Double crossed products of locally compact quantum groups

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Abstract

For a matched pair of locally compact quantum groups, we construct the double crossed product as a locally compact quantum group. This construction generalizes Drinfeld’s quantum double construction. We study the modular theory and the C^* -algebraic properties of these double crossed products, as well as several links between double crossed products and bicrossed products. In an appendix, we study the Radon-Nikodym derivative of a weight under a quantum group action, following Yamanouchi and obtain, as a corollary, a new characterization of closed quantum subgroups.

1 Introduction

In this paper, we study the so-called *double crossed product* construction in the framework of *locally compact quantum groups*, generalizing Drinfeld’s quantum double construction and bringing it into the topological setting.

The history of locally compact quantum groups goes back to the 1960’s, when G. Kac tried to answer the following basic question: is it possible to unify locally compact (non-abelian) groups and their dual objects into one theory? A positive answer was given by Kac, Kac & Vainerman and independently, by Enock & Schwartz. They defined what Enock & Schwartz called *Kac algebras*, see [5] for an overview. In the 1980’s, several new examples of objects certainly to be considered as quantum groups, were constructed and were shown not to be Kac algebras. Henceforth, many efforts were made to enlarge the definition of Kac algebras to include these new examples and in particular, to include Woronowicz’ *compact quantum groups*, see [18, 19]. Many important contributions have been made by Skandalis & the first author [2], Woronowicz [20], Masuda & Nakagami [11] and Van Daele [17]. Finally, the theory of locally compact quantum groups was proposed by Kustermans and the second author in [8, 9].

One of the most celebrated constructions in the early, algebraic theory of quantum groups is *Drinfeld’s quantum double construction* [4]. More generally, Majid [10] defined the double crossed product for mutually matched Hopf algebras, yielding Drinfeld’s quantum double when a Hopf algebra is matched with its dual in the canonical way.

It is a natural idea to perform analogous constructions in the topological framework of locally compact (l.c.) quantum groups. In fact, the double crossed product construction was developed for multiplicative unitaries in [2] (it is called *Z-produit tensoriel*, there). Nevertheless, some extra conditions had to be imposed in order to obtain the double crossed product and it was not clear if they were satisfied for all matchings of two multiplicative unitaries.

In this paper, we show that any matching of l.c. quantum groups allows to perform the double crossed product construction and to obtain again a l.c. quantum group. In particular, the quantum double of a l.c. quantum group is again a l.c. quantum group: a result already shown for Woronowicz algebras by Yamanouchi [23].

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The notion of a matching of l.c. quantum groups was developed by Vainerman and the second author in [15] and the *bicrossed product* was constructed. We explain all these different constructions in a more detailed way for a matching of ordinary l.c. groups. One says that a pair of l.c. groups G_1, G_2 is a matched pair, if there exists a large group G , such that G_1, G_2 are closed subgroups, satisfying $G_1 \cap G_2 = \{e\}$ and such that the complement of $G_1 G_2$ has Haar measure zero in G . In fact, this allows to define an action $(\alpha_g)_{g \in G_1}$ of G_1 by (almost everywhere defined) transformations of the measure space G_2 (not by automorphisms) and an action $(\beta_s)_{s \in G_2}$ of G_2 by transformations of the measure space G_1 such that $\alpha_g(s) \beta_s(g) = gs$ almost everywhere. Conversely, one can say that G_1, G_2 are matched when they act on each other in such a compatible way. Such a compatible pair of actions allows to reconstruct the l.c. group G , this is the *double crossed product*, and to construct a l.c. quantum group with underlying von Neumann algebra $G_1 \rtimes L^\infty(G_2)$, this is the *bicrossed product*.

In the quantum setting, a matching of two l.c. quantum groups was defined in [15] as a compatible pair of actions of the two quantum groups on each other. So, it is a natural question if one can find the double crossed product, i.e. a big l.c. quantum group such that the matched quantum groups are closed quantum subgroups and such that the matching is determined by this inclusion in the double crossed product, as above by the equation $\alpha_g(s) \beta_s(g) = gs$. In our main result, Theorem 5.3 below, we show that there always exists a double crossed product quantum group and we calculate its structural ingredients. In Proposition 7.1 below, we show that the notions of a *matched pair* and a *quantum group with two closed quantum subgroups* are in bijective correspondence.

In [3], Skandalis and the authors studied the C^* -algebraic features of the bicrossed product associated with a matched pair $G_1, G_2 \subset G$ of l.c. groups. The main result was that the bicrossed product is *regular* if and only if $G = G_1 G_2$ homeomorphically, while it is *semi-regular* if and only if $G_1 G_2$ is an open subset of G . Examples of non-semi-regular l.c. quantum groups were given using examples of matched pairs $G_1, G_2 \subset G$ such that $G_1 G_2$ has empty interior in G (but, as needed, $G_1 G_2$ is of full Haar measure in G). The fact that $G = G_1 G_2$ homeomorphically can be expressed in C^* -algebraic terms as $C_0(G) = C_0(G_1) \otimes C_0(G_2)$, while the fact that $G_1 G_2$ is an open subset of G becomes $C_0(G_1) \otimes C_0(G_2) \subset C_0(G)$. In Section 9, we study the same kind of C^* -algebraic questions in the quantum setting and provide, in particular, the same kind of necessary and sufficient condition for the bicrossed product to be (semi-)regular. More specifically, regularity is equivalent with the equation $A_m = A_1 \otimes A_2$ and semi-regularity with $A_1 \otimes A_2 \subset A_m$, where A_1, A_2 and A_m denote the C^* -algebras associated with the two matched quantum groups and the double crossed product, respectively.

The structure of the paper is as follows: we first recall the necessary preliminaries in Section 2, in particular the basics of the theory of l.c. quantum groups and the main results on matchings and bicrossed products from [15]. Sections 3 – 5 are the core of the paper: we define the double crossed product for a matched pair and prove that it is a l.c. quantum group. In order to do so, we have to produce the Haar weights and we will need a continuous interplay between the double crossed product and the bicrossed product, about which we already know a lot from [15]. In the short Section 6, we show how actions of double crossed products on von Neumann algebras and actions of bicrossed products are related by an intermediate von Neumann algebra in the Jones tower. In Section 7, we prove the bijective correspondence between a matched pair and a quantum group with two closed quantum subgroups in a good position with respect to each other. The example of a (generalized) quantum double construction is treated in Section 8 and we deal with the C^* -algebraic features of our constructions in Section 9.

One of our major tools is the *Radon-Nikodym derivative* of a weight under a l.c. quantum group action, as developed originally by Yamanouchi [22]. In an appendix, Section 10, we give a simplified account of Yamanouchi's interesting results. As an application, somehow aside of the main line of our paper, we give a characterization of *closed quantum subgroups* as von Neumann subalgebras invariant under the comultiplication and the antipode.

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2 Preliminaries

We use the following conventions: $\otimes$ will denote, depending on the case, the tensor product of Hilbert spaces, von Neumann algebras or C^* -algebras. There should be no confusion. The domain of an unbounded mapping S is denoted by $D(S)$. We write σ for the flip map on the tensor product $M \otimes M$ of two von Neumann algebras and Σ for the flip map on the tensor product of two Hilbert spaces $H \otimes H$.

2.1 A little bit of weight theory

Let φ be a normal, semi-finite, faithful (n.s.f.) weight on a von Neumann algebra M . Then, we write

$$\mathcal{M}_\varphi^+ = \{x \in M^+ \mid \varphi(x) < \infty\} \quad \text{and} \quad \mathcal{N}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}.$$

We represent M on its GNS-space H and obtain a GNS-map $\Lambda : \mathcal{N}_\varphi \rightarrow H$. Denote by (σ_t) the modular group of φ .

Suppose that k is a strictly positive, self-adjoint operator affiliated with M and satisfying $\sigma_t(k) = \lambda^t k$ for all $t \in \mathbb{R}$ and a number $\lambda > 0$. Then, we can define a new n.s.f. weight φ_k which is formally given by $\varphi_k(x) = \varphi(k^{1/2} x k^{1/2})$. For a precise definition, we refer to [14]. We also get a canonical GNS-map $\Lambda_{\varphi_k} : \mathcal{N}_{\varphi_k} \rightarrow H$ for the weight φ_k , which is formally given by $\Lambda_{\varphi_k}(x) = \Lambda(x k^{1/2})$.

2.2 Locally compact quantum groups

Our references for the theory of l.c. quantum groups are [8, 9].

Definition 2.1. A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group when

- M is a von Neumann algebra and $\Delta : M \rightarrow M \otimes M$ is a normal and unital $*$ -homomorphism satisfying the coassociativity relation : $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- there exist n.s.f. weights φ and ψ on M such that

- φ is left invariant in the sense that $\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1)$ for all $x \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$,
- ψ is right invariant in the sense that $\psi((\iota \otimes \omega)\Delta(x)) = \psi(x)\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$. ▲

Fix such a l.c. quantum group (M, Δ) .

Represent M in the GNS-construction of φ with GNS-map $\Lambda : \mathcal{N}_\varphi \rightarrow H$. We define a unitary W on $H \otimes H$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \quad \text{for all } a, b \in \mathcal{N}_\varphi.$$

Here, $\Lambda \otimes \Lambda$ denotes the canonical GNS-map for the tensor product weight $\varphi \otimes \varphi$. One proves that W satisfies the pentagonal equation: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that W is a *multiplicative unitary*. It is the *left regular representation*. The von Neumann algebra M is the strong closure of the algebra $\{(\iota \otimes \omega)(W) \mid \omega \in B(H)_*\}$ and $\Delta(x) = W^*(1 \otimes x)W$, for all $x \in M$. Next, the l.c. quantum group (M, Δ) has an antipode S , which is the unique σ -strong* closed linear map from M to M satisfying $(\iota \otimes \omega)(W) \in D(S)$ for all $\omega \in B(H)_*$ and $S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*)$ and such that the elements $(\iota \otimes \omega)(W)$ form a σ -strong* core for S . The antipode S has a polar decomposition $S = R\tau_{-i/2}$ where R is an anti-automorphism of M and (τ_t) is a strongly continuous one-parameter group of automorphisms of M . We call R the *unitary antipode* and (τ_t) the *scaling group* of (M, Δ) . From [8], Proposition 5.26 we know that $\sigma(R \otimes R)\Delta = \Delta R$. So φR is a right invariant weight on (M, Δ) and we take $\psi := \varphi R$.

We denote by (σ_t) the modular automorphism group of φ . From [8], Proposition 6.8 we get the existence of a number $\nu > 0$, called the *scaling constant*, such that $\psi \sigma_t = \nu^{-t} \psi$ for all $t \in \mathbb{R}$. Hence, we get the existence of a unique positive, self-adjoint operator δ affiliated to M , such that $\sigma_t(\delta) = \nu^t \delta$ for all $t \in \mathbb{R}$ and $\psi = \varphi_\delta$, see [8], Definition 7.1. The operator δ is called the *modular element* of (M, Δ) and the canonically associated

GNS-map is denoted by $\Gamma : \mathcal{N}_\psi \rightarrow H$. Because of the right invariance of ψ , we find another multiplicative unitary, denoted by V and called the *right regular representation*:

$$V(\Gamma(x) \otimes \Gamma(y)) = (\Gamma \otimes \Gamma)(\Delta(x)(1 \otimes y)) \quad \text{for all } x, y \in \mathcal{N}_\psi.$$

The scaling constant can be characterized as well by the relative invariance $\varphi \tau_t = \nu^{-t} \varphi$.

The *dual l.c. quantum group* $(\hat{M}, \hat{\Delta})$ is defined in [8], Section 8. Its von Neumann algebra $\hat{M}$ is the strong closure of the algebra $\{(\omega \otimes \iota)(W) \mid \omega \in B(H)_*\}$ and the comultiplication is given by $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^* \Sigma$ for all $x \in \hat{M}$. On $\hat{M}$ there exists a canonical left invariant weight $\hat{\varphi}$ with GNS-map $\hat{\Lambda} : \mathcal{N}_{\hat{\varphi}} \rightarrow H$ and the associated multiplicative unitary is denoted by $\hat{W}$. From [8], Proposition 8.16, it follows that $\hat{W} = \Sigma W^* \Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, we can introduce the antipode $\hat{S}$, the unitary antipode $\hat{R}$, the scaling group $(\hat{\tau}_t)$ and the modular element $\hat{\delta}$ exactly as we did it for (M, Δ) . Also, we can again construct the dual of $(\hat{M}, \hat{\Delta})$, starting from the left invariant weight $\hat{\varphi}$ with GNS-construction $(H, \iota, \hat{\Lambda})$. From [8], Theorem 8.29 we have that the bidual l.c. quantum group $(\hat{\hat{M}}, \hat{\hat{\Delta}})$ is isomorphic to (M, Δ) .

We denote by $(\hat{\sigma}_t)$ the modular automorphism groups of the weight $\hat{\varphi}$. The modular conjugations of the weights φ and $\hat{\varphi}$ will be denoted by J and $\hat{J}$ respectively. Then it is worthwhile to mention that

$$R(x) = \hat{J}x^*\hat{J} \quad \text{for all } x \in M \quad \text{and} \quad \hat{R}(y) = Jy^*J \quad \text{for all } y \in \hat{M}.$$

Some important relations involving J and $\hat{J}$ are

$$\hat{J}J = \nu^{i/4}J\hat{J}, \quad W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J), \quad V = (\hat{J} \otimes \hat{J})\Sigma W^* \Sigma(\hat{J} \otimes \hat{J}).$$

The modular operators of φ and $\hat{\varphi}$ are denoted by ∇ and $\hat{\nabla}$, respectively. Finally, we introduce the implementation P of (τ_t) as the unique strictly positive, self-adjoint operator on H such that $P^{it}\Lambda(x) = \nu^{t/2}\Lambda(\tau_t(x))$. We mention that $P = \hat{P}$, i.e., P is as well the implementation of $(\hat{\tau}_t)$. For all kinds of relations between the operators on H introduced so far, we refer to Proposition 2.4 in [16].

