{\lambda} \|R(\omega, (a \otimes 1)x(1 \otimes e_{\lambda}), u)\|_h \\ &= \lim_{\lambda} \|R(\omega, x(1 \otimes e_{\lambda}), u\phi(a))\|_h \\ &\leq \|\omega\| \|x\|_h \|u\phi(a)\|. \end{aligned}$$

Because ϕ is non-degenerate we may conclude that for $\omega \in A^*$, $x \in A \otimes_{mh} A$ and $u \in A \otimes A$ we have $(\iota \otimes \iota \otimes \omega)(u_{13}\psi(x)) \in A \otimes_h A$. We still denote this element with $R(\omega, x, u)$ and we have

$$\|R(\omega, x, u)\|_h \leq \|\omega\| \|x\|_h \|u\|.$$

From the formulas above we can also conclude that for every $\omega \in A^*$ and $u \in A \otimes A$ the map

$$x \mapsto R(\omega, x, u)$$

is strict-norm continuous on bounded sets from $A \otimes_{mh} A$ to $A \otimes_h A$.

Finally let $\omega \in A^*$ and $x \in A \otimes_{mh} A$. Write $\omega = \mu b$ with $\mu \in A^*$ and $b \in A$. Then we get for every $a \in A$

$$(a \otimes 1)(\iota \otimes \iota \otimes \omega)\psi(x) = (\iota \otimes \iota \otimes \mu)((a \otimes b)_{13}\psi(x)) \in A \otimes_h A$$

and

$$\begin{aligned} \|(a \otimes 1)(\iota \otimes \iota \otimes \omega)\psi(x)\|_h &= \|R(\mu, x, a \otimes b)\|_h \\ &= \lim_{\lambda} \|R(\mu, x, a \otimes be_{\lambda})\|_h \\ &= \lim_{\lambda} \|R(\omega, x, a \otimes e_{\lambda})\|_h \\ &\leq \|\omega\| \|x\|_h \|a\|. \end{aligned}$$

Analogously one can prove that $(\iota \otimes \iota \otimes \omega)\psi(x)(1 \otimes a) \in A \otimes_h A$ with norm majorized by $\|\omega\| \|x\|_h \|a\|$. So we get indeed $(\iota \otimes \iota \otimes \omega)\psi(x) \in A \otimes_{mh} A$ with norm majorized by $\|\omega\| \|x\|_h$. With the notation as above we can write

$$(a \otimes 1)(\iota \otimes \iota \otimes \omega)\psi(x) = R(\mu, x, a \otimes b)$$

and this depends strict-norm continuous on bounded x by one of the remarks above. One proves an analogous statement on the other side and gets the strict continuity on bounded sets of $x \mapsto (\iota \otimes \iota \otimes \omega)\psi(x)$ from $A \otimes_{mh} A$ to $A \otimes_{mh} A$.

These results allow us to prove the following basic formula.

Proposition 3.7. *If $a \in A_0$ and $x \in A \otimes_{mh} A$ such that $a \otimes 1 = T_1 x$, then $x \otimes 1 = \psi(x)$.*

Proof. We will prove this formula by applying ω and by showing that $T_1((\iota \otimes \iota \otimes \omega)\psi(x)) = T_1(x)\omega(1)$. Then, the injectivity of T_1 will give us the result.

Now observe that

$$\begin{aligned} T_1((\iota \otimes \iota \otimes \omega)\psi(p \otimes q)) &= T_1((\iota \otimes \iota \otimes \omega)(\phi_{13}(p)\phi_{23}(q))) \\ &= (\iota \otimes \iota \otimes \omega)((\phi \otimes \iota)\phi(p)\phi_{23}(q)). \end{aligned}$$

This can easily be shown by using the concrete formulas for $\psi(p \otimes q)$ obtained in the proof of proposition 3.5. Then, using coassociativity, we get

$$\begin{aligned} T_1((\iota \otimes \iota \otimes \omega)\psi(p \otimes q)) &= (\iota \otimes \iota \otimes \omega)((\iota \otimes \phi)(\phi(p)(1 \otimes q))) \\ &= (\iota \otimes \iota \otimes \omega)((\iota \otimes \phi)T_1(p \otimes q)). \end{aligned}$$

Then, using the strict continuity on bounded sets of all the maps involved, we get

$$T_1((\iota \otimes \iota \otimes \omega)\psi(x)) = (\iota \otimes \iota \otimes \omega)((\iota \otimes \phi)T_1(x)).$$

As $T_1x = a \otimes 1$, the righthand side is

$$\begin{aligned} (\iota \otimes \iota \otimes \omega)((\iota \otimes \phi)(a \otimes 1)) &= (a \otimes 1)\omega(1) \\ &= T_1(x)\omega(1). \end{aligned}$$

Then the proof is complete.

Before we continue, let us make a little excursion to Hopf algebras and see what the above formula really means. We will use the Sweedler notation. So, let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra. For any $a \in H$ we have

$$\begin{aligned} a \otimes 1 &= \sum a_{(1)} \otimes \varepsilon(a_{(2)})1 = \sum a_{(1)} \otimes a_{(2)}S(a_{(3)}) \\ &= \sum \Delta(a_{(1)})(1 \otimes S(a_{(2)})). \end{aligned}$$

So, in this case $a \otimes 1 = T_1(\sum a_{(1)} \otimes S(a_{(2)}))$. Now

$$\begin{aligned} \sum \Delta_{13}(a_{(1)})\Delta_{23}(S(a_{(2)})) &= \sum (a_{(1)} \otimes 1 \otimes a_{(2)})(1 \otimes S(a_{(4)}) \otimes S(a_{(3)})) \\ &= \sum a_{(1)} \otimes S(a_{(4)}) \otimes a_{(2)}S(a_{(3)}) \\ &= \sum a_{(1)} \otimes S(a_{(3)}) \otimes \varepsilon(a_{(2)})1 \\ &= \sum a_{(1)} \otimes S(a_{(2)}) \otimes 1 \end{aligned}$$

and this is precisely the formula that we have, in the C^* -context, in proposition 3.7.

This is very important for understanding what follows. Simply observe that

$$\sum a_{(1)}S(a_{(2)}) = \varepsilon(a)1$$

and

$$\sum (1 \otimes a_{(1)})\Delta(S(a_{(2)})) = \sum S(a_{(3)}) \otimes a_{(1)}S(a_{(2)}) = S(a) \otimes 1.$$

These formulas will motivate the definition of ε and S below.

First we need a little lemma.

Lemma 3.8. *If $x \in M(A)$ and $\phi(x) = x \otimes 1$, then $x \in \mathbb{C}1$.*

Proof. Take $a, b \in A$. Then

$$(a \otimes 1)\phi(xb) = (a \otimes 1)\phi(x)\phi(b) = (ax \otimes 1)\phi(b).$$

We now use the injectivity of T_2 (on $A \odot A$) to find

$$a \otimes xb = ax \otimes b$$

for all $a, b \in A$. This is only possible when $x \in \mathbb{C}1$.

This lemma, together with the formulas in the Hopf algebra context above will lead us to the definition of the counit and the antipode.

Proposition 3.9. *There is a homomorphism $\varepsilon : A_0 \rightarrow \mathbb{C}$, defined by $\varepsilon(a)1 = m(x)$ when $a \in A_0$ and $x \in A \otimes_{mh} A$ satisfy $a \otimes 1 = T_1x$. Here m denotes multiplication as a map from $A \otimes_{mh} A$ to $M(A)$ (see proposition 1.10).*

Proof. Take $a \in A_0$ and $x \in A \otimes_{mh} A$ such that $a \otimes 1 = T_1x$. Take $\omega \in A^*$ and consider the formula $\omega(1)x = (\iota \otimes \iota \otimes \omega)\psi(x)$. Apply multiplication on both sides. For any $p, q \in A$ we have

$$\begin{aligned} m(\iota \otimes \iota \otimes \omega)(\phi_{13}(p)\phi_{23}(q)) &= (\iota \otimes \omega)(\phi(p)\phi(q)) = (\iota \otimes \omega)\phi(pq) \\ &= (\iota \otimes \omega)\phi(m(p \otimes q)). \end{aligned}$$

Again this can be proved by writing out the formula for $(\iota \otimes \iota \otimes \omega)(\phi_{13}(p)\phi_{23}(q))$ that we obtained in the proof of proposition 3.5. We get by continuity that

$$\omega(1)m(x) = m(\iota \otimes \iota \otimes \omega)\psi(x) = (\iota \otimes \omega)\phi(m(x)).$$

As this is true for all ω we must have

$$m(x) \otimes 1 = \phi(m(x)).$$

By the lemma $m(x) \in \mathbb{C}1$. So we can define $\varepsilon : A_0 \rightarrow \mathbb{C}$ by $\varepsilon(a)1 = m(x)$ when $a \otimes 1 = T_1x$ and $x \in A \otimes_{mh} A$.

Now we show that ε is a homomorphism. To do this, take $a, b \in A_0$ and $x, y \in A \otimes_{mh} A$ such that $T_1x = a \otimes 1$ and $T_1y = b \otimes 1$. Use the notation of the proof of proposition 3.2 and define $z = M(x, y)$. We have seen that $z \in A \otimes_{mh} A$ and $T_1z = ab \otimes 1$. Let $c \in A$. Then we have

$$\varepsilon(ab)c = m(z(1 \otimes c)) = m(M(x(1 \otimes c), y)).$$

But for $p, q \in A$ we have

$$m(M(p \otimes q, y)) = m((p \otimes 1)y(1 \otimes q)) = pm(y)q = pq\varepsilon(b) = m(p \otimes q)\varepsilon(b).$$

So we can conclude that

$$\varepsilon(ab)c = m(x(1 \otimes c))\varepsilon(b) = \varepsilon(a)\varepsilon(b)c.$$

This gives the required result.

The antipode is obtained in a similar way. Instead of m , we now need to use the map $p \otimes q \rightarrow (1 \otimes p)\Delta(q)$. This can be seen from the motivation explained before in the Hopf algebra context. We will call this map $\tilde{T}$. It is of the same kind as the maps T_1 and T_2 (with p and q in the correct order). Therefore one can use the same methods to extend $\tilde{T}$ to a continuous map from $A \otimes_{mh} A$ to $M(A \otimes A)$, continuous both for the norm and for the strict topology on bounded sets. It is also possible to view $\tilde{T}$ as

$$\tilde{T}(p \otimes q) = (T_1(q^* \otimes p^*))^*.$$

Now remark that $p \otimes q \mapsto q^* \otimes p^*$ extends to an isometry from $A \otimes_h A$ to itself and further to an isometry from $A \otimes_{mh} A$ to itself that is strictly continuous (see proposition 1.14).

This takes us to the definition of the antipode.

Proposition 3.10. *There is a linear map $S : A_0 \rightarrow A$ defined by $S(a) \otimes 1 = \tilde{T}(x)$ when $a \otimes 1 = T_1(x)$ and $a \in A_0$ and $x \in A \otimes_{mh} A$. This map is an anti-homomorphism. We have $S(a)^* \in A_0$ when $a \in A_0$ and $S(S(a)^*)^* = a$.*

Proof. Let $a \in A_0$ and $x \in A \otimes_{mh} A$ satisfy $a \otimes 1 = T_1x$. Apply the slice map $\iota \otimes \iota \otimes \omega$ and then $\tilde{T}$ to the formula $x \otimes 1 = \psi(x)$ as obtained in proposition 3.7. This yields

$$\omega(1)\tilde{T}(x) = \tilde{T}(\iota \otimes \iota \otimes \omega)\psi(x).$$

Now, when $p, q \in A$ we have with the same technique as in the proof of the previous proposition

$$\begin{aligned} \tilde{T}(\iota \otimes \iota \otimes \omega)\psi(p \otimes q) &= \tilde{T}(\iota \otimes \iota \otimes \omega)(\phi_{13}(p)\phi_{23}(q)) \\ &= (\iota \otimes \iota \otimes \omega)(\phi_{23}(p)(\phi \otimes \iota)\phi(q)) \\ &= (\iota \otimes \iota \otimes \omega)(\iota \otimes \phi)((1 \otimes p)\phi(q)) \\ &= (\iota \otimes \iota \otimes \omega)(\iota \otimes \phi)\tilde{T}(p \otimes q). \end{aligned}$$

By continuity this is still true when we replace $p \otimes q$ by x . We obtain

$$\omega(1)\tilde{T}(x) = \tilde{T}(\iota \otimes \iota \otimes \omega)\psi(x) = (\iota \otimes \iota \otimes \omega)(\iota \otimes \phi)\tilde{T}(x).$$

This gives $\tilde{T}(x) \otimes 1 = (\iota \otimes \phi)\tilde{T}(x)$. Now apply the slice map $\omega \otimes \iota \otimes \iota$ to get

$$(\omega \otimes \iota)\tilde{T}(x) \otimes 1 = \phi((\omega \otimes \iota)\tilde{T}(x)).$$

By lemma 3.8 we get $(\omega \otimes \iota)\tilde{T}(x) \in \mathbb{C}1$. As this is true for all ω , we must have $\tilde{T}(x) \in M(A) \otimes 1$. As in the remark following 3.1 we see that actually $\tilde{T}(x) \in A \otimes 1$. Then we can define $S(a) \in A$ by $\tilde{T}(x) = S(a) \otimes 1$ when $a \in A_0$ and $x \in A \otimes_{mh} A$ satisfy $a = T_1x$.