We write Δ^{op} for the opposite comultiplication $\Delta^{\text{op}} := \sigma\Delta$ and observe that (M, Δ^{op}) is again a l.c. quantum group.

Every l.c. quantum group has an *associated C^* -algebra* of ‘continuous functions tending to zero at infinity’ and we denote them by A and $\hat{A}$:

$$A := [(\iota \otimes \omega)(W) \mid \omega \in B(H)_*], \quad \hat{A} = [(\omega \otimes \iota)(W) \mid \omega \in B(H)_*],$$

where we use the following notation.

Notation 2.2. When X is a subset of a Banach space, we denote by $[X]$ the closed linear span of X . We sometimes use a notation like $[(\iota \otimes \omega)(W)]$ instead of $[(\iota \otimes \omega)(W) \mid \omega \in B(H)_*]$, because ω will always run through $B(H)_*$ for the appropriate Hilbert space H .

For any multiplicative unitary W on a Hilbert space H , we introduce (following [1, 2]) the algebra $\mathcal{C}(W)$ by the formula

$$\mathcal{C}(W) := \{(\iota \otimes \omega)(\Sigma W) \mid \omega \in B(H)_*\}.$$

We say that (M, Δ) is *regular* (see [2]) if $[\mathcal{C}(W)] = \mathcal{K}(H)$, the compact operators on H . We say that (M, Δ) is *semi-regular* (see [1]) if $\mathcal{K}(H) \subset [\mathcal{C}(W)]$. From Proposition 2.6 in [3], we get that

$$[\mathcal{C}(W)] = [JAJ \hat{J}\hat{A}\hat{J}].$$

Not all l.c. quantum groups are semi-regular, see [3].

2.3 Actions of l.c. quantum groups

We use [13] as a reference for actions of l.c. quantum groups, but we repeat the necessary elements of the theory. Let (M, Δ) be a l.c. quantum group and N a von Neumann algebra. A faithful, normal $*$ -homomorphism $\alpha : N \rightarrow M \otimes N$ is called a *(left) action* of (M, Δ) on N if $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$. One can define the crossed product as

$$M \rtimes_{\alpha} N = (\alpha(N) \cup \hat{M} \otimes 1)'' \in \mathcal{B}(H) \otimes N.$$

Given an n.s.f. weight θ on N , one can define the *dual weight* $\tilde{\theta}$ on the crossed product $M \rtimes_{\alpha} N$. Representing N on the GNS-space H_{θ} of θ , we get a GNS-map $\Lambda_{\theta} : \mathcal{N}_{\theta} \rightarrow H_{\theta}$. From [13], Definition 3.4 and Proposition 3.10, we get a canonical GNS-construction for $\tilde{\theta}$ on the Hilbert space $H \otimes H_{\theta}$, with GNS-map determined by

$$(x \otimes 1)\alpha(y) \mapsto \hat{\Lambda}(x) \otimes \Lambda_{\theta}(y) \quad \text{for } x \in \mathcal{N}_{\hat{\varphi}}, y \in \mathcal{N}_{\theta}$$

and where these elements $(x \otimes 1)\alpha(y)$ span a core for the GNS-map, when we equip $M \rtimes_{\alpha} N$ with its σ -strong* topology and $H \otimes H_{\theta}$ with the norm topology.

If we denote by J_{θ} the modular conjugation of θ and by $\tilde{J}_{\theta}$ the modular conjugation of the dual weight $\tilde{\theta}$ in the canonical GNS-construction for $\tilde{\theta}$, we can define the unitary $U_{\alpha} = \tilde{J}_{\theta}(\hat{J} \otimes J_{\theta})$, which is called the *unitary implementation* of α , see [13]. It satisfies

$$U_{\alpha} \in M \otimes \mathcal{B}(H_{\theta}), \quad (\Delta \otimes \iota)(U_{\alpha}) = U_{\alpha,23}U_{\alpha,13} \quad \text{and} \quad \alpha(x) = U_{\alpha}(1 \otimes x)U_{\alpha}^* \quad \text{for } x \in N.$$

Because U_{α}^* is a corepresentation of (M, Δ) , we get that $U_{\alpha} \in M(A \otimes \mathcal{K}(H_{\theta}))$, where $A \subset M$ is the C^* -algebra associated with (M, Δ) .

If $\alpha : N \rightarrow M \otimes N$ is an action, the *fixed point algebra* is denoted by N^{α} and defined as the von Neumann subalgebra of elements $x \in N$ satisfying $\alpha(x) = 1 \otimes x$.

Up to now, we dealt with left actions, but this is only a matter of convention. We can (and will) also consider right actions $\alpha : N \rightarrow N \otimes M$ of (M, Δ) on a von Neumann algebra N , which means that $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$.

2.4 Matchings and bicrossed products of l.c. quantum groups

We give a quick overview of some of the results of [15]. Fix two l.c. quantum groups (M_1, Δ_1) and (M_2, Δ_2) .

Definition 2.3. A normal, faithful $*$ -homomorphism $m : M_1 \otimes M_2 \rightarrow M_1 \otimes M_2$ is called a *matching* of (M_1, Δ_1) and (M_2, Δ_2) if

$$(\Delta_1 \otimes \iota)m = m_{23}m_{13}(\Delta_1 \otimes \iota) \quad \text{and} \quad (\iota \otimes \Delta_2)m = m_{13}m_{12}(\iota \otimes \Delta_2).$$

▲

We fix a matching m of (M_1, Δ_1) and (M_2, Δ_2) . As in [15], we define

$$\alpha : M_2 \rightarrow M_1 \otimes M_2 : \alpha(x) = m(1 \otimes x) \quad \text{and} \quad \beta : M_1 \rightarrow M_1 \otimes M_2 : \beta(y) = m(y \otimes 1).$$

Then, α is a left action of (M_1, Δ_1) on the von Neumann algebra M_2 , while β is a right action of $(M_2, \Delta_2^{\text{op}})$ on M_1 . So, we can define two crossed product von Neumann algebras M and $\hat{M}$ on $H_1 \otimes H_2$:

$$M = M_1 \rtimes_{\alpha} M_2 = (\alpha(M_2) \cup \hat{M}_1 \otimes 1)'' \quad \text{and} \quad \hat{M} = M_1 \rtimes_{\beta} M_2 = (\beta(M_1) \cup 1 \otimes \hat{M}_2)''.$$

In [15], it is proven that we can define a l.c. quantum group structure (M, Δ) on the von Neumann algebra M such that $\hat{M}$ is the dual quantum group. The comultiplications on M and its dual $\hat{M}$ are given by

$$\begin{aligned} \Delta\alpha &= (\alpha \otimes \alpha)\Delta_2, \quad (\iota \otimes \Delta)(W_1 \otimes 1) = W_{1,14}((\iota \otimes \alpha)\beta \otimes \iota)(W_1)_{1452}, \\ \hat{\Delta}\beta &= (\beta \otimes \beta)\Delta_1, \quad (\hat{\Delta} \otimes \iota)(1 \otimes \hat{W}_2) = (\iota \otimes (\beta \otimes \iota)\alpha)(\hat{W}_2)_{2345} \hat{W}_{2,25}. \end{aligned}$$

The l.c. quantum group (M, Δ) is called the *bicrossed product* associated with the matching m of (M_1, Δ_1) and (M_2, Δ_2) .

Define φ to be the dual weight on M of the weight φ_2 on M_2 , with canonical GNS-map determined by

$$\Lambda((a \otimes 1)\alpha(x)) = \hat{\Lambda}_1(a) \otimes \Lambda_2(x) \quad \text{for } a \in \mathcal{N}_{\hat{\varphi}_1}, x \in \mathcal{N}_{\varphi_2}.$$

Then, φ is the left invariant weight on (M, Δ) and the left regular representation is given by

$$W = (\alpha \otimes \iota)(W_2)_{124} (\iota \otimes \beta)(\hat{W}_1)_{134}.$$

Observe that, on $H_1 \otimes H_2$, we have the modular conjugations J and $\hat{J}$ of the invariant weights φ and $\hat{\varphi}$. So, we also have

$$\begin{aligned} U_\alpha &= J(\hat{J}_1 \otimes J_2) \in M_1 \otimes B(H_2), \quad \alpha(x) = U_\alpha(1 \otimes x)U_\alpha^*, \\ U_\beta &= \hat{J}(J_1 \otimes \hat{J}_2) \in B(H_1) \otimes M_2, \quad \beta(y) = U_\beta(y \otimes 1)U_\beta^*. \end{aligned}$$

Notation 2.4. Throughout Sections 3 – 6, we will fix a matching m of two l.c. quantum groups (M_1, Δ_1) and (M_2, Δ_2) . All notations with indices 1, 2 refer to ingredients of these l.c. quantum groups (M_1, Δ_1) , resp. (M_2, Δ_2) . All notations *without indices* refer to the *bicrossed product* quantum group (M, Δ) . Finally, we will discuss an associated *double crossed product* and then, the *index* m will be used.

3 Double crossed products: definition and multiplicative unitary

The matching m of (M_1, Δ_1) and (M_2, Δ_2) leads naturally to a coassociative comultiplication Δ_m in the following definition.

Definition 3.1. Define the von Neumann algebra $M_m = M_1 \otimes M_2$ and

$$\Delta_m : M_m \rightarrow M_m \otimes M_m : \Delta_m = (\iota \otimes \sigma m \otimes \iota)(\Delta_1^{\text{op}} \otimes \Delta_2).$$

Then, Δ_m is a co-associative comultiplication. ▲

One of our main objectives will be to prove that (M_m, Δ_m) is always a l.c. quantum group. We will explicitly calculate all the ingredients of (M_m, Δ_m) in terms of (M_1, Δ_1) , (M_2, Δ_2) and the bicrossed product (M, Δ) .

Notation 3.2. We write $Z = J\hat{J}(\hat{J}_1 J_1 \otimes \hat{J}_2 J_2)$.

Lemma 3.3. We have $m(z) = ZzZ^*$ for all $z \in M_1 \otimes M_2$.

Proof. For $x \in M_1$, we have

$$Z(x \otimes 1)Z^* = \hat{J}U_\alpha(J_1 x J_1 \otimes 1)U_\alpha^* \hat{J} = \hat{J}(J_1 x J_1 \otimes 1)\hat{J} = U_\beta(x \otimes 1)U_\beta^* = \beta(x) = m(x \otimes 1).$$

We analogously have $m(1 \otimes x) = Z(1 \otimes x)Z^*$ for $x \in M_2$. □

It is easy to define a unitary implementing the comultiplication Δ_m , but it is less easy to prove that this unitary is multiplicative.

Definition 3.4. Define on $H_1 \otimes H_2 \otimes H_1 \otimes H_2$, the unitary

$$W_m = (\Sigma V_1^* \Sigma)_{13} Z_{34}^* W_{2,24} Z_{34}.$$
▲

Proposition 3.5. The unitary W_m is multiplicative and $\Delta_m(z) = W_m^*(1 \otimes 1 \otimes z)W_m$ for all $z \in M_m$.

Moreover, we have $(m \otimes \iota \otimes \iota)(W_m) = Z_{34}^* W_{2,24} Z_{34} (\Sigma V_1^* \Sigma)_{13}.$

Proof. Let $z \in M_m$. Then,

$$\begin{aligned} W_m^*(1 \otimes 1 \otimes z)W_m &= Z_{34}^* W_{2,24}^* Z_{34} (\Delta_1^{\text{op}} \otimes \iota)(z)_{134} Z_{34}^* W_{2,24} Z_{34} \\ &= Z_{34}^* ((\iota \otimes \iota \otimes \Delta_2)(\iota \otimes m)(\Delta_1^{\text{op}} \otimes \iota)(z))_{1324} Z_{34} = \Delta_m(z). \end{aligned}$$

We now prove that W_m is a multiplicative unitary.

Because the bicrossed product is a l.c. quantum group, its multiplicative unitary W satisfies $(\Sigma(1 \otimes J\hat{J})W)^3 \in \mathbb{C}$ (we can apply Proposition 6.9 of [2]). In essentially the same way as in the first paragraph of the proof of Proposition 8.14 of [2], it follows that $(\Sigma(1 \otimes \mathcal{U})W_m)^3 \in \mathbb{C}$, where $\mathcal{U} = (J_1 \hat{J}_1 \otimes J_2 \hat{J}_2)Z$. Define $N := (\hat{M}'_1 \otimes 1 \cup Z^*(1 \otimes \hat{M}_2)Z)''$. We claim that $\mathcal{U}N\mathcal{U}^* \subset N'$. Clearly, $\mathcal{U}Z^*(1 \otimes \hat{M}_2)Z\mathcal{U}^* = 1 \otimes \hat{M}'_2$ and this commutes with $\hat{M}'_1 \otimes 1$. To prove that the commutator $[1 \otimes \hat{M}_2, Z^*(1 \otimes \hat{M}_2)Z] = \{0\}$, it suffices to check that $[1 \otimes \hat{M}_2, J\hat{J}(1 \otimes \hat{M}_2)\hat{J}J] = \{0\}$. We know this, because $1 \otimes \hat{M}_2 \subset \hat{M}$, the dual bicrossed product. Observing that, up to a scalar, $\mathcal{U} = Z^*(J_1 \hat{J}_1 \otimes J_2 \hat{J}_2)$, we check in an analogous way that $\mathcal{U}(\hat{M}'_1 \otimes 1)\mathcal{U}^* \subset N'$ and this proves our claim. Because, $(\Sigma(1 \otimes \mathcal{U})W_m)^3 \in \mathbb{C}$, we get

$$\Sigma \hat{V}_m^* \hat{W}_m^* (1 \otimes \mathcal{U})W_m(\mathcal{U} \otimes \mathcal{U}) \in \mathbb{C},$$

where $\hat{W}_m = \Sigma W_m^* \Sigma$ and $\hat{V}_m = (\mathcal{U}^* \otimes 1)W_m^*(\mathcal{U} \otimes 1)$. From our claim, it follows that $(1 \otimes \mathcal{U})W_m(\mathcal{U} \otimes \mathcal{U}) \in B(H_1 \otimes H_2) \otimes N'$ and hence, we get

$$\hat{W}_m^*(1 \otimes 1 \otimes z)\hat{W}_m = \hat{V}_m(z \otimes 1 \otimes 1)\hat{V}_m^* \quad \text{for all } z \in N.$$

So, for $x \in \hat{M}_2$,

$$\begin{aligned} \hat{W}_m^*(1 \otimes 1 \otimes Z^*(1 \otimes x)Z)\hat{W}_m &= \hat{V}_m(Z^*(1 \otimes x)Z \otimes 1 \otimes 1)\hat{V}_m^* \\ &= (\mathcal{U}^* \otimes Z^*)W_{2,24}^*(1 \otimes \hat{J}_2 J_2 x J_2 \hat{J}_2 \otimes 1 \otimes 1)W_{2,24}(\mathcal{U} \otimes Z) \\ &= (Z^* \otimes Z^*)(\hat{\Delta}_2(x))_{24}(Z \otimes Z). \end{aligned}$$

We conclude that $W_m(Z^*(1 \otimes x)Z \otimes 1 \otimes 1)W_m^* = (Z^* \otimes Z^*)(\hat{\Delta}_2^{\text{op}}(x))_{24}(Z \otimes Z)$. It is easy to check that $W_m(y \otimes 1 \otimes 1 \otimes 1)W_m^* = \hat{\Delta}'_1(y)_{13}$ for all $y \in \hat{M}'_1$, where $\hat{\Delta}'_1$ is the natural comultiplication on the commutant $\hat{M}'_1$ given by $\hat{\Delta}'_1(y) = V_1^*(1 \otimes y)V_1$ for $y \in \hat{M}'_1$. Then, we check that

$$W_{m3456} W_{m1234} W_{m3456}^* = (\Sigma V_1^* \Sigma)_{13} (\Sigma V_1^* \Sigma)_{15} Z_{34}^* Z_{56}^* W_{2,24} W_{2,26} Z_{34} Z_{56} = W_{m1234} W_{m1256}.$$

Hence, W_m is a multiplicative unitary. Then also

$$(\Delta_m \otimes \iota \otimes \iota)(W_m) = W_{m1256} W_{m3456}$$

and a small calculation yields the formula $(m \otimes \iota \otimes \iota)(W_m) = Z_{34}^* W_{2,24} Z_{34} (\Sigma V_1^* \Sigma)_{13}$. $\square$

In order to prove later that (M_m, Δ_m) is a l.c. quantum group, we need the following remarkable lemma.

Lemma 3.6. *The following holds.*

- a) m is a $*$ -isomorphism,
- b) $m(\tau_{-t}^1 \otimes \tau_t^2) = (\tau_{-t}^1 \otimes \tau_t^2)m$ for all $t \in \mathbb{R}$,
- c) $m(R_1 \otimes R_2) = (R_1 \otimes R_2)m^{-1}$.

If we write $\eta_t = \tau_{-t}^1 \otimes \tau_t^2$, then the elements $(\iota \otimes \iota \otimes \omega)(W_m)$, $\omega \in B(H_1 \otimes H_2)_*$ form a core for $\eta_{-i/2}$ and

$$(R_1 \otimes R_2)m\eta_{-i/2}(\iota \otimes \iota \otimes \omega)(W_m) = (\iota \otimes \iota \otimes \omega)(W_m^*).$$

Proof. If $\omega = \omega_{\xi, \eta}$, $\xi, \eta \in H_1 \otimes H_2$, we take a basis $(e_i)_{i \in I}$ for $H_1 \otimes H_2$ and get

$$(\iota \otimes \iota \otimes \omega)(W_m) = \sum_{i \in I} (\iota \otimes \omega_{e_i, \eta})((\Sigma V_1^* \Sigma)_{12}) \otimes (\iota \otimes \omega_{\xi, e_i})(Z_{23}^* W_{2,13} Z_{23}).$$

Using the closedness of $\eta_{-i/2}$, it is easy to conclude that $(\iota \otimes \iota \otimes \omega)(W_m) \in D(\eta_{-i/2})$ and

$$(R_1 \otimes R_2)\eta_{-i/2}(\iota \otimes \iota \otimes \omega)(W_m) = (\iota \otimes \iota \otimes \omega)((\Sigma V_1 \Sigma)_{13} Z_{34}^* W_{2,24}^* Z_{34}),$$

for all $\omega \in B(H_1 \otimes H_2)_*$. We claim that the elements $(\iota \otimes \iota \otimes \omega)(W_m)$ provide a core for $\eta_{-i/2}$. Denote by D_0 the domain of the closure of the restriction of $\eta_{-i/2}$ to these elements. We have to prove that $D_0 = D(\eta_{-i/2})$. Taking ω of the form $(x \otimes 1)Z^*\omega(1 \otimes a)$ for $x \in B(H_1)$ and $a \in M_2$, we observe that

$$z := (\iota \otimes \iota \otimes \omega)((\Sigma V_1^* \Sigma)_{13} Z_{34}^* (\alpha(a)(x \otimes 1))_{34} W_{2,24}) \in D_0$$

and

$$(R_1 \otimes R_2)\eta_{-i/2}(z) = (\iota \otimes \iota \otimes \omega)((\Sigma V_1 \Sigma)_{13} Z_{34}^* (\alpha(a)(x \otimes 1))_{34} W_{2,24}^*).$$

Any element of $B(H_1) \otimes M_2$ can be approximated in the strong* topology by a bounded net of elements in $\alpha(M_2)(B(H_1) \otimes 1)$ and in particular the element $1 \otimes b$ for $b \in M_2$. So, we find that for all $b \in M_2$ and $\omega \in B(H_1 \otimes H_2)_*$,

$$z := (\iota \otimes \iota \otimes \omega)((\Sigma V_1^* \Sigma)_{13} Z_{34}^* ((1 \otimes b)W_2)_{24}) \in D_0$$

and

$$(R_1 \otimes R_2)\eta_{-i/2}(z) = (\iota \otimes \iota \otimes \omega)((\Sigma V_1 \Sigma)_{13} Z_{34}^* ((1 \otimes b)W_2^*)_{24}).$$

Taking in the previous formula ω of the form $(1 \otimes x)\omega$, with $x \in \hat{M}_2'$ and using the fact that every element of $B(H_2)$ can be approximated by an element in $M_2 \hat{M}_2'$, we arrive at

$$(\iota \otimes \omega_1)(\Sigma V_1^* \Sigma) \otimes (\iota \otimes \omega_2)(W_2) \in D_0.$$

So, we have proven our claim. In particular, we get that the elements $(\iota \otimes \iota \otimes \omega)(Z_{34}^* W_{2,24} Z_{34} (\Sigma V_1^* \Sigma)_{13})$ are dense in $M_1 \otimes M_2$, because the adjoints of these elements are dense in the range of $(R_1 \otimes R_2)\eta_{-i/2}$, which in its turn is dense in $M_1 \otimes M_2$. Because of Proposition 3.5, they are in the image of m and hence, m is a *-isomorphism.

Combining with Proposition 3.5, we get that

$$m^{-1}(R_1 \otimes R_2)\eta_{-i/2}(\iota \otimes \iota \otimes \omega)(W_m) = (\iota \otimes \iota \otimes \omega)(W_m^*)$$

and the elements $(\iota \otimes \iota \otimes \omega)(W_m)$ form a core. Completely analogously, one shows that

$$m^{-1}(R_1 \otimes R_2)\eta_{i/2}(\iota \otimes \iota \otimes \omega)(W_m^*) = (\iota \otimes \iota \otimes \omega)(W_m)$$

and now, the elements $(\iota \otimes \iota \otimes \omega)(W_m^*)$ form a core for $\eta_{i/2}$. It follows that, as unbounded mappings,

$$m^{-1}(R_1 \otimes R_2)\eta_{-i/2} = (m^{-1}(R_1 \otimes R_2)\eta_{i/2})^{-1} = \eta_{-i/2}(R_1 \otimes R_2)m. \quad (3.1)$$

So, still as unbounded mappings, we have

$$\eta_{-i} = (\eta_{-i/2}(R_1 \otimes R_2)m)(m^{-1}(R_1 \otimes R_2)\eta_{-i/2}) = (m^{-1}(R_1 \otimes R_2)\eta_{-i/2})(\eta_{-i/2}(R_1 \otimes R_2)m)$$

and we conclude that $(R_1 \otimes R_2)m\eta_{-i} = \eta_{-i}(R_1 \otimes R_2)m$. The right hand side, is equal to $(R_1 \otimes R_2)\eta_{-i}m$ and we get $m\eta_{-i} = \eta_{-i}m$. From Sections 4.3 and 4.4 in [6], it follows that $m(\tau_{-t}^1 \otimes \tau_t^2) = (\tau_{-t}^1 \otimes \tau_t^2)m$ for all $t \in \mathbb{R}$. But then, also $\eta_{-i/2}m = m\eta_{-i/2}$ and we conclude from Equation (3.1) that $m^{-1}(R_1 \otimes R_2) = (R_1 \otimes R_2)m$. Hence, we are done. $\square$

4 More about bicrossed products

We emphasize again that all notations without indices refer to the bicrossed product quantum group (M, Δ) associated to our fixed matched pair (M_1, Δ_1) and (M_2, Δ_2) , cf. Notation 2.4.

We give an explicit description of all corepresentations of the bicrossed product (M, Δ) and deduce from this result some useful information on the modular elements δ and $\hat{\delta}$ of (M, Δ) and $(\hat{M}, \hat{\Delta})$.

Proposition 4.1. *Suppose that X is a corepresentation of (M, Δ) on the Hilbert space K , i.e. $X \in M \otimes B(K)$ and $(\Delta \otimes \iota)(X) = X_{125}X_{345}$. Then, there exists unique corepresentations y of (M_2, Δ_2) and z of $(\hat{M}_1, \hat{\Delta}_1)$ on K such that*

$$X = (\alpha \otimes \iota)(y)z_{13}.$$

Conversely, if y and z are corepresentations of (M_2, Δ_2) and $(\hat{M}_1, \hat{\Delta}_1)$ respectively, on the same Hilbert space K , the formula $X = (\alpha \otimes \iota)(y)z_{13}$ defines a corepresentation X of (M, Δ) if and only if

$$z_{13}^* y_{23} Z_{12}^* z_{13} Z_{12} \in M_1' \otimes B(H_2 \otimes K).$$

Proof. Suppose that $X \in M \otimes B(K)$ is unitary and $(\Delta \otimes \iota)(X) = X_{125}X_{345}$. Because we have a morphism from (M, Δ) to $(\hat{M}_1, \hat{\Delta}_1)$, we can define two actions of $(\hat{M}_1, \hat{\Delta}_1)$ on M . Explicitly, we have that $\theta : M \rightarrow M \otimes \hat{M}_1$ is a right action of $(\hat{M}_1, \hat{\Delta}_1)$ while $\mu : M \rightarrow \hat{M}_1 \otimes M$ is a left action and θ, μ are determined by

$$(\theta \otimes \iota \otimes \iota)(W) = W_{1245}(\iota \otimes \beta)(\hat{W}_1)_{345} \quad \text{and} \quad (\mu \otimes \iota \otimes \iota)(W) = (\iota \otimes \beta)(\hat{W}_1)_{145}W_{2345}. \quad (4.1)$$

For a more detailed treatment, see Proposition 3.1 in [15] (but observe that the action θ there corresponds to our $\sigma\theta$ and is, in that way, a left action of $(\hat{M}_1, \hat{\Delta}_1^{\text{op}})$).

Because θ is in fact the dual action on the crossed product $M = M_1 \rtimes M_2$, we know that $M^\theta = \alpha(M_2)$. Further, we observe that $\mu = \sigma(R \otimes \hat{R}_1)\theta R$ and so, $M^\mu = \alpha(M_2)$ as well. From Equation (4.1), we conclude that $(\Delta \otimes \iota)\theta = (\iota \otimes \iota \otimes \theta)\Delta$. So, we calculate

$$(\Delta \otimes \iota \otimes \iota)(X_{124}^*(\theta \otimes \iota)(X)) = X_{346}^*X_{126}^*(\iota \otimes \iota \otimes \theta \otimes \iota)(X_{125}X_{345}) = 1 \otimes 1 \otimes (X_{124}^*(\theta \otimes \iota)(X))$$

and conclude that there exists a unique $z \in \hat{M}_1 \otimes B(K)$ such that $(\theta \otimes \iota)(X) = X_{124}z_{34}$. Because applying $\theta \otimes \iota \otimes \iota$ or $\iota \otimes \iota \otimes \hat{\Delta}_1 \otimes \iota$ to the left hand side of this equation gives the same result, we get $(\hat{\Delta}_1 \otimes \iota)(z) = z_{13}z_{23}$.