Next we show that S is an anti-homomorphism. As before take $a, b \in A_0$ and $x, y \in A \otimes_{mh} A$ such that $a \otimes 1 = T_1x$ and $b \otimes 1 = T_1y$. Use again the notation of the proof of proposition 3.2. Put $z = M(x, y)$. Then $z \in A \otimes_{mh} A$ and $T_1z = ab \otimes 1$. Let $c \in A$. Then we get

$$(S(ab) \otimes 1)\phi(c) = \tilde{T}(z(1 \otimes c)) = \tilde{T}(M(x(1 \otimes c), y)).$$

Now for $p, q \in A$ one has

$$\tilde{T}(M(p \otimes q, y)) = \tilde{T}((p \otimes 1)y(1 \otimes q)) = (1 \otimes p)\tilde{T}(y)\phi(q) = (S(b) \otimes 1)\tilde{T}(p \otimes q).$$

This gives

$$(S(ab) \otimes 1)\phi(c) = (S(b) \otimes 1)\tilde{T}(x(1 \otimes c)) = (S(b)S(a) \otimes 1)\phi(c)$$

and so $S(ab) = S(b)S(a)$. Finally

$$S(a)^* \otimes 1 = (S(a) \otimes 1)^* = T_1(\Gamma(x))$$

where Γ is the map from proposition 1.14. As $\Gamma(x) \in A \otimes_{mh} A$, we have $S(a)^* \in A_0$. Also

$$S(S(a)^*) \otimes 1 = \tilde{T}(\Gamma(x)) = T_1(x)^* = a^* \otimes 1$$

and this proves the result.

In the case of a multiplier Hopf algebra we have essentially the same formulas. The counit in a multiplier Hopf algebra has been defined in just the same way as we do it here, see section 3 of [23]. The way to define the antipode for multiplier Hopf algebras was not usable however. The formulas in lemma 5.4 and lemma 5.5 of [23] are nevertheless very similar to the formulas we use to define the antipode.

We would now like to prove some of the well-known formulas involving the counit and the antipode. We will refer to the discussion above in the Hopf algebra context. Recall that

$$a \otimes 1 = \sum \Delta(a_{(1)})(1 \otimes S(a_{(2)})) = T_1((\iota \otimes S)\Delta(a)).$$

So we expect that the element $x \in A \otimes_{mh} A$ satisfying $T_1(x) = a \otimes 1$ should be, in some sense, $(\iota \otimes S)\Delta(a)$. This is the content of the following proposition.

Proposition 3.11. *Let $a \in A_0, x \in A \otimes_{mh} A$ and $a \otimes 1 = T_1x$. For all $\omega \in A^*$ we have that $(\omega \otimes \iota)\phi(a) \in A_0$ and $S((\omega \otimes \iota)\phi(a)) = (\omega \otimes \iota)(x)$.*

Proof. Before we can really start the proof we have to make some introductory remarks. Let $p, q \in A$ and $\omega \in A^*$. Then we have

$$\begin{aligned} (\omega \otimes \iota \otimes \iota)\psi(p \otimes q) &= (\omega \otimes \iota \otimes \iota)(\phi_{13}(p)\phi_{23}(q)) \\ &= (1 \otimes (\omega \otimes \iota)\phi(p))\phi(q) \\ &= \tilde{T}((\omega \otimes \iota)\phi(p) \otimes q) \\ &= \tilde{T}(((\omega \otimes \iota)\phi \otimes \iota)(p \otimes q)). \end{aligned}$$

Now we would like to have the same formula for arbitrary $x \in A \otimes_{mh} A$ instead of $p \otimes q$. So we have to extend the map $(\omega \otimes \iota)\phi \otimes \iota$. Because $\omega \otimes \iota$ is completely bounded as a map from $M(A \otimes A)$ to $M(A)$ with norm $\|\omega\|$ we get that $(\omega \otimes \iota)\phi \otimes \iota$ can be uniquely extended to a continuous map from $A \otimes_h A$ to itself with norm majorized by $\|\omega\|$. See the discussion in the beginning of section 4. For every $\omega \in A^*$ and $u \in A \otimes A$ we can, by the same argument, extend

$$p \otimes q \mapsto (\omega \otimes \iota)(u\phi(p)) \otimes q$$

to a continuous map from $A \otimes_h A$ to itself with norm majorized by $\|\omega\| \|u\|$. We write

$$Q(\omega, u, x) = ((\omega \otimes \iota)(u\phi(\cdot)) \otimes \iota)(x)$$

for this extension applied to $x \in A \otimes_h A$. If now $x \in A \otimes_{mh} A$, $u \in A \otimes A$, $\omega \in A^*$ and $a \in A$ we get

$$\begin{aligned} \|Q(\omega, u, (a \otimes 1)x)\|_h &= \lim_{\lambda} \|Q(\omega, u, (a \otimes 1)x(1 \otimes e_{\lambda}))\|_h \\ &= \lim_{\lambda} \|Q(\omega, u\phi(a), x(1 \otimes e_{\lambda}))\|_h \\ &\leq \|\omega\| \|u\phi(a)\| \|x\|_h \end{aligned}$$

So by the non-degenerateness of ϕ it is possible to extend Q uniquely to a map

$$A^* \times (A \otimes A) \times (A \otimes_{mh} A) \rightarrow A \otimes_h A$$

such that

$$Q(\omega, u\phi(a), x) = Q(\omega, u, (a \otimes 1)x)$$

for $\omega \in A^*$, $u \in A \otimes A$, $a \in A$ and $x \in A \otimes_{mh} A$. It is clear from the formulas above that for every $\omega \in A^*$ and $u \in A \otimes A$ the map $x \mapsto Q(\omega, u, x)$ from $A \otimes_{mh} A$ to $A \otimes_h A$ is strict-norm continuous on bounded sets.

Let now $\omega \in A^*$. Write $\omega = \mu b$ with $\mu \in A^*$ and $b \in A$. Then define

$$\rho_2(a, x) = Q(\mu, b \otimes a, x) \in A \otimes_h A$$

for every $a \in A$ and $x \in A \otimes_{mh} A$. Then for every $a \in A$ the map $x \mapsto \rho_2(a, x)$ is strict-norm continuous on bounded sets and

$$\rho_2(a, p \otimes q) = (a \otimes 1)((\omega \otimes \iota)\phi \otimes \iota)(p \otimes q).$$

Analogously the map

$$p \otimes q \mapsto ((\omega \otimes \iota)\phi \otimes \iota)(p \otimes q)(1 \otimes a)$$

can be extended uniquely to a map from $A \otimes_{mh} A$ to $A \otimes_h A$ which is strict-norm continuous on bounded sets. So we have proved that $(\omega \otimes \iota)\phi \otimes \iota$ can be extended to a bounded map from $A \otimes_{mh} A$ to itself which is strictly continuous on bounded sets. We denote this (unique) extension by the same symbols.

Now we start the real proof. Let $a \in A_0$, $x \in A \otimes_{mh} A$ and $T_1 x = a \otimes 1$. Then we have $x \otimes 1 = \psi(x)$ so that for all $\omega \in A^*$ we get $(\omega \otimes \iota)(x) \otimes 1 = (\omega \otimes \iota \otimes \iota)\psi(x)$. Now the computation in the beginning of this proof, together with the strict continuity of the maps involved gives

$$(\omega \otimes \iota)(x) \otimes 1 = (\omega \otimes \iota \otimes \iota)\psi(x) = \tilde{T}(((\omega \otimes \iota)\phi \otimes \iota)(x)).$$

It follows from this that $(\omega \otimes \iota)(x) = S(b)$ for $b \in A_0$ satisfying

$$b \otimes 1 = T_1(((\omega \otimes \iota)\phi \otimes \iota)(x)).$$

Now

$$\begin{aligned} T_1(((\omega \otimes \iota)\phi \otimes \iota)(p \otimes q)) &= (\omega \otimes \iota \otimes \iota)((\iota \otimes \phi)\phi(p)(1 \otimes 1 \otimes q)) \\ &= ((\omega \otimes \iota)\phi \otimes \iota)(T_1(p \otimes q)). \end{aligned}$$

So if we replace $p \otimes q$ by x we get

$$b \otimes 1 = ((\omega \otimes \iota)\phi \otimes \iota)(T_1(x)) = ((\omega \otimes \iota)\phi \otimes \iota)(a \otimes 1)$$

and so $b = (\omega \otimes \iota)\phi(a)$.

So formally $x = (\iota \otimes S)\phi(a)$ when $a \otimes 1 = T_1(x)$. Now, by definition $\varepsilon(a)1 = m(x)$ and we recover the formula $m(\iota \otimes S)\phi(a) = \varepsilon(a)1$. Actually we can now give a meaning to this formula because we take elements a such that $(\iota \otimes S)\Delta(a)$ belongs to $A \otimes_{mh} A$, where we can apply multiplication (recall the discussion in the introduction).

From the proof of the previous proposition we also find $\varepsilon((\omega \otimes \iota)\phi(a))1 = m(((\omega \otimes \iota)\phi \otimes \iota)(x))$, when $a \otimes 1 = T_1x$ and $\omega \in A^*$. Now

$$m(((\omega \otimes \iota)\phi \otimes \iota)(p \otimes q)) = (\omega \otimes \iota)(\phi(p)(1 \otimes q)) = (\omega \otimes \iota)T_1(p \otimes q).$$

If we use this formula for x in the place of $p \otimes q$, we arrive at

$$\varepsilon((\omega \otimes \iota)\phi(a))1 = \omega(a)1$$

which precisely stands for $(\iota \otimes \varepsilon)\phi(a) = a$ when $a \in A_0$.

Also the following formula is very familiar:

Proposition 3.12. *Let $a \in A_0$. Then $S(a)^* \in A_0$ and $\varepsilon(S(a)^*) = \overline{\varepsilon(a)}$.*

Proof. Take $x \in A \otimes_{mh} A$ such that $T_1(x) = a \otimes 1$. It follows from the proof of proposition 3.10 that $S(a)^* \otimes 1 = T_1(\Gamma(x))$. Hence we have $\varepsilon(S(a)^*)1 = m(\Gamma(x))$. Now for $p, q \in A$ we get

$$m(\Gamma(p \otimes q)) = m(q^* \otimes p^*) = q^*p^* = (m(p \otimes q))^*$$

so that $m(\Gamma(x)) = (m(x))^* = \overline{\varepsilon(a)}1$. This gives the result.

There is one more formula in Hopf algebra theory that we can give a meaning here, namely

$$\Delta(S(a)) = \sigma(S \otimes S)\Delta(a).$$

Proposition 3.13. *If $a \in A_0$ and $x \in A \otimes_{mh} A$ such that $a \otimes 1 = T_1x$, then $(\iota \otimes \omega)(x) \in A_0$ for all $\omega \in A^*$ and $S((\iota \otimes \omega)(x)) = (\omega \otimes \iota)\phi(S(a))$.*

The formula mentioned before the proposition is obtained by writing formally that $x = (\iota \otimes S)\phi(a)$.