Define $\gamma_1 : \hat{M}_1 \rightarrow M : \gamma_1(a) = a \otimes 1$. It is easy to check that $\theta\gamma_1 = (\gamma_1 \otimes \iota)\hat{\Delta}_1$. If we apply $\iota \otimes \iota \otimes \theta\gamma_1 \otimes \iota$ to both sides of the equation $(\theta \otimes \iota)(X) = X_{124}z_{34}$, we conclude that $(\theta \otimes \iota)(z_{13}) = z_{14}z_{34}$. But then, it follows that $(\theta \otimes \iota)(Xz_{13}^*) = (Xz_{13}^*)_{124}$ and, because $M^\theta = \alpha(M_2)$, we can take $y \in M_2 \otimes B(K)$ such that $Xz_{13}^* = (\alpha \otimes \iota)(y)$. We conclude that $X = (\alpha \otimes \iota)(y)z_{13}$. We want to prove that $(\Delta_2 \otimes \iota)(y) = y_{13}y_{23}$.

Because X is a corepresentation and because of the formula for $(\theta \otimes \iota)(X)$, we get $((\theta \otimes \iota \otimes \iota)\Delta \otimes \iota)(X) = X_{126}z_{36}X_{456}$. Using Equation (4.1), we get that $(\theta \otimes \iota \otimes \iota)\Delta = (\iota \otimes \iota \otimes \mu)\Delta$ and so, we conclude that $(\mu \otimes \iota)(X) = z_{14}X_{234}$. Because $X = (\alpha \otimes \iota)(y)z_{13}$ and $\alpha(M_2) = M^\mu$, it follows that

$$(\mu \otimes \iota)(z_{13}) = (\alpha \otimes \iota)(y^*)_{234}z_{14}(\alpha \otimes \iota)(y)_{234}z_{24}. \quad (4.2)$$

Next, we get from Equation (4.1) that $\mu(x) = (\iota \otimes \beta)(\hat{W}_1^*)(1 \otimes x)(\iota \otimes \beta)(\hat{W}_1)$ for all $x \in M$. In particular, it follows that $\Delta\gamma_1 = (\gamma_1 \otimes \iota \otimes \iota)\mu\gamma_1$. Applying this to Equation (4.2), it follows that

$$(\Delta \otimes \iota)(z_{13}) = (\alpha \otimes \iota)(y^*)_{345}z_{15}(\alpha \otimes \iota)(y)_{345}z_{35}.$$

Because X is a corepresentation and $X = (\alpha \otimes \iota)(y)z_{13}$, we get

$$\begin{aligned} (\alpha \otimes \iota)(y)_{125}z_{15} (\alpha \otimes \iota)(y)_{345}z_{35} &= X_{125}X_{345} = (\Delta \otimes \iota)(X) \\ &= ((\alpha \otimes \alpha)\Delta_2 \otimes \iota)(y) (\alpha \otimes \iota)(y^*)_{345}z_{15}(\alpha \otimes \iota)(y)_{345}z_{35}. \end{aligned}$$

It follows that $((\alpha \otimes \alpha)\Delta_2 \otimes \iota)(y) = (\alpha \otimes \iota)(y)_{125}(\alpha \otimes \iota)(y)_{345}$, which finally yields $(\Delta_2 \otimes \iota)(y) = y_{13}y_{23}$.

The uniqueness statement is obvious from the given construction. To prove the converse statement, the formula $z_{13}^* y_{23} Z_{12}^* z_{13} Z_{12} \in M_1' \otimes B(H_2 \otimes K)$ is equivalent with

$$z_{24}^* y_{34} Z_{23}^* z_{24} Z_{23} \quad \text{commutes with} \quad \hat{W}_{1,12},$$

which is equivalent with

$$z_{24} \hat{W}_{1,12}^* z_{24}^* y_{34} Z_{23}^* z_{24} Z_{23} \hat{W}_{1,12} = y_{34} Z_{23}^* z_{24} Z_{23}.$$

Using that z is a corepresentation of $(\hat{M}_1, \hat{\Delta}_1)$, the last formula is equivalent with

$$\hat{W}_{1,12}^* Z_{23}^* z_{24} Z_{23} \hat{W}_{1,12} = y_{34}^* z_{14} y_{34} Z_{23}^* z_{24} Z_{23}.$$

Applying $\text{Ad } Z_{23}$ and using the fact that $\Delta(a \otimes 1) = (\iota \otimes \beta)(\hat{W}_1^*)_{134}(1 \otimes 1 \otimes a \otimes 1)(\iota \otimes \beta)(\hat{W}_1)_{134}$ for $a \in \hat{M}_1$, the last formula is equivalent with

$$(\Delta \otimes \iota)(z_{13}) = (\alpha \otimes \iota)(y^*)_{345} z_{15} (\alpha \otimes \iota)(y)_{345} z_{35}.$$

Because y is a corepresentation of (M_2, Δ_2) and $\Delta\alpha = (\alpha \otimes \alpha)\Delta_2$, it is clear that the final formula is equivalent with $X = (\alpha \otimes \iota)(y)z_{13}$ being a corepresentation of (M, Δ) . $\square$

Proposition 4.2. *There exist strictly positive, self-adjoint operators ρ_1, ρ_2 affiliated with M_1, M_2 such that*

- $\delta^{it} = \alpha(\rho_2^{it})(\hat{\delta}_1^{it} \otimes 1)$ and both factors commute;
- $\hat{\delta}^{it} = \beta(\rho_1^{it})(1 \otimes \hat{\delta}_2^{it})$ and both factors commute;
- $\Delta_1(\rho_1) = \rho_1 \otimes \rho_1$ and $\Delta_2(\rho_2) = \rho_2 \otimes \rho_2$;
- $\nu = \frac{\nu_2}{\nu_1}$;
- we have

$$\begin{aligned} J\delta^{it}J &= \hat{J}_1\hat{\delta}_1^{it}\hat{J}_1 \otimes \rho_2^{-it}\delta_2^{it}J_2\delta_2^{it}J_2, \\ \hat{J}\hat{\delta}^{it}\hat{J} &= \rho_1^{-it}\delta_1^{it}J_1\delta_1^{it}J_1 \otimes \hat{J}_2\hat{\delta}_2^{it}\hat{J}_2, \\ [D\varphi_2 \circ \alpha : D\varphi_2]_t &= m(\rho_1^{-it/2} \otimes \rho_2^{-it/2})(\rho_1^{-it/2}\delta_1^{it} \otimes \rho_2^{it/2}), \end{aligned}$$

where we used the Radon-Nikodym derivative introduced in Definition 10.2 of the appendix;

- m leaves invariant $\rho_1^{-it/2}\delta_1^{it} \otimes \rho_2^{it/2}\delta_2^{-it}$.

There exist positive numbers $\lambda_{1,2} > 0$ such that $\sigma_t^1(\rho_1^{is}) = \lambda_1^{ist}\rho_1^{is}$ and analogously for λ_2 and they satisfy $\frac{\lambda_2}{\lambda_1} = \frac{\nu_2}{\nu_1}$.

Proof. Because δ^{it} is a one-dimensional corepresentation of (M, Δ) , there exist unique one-dimensional corepresentations (i.e. group-like unitaries) $v_t \in \hat{M}_1$ and $u_t \in M_2$ such that $\delta^{it} = (v_t \otimes 1)\alpha(u_t)$. Because α and β are morphisms we know that

$$\begin{aligned} \text{Ad}(\delta^{it}) \circ \alpha &= \alpha \circ \text{Ad}(\delta_2^{it}), & \text{Ad}(\hat{\delta}^{it}) \circ \alpha &= \alpha \circ \text{Ad}(\hat{\delta}_2^{it}) \quad \text{on } M_2, \\ \text{Ad}(\delta^{it}) \circ \beta &= \beta \circ \text{Ad}(\hat{\delta}_1^{it}), & \text{Ad}(\hat{\delta}^{it}) \circ \beta &= \beta \circ \text{Ad}(\delta_1^{it}) \quad \text{on } M_1. \end{aligned}$$

Then,

$$\begin{aligned} \Delta^{\text{op}}(v_t \otimes 1) &= (\beta \otimes \iota)(W_1)(v_t \otimes 1 \otimes 1)(\beta \otimes \iota)(W_1^*) \otimes 1 \\ &= ((\beta \otimes \iota)(W_1)(\delta^{it} \otimes 1)(\beta \otimes \iota)(W_1^*) \otimes 1)(\alpha(u_t^*) \otimes 1 \otimes 1) \\ &= ((\beta \otimes \iota)(W_1)(\beta \circ \text{Ad}(\hat{\delta}_1^{it}) \otimes \iota)(W_1^*)(v_t \otimes 1 \otimes 1)) \otimes 1 = v_t \otimes 1 \otimes \hat{\delta}_1^{it} \otimes 1. \end{aligned}$$

Hence, $\Delta^{\text{op}}(\delta^{it}) = \Delta^{\text{op}}((v_t \otimes 1)\alpha(u_t)) = (v_t \otimes 1)\alpha(u_t) \otimes (\hat{\delta}_1^{it} \otimes 1)\alpha(u_t)$. Because δ^{it} is group-like, we conclude that $v_t = \hat{\delta}_1^{it}$.

At the end of this proof, we will argue that in a l.c. quantum group, the modular element commutes with any group-like unitary. So, δ^{it} commutes with $\alpha(u_s)$ for all $s, t \in \mathbb{R}$. As δ^{it} commutes with δ^{is} , we get that δ^{it} commutes with $\hat{\delta}_1^{is} \otimes 1$ for all $s, t \in \mathbb{R}$. This means that $((\hat{\delta}_1^{-it} \otimes 1)\delta^{it})_t$ is a one-parameter group of unitaries. Then, there exists a unique strictly positive, self-adjoint operator ρ_2 affiliated with M_2 such that $\delta^{it} = (\hat{\delta}_1^{it} \otimes 1)\alpha(\rho_2^{it})$. By symmetry, we find ρ_1 and we have proven the first three statements of the proposition.

Let $a \in \mathcal{N}_{\hat{\varphi}_1}$ and $x \in \mathcal{N}_{\varphi_2}$. Then, we have

$$(a \otimes 1)\alpha(x)\delta^{it} = (a \otimes 1)\delta^{it}\alpha(\delta_2^{-it}x\delta_2^{it}) = (a\hat{\delta}_1^{it} \otimes 1)\alpha(\rho_2^{it}\delta_2^{-it}x\delta_2^{it}).$$

We conclude that

$$\Lambda((a \otimes 1)\alpha(x)\delta^{it}) = \left(\frac{\nu_2}{\nu_1}\right)^{-t/2}(\hat{J}_1\hat{\delta}_1^{-it}\hat{J}_1 \otimes \rho_2^{it}\delta_2^{-it}J_2\delta_2^{-it}J_2)\Lambda((a \otimes 1)\alpha(x)).$$

On the other hand, we know that

$$\Lambda((a \otimes 1)\alpha(x)\delta^{it}) = \nu^{-t/2}J\delta^{-it}J\Lambda((a \otimes 1)\alpha(x)).$$

So, $\nu = \frac{\nu_2}{\nu_1}$ and $J\delta^{it}J = \hat{J}_1\hat{\delta}_1^{it}\hat{J}_1 \otimes \rho_2^{-it}\delta_2^{it}J_2\delta_2^{it}J_2$. We find the formula for $\hat{J}\delta^{it}\hat{J}$ in an analogous way.

From [16], Proposition 2.4, we know that

$$\nabla^{it} = \hat{\delta}^{-it/2}\delta^{-it/2}J\delta^{-it/2}J\hat{J}\hat{\delta}^{it/2}\hat{J}.$$

We can take positive numbers $\lambda_{1,2} > 0$ such that $\sigma_t^1(\rho_1^{is}) = \lambda_1^{ist}\rho_1^{is}$ and analogously for λ_2 (see the last paragraph of this proof). Because in any l.c. quantum group, $\text{Ad } \hat{\delta}^{it} = \tau_{-t} \text{Ad}(\delta^{-it})\sigma_{-t}$ on M , we easily find the required formula for $[D\varphi_2 \circ \alpha : D\varphi_2]_t := \nabla^{it}(\hat{\nabla}_1^{-it} \otimes \nabla_2^{-it})$. We also get that $\hat{\delta}^{-is}\alpha(\rho_2^{it})\hat{\delta}^{is} = \lambda_2^{ist}\alpha(\rho_2^{it})$ and $\delta^{it}\beta(\rho_1^{is})\delta^{-it} = \lambda_1^{-ist}\beta(\rho_1^{is})$. A short calculation yields that $\delta^{it}\hat{\delta}^{is} = (\frac{\lambda_2}{\lambda_1})^{ist}\hat{\delta}^{is}\delta^{it}$. But, from the general theory, we know that $\delta^{it}\hat{\delta}^{is} = \nu^{ist}\hat{\delta}^{is}\delta^{it}$, giving us the equality $\frac{\lambda_2}{\lambda_1} = \frac{\nu_2}{\nu_1}$.