Proof. Start again from the basic formula $x \otimes 1 = \psi(x)$ and now apply the slice map $\iota \otimes \omega \otimes \iota$. For $p, q \in A$ we have

$$\begin{aligned} (\iota \otimes \omega \otimes \iota)(\phi_{13}(p)\phi_{23}(q)) &= \phi(p)(1 \otimes (\omega \otimes \iota)\phi(q)) \\ &= T_1(\iota \otimes (\omega \otimes \iota)\phi)(p \otimes q). \end{aligned}$$

Exactly as in the proof of the previous proposition we can extend $\iota \otimes (\omega \otimes \iota)\phi$ to a map from $A \otimes_{mh} A$ to itself which is strictly continuous on bounded sets. So applying the formula above to x in the place of $p \otimes q$ we get

$$(\iota \otimes \omega)(x) \otimes 1 = (\iota \otimes \omega \otimes \iota)\psi(x) = T_1((\iota \otimes (\omega \otimes \iota)\phi)(x)).$$

Hence $(\iota \otimes \omega)(x) \in A_0$ and

$$S((\iota \otimes \omega)(x)) \otimes 1 = \tilde{T}((\iota \otimes (\omega \otimes \iota)\phi)(x)).$$

Again, with x replaced by $p \otimes q$ we get

$$\begin{aligned} \tilde{T}((\iota \otimes (\omega \otimes \iota)\phi)(p \otimes q)) &= (\omega \otimes \iota \otimes \iota)((1 \otimes 1 \otimes p)(\iota \otimes \phi)\phi(q)) \\ &= ((\omega \otimes \iota)\phi \otimes \iota)\tilde{T}(p \otimes q). \end{aligned}$$

So

$$\begin{aligned} S((\iota \otimes \omega)(x)) \otimes 1 &= ((\omega \otimes \iota)\phi \otimes \iota)(\tilde{T}(x)) \\ &= (\omega \otimes \iota)\phi(S(a)) \otimes 1. \end{aligned}$$

This proves the result.

Again we also get

$$\varepsilon((\iota \otimes \omega)(x))1 = m((\iota \otimes (\omega \otimes \iota)\phi)(x)).$$

With x replaced by $p \otimes q$ we get

$$m((\iota \otimes (\omega \otimes \iota)\phi)(p \otimes q)) = (\omega \otimes \iota)((1 \otimes p)\phi(q)) = (\omega \otimes \iota)\tilde{T}(p \otimes q)$$

and so

$$\varepsilon((\iota \otimes \omega)(x))1 = (\omega \otimes \iota)(S(a) \otimes 1).$$

Thus $\varepsilon((\iota \otimes \omega)(x)) = \omega(S(a))$. This essentially is the formula

$$(\varepsilon \otimes \iota)((\iota \otimes S)\Delta(a)) = S(a).$$

Remark 3.14. In this section we have worked with T_1 and $\tilde{T}$ to define A_0 , S and ε . It is for reasons of symmetry clear that we could also work with A'_0 , consisting of those elements $a \in A$ such that $1 \otimes a$ belongs to the range of T_2 on $A \otimes_{mh} A$. We can then extend the map $p \otimes q \mapsto \phi(p)(q \otimes 1)$ to a map $\hat{T}$ from $A \otimes_{mh} A$ to $M(A \otimes A)$ and define S' and ε' on A'_0 such that

$$1 \otimes S'(a) = \hat{T}(x) \quad \varepsilon'(a)1 = m(x)$$

when $1 \otimes a = T_2(x)$. In the next section and in section 5 we will describe natural subspaces of $A_0 \cap A'_0$ on which S, S' and $\varepsilon, \varepsilon'$ coincide.

4 The two-sided counit and antipode

When A is a C^* -algebra there is a natural way to norm the linear space $M_{nm}(A)$ of n by m matrices over A . When A is a closed $*$ -subalgebra of $B(H)$ this norm is obtained by looking at $M_{nm}(A) \subseteq B(H^m, H^n)$, where H^m denotes the m -fold direct sum of H . Now when $(p_i) \in M_{1n}(A)$ and $(q_i) \in M_{n1}(A)$ we have

$$\|(p_i)\| = \left\| \sum_{i=1}^n p_i p_i^* \right\|^{1/2} \quad \text{and} \quad \|(q_i)\| = \left\| \sum_{i=1}^n q_i^* q_i \right\|^{1/2}.$$

Compared to definition 1.1 the following definition should not be too surprising:

Definition 4.1. Let A be a C^* -algebra and $x \in A \odot A \odot A$. Then we define

$$\|x\|_h = \inf \left\{ \|(p_i)\| \|(q_{ij})\| \|(r_j)\| \mid (p_i) \in M_{1n}(A), (q_{ij}) \in M_{nm}(A), (r_j) \in M_{m1}(A) \right. \\ \left. \text{and } x = \sum_{i,j=1}^{n,m} p_i \otimes q_{ij} \otimes r_j \right\}.$$

So again we take the infimum over all such representations of x in the algebraic tensor product and one can again prove that this is a norm on $A \odot A \odot A$. The completion for this norm will be denoted by $A \otimes_h A \otimes_h A$.

Also here we will regularly want to extend maps defined on the algebraic tensor product to the triple Haagerup tensor product. The following obvious lemma is crucial.

Lemma 4.2. Let B be a C^* -algebra, $(p_i) \in M_{1n}(B)$, $(q_{ij}) \in M_{nm}(B)$ and $(r_j) \in M_{m1}(B)$. Then

$$\left\| \sum_{i,j=1}^{n,m} p_i q_{ij} r_j \right\| \leq \|(p_i)\| \|(q_{ij})\| \|(r_j)\|.$$

Proof. This follows immediately when one takes $B \subseteq B(H)$ and $M_{kl}(B) \subseteq B(H^k, H^l)$.

We recall that a map F from a C^* -algebra A to a C^* -algebra B is called completely bounded when there exists a positive number M such that $\|(F(a_{ij}))\| \leq M\|(a_{ij})\|$ for all $(a_{ij}) \in M_{nm}(A)$ and all $n, m \in \mathbb{N}$. The smallest such number M is denoted by $\|F\|_{cb}$. When $\|F\|_{cb} \leq 1$ we say F is a complete contraction. Completely bounded maps can be tensored freely on the Haagerup tensor product. For instance, when $F, G, H : A \rightarrow B$ are completely bounded, it is clear nearly by definition that $F \otimes G \otimes H$ defined from $A \odot A \odot A$ to $B \odot B \odot B$ is bounded for the Haagerup norm. We also mention that every $\omega \in A^*$ is completely bounded when considered as a map from A to $\mathbb{C}$ and $\|\omega\|_{cb} = \|\omega\|$. So it is clear how to define slice maps $(\iota \otimes \iota \otimes \omega)$ or $(\omega \otimes \iota \otimes \iota)$ from $A \otimes_h A \otimes_h A$ to $A \otimes_h A$. Finally it is also possible to define norms on the spaces of matrices over the Haagerup tensor product. If for instance $x \in M_{nm}(A \odot A)$ we define

$$\|x\|_h = \inf \left\{ \|(p_{ik})\| \|(q_{kj})\| \mid r \in \mathbb{N}, (p_{ik}) \in M_{nr}(A), (q_{kj}) \in M_{rm}(A) \right. \\ \left. \text{and } x_{ij} = \sum_{k=1}^r p_{ik} \otimes q_{kj} \text{ for all } i, j \right\}.$$

So it also makes sense to speak about completely bounded maps from $A \otimes_h A$ to A . When F is such a map, of course $F \otimes \iota$ will again be a completely bounded map from $A \otimes_h A \otimes_h A$ to $A \otimes_h A$. All this is clearly worked out in the theory of *operator spaces* and we refer to e.g. [4], [5] and [9] for this.

We will now treat the two-sided approach somewhat different from the one-sided approach. There we worked with the multiplier Haagerup tensor product $A \otimes_{mh} A$ which could be characterized as the space of elements $x \in M(A \otimes A)$ such that both $(a \otimes 1)x$ and $x(1 \otimes a)$

belong to $A \otimes_h A$ for all $a \in A$. We could also proceed here in a similar way. However, we will stick to the triple Haagerup tensor product.

First we need the map

$$T : p \otimes q \otimes r \mapsto (p \otimes 1 \otimes 1)\phi^{(2)}(q)(1 \otimes 1 \otimes r)$$

where $\phi^{(2)}(q) = (\phi \otimes \iota)\phi(q) = (\iota \otimes \phi)\phi(q)$. Remark that

$$\begin{aligned} T(p \otimes q \otimes r) &= (\iota \otimes \phi)((p \otimes 1)\phi(q))(1 \otimes 1 \otimes r) \\ T(p \otimes q \otimes r) &= (p \otimes 1 \otimes 1)(\phi \otimes \iota)(\phi(q)(1 \otimes r)) \end{aligned}$$

so that formally $T = (\iota \otimes T_1)(T_2 \otimes \iota) = (T_2 \otimes \iota)(\iota \otimes T_1)$. Therefore, the following should be no surprise.

Proposition 4.3. *The map T can be uniquely extended to a contraction from $A \otimes_h A \otimes_h A$ to $A \otimes A \otimes A$. The extended T is injective.*

Proof. First observe that $T(p \otimes q \otimes s) \in A \otimes A \otimes A$ as e.g. $\phi(q)(1 \otimes r) \in A \otimes A$ and $(p \otimes 1)\phi(s) \in A \otimes A$ (for all $p, q, r, s \in A$). Let (p_i) , (q_{ij}) and (r_j) be as before. Then by lemma 4.2 we have

$$\left\| \sum_{i,j=1}^{n,m} (p_i \otimes 1 \otimes 1)\phi^{(2)}(q_{ij})(1 \otimes 1 \otimes r_j) \right\| \leq \|(p_i)\| \|(\phi^{(2)}(q_{ij}))\| \|(r_j)\| \leq \|(p_i)\| \|(q_{ij})\| \|(r_j)\|.$$

So T is a contraction for the Haagerup norm. Now suppose $x \in A \otimes_h A \otimes_h A$ and $T(x) = 0$. Let $\omega \in A^*$ and apply $(\omega \otimes \iota \otimes \iota)$. This formally gives $T_1((\omega \otimes \iota)T_2 \otimes \iota)(x) = 0$. To give a meaning to this we have to verify that the map $(\omega \otimes \iota)T_2$ from $A \otimes_h A$ to A is completely bounded. But this is true because T_2 is completely bounded from $A \otimes_h A$ to $A \otimes A$ and $(\omega \otimes \iota)$ from $A \otimes A$ to A . Then the formula

$$(\omega \otimes \iota \otimes \iota)T(y) = T_1((\omega \otimes \iota)T_2 \otimes \iota)(y)$$

is easily checked for $y \in A \odot A \odot A$ and then extended by continuity. So we get indeed $((\omega \otimes \iota)T_2 \otimes \iota)(x) = 0$ for all $\omega \in A^*$ because T_1 is injective. But then

$$(\omega \otimes \iota)T_2((\iota \otimes \iota \otimes \mu)(x)) = 0$$

for all $\omega, \mu \in A^*$. Because T_2 is injective we get $(\iota \otimes \iota \otimes \mu)(x) = 0$ for all $\mu \in A^*$ and hence $x = 0$.

Now, we will use this triple Haagerup tensor product mainly to prove that the left and right counit and antipode, as obtained in the previous section coincide on a certain subalgebra, contained in $A_0 \cap A'_0$. It will be shown in the examples that this subalgebra is still dense in most known cases. We will also obtain some more formulas for the counit and antipode.

We start with the following notation.

Notation 4.4. Let $\mathcal{A}$ denote the space of elements $a \in A$ such that $b \otimes a \otimes c$ belongs to the range of T on $A \otimes_h A \otimes_h A$ for all $b, c \in A$.

So when $a \in \mathcal{A}$ and $b, c \in A$ we can find $x \in A \otimes_h A \otimes_h A$ such that $b \otimes a \otimes c = T(x)$. If now $\omega \in A^*$ we see, using the formula in the proof of proposition 4.3, that

$$\omega(b)a \otimes c = (\omega \otimes \iota \otimes \iota)T(x) = T_1((\omega \otimes \iota)T_2 \otimes \iota)(x).$$

So we get that $a \otimes c$ belongs to the range of T_1 on $A \otimes_h A$. This is not sufficient for a to be in A_0 . In order to have this one would need $T_1^{-1}(a \otimes c) = y(1 \otimes c)$ for some element $y \in A \otimes_{mh} A$ and all $c \in A$, and this involves an extra condition. If we assume however that $a \in A_0$ then we get the following result.

Proposition 4.5. Let $a \in \mathcal{A} \cap A_0$. Let $b, c \in A$ and $x \in A \otimes_h A \otimes_h A$ such that $b \otimes a \otimes c = T(x)$. Then

$$\varepsilon(a)(b \otimes c) = E(x) \quad \text{with } E(p \otimes q \otimes r) = (p \otimes 1)\phi(q)(1 \otimes r).$$

For the antipode we get the following formulas.

$$\begin{aligned} (b \otimes 1 \otimes 1)(1 \otimes S(a) \otimes 1)(1 \otimes \phi(c)) &= F_1(x) \quad \text{with } F_1(p \otimes q \otimes r) = (p \otimes 1 \otimes 1)\phi_{13}(q)\phi_{23}(r) \\ (\phi(b) \otimes 1)(1 \otimes S(a) \otimes 1)(1 \otimes \phi(c)) &= F_2(x) \quad \text{with } F_2(p \otimes q \otimes r) = (\phi(p) \otimes 1)\phi^{(2)}(q)(1 \otimes \phi(r)) \\ bS(a) \otimes c &= F_3(x) \quad \text{with } F_3(p \otimes q \otimes r) = p \otimes qr \\ bS(a)c &= F_4(x) \quad \text{with } F_4(p \otimes q \otimes r) = pqr \end{aligned}$$

Remark that before the proposition makes sense, we have to show that E, F_1, F_2, F_3 and F_4 are all bounded for the Haagerup norm, with the range in A for F_4 , in $A \otimes A$ for E and F_3 and in $A \otimes A \otimes A$ for F_1 and F_2 . This can be done completely analogously as the extension of T (see proposition 4.3).

The formulas in the proposition can be easily illustrated for Hopf algebras. So let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra and $x \in H$. Then we clearly have, using Sweedler notation,

$$1 \otimes x \otimes 1 = T\left(\sum S(x_{(1)}) \otimes x_{(2)} \otimes S(x_{(3)})\right).$$

To illustrate the second formula for the antipode, we compute:

$$\begin{aligned} \sum \Delta_{12}(S(x_{(1)}))\Delta^{(2)}(x_{(2)})\Delta_{23}(S(x_{(3)})) &= \sum S(x_{(2)})x_{(3)} \otimes S(x_{(1)})x_{(4)}S(x_{(7)}) \otimes x_{(5)}S(x_{(6)}) \\ &= \sum 1 \otimes \varepsilon(x_{(2)})S(x_{(1)})x_{(3)}S(x_{(5)})\varepsilon(x_{(4)}) \otimes 1 \\ &= \sum 1 \otimes S(x_{(1)})x_{(2)}S(x_{(3)}) \otimes 1 \\ &= 1 \otimes \varepsilon(x_{(1)})S(x_{(2)}) \otimes 1 = 1 \otimes S(x) \otimes 1. \end{aligned}$$

Analogously one can compute the other formulas.

Proof. Before this proposition we already remarked that

$$\omega(b)a \otimes c = T_1((\omega \otimes \iota)T_2 \otimes \iota)(x). \quad (4)$$

Because $a \in A_0$ we get from this that

$$\omega(b)\varepsilon(a)c = m((\omega \otimes \iota)T_2 \otimes \iota)(x).$$

Now for $p, q, r \in A$ we have

$$m((\omega \otimes \iota)T_2 \otimes \iota)(p \otimes q \otimes r) = (\omega \otimes \iota)E(p \otimes q \otimes r)$$

and so by continuity we have $\varepsilon(a)b \otimes c = E(x)$. This proves the formula for the counit.

Now starting again with formula 4 and using the definition of the antipode (see 3.10) we get

$$\omega(b)(S(a) \otimes 1)\phi(c) = \tilde{T}((\omega \otimes \iota)T_2 \otimes \iota)(x).$$

But for all $p, q, r \in A$

$$\tilde{T}((\omega \otimes \iota)T_2 \otimes \iota)(p \otimes q \otimes r) = (\omega \otimes \iota \otimes \iota)F_1(p \otimes q \otimes r)$$

and so we get

$$(b \otimes 1 \otimes 1)(1 \otimes S(a) \otimes 1)(1 \otimes \phi(c)) = F_1(x).$$

This gives the first formula for the antipode. Now choose $\omega \in A^*$ and apply $(\iota \otimes \iota \otimes \omega)$ to this equation. This gives

$$b \otimes S(a)(\iota \otimes \omega)\phi(c) = (\iota \otimes \iota \otimes \omega)F_1(x).$$

But for $p, q, r \in A$ we have

$$(\iota \otimes \iota \otimes \omega)F_1(p \otimes q \otimes r) = (p \otimes 1)((\iota \otimes \iota \otimes \omega)(\phi_{13}(q)\phi_{23}(r))).$$

In proposition 3.5 we have seen that

$$q \otimes r \mapsto (\iota \otimes \iota \otimes \omega)(\phi_{13}(q)\phi_{23}(r))$$

extends to a bounded map from $A \otimes_h A$ to itself. A slight modification of that proof gives that this map is even completely bounded. But this allows us to conclude that the map

$$p \otimes q \otimes r \mapsto (p \otimes 1)((\iota \otimes \iota \otimes \omega)(\phi_{13}(q)\phi_{23}(r)))$$

can be extended to a bounded map from $A \otimes_h A \otimes_h A$ to $A \otimes_h A$. We will denote this extension by G . So we get in $A \otimes_h A$ the equality

$$b \otimes S(a)(\iota \otimes \omega)\phi(c) = G(x). \quad (5)$$

Then we may conclude that

$$\phi(b)(1 \otimes S(a)(\iota \otimes \omega)\phi(c)) = T_1(G(x)). \quad (6)$$

Observe now that

$$\begin{aligned} T_1(G(p \otimes q \otimes r)) &= \phi(p)T_1((\iota \otimes \iota \otimes \omega)(\phi_{13}(q)\phi_{23}(r))) \\ &= \phi(p)(\iota \otimes \iota \otimes \omega)(\phi^{(2)}(q)\phi_{23}(r)) \end{aligned}$$

as we already observed in the proof of proposition 3.7

$$= (\iota \otimes \iota \otimes \omega)F_2(p \otimes q \otimes r)$$

By continuity we conclude from formula 6 that

$$\phi_{12}(b)(1 \otimes S(a) \otimes 1)\phi_{23}(c) = F_2(x).$$

This is the second formula for the antipode we had to prove. Next we observe that

$$\begin{aligned} m(G(p \otimes q \otimes r)) &= pm((\iota \otimes \iota \otimes \omega)(\phi_{13}(q)\phi_{23}(r))) \\ &= p(\iota \otimes \omega)\phi(qr) \end{aligned}$$

as we already observed in the proof of proposition 3.9

$$= (\iota \otimes \omega)T_2(p \otimes qr)$$

Now we have to remark that F_3 is a contraction from $A \otimes_h A \otimes_h A$ to $A \otimes_h A$ and this allows us to write

$$m(G(x)) = (\iota \otimes \omega)(T_2(F_3(x))).$$

Combined with formula 5 we get

$$T_2(bS(a) \otimes c) = T_2(F_3(x))$$

and so $bS(a) \otimes c = F_3(x)$. We already observed that $F_3(x) \in A \otimes_h A$ and so we get $bS(a)c = m(F_3(x))$. Clearly $m(F_3(x)) = F_4(x)$ and this concludes the proof of the proposition.

Now, some of these formulas are nicely symmetric. This is the case e.g. for the formula with ε and for the second and the fourth formula involving S . Using the notation of remark 3.14 this implies the following.

Proposition 4.6. *If $a \in \mathcal{A} \cap A_0 \cap A'_0$, then $\varepsilon(a) = \varepsilon'(a)$ and $S(a) = S'(a)$.*

Proof. In a completely similar fashion we get for elements $a \in \mathcal{A} \cap A'_0$ formulas for $\varepsilon'(a)$ and $S'(a)$ as in the previous proposition. From these formulas, it immediately follows that $\varepsilon(a) = \varepsilon'(a)$ and $S(a) = S'(a)$.

One might guess from the formulas in 4.5 that it is possible to define ε and S on all of $\mathcal{A}$. This is indeed the case. It requires some more arguments. But it is not very relevant for the theory. In some sense, the space $\mathcal{A}$ is too big. Moreover, there seems to be no argument to show that $\mathcal{A}$ is an algebra which would be a natural requirement.

Now, also the set $\mathcal{A} \cap A_0 \cap A'_0$ will probably not be an algebra. Of course, as A_0 and A'_0 are algebras, the subalgebra of A generated by $\mathcal{A} \cap A_0 \cap A'_0$ will be a subalgebra of A_0 and of A'_0 where $\varepsilon = \varepsilon'$ and $S = S'$.

From the theoretical point it would certainly be nice to have a more precise understanding of all these sets. On the other hand in practice it is just important to have enough elements. This is what we will show in the examples in the next section.

5 Special cases and examples

In this section, we will consider some special cases and examples. Essentially, there are always three steps to be taken. First there is the construction of the comultiplication on the C^* -algebra. Then one must try to prove the injectivity of the maps T_1 and T_2 so that we do have a Hopf C^* -algebra. Finally one has to find the counit and antipode, as constructed in section 3 (one-sided) and section 4 (two-sided) and hopefully show that the respective domains are dense.

5.1 Locally compact groups

The main two examples to start with are $C_0(G)$ and $C_r^*(G)$ for a locally compact group G . The first result is obvious.

Proposition 5.1. *Let A be the C^* -algebra $C_0(G)$ of continuous complex functions tending to 0 at infinity on a locally compact group G . Let ϕ be defined as before by $\phi(f)(p, q) = f(pq)$ for $f \in C_0(G)$ and $p, q \in G$. The ϕ is a comultiplication on A and (A, ϕ) is a Hopf C^* -algebra.*

The injectivity of the maps T_1 and T_2 follows as $A \otimes_h A$ is contained in $A \otimes A$ and here T_1 and T_2 have continuous extensions to $A \otimes A$ and these extensions are even C^* -automorphisms.

We give the following result mainly for reasons of presentation. We will formulate and prove more general results in proposition 5.6 and 5.11.

Proposition 5.2. *Let $A = C_0(G)$ and ϕ as before. Consider a left Haar measure on G , take two functions f and g in $K(G)$, the space of continuous complex functions with compact support in G and consider $h \in K(G)$ defined by*

$$h(p) = \int f(pq)g(q)dq.$$

Then $h \in A_0$ (as defined in 3.1) and $\varepsilon(h) = h(e)$ and $S(h)(p) = h(p^{-1})$.

We will not give the proof here in detail as we will come back to this later. But we can define a function k of two variables by

$$k(p, q) = \int f(pr)g(qr)dr.$$

This is a bounded continuous function and one can show that it belongs to $A \otimes_{mh} A$. If we apply T_1 to it, we obtain

$$\begin{aligned} (T_1 k)(p, q) &= k(pq, q) = \int f(pqr)g(qr)dr \\ &= \int f(pr)g(r)dr = h(p). \end{aligned}$$

It follows that $T_1(k) = h \otimes 1$ and so $h \in A_0$. Then $\varepsilon(h) = k(p, p) = \int f(pr)g(pr)dr = \int f(r)g(r)dr = h(e)$ for all p . Moreover

$$\begin{aligned} k(q, pq) &= \int f(qr)g(pqr)dr \\ &= \int f(r)g(pr)dr \\ &= \int f(p^{-1}r)g(r)dr = h(p^{-1}) \end{aligned}$$

and this illustrates the formula $S(h)(p) = h(p^{-1})$.

The next case is $C_r^*(G)$. Here, the first step is well-known.

Proposition 5.3. *Let G be a locally compact group. The reduced C^* -algebra $C_r^*(G)$ of G is the C^* -algebra generated by the left convolution operators on $L^2(G)$, where G is considered with the left Haar measure. If we define W on $L^2(G) \otimes L^2(G)$ by*

$$(W\xi)(p, q) = \xi(p, pq)$$

then $\phi(x) = W^(x \otimes 1)W$ defines a comultiplication on $C_r^*(G)$ and $(C_r^*(G), \phi)$ is a Hopf C^* -algebra.*

Proof. That W induces a comultiplication on $C_r^*(G)$ is proved in e.g. [22], proposition 4.2. The only thing that remains to be shown is that the maps T_1 and T_2 are injective on $C_r^*(G) \otimes_h C_r^*(G)$. We will prove it for T_2 . Then the injectivity of T_1 can be obtained easily as $\phi(x) = \sigma\phi(x)$ (where σ is the flip) so that

$$\begin{aligned} T_1(a \otimes b) &= \phi(a)(1 \otimes b) \\ &= ((1 \otimes b^*)\phi(a^*))^* \\ &= (\sigma(T_2(b^* \otimes a^*)))^* \end{aligned}$$

and we have seen that $a \otimes b \rightarrow b^* \otimes a^*$ is an isometry of the Haagerup tensor product to itself.