Consider Δ_2^{op} as an action of $(M_2, \Delta_2^{\text{op}})$ on M_2 . Then, $[D\varphi_2 \circ \Delta_2^{\text{op}} : D\varphi_2]_t = \delta_2^{-it} \otimes 1$. Because $(\iota \otimes \Delta_2^{\text{op}})\alpha = (\text{m}\sigma \otimes \iota)(\iota \otimes \alpha)\Delta_2^{\text{op}}$ and because, by Lemma 3.6, $\text{m}(\tau_{-t}^1 \otimes \tau_t^2) = (\tau_{-t}^1 \otimes \tau_t^2)\text{m}$, we can apply Lemma 10.4 of our Appendix to conclude that

$$(\iota \otimes \Delta_2^{\text{op}})([D\varphi_2 \circ \alpha : D\varphi_2]_t) = (\text{m}\sigma \otimes \iota)(\delta_2^{-it} \otimes [D\varphi_2 \circ \alpha : D\varphi_2]_t)(1 \otimes \delta_2^{it} \otimes 1).$$

Filling in the already obtained formula for $[D\varphi_2 \circ \alpha : D\varphi_2]_t$ and carefully rewriting, we arrive at the final statement of the proposition.

As we promised above, we consider now a group-like unitary $u \in M$ for some l.c. quantum group (M, Δ) . By the uniqueness of the Haar measure, we find $\lambda > 0$ such that $\varphi(uxu^*) = \lambda\varphi(x)$ for all $x \in M^+$. It follows that $\sigma_t(u) = \lambda^{it}u$ for all $t \in \mathbb{R}$. Because a group-like unitary is a one-dimensional corepresentation, we have $u \in D(S)$ and $S(u) = u^*$. But, also u^* is a group-like element and so $S^2(u) = u$. It follows that $\tau_t(u) = u$ for all $t \in \mathbb{R}$ and $R(u) = u^*$. Then $\sigma'_t(u) = R\sigma_{-t}R(u) = \lambda^{it}u$, which gives $\sigma'_t\sigma_{-t}(u) = u$ and so, u commutes with δ^{it} for all $t \in \mathbb{R}$. $\square$

5 Double crossed products as locally compact quantum groups

In this section, we compute the invariant weights on $(M_{\text{m}}, \Delta_{\text{m}})$ and obtain that $(M_{\text{m}}, \Delta_{\text{m}})$ is a l.c. quantum group. We compute all the operators associated with $(M_{\text{m}}, \Delta_{\text{m}})$.

Notation 5.1. We use ρ_1, ρ_2 to denote the unbounded operators introduced in Proposition 4.2. We define strictly positive, self-adjoint operators k_1, k_2 such that

$$k_1^{it} = \rho_1^{it}\delta_1^{-it} \quad \text{and} \quad k_2^{it} = \rho_2^{it}\delta_2^{-it}.$$

In order to obtain invariant weights for (M_m, Δ_m) , we need the following relative invariance result of the weight ψ_2 under the action α .

Proposition 5.2. *For all $x \in \mathcal{N}_{\psi_2}$, $\xi \in H_1$ and $\eta \in D(k_1^{1/2})$, we have $(\omega_{\xi, \eta} \otimes \iota)(U_\beta^* \alpha(x)) \in \mathcal{N}_{\psi_2}$ and*

$$\Gamma_2((\omega_{\xi, \eta} \otimes \iota)(U_\beta^* \alpha(x))) = (\omega_{\xi, k_1^{1/2} \eta} \otimes \iota)(U_\beta^* U_\alpha) \Gamma_2(x) .$$

On the other hand, for $x \in \mathcal{N}_{\psi_1}$, $\xi \in H_2$ and $\eta \in D(k_2^{1/2})$, we have $(\iota \otimes \omega_{\xi, \eta})(U_\alpha^ \beta(x)) \in \mathcal{N}_{\psi_1}$ and*

$$\Gamma_1((\iota \otimes \omega_{\xi, \eta})(U_\alpha^* \beta(x))) = (\iota \otimes \omega_{\xi, k_2^{1/2} \eta})(U_\alpha^* U_\beta) \Gamma_1(x) .$$

Proof. Consider the dual weight $\tilde{\psi}_2$ on M with a canonical GNS-map $\tilde{\Gamma}$ determined by

$$\tilde{\Gamma}((a \otimes 1)\alpha(x)) = \hat{\Lambda}_1(a) \otimes \Gamma_2(x) \quad \text{for all } a \in \mathcal{N}_{\hat{\varphi}_1}, x \in \mathcal{N}_{\psi_2} .$$

We denote by $\tilde{\nabla}$ the modular operator of $\tilde{\psi}_2$ in this GNS-construction. Observe that $\psi_2 = (\varphi_2)_{\delta_2}$ and hence, $\tilde{\psi}_2 = \varphi_{\alpha(\delta_2)}$. It then follows from [14], Proposition 2.5 that

$$\begin{aligned} \tilde{\nabla}^{it} &= J \alpha(\delta_2^{it}) J \alpha(\delta_2^{it}) \nabla^{it} , \\ \tilde{T} &= \nu_2^{i/4} J \tilde{\nabla}^{1/2} , \end{aligned}$$

where $\tilde{T}$ is determined by $\tilde{T}\tilde{\Gamma}(z) = \tilde{\Gamma}(z^*)$. From Proposition 4.2, it follows that

$$\begin{aligned} \tilde{\nabla}^{it} &= \alpha(\delta_2^{it})(1 \otimes J_2 \delta_2^{it} J_2) \nabla^{it} = \alpha(\delta_2^{it})(1 \otimes J_2 \delta_2^{it} J_2) [D\varphi_2 \circ \alpha : D\varphi_2]_t (\hat{\nabla}_1^{it} \otimes \nabla_2^{it}) \\ &= m(\rho_1^{-it/2} \otimes \rho_2^{-it/2} \delta_2^{it}) (\rho_1^{-it/2} \delta_1^{it} \otimes \rho_2^{it/2} \delta_2^{-it}) (\hat{\nabla}_1^{it} \otimes \nabla_2^{it}) = \beta(k_1^{-it}) (\hat{\nabla}_1^{it} \otimes \nabla_2^{it}) , \end{aligned}$$

where ∇_2 denotes the modular operator of ψ_2 .

We know that the elements $\alpha(x^*)(\mu \otimes \Gamma_2(y))$, $x, y \in \mathcal{N}_{\psi_2}$, $\mu \in D(\hat{\nabla}_1^{1/2})$ span a core for $\tilde{T}$ and

$$\tilde{T}\alpha(x^*)(\mu \otimes \Gamma_2(y)) = \alpha(y^*)(\hat{J}_1 \hat{\nabla}_1^{1/2} \mu \otimes \Gamma_2(x)) .$$

It follows that

$$\hat{\nabla}_1^{1/2} \alpha(x^*)(\mu \otimes \Gamma_2(y)) = \nu_2^{i/4} (1 \otimes J_2 y^* J_2) U_\alpha(\hat{\nabla}_1^{1/2} \mu \otimes J_2 \Gamma_2(x)) .$$

Because the product of $\beta(k_1^{-it})$ and $\hat{\nabla}_1^{it} \otimes \nabla_2^{it}$ yields the one-parameter group $\tilde{\nabla}^{it}$, we get that $\beta(k_1)$ and $\hat{\nabla}_1 \otimes \nabla_2$ commute strongly. If we assume now that x is analytic w.r.t. the modular group $(\sigma_t^{2'})$ of ψ_2 and $y \in \mathcal{N}_{\psi_2} \cap \mathcal{N}_{\psi_2}^*$, we conclude that the left hand side of the previous equality is equal to

$$\nu_2^{i/4} \beta(k_1^{-1/2}) \alpha(\sigma_{-i/2}^{2'}(x^*)) (\hat{\nabla}_1^{1/2} \mu \otimes J_2 \Gamma_2(y^*)) .$$

By a typical density argument, we may conclude that for $x \in \mathcal{N}_{\psi_2}$, analytic w.r.t. $(\sigma_t^{2'})$, $y \in \mathcal{N}_{\psi_2}$ and $\xi \in H_1$,

$$\beta(k_1^{-1/2}) \alpha(\sigma_{-i/2}^{2'}(x^*)) (\xi \otimes J_2 \Gamma_2(y)) = (1 \otimes J_2 y J_2) U_\alpha(\xi \otimes J_2 \Gamma_2(x)) .$$

Applying $U_\beta^* \in B(H_1) \otimes M_2$, it follows that for x, y, ξ as above and $\eta \in D(k_1^{1/2})$,

$$(\omega_{\xi, \eta} \otimes \iota)(U_\beta^* \alpha(\sigma_{-i/2}^{2'}(x^*))) J_2 \Gamma_2(y) = J_2 y J_2 (\omega_{\xi, k_1^{1/2} \eta} \otimes \iota)(U_\beta^* U_\alpha) J_2 \Gamma_2(x) .$$

Using $J_2 \Gamma_2(z) = \Gamma_2(\sigma_{-i/2}^{2'}(z^*))$, we get, for $x, y \in \mathcal{N}_{\psi_2}$, y analytic w.r.t. $(\sigma_t^{2'})$, $\xi \in H_1$ and $\eta \in D(k_1^{1/2})$,

$$\Gamma_2((\omega_{\xi, \eta} \otimes \iota)(U_\beta^* \alpha(x)) y) = J_2 \sigma_{i/2}^{2'}(y)^* J_2 (\omega_{\xi, k_1^{1/2} \eta} \otimes \iota)(U_\beta^* U_\alpha) \Gamma_2(x) .$$

From this formula, our first result follows. The second result is analogous. $\square$

We now prove our main result: the pair (M_m, Δ_m) is a l.c. quantum group and we calculate all its structural ingredients. Observe that in the case of ordinary l.c. group matchings, the same kind of formulas were obtained in [15], Propositions 4.14 and 4.15.

Theorem 5.3. *The pair (M_m, Δ_m) is a l.c. quantum group. The weight $\varphi_m := \psi_1 \otimes (\varphi_2)_{k_2}$ is left invariant and*

$$W_m = (\Sigma V_1^* \Sigma)_{13} Z_{34}^* W_{2,24} Z_{34} = (\Sigma V_1^* \Sigma)_{13} U_{\beta,32}^* (\alpha \otimes \iota)(W_2)_{324}$$

is the left regular representation when we use the canonical GNS-construction for φ_m .

The canonical right invariant weight ψ_m is given by $(\varphi_1)_{k_1} \otimes \psi_2$ and the right regular representation by

$$V_m = Z_{12}^* \hat{W}_{1,13} Z_{12} V_{2,24} = (\iota \otimes \beta)(\hat{W}_1)_{132} U_{\alpha,32} V_{2,24} .$$

Further, we have

$$\begin{aligned} R_m &= m^{-1}(R_1 \otimes R_2) = (R_1 \otimes R_2)m , & \nu_m &= \frac{\nu_2}{\nu_1} , \\ \tau_t^m &= \tau_{-t}^1 \otimes \tau_t^2 , & P_m &= P_1^{-1} \otimes P_2 , \\ J_m &= \lambda_1^{i/4} (J_1 \otimes J_2) , & \hat{\delta}_m^{it} &= (\hat{J}_1 \hat{\delta}_1^{-it} \hat{J}_1 \otimes 1) Z^* (1 \otimes \hat{\delta}_2^{it}) Z , \\ \hat{J}_m &= \lambda_1^{i/4} (\hat{J}_1 \otimes \hat{J}_2) J \hat{J} (\hat{J}_1 J_1 \otimes \hat{J}_2 J_2) , & \hat{J}_m \hat{\delta}_m^{it} \hat{J}_m &= (1 \otimes \hat{J}_2 \hat{\delta}_2^{it} \hat{J}_2) Z^* (\hat{\delta}_1^{-it} \otimes 1) Z , \\ \delta_m^{it} &= \rho_1^{it} \delta_1^{-2it} \otimes \rho_2^{-it} \delta_2^{2it} , \end{aligned}$$

where $\lambda_1 > 0$ is defined in Proposition 4.2. The dual of (M_m, Δ_m) is given by

$$\hat{M}_m = (\hat{M}'_1 \otimes 1 \cup Z^*(1 \otimes \hat{M}_2)Z)'' \quad \text{and} \quad \hat{M}'_m = (Z^*(\hat{M}_1 \otimes 1)Z \cup 1 \otimes \hat{M}'_2)'' ,$$

and in this picture, $(\hat{M}'_1, \hat{\Delta}'_1)$ and $(\hat{M}_2, \hat{\Delta}_2)$ are closed quantum subgroups of $(\hat{M}_m, \hat{\Delta}_m)$.

Finally, we have the following commutation relations

$$J \hat{J} (J_1 \otimes J_2) = \left(\frac{\nu_1}{\nu_2}\right)^{i/4} (J_1 \otimes J_2) J \hat{J} ,$$

$$(\iota \otimes m)(U_{\beta,13} V_{1,12}) = V_{1,12} U_{\beta,13} , \tag{5.1}$$

$$(m \otimes \iota)(U_{\alpha,13} (\Sigma V_2 \Sigma)_{23}) = (\Sigma V_2 \Sigma)_{23} U_{\alpha,13} . \tag{5.2}$$

Recall that the notations Z and W_m are introduced in Notation 3.2 and Proposition 3.5.