Now remark that for $p, q \in C_r^*(G)$, $\omega \in B(L^2(G))_*$ and any right convolution operator y on $L^2(G)$ we have

$$y((\iota \otimes \omega)(T_2(p \otimes q)W^*)) = y((\iota \otimes \omega)((p \otimes 1)W^*(q \otimes 1))) = py((\iota \otimes \omega)(W^*))q$$

where we used the obvious fact that left and right convolution operators commute. It is clear that for any $z \in B(L^2(G))$ we can define a bounded map M_z from $C_r^*(G) \otimes_h C_r^*(G)$ to $B(L^2(G))$, with norm majorized by $\|z\|$ and such that $M_z(p \otimes q) = p z q$. If now $x \in C_r^*(G) \otimes_h C_r^*(G)$ and $T_2(x) = 0$ we conclude by continuity from the computation above that $M_z(x) = 0$ for all z of the form $y((\iota \otimes \omega)(W^*))$. But the space $\{(\iota \otimes \omega)(W^*) \mid \omega \in B(L^2(G))_*\}$ is norm dense in the space of multiplication operators by functions of $C_0(G)$. So the linear span of the elements $y((\iota \otimes \omega)(W^*))$ with $\omega \in B(L^2(G))_*$ and y a right convolution operator,

is norm dense in the compact operators on $L^2(G)$. So we can conclude that $M_k(x) = 0$ for all compact operators k on $L^2(G)$. In particular we get, for all $\xi_1, \xi_2, \eta_1, \eta_2 \in L^2(G)$

$$0 = \langle M_{\theta_{\xi_1} \theta_{\eta_2}^*}(x) \eta_1, \xi_2 \rangle = (\omega_{\xi_1, \xi_2} \otimes \omega_{\eta_1, \eta_2})(x)$$

where we used the notation θ_ξ for the rank one operator from $\mathbb{C}$ to $L^2(G)$ given by $\theta_\xi(\lambda) = \lambda\xi$. From this it follows that $x = 0$ and that completes the proof.

Also here we can get enough elements in A_0 . Again this follows from a more general result that we will prove later (see propositions 5.6 and 5.11). But let us once more illustrate the theory by looking closer to this special case. We have the following result.

Proposition 5.4. *Let $f \in L^1(G)$ and let $\pi(f)$ denote left convolution with f on $L^2(G)$, then $\pi(f) \in A_0$ and $\varepsilon(\pi(f)) = \int f(p)dp$ and $S(\pi(f)) = \pi(\tilde{f})$ where $\tilde{f}(p) = \Delta(p)^{-1}f(p^{-1})$ and Δ is the modular function of G .*

The argument here is essentially the following. Denote by λ_p the left translation operator on L^2 by p given by $(\lambda_p \xi)(q) = \xi(p^{-1}q)$. Put $h = \int f(p)(\lambda_p \otimes \lambda_{p^{-1}})dp$. This will be in $C_r^*(G) \otimes_{mh} C_r^*(G)$ and

$$\begin{aligned} T_1(h) &= \int f(p)(\lambda_p \otimes \lambda_p \lambda_{p^{-1}})dp \\ &= \left(\int f(p) \lambda_p dp \right) \otimes 1. \end{aligned}$$

So $\int f(p) \lambda_p dp \in A_0$ and $\varepsilon(\int f(p) \lambda_p dp)1 = \int f(p) \lambda_p \lambda_{p^{-1}} dp = \int f(p) dp 1$ and

$$\int f(p)(\lambda_{p^{-1}} \otimes \lambda_p \lambda_{p^{-1}})dp = \int f(p) \lambda_{p^{-1}} dp = \int f(p^{-1}) \Delta(p^{-1}) \lambda_p dp.$$

For these two basic examples it is also possible to give elements in the space $\mathcal{A} \cap A_0 \cap A'_0$ (cf. previous section). This is not so relevant here, see propositions 5.8 and 5.12.

5.2 Hopf C^* -algebras with corepresentations

Now we will pass to a much more general situation and we will show how corepresentations give rise to elements in the domain of the counit and the antipode.

First recall the notion of a corepresentation as we will use it here. In this section, we will fix a Hopf C^* -algebra (A, ϕ) .

Definition 5.5. *By a unitary corepresentation of (A, ϕ) on a Hilbert space H we mean a unitary element u in $M(A \otimes B_0(H))$, where $B_0(H)$ stands for the compact operators on H , satisfying $(\phi \otimes \iota)(u) = u_{13}u_{23}$.*

We again use the leg numbering notation here : u_{23} is $1 \otimes u$ in $M(A \otimes A \otimes B_0(H))$ while u_{13} is obtained from $1 \otimes u$ by applying the flip on the first two factors.

Proposition 5.6. *Let u be a unitary corepresentation such that $(\iota \otimes \omega)(u) \in A$ for all $\omega \in B(H)_*$. Then $(\iota \otimes \omega)(u) \in A_0$ for all $\omega \in B(H)_*$ and*

$$\varepsilon(\iota \otimes \omega)(u) = \omega(1) \quad S(\iota \otimes \omega)(u) = (\iota \otimes \omega)(u^*)$$

Proof. First remark that it is sufficient to look at the case $\omega = \omega_{\xi, \eta}$ with $\xi, \eta \in H$. Because $\omega \in B_0(H)^*$ we can take a representation π of $B_0(H)$ on another Hilbert space K such that $\omega = \omega_{\xi, \eta} \circ \pi$ with $\xi, \eta \in K$. Then we replace u by $(\iota \otimes \pi)(u)$. So we may suppose $\omega = \omega_{\xi, \eta}$ with $\xi, \eta \in H$. Let $(\rho_i)_{i \in I}$ be an orthonormal basis for H and define

$$p_i = (\iota \otimes \omega_{\rho_i, \eta})(u) \quad q_i = (\iota \otimes \omega_{\xi, \rho_i})(u^*).$$

We have $p_i, q_i \in A$ for all i and by a slight modification of lemma 3.4 (or by lemma 9.5 in [14]) we get that $\sum_{i \in I} p_i p_i^*$ converges strictly to $(\iota \otimes \omega_{\eta, \eta})(uu^*)$. Also $\sum_{i \in I} q_i^* q_i$ is a bounded strictly converging net. So by proposition 1.15 we can define an element $x \in A \otimes_{mh} A$ by

$$x = \sum_{i \in I} p_i \otimes q_i,$$

the sum being strictly convergent. Then we have

$$\begin{aligned} T_1(x) &= \sum_{i \in I} \phi(p_i)(1 \otimes q_i) \\ &= (\iota \otimes \iota \otimes \omega_{\xi, \eta})((\phi \otimes \iota)(u)u_{23}^*) \\ &= (\iota \otimes \iota \otimes \omega)(u_{13}u_{23}u_{23}^*) \\ &= (\iota \otimes \iota \otimes \omega)(u_{13}) \\ &= (\iota \otimes \omega)(u) \otimes 1. \end{aligned}$$

So indeed $(\iota \otimes \omega)(u) \in A_0$. Next we have $\varepsilon((\iota \otimes \omega)(u))1 = \sum_{i \in I} p_i q_i = (\iota \otimes \omega)(uu^*) = \omega(1)1$ and

$$\begin{aligned} S((\iota \otimes \omega)(u)) \otimes 1 &= \sum_{i \in I} (1 \otimes p_i) \phi(q_i) \\ &= (\iota \otimes \iota \otimes \omega_{\xi, \eta})(u_{23}(\phi \otimes \iota)(u^*)) \\ &= (\iota \otimes \iota \otimes \omega)(u_{23}u_{23}^*u_{13}^*) \\ &= (\iota \otimes \iota \otimes \omega)(u_{13}^*) \\ &= (\iota \otimes \omega)(u^*) \otimes 1 \end{aligned}$$

so that $S((\iota \otimes \omega)(u)) = (\iota \otimes \omega)(u^*)$. This completes the proof.

We would like to comment a little on this result and the proof of it.

Observe that the left regular representation of a locally compact quantum group will satisfy the requirement that $(\iota \otimes \omega)(u) \in A$ for all ω . Remark that the trivial corepresentation ($1 \in M(A)$) does not have this property when A has no identity.

Formally, we can write

$$(\varepsilon \otimes \iota)(u) = 1 \quad (S \otimes \iota)(u) = u^*$$

If u and v are corepresentations as in definition 5.5, respectively on H and K , we can define a corepresentation w on $H \otimes K$ by $w = u_{12}v_{13}$. When $\omega_1 \in B(H)_*$ and $\omega_2 \in B(K)_*$ we find

$$(\iota \otimes \omega_1 \otimes \omega_2)(w) = (\iota \otimes \omega_1)(u)(\iota \otimes \omega_2)(v)$$

and we see that we obtain a subalgebra of elements in A_0 . It is easy to verify that ε is a homomorphism and S a antihomomorphism on this subalgebra. Also $S(a)^* = (\iota \otimes \bar{\omega})(u)$ when $a = (\iota \otimes \omega)(u)$ so that again $S(a)^*$ is of this form. And the formula $S(S(a)^*)^* = a$ is obvious here.

Observe that the same formula for the antipode is obtained in [26], Theorem 1.6, in the framework of manageable multiplicative unitaries. Also for locally compact quantum groups we have such a formula, but that is not so surprising because any locally compact quantum group is a Hopf C*-algebra. See proposition 5.13 and the remark following it for a more precise result. In the framework of Hopf algebras (or multiplier Hopf algebras) these formulas are again well known, see § 11.1.1 of [12] and [25]. So far about the left counit and antipode.

In a completely similar way, we can work with unitary elements v in $M(B_0(H) \otimes A)$ satisfying $(\iota \otimes \phi)(v) = v_{12}v_{13}$. Then $(\omega \otimes \iota)(v) \in A'_0$ and $\varepsilon'((\omega \otimes \iota)(v)) = \omega(1)$ and $S'((\omega \otimes \iota)(v)) = (\omega \otimes \iota)(v^*)$, when $(\omega \otimes \iota)(v) \in A$ for all ω . Now, because we can simply go from one type corepresentation to the other by applying the flip, we get the following result.

Proposition 5.7. *As before, let u be a unitary corepresentation such that $(\iota \otimes \omega)(u) \in A$ for all $\omega \in B(H)_*$. Then $(\iota \otimes \omega)(u) \in A_0 \cap A'_0$ and ε and ε' , as well as S and S' coincide on these elements.*

Proof. Let $v = \sigma u$ where σ denotes the flip from $M(A \otimes B_0(H))$ to $M(B_0(H) \otimes A)$. Then $(\iota \otimes \phi)(v) = v_{12}v_{13}$. If $a = (\omega \otimes \iota)(v)$ and $x = (\omega \otimes \iota \otimes \iota)(v_{12}^*v_{13})$ one verifies, as in the proof of 5.6, that $x \in A \otimes_{mh} A$ and that

$$\begin{aligned} T_2(x) &= (\omega \otimes \iota \otimes \iota)(v_{12}^*(\iota \otimes \phi)(v)) \\ &= (\omega \otimes \iota \otimes \iota)(v_{12}^*v_{12}v_{13}) \\ &= (\omega \otimes \iota \otimes \iota)(v_{13}) = 1 \otimes a. \end{aligned}$$

Hence $\varepsilon'(a)1 = (\omega \otimes \iota)(v^*v) = \omega(1)1$. Also

$$\begin{aligned} 1 \otimes S'(a) &= (\omega \otimes \iota \otimes \iota)((\iota \otimes \phi)(v^*)v_{12}) \\ &= (\omega \otimes \iota \otimes \iota)(v_{13}^*v_{12}^*v_{12}) \\ &= (\omega \otimes \iota \otimes \iota)(v_{13}^*) = 1 \otimes (\omega \otimes \iota)(v^*). \end{aligned}$$

Together with the previous proposition this gives the result.

As we have remarked before, taking into account all such corepresentations, we get a subalgebra of $A_0 \cap A'_0$ where $\varepsilon = \varepsilon'$ and $S = S'$.

Also remark that here we need $v^*v = 1$ and hence $u^*u = 1$ whereas in the previous proposition we needed $uu^* = 1$.

Now we come to the two-sided approach. Here we can prove the following.

Proposition 5.8. *Let u and w be unitary corepresentations on H and K respectively with the property that $(\iota \otimes \omega)(u)$ and $(\iota \otimes \omega')(w)$ are in A for $\omega \in B(H)_*$ and $\omega' \in B(K)_*$ respectively. Then the product of two such elements belongs to $\mathcal{A} \cap A_0 \cap A'_0$, as considered in 4.4.*

Proof. Take $\omega \in B(H)_*$ and $\omega' \in B(K)_*$. Just as in the proof of proposition 5.6 we may suppose that $\omega = \omega_{\xi, \eta}$ and $\omega' = \omega_{\xi', \eta'}$. Write $a = (\iota \otimes \omega)(u)$ and $b = (\iota \otimes \omega')(w)$. Use the flip map σ and denote $v = \sigma w \in M(B_0(K) \otimes A)$. Then $b = (\omega' \otimes \iota)(v)$ and

$$ab = (\omega' \otimes \iota \otimes \omega)(u_{23}v_{12}).$$

From the previous proposition it follows that $a, b \in A_0 \cap A'_0$, and because both are subalgebras of A we get $ab \in A_0 \cap A'_0$. To show that $ab \in \mathcal{A}$ we consider

$$x = (\omega_{\xi', \eta'} \otimes \iota \otimes \iota \otimes \iota \otimes \omega_{\xi, \eta})(v_{12}^* u_{25}^* u_{35} v_{13} v_{14}^* u_{45}^*).$$

We claim that for all $c, d \in A$ we have

$$(c \otimes 1 \otimes 1)x(1 \otimes 1 \otimes d) \in A \otimes_h A \otimes_h A$$

and

$$T((c \otimes 1 \otimes 1)x(1 \otimes 1 \otimes d)) = c \otimes ab \otimes d.$$

Let $(\rho_i)_{i \in I}$ be an orthonormal basis for H and $(\rho'_j)_{j \in J}$ for K . Define

$$\begin{aligned} p'_k &= (\omega_{\rho'_k, \eta'} \otimes \iota)(v^*) & p_i &= (\iota \otimes \omega_{\rho_i, \eta})(u^*) \\ q'_{kl} &= (\omega_{\rho'_l, \rho'_k} \otimes \iota)(v) & q_{ij} &= (\iota \otimes \omega_{\rho_j, \rho_i})(u) \\ r'_l &= (\omega_{\xi', \rho'_l} \otimes \iota)(v^*) & r_j &= (\iota \otimes \omega_{\xi, \rho_j})(u^*). \end{aligned}$$

As in the proof of proposition 5.6 we get that the nets $\sum_{k \in J} p'_k p'_k^*$ and $\sum_{i \in I} p_i p_i^*$ are both bounded and strictly converging. So also

$$\sum_{(i,k) \in I \times J} p'_k p_i p_i^* p'_k^*$$

will be bounded and strictly converging. Analogously

$$\sum_{(j,l) \in I \times J} r_j^* r'^*_{l'} r_l r_j$$

is bounded and strictly converging. It is clear that $(q_{ij} q'_{kl})_{(i,k), (j,l)}$ is a bounded (infinite) $I \times J$ by $I \times J$ matrix over A . Then it is not hard to see that for every $c, d \in A$ the net

$$\sum_{i,k,j,l} c p'_k p_i \otimes q_{ij} q'_{kl} \otimes r'_l r_j d$$

is norm convergent in $A \otimes_h A \otimes_h A$ (see also the estimate in the proof of proposition 5.12). The limit in $A \otimes A \otimes A$ is given by $(c \otimes 1 \otimes 1)x(1 \otimes 1 \otimes d)$. Applying T to this element gives, after some computations,

$$(c \otimes 1 \otimes 1)(\omega_{\xi', \eta'} \otimes \iota \otimes \iota \otimes \iota \otimes \omega_{\xi, \eta})(v_{12}^* u_{25}^* u_{25} u_{35} u_{45} v_{12} v_{13} v_{14} v_{14}^* u_{45}^*)(1 \otimes 1 \otimes d) \quad (7)$$

where we used

$$(\iota \otimes \phi^{(2)} \otimes \iota)(1 \otimes u) = u_{25} u_{35} u_{45} \quad (\iota \otimes \phi^{(2)} \otimes \iota)(v \otimes 1) = v_{12} v_{13} v_{14}.$$

But now observe that the element in formula 7 equals

$$\begin{aligned} (c \otimes 1 \otimes 1)(\omega' \otimes \iota \otimes \iota \otimes \iota \otimes \omega)(v_{12}^* u_{35} u_{45} v_{12} v_{13} u_{45}^*)(1 \otimes 1 \otimes d) \\ = (c \otimes 1 \otimes 1)(\omega' \otimes \iota \otimes \iota \otimes \iota \otimes \omega)(u_{35} u_{45} u_{45}^* v_{12}^* v_{12} v_{13})(1 \otimes 1 \otimes d) \\ = (c \otimes 1 \otimes 1)(\omega' \otimes \iota \otimes \iota \otimes \iota \otimes \omega)(u_{35} v_{13})(1 \otimes 1 \otimes d) \\ = c \otimes (\omega' \otimes \iota \otimes \omega)(u_{23} v_{12}) \otimes d \\ = c \otimes ab \otimes d. \end{aligned}$$

So we get that $c \otimes ab \otimes d$ belongs to the range of T on $A \otimes_h A \otimes_h A$ for all $c, d \in A$. This gives $ab \in \mathcal{A}$.

Remark that here we need both $u^*u = 1$ and $uu^* = 1$, and similarly for w .

Let us illustrate some of the formulas that we obtained in section 4.

If we apply the map $p \otimes q \otimes r \mapsto (p \otimes 1)\phi(q)(1 \otimes r)$ to the element x in the proof of the proposition, we get

$$\begin{aligned} (\omega' \otimes \iota \otimes \iota \otimes \omega)(v_{12}^* u_{24}^* u_{24} u_{34} v_{12} v_{13} v_{13}^* v_{34}^*) &= (\omega' \otimes \iota \otimes \iota \otimes \omega)(v_{12}^* u_{34} v_{12} u_{34}^*) \\ &= (\omega' \otimes \iota \otimes \iota \otimes \omega)(v_{12}^* v_{12} u_{34} u_{34}^*) \\ &= \omega'(1)\omega(1)1 \otimes 1 \end{aligned}$$

and this illustrates the formula

$$\varepsilon(a)(b \otimes c) = \sum (p_i \otimes 1)\phi(q_{ij})(1 \otimes r_j)$$

we obtained in proposition 4.5.

If we apply $p \otimes q \otimes r \rightarrow (\phi(p) \otimes 1)\phi^{(2)}(q)(1 \otimes \phi(r))$ to x we get

$$\begin{aligned} (\omega' \otimes \iota \otimes \iota \otimes \iota \otimes \omega)(v_{13}^* v_{12}^* u_{35}^* u_{25}^* u_{25} u_{35} u_{45} v_{12} v_{13} v_{14} v_{14}^* v_{13}^* u_{45}^* u_{35}^*) \\ = (\omega' \otimes \iota \otimes \iota \otimes \iota \otimes \omega)(v_{13}^* v_{12}^* u_{45} v_{12} u_{45}^* u_{35}^*) \\ = (\omega' \otimes \iota \otimes \iota \otimes \iota \otimes \omega)(v_{13}^* u_{35}^*) \\ = 1 \otimes (\omega' \otimes \iota)(v^*)(\iota \otimes \omega)(u^*) \otimes 1 \\ = 1 \otimes S(b)S(a) \otimes 1 = 1 \otimes S(ab) \otimes 1 \end{aligned}$$

and this illustrates the formula

$$(\phi(b) \otimes 1)(1 \otimes S(a) \otimes 1)(1 \otimes \phi(c)) = \sum (\phi(p_i) \otimes 1) \phi^{(2)}(q_{ij})(1 \otimes \phi(r_j)).$$

Finally, look at the map $p \otimes q \otimes r \rightarrow p \otimes qr$ and apply it to x . This gives

$$\begin{aligned} (\omega' \otimes \iota \otimes \iota \otimes \omega)(v_{12}^* u_{24}^* u_{34} v_{13}^* u_{34}^*) &= (\omega' \otimes \iota \otimes \iota \otimes \omega)(v_{12}^* u_{24}^*) \\ &= (\omega' \otimes \iota)(v^*)(\iota \otimes \omega)(u^*) \otimes 1 \\ &= S(b)S(a) \otimes 1 = S(ab) \otimes 1. \end{aligned}$$

which illustrates the formula

$$bS(a) \otimes c = \sum p_i \otimes q_{ij} r_j.$$

So we can conclude that for Hopf C^* -algebras (A, ϕ) with enough corepresentations u satisfying $(\iota \otimes \omega)(u) \in A$ for all $\omega \in B(H)_*$, we get a dense subalgebra A_1 , namely the set of such elements $(\iota \otimes u)(u)$, contained in $A_0 \cap A'_0$. By taking A_1^2 , we even get a dense subalgebra contained in $\mathcal{A} \cap A_0 \cap A'_0$. But already $\varepsilon = \varepsilon'$ and $S = S'$ on A_1 .

5.3 Locally compact quantum groups

The last class of Hopf C^* -algebras we will consider here are the ones admitting faithful Haar measures.

Let us first concentrate on the problem of the injectivity of the maps T_1 and T_2 . Clearly we have the following result.

Proposition 5.9. *Let (A, ϕ) be a pair of a C^* -algebra A with identity and a comultiplication ϕ . Assume there is a faithful state h on A which is both left and right invariant (i.e. $(\iota \otimes h)\phi(a) = h(a)1$ and $(h \otimes \iota)\phi(a) = h(a)1$ for all $a \in A$), then the maps T_1 and T_2 are injective on $A \otimes_h A$.*

Proof. Let $x \in A \otimes_h A$ and assume that $T_1(x) = 0$. Now observe that for all $a \in A$, $\omega \in A^*$ and $p, q \in A$

$$(h \otimes \omega)(\phi(a)T_1(p \otimes q)) = (h \otimes \omega)(\phi(ap)(1 \otimes q)) = h(ap\omega(q)) = h(a(\iota \otimes \omega)(p \otimes q)).$$

By continuity we may conclude that

$$h(a(\iota \otimes \omega)(x)) = (h \otimes \omega)(\phi(a)T_1(x)) = 0.$$

By the faithfulness of h we get $(\iota \otimes \omega)(x) = 0$ for all $\omega \in A^*$. So $x = 0$.

This is the case of a compact quantum group with a faithful Haar measure. Remark that the Hopf $*$ -algebra obtained as matrix elements of finite-dimensional unitary corepresentations will be (properly) contained in the algebra that we obtained previously in this section and

that of course the counit and antipode, when restricted to this Hopf * -algebra gives the counit and the antipode of the Hopf * -algebra.

This proposition indicates the relation between the faithfulness of the Haar measures and the injectivity of the maps T_1 and T_2 . Unfortunately, in the general case, where we have to work with weights (instead of states), the situation is much more complicated.

We will use chapter 1 of [14] as a reference for weights on C^* -algebras and use freely the notations and results introduced there. If φ is a weight on a C^* -algebra A we will use the notation $\mathcal{N}_\varphi = \{x \in A \mid \varphi(x^*x) < \infty\}$ and $\mathcal{M}_\varphi^+ = \{x \in A^+ \mid \varphi(x) < \infty\}$. We use the abbreviation l.s.c. for lower semicontinuous. As an analogue for the left invariance of a measure we introduce the following:

Definition 5.10. *Let A be a C^* -algebra and ϕ a comultiplication on A . A l.s.c. weight φ on A is called left invariant if $\varphi((\omega \otimes \iota)\phi(a)) = \varphi(a)\omega(1)$ whenever $a \in \mathcal{M}_\varphi^+$ and $\omega \in A_+^*$.*

Observe that we use the canonical extension of the l.s.c. weight φ to a weight on $M(A)$ and still denote it with φ .

Once we have a left invariant weight on a Hopf C^* -algebra (A, ϕ) we can construct elements in A_0 . The result we obtain is closely related to proposition 5.6: with this left invariant weight one can also construct the left regular corepresentation W and then the elements in A_0 given in the following proposition are of the form $(\iota \otimes \omega)(W)$. Nevertheless we cannot simply apply proposition 5.6 because with the given assumptions it seems impossible to prove that W is a unitary (and not only a partial isometry) and that W belongs to $M(A \otimes B_0(H))$.