Proof. Define $\varphi_m := \psi_1 \otimes (\varphi_2)_{k_2}$ and denote by Λ_m its canonical GNS-map. Let $x \in \mathcal{N}_{\psi_1}$, $y, z \in \mathcal{N}_{(\varphi_2)_{k_2}}$ where z is analytic w.r.t. the modular group of $(\varphi_2)_{k_2}$. Take $\xi, \eta \in H_1$ and $\mu, \rho \in H_2$. Then,

$$(\omega_{\xi \otimes \mu, \eta \otimes \rho} \otimes \iota \otimes \iota) \Delta_m(x \otimes y)(1 \otimes z) = (\iota \otimes \omega_{\mu, \rho} \otimes \iota) ((U_\alpha \otimes 1)(1 \otimes \Delta_2(y)) (U_\alpha^* \beta((\iota \otimes \omega_{\xi, \eta}) \Delta_1(x)) \otimes z)) .$$

Let $\chi_{[1/n, n]}$ be the characteristic function of the interval $[1/n, n]$ and define $P_n = \chi_{[1/n, n]}(k_2)$. Let (e_i) be an orthonormal basis for H_2 . Consider the following element in $M_1 \otimes M_2$:

$$\begin{aligned} r &:= (\iota \otimes \omega_{\mu, \rho} \otimes \iota) ((U_\alpha \otimes 1)(1 \otimes \Delta_2(y))(1 \otimes P_n \otimes 1) (U_\alpha^* \beta((\iota \otimes \omega_{\xi, \eta}) \Delta_1(x)) \otimes z)) \\ &= \sum_i (\iota \otimes \omega_{e_i, \rho} \otimes \iota) ((U_\alpha \otimes 1)(1 \otimes \Delta_2(y))) ((\iota \otimes \omega_{\mu, P_n e_i}) (U_\alpha^* \beta((\iota \otimes \omega_{\xi, \eta}) \Delta_1(x))) \otimes z) . \end{aligned}$$

Denote every individual term of this sum by r_i . Because of Proposition 5.2, every r_i belongs to $\mathcal{N}_{\varphi_m}$ and

$$\Lambda_m(r_i) = (\iota \otimes \omega_{e_i, \rho} \otimes \iota) ((U_\alpha \otimes 1)(1 \otimes \Delta_2(y))) ((\iota \otimes \omega_{\mu, k_2^{1/2} P_n e_i}) (U_\alpha^* U_\beta) (\iota \otimes \omega_{\xi, \eta}) (V_1) \Gamma_1(x) \otimes \Lambda_{(\varphi_2)_{k_2}}(z)) .$$

This expression is summable over i and we conclude that $r \in \mathcal{N}_{\varphi_m}$ and

$$\Lambda_m(r) = (\iota \otimes \omega_{\mu, \rho} \otimes \iota) ((\alpha \otimes \iota) (\Delta_2(y) ((k_2^{1/2} P_n) \otimes 1)) (U_\beta \otimes 1)) ((\iota \otimes \omega_{\xi, \eta}) (V_1) \Gamma_1(x) \otimes \Lambda_{(\varphi_2)_{k_2}}(z)) .$$

Write $v := (\iota \otimes \omega_{\xi, \eta})(V_1)\Gamma_1(x)$ and take $w \in H_1$. Then, denoting by $(\tilde{\sigma}_t)$ the modular group of $(\varphi_2)_{k_2}$,

$$\begin{aligned} (\theta_w^* \otimes \iota)\Lambda_m(r) &= \Lambda_{(\varphi_2)_{k_2}}((\omega_{U_\beta(v \otimes \mu), w \otimes \rho} \circ \alpha \otimes \iota)(\Delta_2(y)((k_2^{1/2}P_n) \otimes 1))z) \\ &= J_2 \tilde{\sigma}_{i/2}(z)^* J_2 (\omega_{\alpha(P_n)U_\beta(v \otimes \mu), w \otimes \rho} \circ \alpha \otimes \iota)(W_2^*) \Lambda_{(\varphi_2)_{k_2}}(y), \end{aligned}$$

where we used that k_2 is a group-like. Hence,

$$\Lambda_m(r) = (1 \otimes J_2 \tilde{\sigma}_{i/2}(z)^* J_2) (\iota \otimes \omega_{\mu, \rho} \otimes \iota)((\alpha \otimes \iota)(W_2^*)(\alpha(P_n)U_\beta \otimes 1)) ((\iota \otimes \omega_{\xi, \eta})(V_1)\Gamma_1(x) \otimes \Lambda_{(\varphi_2)_{k_2}}(y)).$$

Taking limits over $n \rightarrow \infty$ and $z \rightarrow 1$, we get that

$$(\omega_{\xi \otimes \mu, \eta \otimes \rho} \otimes \iota \otimes \iota)\Delta_m(x \otimes y) \in \mathcal{N}_{\varphi_m}$$

and

$$\Lambda_m((\omega_{\xi \otimes \mu, \eta \otimes \rho} \otimes \iota \otimes \iota)\Delta_m(x \otimes y)) = (\omega_{\xi \otimes \mu, \eta \otimes \rho} \otimes \iota \otimes \iota)((\alpha \otimes \iota)(W_2^*)_{324} U_{\beta, 32}(\Sigma V_1 \Sigma)_{13})\Lambda_m(x \otimes y).$$

It follows that φ_m is left invariant and that its associated left regular representation is given by

$$\mathcal{W}_m := (\Sigma V_1^* \Sigma)_{13} U_{\beta, 32}^* (\alpha \otimes \iota)(W_2)_{324}.$$

Define $R_m := (R_1 \otimes R_2)_m$. From Lemma 3.6, we know that $R_m = m^{-1}(R_1 \otimes R_2)$ as well and then, it is easy to verify that $\Delta_m R_m = (R_m \otimes R_m)\Delta_m^{\text{op}}$.

So, $\varphi_m R_m$ is a right invariant weight and (M_m, R_m) is a l.c. quantum group. It follows that $\mathcal{W}_m$ is a multiplicative unitary and we want to prove that $\mathcal{W}_m = W_m$.

An analogous computation as above shows that $(\varphi_1)_{k_1} \otimes \psi_2$ is a right invariant weight with associated right regular representation

$$V_m := (\iota \otimes \beta)(\hat{W}_1)_{132} U_{\alpha, 32} V_{2, 24}.$$

Observe that

$$\begin{aligned} (\Delta_m \otimes \iota \otimes \iota)(\mathcal{W}_m) &= \sigma_{23} m_{23} ((\Sigma V_1^* \Sigma)_{15} (\Sigma V_1^* \Sigma)_{25} U_{\beta, 53}^* U_{\beta, 54}^* ((\iota \otimes \Delta_2) \alpha \otimes \iota)(W_2)_{5346}) \\ &= \sigma_{23} m_{23} ((\Sigma V_1^* \Sigma)_{15} (\Sigma V_1^* \Sigma)_{25} U_{\beta, 53}^* U_{\beta, 54}^* (m_{13}(\alpha \otimes \iota \otimes \iota)(W_{2, 13} W_{2, 23}))_{5346}) \\ &= (\Sigma V_1^* \Sigma)_{15} ((\iota \otimes m)(V_{1, 12}^* U_{\beta, 13}^*))_{532} ((\alpha \otimes \iota) \alpha \otimes \iota)(W_2)_{5326} U_{\beta, 54}^* (\alpha \otimes \iota)(W_2)_{546}. \end{aligned}$$

On the other hand, we know that

$$(\Delta_m \otimes \iota \otimes \iota)(\mathcal{W}_m) = (\mathcal{W}_m)_{1256} (\mathcal{W}_m)_{3456} = (\Sigma V_1^* \Sigma)_{15} U_{\beta, 52}^* (\alpha \otimes \iota)(W_2)_{526} (\Sigma V_1^* \Sigma)_{35} U_{\beta, 54}^* (\alpha \otimes \iota)(W_2)_{546}.$$

We conclude that

$$(\iota \otimes m)(U_{\beta, 13} V_{1, 12}) = V_{1, 12} U_{\beta, 13}.$$

Making an analogous computation with V_m , we arrive at the commutation relations (5.1) and (5.2).

We now claim that

$$Z_{23}^* W_{2, 13}^* Z_{23} = (\alpha \otimes \iota)(W_2^*)_{213} U_{\beta, 21},$$

yielding the equality $\mathcal{W}_m = W_m$. Analogously, one proves the second formula for the right regular representation: $V_m = Z_{12}^* \hat{W}_{1, 13} Z_{12} V_{2, 24}$. To prove the claim, we make the following calculation:

$$\begin{aligned} Z_{23}^* W_{2, 13}^* Z_{23} &= (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) U_{\alpha, 23}^* U_{\beta, 23} (\iota \otimes \Delta_2^{\text{op}})(U_{\beta}^*)_{213} (\Sigma V_2 \Sigma)_{13} U_{\alpha, 23} (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) \\ &= (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) U_{\alpha, 23}^* U_{\beta, 21}^* (\Sigma V_2 \Sigma)_{13} U_{\alpha, 23} (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2). \end{aligned}$$

Observe that

$$Z^* = \nu_1^{-i/4} \tilde{U}_\alpha^* U_\beta^* \quad \text{where} \quad \tilde{U}_\alpha = (J_1 \otimes \hat{J}_2) U_\alpha (J_1 \otimes \hat{J}_2) \in M'_1 \otimes B(H_2).$$

We combine Lemma 3.3 and Equation (5.2) with the calculation above, to get

$$\begin{aligned}
Z_{23}^* W_{2,13}^* Z_{23} &= \nu_1^{-i/4} (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) U_{\alpha,23}^* \tilde{U}_{\alpha,21} Z_{21}^* (\Sigma V_2 \Sigma)_{13} U_{\alpha,23} (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) \\
&= \nu_1^{-i/4} (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) U_{\alpha,23}^* \tilde{U}_{\alpha,21} U_{\alpha,23} (\Sigma V_2 \Sigma)_{13} Z_{21}^* (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) \\
&= \nu_1^{-i/4} (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) \tilde{U}_{\alpha,21} (\Sigma V_2 \Sigma)_{13} Z_{21}^* (\hat{J}_2 \otimes J_1 \otimes \hat{J}_2) \\
&= U_{\alpha,21} W_{2,13}^* U_{\alpha,21}^* U_{\beta,21} = (\alpha \otimes \iota) (W_2^*)_{213} U_{\beta,21} .
\end{aligned}$$

This proves our claim.

Defining $\tau_t^m := \tau_{-t}^1 \otimes \tau_t^2$, it follows from Lemma 3.6 that for all $\omega \in B(H_1 \otimes H_2)_*$,

$$R_m \tau_{-i/2}^m (\iota \otimes \iota \otimes \omega) (W_m) = (\iota \otimes \iota \otimes \omega) (W_m^*) .$$

From Lemma 3.6, we also know that R_m and τ_t^m commute. So, we can conclude that R_m is the unitary antipode of (M_m, Δ_m) and (τ_t^m) is its scaling group.

Taking the it -th power of the Radon-Nikodym derivative of $(\varphi_1)_{k_1} \otimes \psi_2$ w.r.t. φ_m , we get $\rho_1^{it} \delta_1^{-2it} \otimes \rho_2^{-it} \delta_2^{2it}$. These unitaries are group-like because of Proposition 4.2. By uniqueness of the Haar weights, $\varphi_m R_m$ is proportional to $(\varphi_1)_{k_1} \otimes \psi_2$ and also the Radon-Nikodym derivative of $\varphi_m R_m$ w.r.t. φ_m is group-like. Because a non-trivial multiple of a group-like is no longer group-like, we conclude that $\varphi_m R_m = (\varphi_1)_{k_1} \otimes \psi_2$. So, $(\varphi_1)_{k_1} \otimes \psi_2$ is indeed the canonical right invariant weight of (M_m, Δ_m) .

We also conclude that $\delta_m = \rho_1^{it} \delta_1^{-2it} \otimes \rho_2^{-it} \delta_2^{2it}$. Because clearly $\varphi_m \tau_t^m = (\nu_1/\nu_2)^t \varphi_m$, we get that $\nu_m = \frac{\nu_2}{\nu_1}$.

From [14], Proposition 2.5 and from Proposition 4.2 above, we know that $J_m = \lambda_1^{i/4} (J_1 \otimes J_2)$. Denote by $\hat{J}_m$ the modular conjugation of the canonical left invariant weight on the dual of (M_m, Δ_m) . Write $U_m := \hat{J}_m J_m$. Then, we know that

$$\begin{aligned}
(1 \otimes 1 \otimes U_m) (\Sigma V_1^* \Sigma)_{13} Z_{34}^* W_{2,24} Z_{34} (1 \otimes 1 \otimes U_m^*) &= (1 \otimes 1 \otimes U_m) W_m (1 \otimes 1 \otimes U_m^*) \\
&= \Sigma V_m \Sigma = Z_{34}^* W_{1,13}^* Z_{34} (\Sigma V_2 \Sigma)_{24} .
\end{aligned} \tag{5.3}$$

Taking slices on the first two legs, it follows that $U_m(x \otimes 1) U_m^* = Z^*(u_1 x u_1^* \otimes 1) Z$ for all $x \in \hat{M}'_1$, where we use the notation $u_i = \hat{J}_i J_i$. Applying $\text{Ad}(1 \otimes 1 \otimes U_m^*)$ to Equation (5.3), realizing that U_m and U_m^* only differ up to a scalar and again taking slices, we also conclude that $U_m(1 \otimes y) U_m^* = Z^*(1 \otimes u_2 y u_2^*) Z$ for all $y \in \hat{M}'_2$. We conclude that $\text{Ad } U_m = \text{Ad}(Z^*(u_1 \otimes u_2))$ on $\hat{M}'_1 \otimes \hat{M}'_2$. As we know J_m , it follows that

$$\hat{J}_m z \hat{J}_m = Z^*(\hat{J}_1 \otimes \hat{J}_1) z (\hat{J}_1 \otimes \hat{J}_2) Z ,$$

for all $z \in \hat{M}'_1 \otimes \hat{M}'_2$. If next $z \in M_1 \otimes M_2$, the left hand side equals $R_m(z^*)$, but the right hand side as well because $m^{-1}(R_1 \otimes R_2)(z^*) = R_m(z^*)$. Because their quotient commutes with both $\hat{M}'_1 \otimes \hat{M}'_2$ and $M_1 \otimes M_2$, we conclude that $\hat{J}_m$ and $Z^*(\hat{J}_1 \otimes \hat{J}_2)$ only differ up to a scalar.

Because $\hat{J}_m^2 = 1$, we also get $(Z^*(\hat{J}_1 \otimes \hat{J}_2))^2 = 1$. Writing this out, it follows that

$$(J_1 \otimes J_2) J \hat{J} = \hat{J} J (J_1 \otimes J_2) .$$

Using this commutation relation, we arrive at

$$J_m Z^*(\hat{J}_1 \otimes \hat{J}_2) = \lambda_1^{i/2} \left(\frac{\nu_1}{\nu_2} \right)^{i/4} Z^*(\hat{J}_1 \otimes \hat{J}_2) J_m .$$

On the other hand,

$$J_m \hat{J}_m = \nu_m^{-i/4} \hat{J}_m J_m = \left(\frac{\nu_1}{\nu_2} \right)^{i/4} \hat{J}_m J_m .$$

We conclude that

$$\hat{J}_m = \lambda_1^{i/4} Z^*(\hat{J}_1 \otimes \hat{J}_2) = \lambda_1^{i/4} (J_1 \hat{J}_1 \otimes J_2 \hat{J}_2) \hat{J} J (\hat{J}_1 \otimes \hat{J}_2) = \lambda_1^{i/4} (\hat{J}_1 \otimes \hat{J}_2) J \hat{J} (\hat{J}_1 J_1 \otimes \hat{J}_2 J_2) .$$

Because we know (τ_t^m) and the GNS-construction for φ_m , we immediately get that $P_m = P_1^{-1} \otimes P_2$. Because

$$\nabla_m^{it} = \nabla_1^{it} \otimes \nabla_2^{it} k_2^{it} J_2 k_2^{it} J_2 \quad \text{and} \quad \nabla_m^{it} = \nabla_1^{it} k_1^{it} J_1 k_1^{it} J_1 \otimes \nabla_2^{it},$$

where ∇_1, ∇_2 and ∇_m denote the modular operators of ψ_1, ψ_2 and ψ_m respectively, it is easy to find, out of Proposition 2.4 in [16], the formula

$$\hat{\delta}_m^{it} = \hat{J}_1 \hat{\delta}_1^{-it} \hat{J}_1 k_1^{-it} J_1 k_1^{-it} J_1 \otimes \hat{\delta}_2^{it}.$$

But, from Proposition 4.2, it follows that $\hat{J}(1 \otimes \hat{\delta}_2^{it})\hat{J} = k_1^{-it} J_1 k_1^{-it} J_1 \otimes \hat{J}_2 \hat{\delta}_2^{it} \hat{J}_2$. Because $1 \otimes \hat{\delta}_2^{it}$ is a group-like unitary of $(\hat{M}, \hat{\Delta})$, we know that $J(1 \otimes \hat{\delta}_2^{it})J = 1 \otimes \hat{\delta}_2^{it}$ and so $\hat{\delta}_m^{it} = (\hat{J}_1 \hat{\delta}_1^{-it} \hat{J}_1 \otimes 1)Z^*(1 \otimes \hat{\delta}_2^{it})Z$. The formula for $\hat{J}_m \hat{\delta}_m^{it} \hat{J}_m$ is proven analogously. $\square$

6 Actions of double crossed products and bicrossed products

Suppose that $\rho : N \rightarrow M_m \otimes N$ is an action of the double crossed product (M_m, Δ_m) on the von Neumann algebra N . We will show how to construct another von Neumann algebra L with an action of the bicrossed product (M, Δ) and we show that the strict outerness of one action is equivalent to the strict outerness of the other action. As a definition of strict outerness, we take Definition 5.5 in [13]: $M_m \rho \ltimes N \cap \rho(N)' = \mathbb{C}$. In [12], the second author proves that every l.c. quantum group can act strictly outerly in this strong sense on a factor.

It is clear that we have morphisms $(M_m, \Delta_m) \rightarrow (M_1, \Delta_1^{\text{op}})$ and $(M_m, \Delta_m) \rightarrow (M_2, \Delta_2)$. Although these morphisms do not exist on the von Neumann algebra level, they can nevertheless be used to *restrict* the action ρ to an action of $(M_1, \Delta_1^{\text{op}})$, resp. (M_2, Δ_2) . We briefly explain how to do this.

In the von Neumann algebra language, a morphism appears rather as a special type of action (the *restricted* left or right translation), as was pointed out by J. Kustermans in [7]. We clearly have a left action $\zeta_1 : M_m \rightarrow M_1 \otimes M_m : \zeta_1 = \Delta_1^{\text{op}} \otimes \iota$ of $(M_1, \Delta_1^{\text{op}})$ on M_m satisfying $(\iota \otimes \Delta_m)\zeta_1 = (\zeta_1 \otimes \iota \otimes \iota)\Delta_m$. Analogously, we have the right action $\iota \otimes \Delta_2 : M_m \rightarrow M_m \otimes M_2$, which can be composed with the unitary antipodes to yield the left action of (M_2, Δ_2) on M_m given by $\zeta_2 : M_m \rightarrow M_2 \otimes M_m : \zeta_2 = \sigma_{12} m_{12}(\iota \otimes \Delta_2)$.

This allows us to define the action $\eta_1 : N \rightarrow M_1 \otimes N$ of $(M_1, \Delta_1^{\text{op}})$ on N such that $(\iota \otimes \rho)\eta_1 = (\zeta_1 \otimes \iota)\rho$ and the action $\eta_2 : N \rightarrow M_2 \otimes N$ of (M_2, Δ_2) on N such that $(\iota \otimes \rho)\eta_2 = (\zeta_2 \otimes \iota)\rho$. Denote

$$L := M_1 \eta_1 \ltimes N = (\eta_1(N) \cup \hat{M}_1' \otimes 1)''.$$

It is easy to check that $(\iota \otimes \eta_2)\eta_1 = \rho$ and applying $\iota \otimes \eta_2$ to L , we arrive at another representation of L : $(\rho(N) \cup \hat{M}_1' \otimes 1 \otimes 1)''$. This shows how L is an intermediate von Neumann algebra

$$N \hookrightarrow L \hookrightarrow M_m \rho \ltimes N.$$

On L , we will define a natural action η of the bicrossed product (M, Δ) such that

$$N \hookrightarrow L \hookrightarrow M_m \rho \ltimes N \hookrightarrow M \eta \ltimes L \hookrightarrow \hat{M}_m \hat{\rho} \ltimes (M_m \rho \ltimes N). \quad (6.1)$$

Proposition 6.1. *There exists a unique action $\eta : L \rightarrow M \otimes L$ of (M, Δ) on L such that*

$$\eta(\eta_1(x)) = (\alpha \otimes \eta_1)\eta_2(x) \quad \text{for all } x \in N \quad \eta(\hat{J}_1 y \hat{J}_1 \otimes 1) = (\hat{J} \otimes \hat{J}_1)\mu(y)(\hat{J} \otimes \hat{J}_1) \otimes 1 \quad \text{for all } y \in \hat{M}_1,$$

where $\mu : \hat{M}_1 \rightarrow M \otimes \hat{M}_1$ is the left action of (M, Δ^{op}) on $\hat{M}_1$ defined by $\mu(y) \otimes 1 = \Delta^{\text{op}}(y \otimes 1)$.

Moreover, the action η is strictly outer if and only if the action ρ is strictly outer. Finally, the inclusion of Equation (6.1) holds.

Proof. During the proof, we replace L by its representation $(\rho(N) \cup \hat{M}_1' \otimes 1 \otimes 1)''$ in $B(H_1 \otimes H_2) \otimes N$. Define for $z \in L$,

$$\eta(z) = \mathcal{U}_{123}(\iota \otimes \iota \otimes \rho)(z)\mathcal{U}_{123}^* \quad \text{where} \quad \mathcal{U} = (\hat{J} \otimes \hat{J}_1)(\beta \otimes \iota)(W_1)(\hat{J}_1 \otimes \hat{J}_2 \otimes \hat{J}_1).$$

Represent N on its standard Hilbert space, with anti-unitary J_N and use the notation

$$\tilde{\rho}(y) = (\hat{J}_1 \otimes \hat{J}_2 \otimes J_N) \rho(J_N y J_N) (\hat{J}_1 \otimes \hat{J}_2 \otimes J_N) \quad \text{for } y \in N'.$$

Let $x \in N$. We have

$$\begin{aligned} \eta(\rho(x)) &= \mathcal{U}_{123}(\Delta_m \otimes \iota) \rho(x) \mathcal{U}_{123}^* \\ &= (\hat{J} \otimes \hat{J}_1 \otimes \hat{J}_2 \otimes J_N) (\beta \otimes \iota)(W_1)_{123} (\sigma_{23} m_{23}^{-1} (\Delta_1 \otimes \Delta_2^{\text{op}} \otimes \iota) \tilde{\rho}(J_N x J_N)) (\beta \otimes \iota)(W_1^*)_{123} (\hat{J} \otimes \hat{J}_1 \otimes \hat{J}_2 \otimes J_N) \\ &= (\hat{J} \otimes \hat{J}_1 \otimes \hat{J}_2 \otimes J_N) (\alpha \otimes \iota \otimes \iota \otimes \iota) (\sigma_{12} m_{12}^{-1} (\iota \otimes \Delta_2^{\text{op}} \otimes \iota) \tilde{\rho}(J_N x J_N)) (\hat{J} \otimes \hat{J}_1 \otimes \hat{J}_2 \otimes J_N) \\ &= (\alpha \otimes \iota \otimes \iota \otimes \iota) (\sigma_{12} m_{12} (\iota \otimes \Delta_2 \otimes \iota) \rho(x)) = (\alpha \otimes \rho) \eta_2(x). \end{aligned}$$

On the other hand, it is obvious that $\eta(\hat{J}_1 y \hat{J}_1 \otimes 1 \otimes 1) = (\hat{J} \otimes \hat{J}_1) \mu(y) (\hat{J} \otimes \hat{J}_1) \otimes 1 \otimes 1$ for $y \in \hat{M}_1$. Then, it is also clear that $\eta : L \rightarrow M \otimes L$ and that η is an action of (M, Δ) on L .

By definition, $M_{\eta \ltimes} L = (\eta(L) \cup \hat{M} \otimes 1 \otimes 1 \otimes 1)''$, but using $\mathcal{U}$, we easily identify

$$M_{\eta \ltimes} L = (L \cup (\hat{J}_1 \otimes \hat{J}_2) \hat{J} \hat{M} \hat{J} (\hat{J}_1 \otimes \hat{J}_2) \otimes 1)'' \subset B(H_1 \otimes H_2) \otimes N.$$

In this way, we see that the inclusion of Equation (6.1) is given by

$$\rho(N) \subset L \subset (\rho(N) \cup \hat{M}_m \otimes 1)'' \subset (L \cup (\hat{J}_1 \otimes \hat{J}_2) \hat{J} \hat{M} \hat{J} (\hat{J}_1 \otimes \hat{J}_2) \otimes 1)'' \subset B(H_1 \otimes H_2) \otimes N.$$

Suppose first that ρ is strictly outer and let $z \in (M_{\eta \ltimes} L) \cap L'$. Then, we observe that $z \in B(H_1 \otimes H_2) \otimes N \cap \rho(N)'$, which equals $M'_1 \otimes M'_2 \otimes 1$ by the strict outerity of ρ . So, $z = y \otimes 1$ with $y \in M'_1 \otimes M'_2$. But z also commutes with $\hat{M}'_1 \otimes 1 \otimes 1$, which yields $z = 1 \otimes a \otimes 1$ with $a \in M'_2$. Because

$$(\iota \otimes \iota \otimes \eta_2) \rho = (\iota \otimes \iota \otimes \eta_2) (\iota \otimes \eta_2) \eta_1 = (\iota \otimes \Delta_2 \otimes \iota) \rho,$$

we see that $(\iota \otimes \iota \otimes \eta_2)(z) = V_{2,23} z_{124} V_{2,23}^*$ for all $z \in M_{\eta \ltimes} L$. So, $a \in \hat{M}'_2$ and hence, $a \in \mathbb{C}$. We get that η is strictly outer.