Proposition 5.11. *Let (A, ϕ) be a Hopf C^* -algebra and suppose that φ is a l.s.c. densely defined left invariant weight on A . Then for every $a, b \in \mathcal{N}_\varphi$ we have*

$$(\iota \otimes \varphi)(\phi(a^*)(1 \otimes b)) \in A_0$$

and

$$S((\iota \otimes \varphi)(\phi(a^*)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a^*)\phi(b)) \quad \varepsilon((\iota \otimes \varphi)(\phi(a^*)(1 \otimes b))) = \varphi(a^*b).$$

When moreover $\{(\iota \otimes \omega)\phi(a) \mid \omega \in A^*, a \in A\}$ spans a dense subspace of A and when φ is faithful, then A_0 is dense in A .

Proof. Let $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ be a GNS-construction for φ . Let $(\xi_i)_{i \in I}$ be an orthonormal basis for H_φ and define for $\eta \in H_\varphi$ the operator $\theta_\eta(\lambda) = \lambda\eta$ for $\lambda \in \mathbb{C}$. Now we will use the map $\iota \otimes \Lambda_\varphi$ introduced in proposition 1.26 of [14]. This map takes values in $\mathcal{L}(A, A \otimes H_\varphi)$, the space of adjointable maps between the Hilbert A -modules A and $A \otimes H_\varphi$. Of course, for $a \in M(A)$ and $b \in \mathcal{N}_\varphi$ we have $(\iota \otimes \Lambda_\varphi)(a \otimes b)(x) = ax \otimes \Lambda_\varphi(b)$ for all $x \in A$, but it is possible to extend $\iota \otimes \Lambda_\varphi$ to a much larger domain, see [14]. Then one has $(\iota \otimes \Lambda_\varphi)(y)^*(\iota \otimes \Lambda_\varphi)(x) = (\iota \otimes \varphi)(y^*x)$ for all x and y in the domain of $\iota \otimes \Lambda_\varphi$. With the help of this we define

$$p_i^* = (1 \otimes \theta_{\xi_i}^*)(\iota \otimes \Lambda_\varphi)\phi(a) \quad \text{and} \quad q_i = (1 \otimes \theta_{\xi_i})(\iota \otimes \Lambda_\varphi)\phi(b).$$

Then $p_i, q_i \in \mathcal{L}(A) = M(A)$ for all $i \in I$. But if we fix $i \in I$ we can approximate q_i in norm by elements of the form

$$(1 \otimes \theta_{\Lambda_\varphi(x)}^*)(\iota \otimes \Lambda_\varphi)\phi(b)$$

with $x \in \mathcal{N}_\varphi$. This equals $(\iota \otimes \varphi)((1 \otimes x^*)\phi(b))$ and hence belongs to A by 2.5 in [14]. So $p_i, q_i \in A$ for all $i \in I$. Now consider the net

$$\sum_{i \in I} p_i p_i^* = ((\iota \otimes \Lambda_\varphi)\phi(a))^* \left(1 \otimes \sum_{i \in I} \theta_{\xi_i} \theta_{\xi_i}^*\right) (\iota \otimes \Lambda_\varphi)\phi(a).$$

This net is strictly convergent to

$$((\iota \otimes \Lambda_\varphi)\phi(a))^* (\iota \otimes \Lambda_\varphi)\phi(a) = (\iota \otimes \varphi)\phi(a^*a) = \varphi(a^*a)1.$$

Analogously $\sum_{i \in I} q_i^* q_i$ is bounded and strictly convergent. So by proposition 1.15 we can define, with strict convergence

$$x = \sum_{i \in I} p_i \otimes q_i \in A \otimes_{mh} A.$$

Then we get

$$\begin{aligned} T_1(x) &= \sum_{i \in I} \phi(p_i)(1 \otimes q_i) \\ &= \sum_{i \in I} (\iota \otimes \iota \otimes \Lambda_\varphi)(\phi^{(2)}(a))^* (1 \otimes 1 \otimes \theta_{\xi_i} \theta_{\xi_i}^*) (\iota \otimes \iota \otimes \Lambda_\varphi)(1 \otimes \phi(b)) \\ &= (\iota \otimes \iota \otimes \varphi)(\phi^{(2)}(a^*)(1 \otimes \phi(b))) \\ &= (\iota \otimes \varphi)(\phi(a^*)(1 \otimes b)) \otimes 1 \end{aligned}$$

by left invariance. All these manipulations can be made precise when interpreted in the sense of [14]. So we may conclude that $(\iota \otimes \varphi)(\phi(a^*)(1 \otimes b)) \in A_0$. Then we can make analogous computations as above to get the stated formulas for S and ε .

Assume now the extra conditions stated in the proposition. Let $\mu \in A^*$ such that $\mu(a) = 0$ for all $a \in A_0$. By the Hahn-Banach theorem it is enough to prove $\mu = 0$. By the result above we have $\varphi((\mu \otimes \iota)\phi(a^*)b) = 0$ for all $a, b \in \mathcal{N}_\varphi$. Because φ is faithful we conclude that $(\mu \otimes \iota)\phi(a^*) = 0$ for all $a \in \mathcal{N}_\varphi$ and hence for all $a \in A$ by density and continuity. Then we get $\mu((\iota \otimes \omega)\phi(a)) = 0$ for all $a \in A$ and $\omega \in A^*$. By the assumed density we get $\mu = 0$.

The density of A_0 implies that the range of T_1 on $A \otimes_h A$ is dense in $A \otimes A$. And this implies that $\phi(A)(1 \otimes A)$ is dense in $A \otimes A$. On the other hand, the density of $\phi(A)(1 \otimes A)$ in $A \otimes A$ will imply that $\{(\iota \otimes \omega)\phi(a) | \omega \in A^*, a \in A\}$ will be dense in $A \otimes A$. So we find that under the given circumstances, these two density conditions are in some sense equivalent.

A similar result holds when there is a right invariant weight and this will give the density of A'_0 .

If moreover the left or the right regular representation is unitary, it will follow from 5.8 that also $\mathcal{A} \cap A_0 \cap A'_0$ is dense in A . This is the case when (A, ϕ) is a locally compact quantum

group in the sense of [13] and [14], see proposition 5.13. Roughly speaking, a locally compact quantum group is a pair (A, ϕ) of a C^* -algebra A with a comultiplication ϕ such that the spaces $\phi(A)(1 \otimes A)$ and $\phi(A)(A \otimes 1)$ are dense in $A \otimes A$ and such that there exist left and right invariant faithful l.s.c. weights that are KMS.

However, to obtain the density of $\mathcal{A} \cap A_0 \cap A'_0$ in A , one can do with weaker assumptions. We have the following result.

Proposition 5.12. *Let (A, ϕ) be a Hopf C^* -algebra. Assume there is a densely defined left invariant l.s.c. weight φ and a densely defined right invariant l.s.c. weight ψ . Then, for all $a, b \in \mathcal{N}_\varphi$ and $c, d \in \mathcal{N}_\psi$ we have that*

$$x := (\psi \otimes \iota \otimes \varphi)((c^* \otimes 1 \otimes 1)\phi^{(2)}(a^*d)(1 \otimes 1 \otimes b))$$

belongs to $\mathcal{A} \cap A_0 \cap A'_0$ and $\varepsilon(x) = \psi(c^*d)\varphi(a^*b)$ and

$$\begin{aligned} S(x) &= (\psi \otimes \iota \otimes \varphi)((c^* \otimes 1 \otimes 1)\phi_{13}(a^*d)\phi_{23}(b)) \\ &= (\psi \otimes \iota \otimes \varphi)(\phi_{12}(c^*)\phi_{13}(a^*d)(1 \otimes 1 \otimes b)) \\ &= (\psi \otimes \iota \otimes \varphi)(\phi_{12}(c^*)\phi^{(2)}(a^*d)\phi_{23}(b)). \end{aligned}$$

Proof. Again, as in the proof of the previous proposition, we will use the results of chapter 1 of [14]. Implicitly a lot of weight theory techniques that appeared in that paper will be used. Also remark that a more elementary but very long proof for this result is given in [21]. So we can consider

$$(\Lambda_\psi \otimes \iota \otimes \iota)(\phi^{(2)}(d)) \in \mathcal{L}(A \otimes A, H_\psi \otimes A \otimes A).$$

If now $T \in \mathcal{L}(A \otimes A, H_\psi \otimes A \otimes A)$ it is possible to define

$$(\iota \otimes \iota \otimes \pi_\varphi)(T) \in \mathcal{L}(A \otimes H_\varphi, H_\psi \otimes A \otimes H_\varphi)$$

such that

$$(\iota \otimes \iota \otimes \pi_\varphi)(T)(a \otimes \pi_\varphi(b)\xi) = (\iota \otimes \iota \otimes \pi_\varphi(\cdot)\xi)(T(a \otimes b))$$

where

$$(\iota \otimes \iota \otimes \pi_\varphi(\cdot)\xi)(\eta \otimes x \otimes y) = \eta \otimes x \otimes \pi_\varphi(y)\xi.$$

This should not be too surprising. If $\eta \in H_\psi$ we have on the one hand

$$(\iota \otimes \pi_\varphi)((\theta_\eta^* \otimes 1 \otimes 1)T) \in M(A \otimes B_0(H_\varphi))$$

and on the other hand

$$(\theta_\eta^* \otimes 1 \otimes 1)(\iota \otimes \iota \otimes \pi_\varphi)(T) \in \mathcal{L}(A \otimes H_\varphi)$$

and this agrees with the usual identification of $M(A \otimes B_0(H_\varphi))$ and $\mathcal{L}(A \otimes H_\varphi)$. So we can consider the element $P \in \mathcal{L}(A \otimes H_\varphi, H_\psi \otimes A)$ given by

$$P = [(\pi_\psi \otimes \iota \otimes \iota)(\iota \otimes \iota \otimes \Lambda_\varphi)(\phi^{(2)}(a))]^*[(\iota \otimes \iota \otimes \pi_\varphi)(\Lambda_\psi \otimes \iota \otimes \iota)(\phi^{(2)}(d))].$$

Now let $(e_j)_{j \in J}$ and $(f_i)_{i \in I}$ be orthonormal bases for H_φ and H_ψ respectively. Define for $i \in I$ and $j \in J$:

$$q_{ij} = (\theta_{f_i}^* \otimes 1)P(1 \otimes \theta_{e_j}).$$

Then we have that $q_{ij} \in \mathcal{L}(A) = M(A)$. But for $x \in \mathcal{N}_\varphi$ and $y \in \mathcal{N}_\psi$ we have

$$\begin{aligned} & (\theta_{\Lambda_\psi(y)}^* \otimes 1)P(1 \otimes \Lambda_\varphi(x)) \\ &= [(\pi_\psi \otimes \iota \otimes \iota)(\iota \otimes \iota \otimes \Lambda_\varphi)(\phi^{(2)}(a))(\theta_{\Lambda_\psi(y)}^* \otimes 1)]^* \\ & \quad [(\iota \otimes \iota \otimes \pi_\varphi)(\Lambda_\psi \otimes \iota \otimes \iota)(\phi^{(2)}(d))(1 \otimes \theta_{\Lambda_\varphi(x)})] \\ &= (\Lambda_\psi \otimes \iota \otimes \Lambda_\varphi)(\phi^{(2)}(a)(y \otimes 1 \otimes 1))^*(\Lambda_\psi \otimes \iota \otimes \Lambda_\varphi)(\phi^{(2)}(d)(1 \otimes 1 \otimes x)) \\ &= (\psi \otimes \iota \otimes \varphi)((y^* \otimes 1 \otimes 1)\phi^{(2)}(a^*d)(1 \otimes 1 \otimes x)) \\ &= (\psi \otimes \iota)((y^* \otimes 1)\phi((\iota \otimes \varphi)(\phi(a^*d)(1 \otimes x)))). \end{aligned}$$

By the results of [14] one has $(\iota \otimes \varphi)(\phi(a^*d)(1 \otimes x)) \in \mathcal{N}_\psi$ and so the expression above belongs to A . Then we get $q_{ij} \in A$ for all i and j . From its definition it follows now immediately that $(q_{ij})_{i \in I, j \in J}$ is a bounded matrix over A . Define also

$$p_i^* = (\theta_{f_i}^* \otimes 1)(\Lambda_\psi \otimes \iota)(\phi(c)) \quad r_j = (1 \otimes \theta_{e_j}^*)(\iota \otimes \Lambda_\varphi)(\phi(b)).$$

As in the proof of the previous proposition we get $\sum_{i \in I} p_i p_i^*$ and $\sum_{j \in J} r_j^* r_j$ strictly converging and bounded. But then we have for any $u, v \in A$ that

$$\sum_{i \in I, j \in J} u p_i \otimes q_{ij} \otimes r_j v$$

converges in norm in $A \otimes_h A \otimes_h A$, because for $I_0 \subseteq I$ and $J_0 \subseteq J$ finite we have

$$\left\| \sum_{i \in I_0, j \in J_0} u p_i \otimes q_{ij} \otimes r_j v \right\|_h \leq \left\| \sum_{i \in I_0} u p_i p_i^* u^* \right\|^{1/2} \|(q_{ij})_{i \in I, j \in J}\| \left\| \sum_{j \in J_0} v^* r_j^* r_j v \right\|^{1/2}.$$

In order to apply T to this we have to realize that (use the abbreviation ι^4 for $\iota \otimes \iota \otimes \iota \otimes \iota$)
 $\phi^{(2)}(q_{ij}) = (\theta_{f_i}^* \otimes 1 \otimes 1 \otimes 1)[(\pi_\psi \otimes \iota^4)(\iota^4 \otimes \Lambda_\varphi)(\phi^{(4)}(a))]^*[(\iota^4 \otimes \pi_\varphi)(\Lambda_\psi \otimes \iota^4)(\phi^{(4)}(d))](1 \otimes 1 \otimes 1 \otimes \theta_{e_j}^*).$

Then it is not too hard to see that with strict convergence

$$\begin{aligned} & \sum_{i \in I, j \in J} (p_i \otimes 1 \otimes 1) \phi^{(2)}(q_{ij})(1 \otimes 1 \otimes r_j) \\ &= [(\pi_\psi \otimes \iota^4)((\iota^4 \otimes \Lambda_\varphi)(\phi^{(4)}(a)))(\Lambda_\psi \otimes \iota^3)(\phi_{12}(c))]^* \\ & \quad [(\iota^4 \otimes \pi_\varphi)((\Lambda_\psi \otimes \iota^4)(\phi^{(4)}(d)))(\iota^3 \otimes \Lambda_\varphi)(\phi_{34}(b))] \\ &= (\Lambda_\psi \otimes \iota \otimes \iota \otimes \iota \otimes \Lambda_\varphi)(\phi^{(4)}(a)\phi_{12}(c))^*(\Lambda_\psi \otimes \iota \otimes \iota \otimes \iota \otimes \Lambda_\varphi)(\phi^{(4)}(d)\phi_{45}(b)) \\ &= (\psi \otimes \iota \otimes \iota \otimes \iota \otimes \varphi)(\phi_{12}(c^*)\phi^{(4)}(a^*d)\phi_{45}(b)) \\ &= (\psi \otimes \iota \otimes \iota \otimes \iota \otimes \varphi)(\phi \otimes \iota \otimes \phi)((c^* \otimes 1 \otimes 1)\phi^{(2)}(a^*d)(1 \otimes 1 \otimes b)) \\ &= 1 \otimes (\psi \otimes \iota \otimes \varphi)((c^* \otimes 1 \otimes 1)\phi^{(2)}(a^*d)(1 \otimes 1 \otimes b)) \otimes 1 \\ &= 1 \otimes x \otimes 1. \end{aligned}$$

So we get indeed that $x \in \mathcal{A}$ and for all $u, v \in A$ we have

$$u \otimes x \otimes v = T\left(\sum_{i,j} up_i \otimes q_{ij} \otimes r_j z\right).$$

Because we can write $x = (\iota \otimes \varphi)(\phi((\psi \otimes \iota)((c^* \otimes 1)\phi(a^*d)))(1 \otimes b))$ and because $(\psi \otimes \iota)((c^* \otimes 1)\phi(a^*d)) \in \mathcal{N}_\varphi^*$ we get $x \in A_0$ by the previous proposition. Analogously $x \in A'_0$. The formulas to be proven for $S(x)$ can be obtained from proposition 4.5 with a similar calculation as above.

We finish this section by the following result:

Proposition 5.13. *If (A, ϕ) is a (reduced) locally compact quantum group in the sense of [13] and [14], then (A, ϕ) is a Hopf C^* -algebra and the space $\mathcal{A} \cap A_0 \cap A'_0$ is a core for the antipode obtained in the theory of locally compact quantum groups.*

So, in particular the discrete and compact quantum groups will be Hopf C^* -algebras and also the C^* -Kac algebras will be.

Proof. We use of course the notations and conventions from [13] and [14]. So let (A, ϕ) be a reduced locally compact quantum group. We let A act on the Hilbert space H , which is the GNS-space of the left Haar weight φ . We first proof that (A, ϕ) is a Hopf C^* -algebra. We will work along the same lines as in the proof of proposition 5.3. Let $W \in B(H \otimes H)$ be the multiplicative unitary associated with (A, ϕ) and observe that for any $\omega \in B(H)_*$ and any $y \in A'$ we have

$$y(\omega \otimes \iota)(WT_1(p \otimes q)) = y(\omega \otimes \iota)((1 \otimes p)W(1 \otimes q)) = py(\omega \otimes \iota)(W)q.$$

We can define for any $z \in B(H)$ a bounded map M_z from $A \otimes_h A$ to $B(H)$ such that

$$M_z(p \otimes q) = pzq$$

for all $p, q \in A$. Then it is clear that for any $x \in A \otimes_h A$ the map

$$z \mapsto M_z(x)$$

is strongly continuous on bounded subsets of $B(H)$. Let now $x \in A \otimes_h A$ and suppose $T_1(x) = 0$. Then it follows from the computation above that $M_z(x) = 0$ for all z of the form $y(\omega \otimes \iota)(W)$, with $y \in A'$ and $\omega \in B(H)_*$. Now we claim two things. First, the closed linear span of $\{ya \mid y \in JAJ, a \in \hat{A}\}$ is a C^* -algebra that acts non-degenerately on H . Secondly this C^* -algebra is strongly dense in $B(H)$. Here J denotes the modular conjugation of the weight φ , and so $JAJ \subseteq A'$. After these two claims have been proven we may conclude that $M_z(x) = 0$ for all $z \in B(H)$ and exactly as in the proof of proposition 5.3 this is enough to conclude $x = 0$.

To prove the first claim denote the stated closed linear space by L . It is then enough to prove that for all $\omega, \mu \in B(H)_*$

$$(\mu \otimes \iota)(W)J(\iota \otimes \omega)(W)J \in L. \tag{8}$$

But this expression equals

$$J(\rho \otimes \iota)(W^*)(\iota \otimes \omega)(W)J$$

where $\rho(x) = \overline{\mu(\hat{J}x\hat{J})}$ and $\hat{J}$ is the modular conjugation of the dual weight $\hat{\varphi}$. Here we used the equality $W(\hat{J} \otimes J) = (\hat{J} \otimes J)W^*$. Now observe that

$$\begin{aligned} (\rho \otimes \iota)(W^*)(\iota \otimes \omega)(W) &= (\rho \otimes \iota \otimes \omega)(W_{12}^* W_{23}) \\ &= (\rho \otimes \iota \otimes \omega)(W_{13} W_{23} W_{12}^*) \end{aligned}$$

where we used the pentagon equation $W_{12} W_{13} W_{23} = W_{23} W_{12}$. So we get

$$\begin{aligned} (\rho \otimes \iota)(W^*)(\iota \otimes \omega)(W) &\in \text{Closed span}\{(\rho \otimes \iota \otimes \omega)(W_{23} W_{12}^*) \mid \rho, \omega \in B(H)_*\} \\ &= \text{Closed span}\{xy \mid x \in A, y \in \hat{A}\}. \end{aligned}$$

Hence we can conclude that the expression in 8 belongs to the closed linear span of

$$\{JxyJ \mid x \in A, y \in \hat{A}\} = \{JxJ JyJ \mid x \in A, y \in \hat{A}\} = L$$

because $J\hat{A}J = \hat{A}$. So we have proved the first claim.

To prove the second claim first observe that $A'' \cap \hat{A}' = \mathbb{C}1$. If $x \in A'' \cap \hat{A}'$ then

$$\tilde{\phi}(x) = W^*(1 \otimes x)W = 1 \otimes x$$

because $W \in M(A \otimes \hat{A})$, and with $\tilde{\phi}$ the extension of ϕ to the von Neumann algebra A'' . But this implies $x \in \mathbb{C}1$. By taking the commutant we get $(JAJ \cup \hat{A})'' = B(H)$ and from this follows the second claim. So we have proven the injectivity of T_1 .

The injectivity of T_2 follows easily by using the unitary antipode R . Just observe that for $p, q \in A$

$$\chi(R \otimes R)(T_2(p \otimes q)) = T_1(\chi(R \otimes R)(p \otimes q))$$

where χ denotes the flip map extended to $M(A \otimes A)$. Because R is an anti-automorphism of A we can extend $\chi(R \otimes R)$ to an isomorphism of $A \otimes_h A$ (compare to proposition 1.14). Then the injectivity of T_2 follows from the already proven injectivity of T_1 .

From proposition 8.3 in [14] it follows that $\{(\iota \otimes \omega)(W) \mid \omega \in B(H)_*\}$ is a core for the antipode of the locally compact quantum group (A, ϕ) , which we shall denote by S_1 . Because S_1 is an anti-homomorphism also the linear span of the products of two such elements will give a core. It is not hard to finetune the proofs of propositions 5.33 and 5.45 in [14] to obtain that A_0 is a subspace of the domain of S_1 . All these remarks, together with proposition 5.8 give that $\mathcal{A} \cap A_0 \cap A'_0$ is a core for S_1 .

Finally remark that we have, in case of a locally compact quantum group, that both A_0 and A'_0 are subspaces of the domain of S_1 and S_1 coincides with S and S' on respectively A_0 and A'_0 . In particular S and S' coincide on $A_0 \cap A'_0$, although in the general situation we could only prove this on $\mathcal{A} \cap A_0 \cap A'_0$.

5.4 Further examples

We will now discuss some well known examples of C^* -algebras with comultiplication and explain why they are Hopf C^* -algebras.

Combining the results of [27] and [3] one gets that Woronowicz' quantum $E(2)$ group is a locally compact quantum group in the sense of [13] and [14] and hence is a Hopf C^* -algebra. Also the quantum Heisenberg group will be a Hopf C^* -algebra because it is even a C^* -Kac algebra.

Finally the Kac systems of [2] give rise to Hopf C^* -algebras and this can be proved as follows. Let (H, V, U) be a Kac system in the sense of [2], i.e. H is a Hilbert space, V a multiplicative unitary on $H \otimes H$, U a unitary on H and (H, V, U) satisfies the axioms of *biregularity* and *irreducibility*. With this is associated a pair (S, δ) of a C^* -algebra with comultiplication by [2, Théorème 3.8], satisfying $\delta(x) = V(x \otimes 1)V^*$ for all $x \in S$. One also defines $\hat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$, where Σ denotes the flip map on $H \otimes H$. Then one proves that $\hat{V}$ is again a biregular multiplicative unitary and $\delta(x) = \hat{V}^*(1 \otimes x)\hat{V}$ for all $x \in S$ (see [2, Définition 6.5 and Proposition 6.8.c]). Observe now that for all $p, q \in S$

$$V^*T_1(p \otimes q)\Sigma = (p \otimes 1)V^*\Sigma(q \otimes 1).$$

So if $x \in S \otimes_h S$ and $T_1(x) = 0$ we have, with the notation of the proof of proposition 5.13, $M_z(x) = 0$ for all $z = (\iota \otimes \omega)(V^*\Sigma)$ and $\omega \in B(H)_*$. By definition of regularity the closure of the set of these elements z is the C^* -algebra of compact operators on H . So we may conclude, as in the proof of proposition 5.13 that $x = 0$. To prove the injectivity of T_2 we argue analogously, but using now the formula $\delta(q) = \hat{V}^*(1 \otimes q)\hat{V}$ for all $q \in S$ and the regularity of $\hat{V}$.

These Kac systems provide us with nice examples as the quantum double of a locally compact group and the quantum Lorentz group as studied by Podleś and Woronowicz in [19].

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