Suppose next that η is strictly outer and let $z \in B(H_1 \otimes H_2) \otimes N \cap (M_m \rho \ltimes N)'$. We claim that,

$$\mathcal{U}_{123}(\iota \otimes \iota \otimes \rho)(y) \mathcal{U}_{123}^* \in B(H_1 \otimes H_2) \otimes L \quad \text{for all } y \in B(H_1 \otimes H_2) \otimes N.$$

When $y \in \rho(N)$, this is clear and when $y \in B(H_1 \otimes H_2) \otimes 1$, it follows because $\mathcal{U} \in B(H_1 \otimes H_2) \otimes \hat{M}'_1$. Because ρ is an action, we know that

$$B(H_1 \otimes H_2) \otimes N = (\rho(N) \cup B(H_1 \otimes H_2) \otimes 1)''$$

and our claim is proven. So, we find in particular that $\mathcal{U}_{123}(\iota \otimes \iota \otimes \rho)(z) \mathcal{U}_{123}^* \in B(H_1 \otimes H_2) \otimes L \cap \eta(L)'$. From the strict outerity of η , it follows that $B(H_1 \otimes H_2) \otimes L \cap \eta(L)' = M' \otimes 1 \otimes 1 \otimes 1$ and we can take $a \in M'$ such that

$$(\iota \otimes \iota \otimes \rho)(z) = \mathcal{U}_{123}^*(a \otimes 1 \otimes 1 \otimes 1) \mathcal{U}_{123}.$$

But, $\rho(N) \subset M_1 \otimes M_2 \otimes N$, while the last leg of $\mathcal{U}$ sits in $\hat{M}'_1$. Hence, $z = y \otimes 1$ for some $y \in B(H_1 \otimes H_2)$. Writing $x = (\hat{J}_1 \otimes \hat{J}_2) y (\hat{J}_1 \otimes \hat{J}_2)$ and $b = \hat{J}_a \hat{J} \in M'$, we get

$$(\beta \otimes \iota)(W_1)(x \otimes 1)(\beta \otimes \iota)(W_1^*) = b \otimes 1.$$

If we write $\zeta(c) = (\beta \otimes \iota)(W_1)(c \otimes 1)(\beta \otimes \iota)(W_1^*)$ for $c \in B(H_1 \otimes H_2)$, we see that $\zeta : B(H_1 \otimes H_2) \rightarrow B(H_1 \otimes H_2) \otimes \hat{M}_1$ is a right action of $(\hat{M}_1, \hat{\Delta}_1^{\text{op}})$ on $B(H_1 \otimes H_2)$. Because $\zeta(x) = b \otimes 1$, we find that $\zeta(b) \otimes 1 = (\zeta \otimes \iota) \zeta(x) = (\iota \otimes \iota \otimes \hat{\Delta}_1^{\text{op}}) \zeta(x) = b \otimes 1 \otimes 1$. Hence, $\zeta(b) = b \otimes 1$ and $x = b$. We arrive at the conclusion that $x \in M' \cap \beta(M_1)' = \alpha(M_2)' \cap (\hat{M}_1 \otimes 1)' \cap \beta(M_1)' = M'_1 \otimes M'_2 \cap (\hat{M}_1 \otimes 1)' = 1 \otimes M'_2$. This gives us that $z = 1 \otimes d \otimes 1$ with $d \in M'_2$. Because $z \in (M_m \rho \ltimes N)'$, $1 \otimes d$ must commute with $Z^*(1 \otimes \hat{M}_2)Z$. By Theorem 5.3, Z^* equals $(\hat{J}_1 J_1 \otimes 1) U_{\beta}^* J(J_1 \otimes J_2)$, up to a scalar. The first part belongs to $B(H_1) \otimes M_2$ and we find that $1 \otimes d$ must commute with $J(1 \otimes \hat{M}_2)J$. This means that $\alpha(J_2 d J_2)$ commutes with $1 \otimes \hat{M}_2$. Hence, $\alpha(J_2 d J_2) \in M_1 \otimes 1$. Because α is an action of (M_1, Δ_1) on M_2 , it follows that $J_2 d J_2 \in \mathbb{C}$. So, $z \in \mathbb{C}$ and we are done. $\square$

7 A characterization of double crossed products

In Definition 2.3, we defined a matching m of l.c. quantum groups (M_1, Δ_1) , (M_2, Δ_2) and we constructed the associated double crossed product (M_m, Δ_m) as a l.c. quantum group in Section 5. It is clear that the mappings $\Delta_1^{\text{op}} \otimes \iota$ and $\iota \otimes \Delta_2$ define, respectively, a left action of $(M_1, \Delta_1^{\text{op}})$ and a right action of (M_2, Δ_2) on the von Neumann algebra $M_m \cong M_1 \otimes M_2$. In fact, these actions are precisely the actions that correspond to the natural morphisms from (M_m, Δ_m) to $(M_1, \Delta_1^{\text{op}})$ and (M_2, Δ_2) .

We claim that this characterizes double crossed products: whenever we have a l.c. quantum group (N, Γ) and morphisms from (N, Γ) to $(M_1, \Delta_1^{\text{op}})$ and (M_2, Δ_2) , such that N , equipped with the naturally associated left action of $(M_1, \Delta_1^{\text{op}})$ and right action of (M_2, Δ_2) , is isomorphic to $M_1 \otimes M_2$ as an M_1, M_2 -bicomodule, we get that (N, Γ) is a double crossed product.

We can interpret this in the classical, commutative case as follows. We are given l.c. groups G, G_1 and G_2 . We suppose that G_1^{op} acts on the left and G_2 on the right on G such that, as a measure space, G is isomorphic to $G_1 \times G_2$ where the isomorphism intertwines the actions of G_1^{op} and G_2 on G with the natural actions of G_1^{op} and G_2 on $G_1 \times G_2$. So, we assume in fact that G_1^{op} and G_2 are closed subgroups of G and that there exists an element $x \in G$ such that the map $G_1 \times G_2 \rightarrow G : (g, s) \mapsto gxs$ is a measure class isomorphism. Then, it is clear that, considering $x^{-1}G_1x$ and G_2 as closed subgroups of G , we arrive at a matched pair (in the sense of [3], Definition 3.1) of l.c. groups with G as the double crossed product. It is obvious that it is really necessary to allow for the presence of $x \in G$ and this explains the presence of the group-like unitary $u \in \hat{N}$ in Proposition 7.1.

Proposition 7.1. *Let (N, Γ) , (M_1, Δ_1) and (M_2, Δ_2) be l.c. quantum groups. Suppose that there are morphisms from (N, Γ) to $(M_1, \Delta_1^{\text{op}})$ and (M_2, Δ_2) respectively, with associated actions $\zeta_1 : N \rightarrow M_1 \otimes N$, $\zeta_2 : N \rightarrow N \otimes M_2$ of $(M_1, \Delta_1^{\text{op}})$ on the left and (M_2, Δ_2) on the right on N (in the sense of Proposition 12.1 in [7]). Suppose that there exists an isomorphism of von Neumann algebras $\pi : N \rightarrow M_1 \otimes M_2$ intertwining ζ_1 with $\Delta_1^{\text{op}} \otimes \iota$ and ζ_2 with $\iota \otimes \Delta_2$.*

Then, there exists a matching $m : M_1 \otimes M_2 \rightarrow M_1 \otimes M_2$ and a group-like unitary $u \in \hat{N}$ such that $\pi \circ \text{Ad } u : N \rightarrow M_1 \otimes M_2$ is an isomorphism between the l.c. quantum groups (N, Γ) and the double crossed product (M_m, Δ_m) .

Proof. Denote by W_N the left regular representation of (N, Γ) . Because the actions ζ_1, ζ_2 are defined by morphisms, we know that

$$\begin{aligned} (\zeta_1 \otimes \iota)\Gamma &= (\iota \otimes \Gamma)\zeta_1, & (\iota \otimes \zeta_2)\Gamma &= (\Gamma \otimes \iota)\zeta_2, \\ (\zeta_1 \otimes \iota)(W_N) &= X_{1,13}W_{N,23}, & (\zeta_2 \otimes \iota)(W_N) &= W_{N,13}X_{2,23}, \end{aligned} \quad (7.1)$$

where $X_1 \in M_1 \otimes \hat{N}$ is a corepresentation of $(M_1, \Delta_1^{\text{op}})$ and $X_2 \in M_2 \otimes \hat{N}$ is a corepresentation of (M_2, Δ_2) . Defining $\tilde{W}_N = (\pi \otimes \iota)(W_N) \in M_1 \otimes M_2 \otimes \hat{N}$, we deduce that

$$(\Delta_1^{\text{op}} \otimes \iota \otimes \iota)(\tilde{W}_N) = X_{1,14}\tilde{W}_{N,234}, \quad (\iota \otimes \Delta_2 \otimes \iota)(\tilde{W}_N) = \tilde{W}_{N,124}X_{2,34}.$$

It then follows that $X_{1,13}^*\tilde{W}_N$ is invariant under $\Delta_1^{\text{op}} \otimes \iota \otimes \iota$ and we find $Y \in M_2 \otimes \hat{N}$ such that $\tilde{W}_N = X_{1,13}Y_{23}$. Using next the formula for $(\iota \otimes \Delta_2 \otimes \iota)(\tilde{W}_N)$, we get $(\Delta_2 \otimes \iota)(Y) = Y_{13}X_{2,23}$. Hence, YX_2^* is invariant under $\Delta_2 \otimes \iota$ and we find a unitary $u \in \hat{N}$ such that

$$\tilde{W}_N = X_{1,13}(1 \otimes 1 \otimes u)X_{2,23}.$$

We know that $(\iota \otimes \iota \otimes \hat{\Delta}_N)(\tilde{W}_N) = \tilde{W}_{N,124}\tilde{W}_{N,123}$, from which we conclude that $(\iota \otimes \hat{\Delta}_N)(X_i) = X_{i,13}X_{i,12}$ for $i = 1, 2$. Writing this out, it follows that $\hat{\Delta}(u) = u \otimes u$.

The equation $\hat{\Delta}(u) = u \otimes u$ gives us that $(\text{Ad } u \otimes \iota)(W_N) = (1 \otimes u^*)W_N$. From Equation (7.1), it is clear that ζ_2 intertwines $\text{Ad } u$ and $\text{Ad}(u \otimes 1)$, while defining $\tilde{\zeta}_1 = (\iota \otimes \text{Ad } u^*) \circ \zeta_1 \circ \text{Ad } u$ and $\tilde{X}_1 = (1 \otimes u^*)X_1(1 \otimes u)$, we get

$$(\tilde{\zeta}_1 \otimes \iota)(W_N) = \tilde{X}_{1,13}W_{N,23}.$$

Finally, we put $\tilde{\pi} = \pi \circ \text{Ad } u$ and observe that $(\tilde{\pi} \otimes \iota)(W_N) = \tilde{X}_{1,13}X_{2,23}$. Also, $\tilde{\pi}$ intertwines $\tilde{\zeta}_1$ with $\Delta_1^{\text{op}} \otimes \iota$ and ζ_2 with $\iota \otimes \Delta_2$. As a conclusion, we see that changing the morphism from (N, Γ) to $(M_1, \Delta_1^{\text{op}})$ by using u , we may in fact suppose from the beginning that $u = 1$ and now prove the existence of a matching $m : M_1 \otimes M_2 \rightarrow M_1 \otimes M_2$ such that π is an isomorphism between the l.c. quantum groups (N, Γ) and (M_m, Δ_m) .

So, we assume that $u = 1$ and get $\tilde{W}_N = X_{1,13}X_{2,23}$. Because a morphism commutes with the antipode (Remark 12.1 in [7]), we know that $\zeta_1 \tau_t^N = (\tau_{-t}^1 \otimes \tau_t^N) \zeta_1$ and $\zeta_2 \tau_t^N = (\tau_t^N \otimes \tau_t^2) \zeta_2$. Combining this with Equation (7.1), we get that X_1 is invariant under $\tau_{-t}^1 \otimes \hat{\tau}_t^N$, while X_2 is invariant under $\tau_t^2 \otimes \hat{\tau}_t^N$. Because $\tilde{W}_N = X_{1,13}X_{2,23}$, it follows that $\pi \tau_t^N = (\tau_{-t}^1 \otimes \tau_t^2) \pi$. Moreover, because X_1 and X_2 are corepresentations, it follows that, for all $\omega \in \hat{N}_*$, the element $y := (\iota \otimes \iota \otimes \omega)(X_{1,13}X_{2,23}) \in D(S_1^{-1} \otimes S_2)$ and

$$(S_1^{-1} \otimes S_2)(y) = (\iota \otimes \iota \otimes \omega)(X_{1,13}^* X_{2,23}^*) .$$

Define $m := (R_1 \otimes R_2) \pi R_N \pi^{-1}$, use that $S_N(\iota \otimes \omega)(W_N) = (\iota \otimes \omega)(W_N^*)$ and combine this with the formulas above, to conclude that

$$(m \otimes \iota)(X_{1,13}X_{2,23}) = X_{2,23}X_{1,13} .$$

It follows that $((\iota \otimes \sigma m \otimes \iota)(\Delta_1^{\text{op}} \otimes \Delta_2) \otimes \iota)(\tilde{W}_N) = \tilde{W}_{N,125} \tilde{W}_{N,345}$ and hence, $(\iota \otimes \sigma m \otimes \iota)(\Delta_1^{\text{op}} \otimes \Delta_2) \pi = (\pi \otimes \pi) \Gamma$. We also verify easily that

$$\begin{aligned} ((\Delta_1 \otimes \iota) m \otimes \iota)(X_{1,13}X_{2,23}) &= X_{2,34}X_{1,24}X_{1,14} \\ \text{and } m_{23} m_{13} (\Delta_1 \otimes \iota \otimes \iota)(X_{1,13}X_{2,23}) &= m_{23} m_{13} (X_{1,24}X_{1,14}X_{2,34}) = X_{2,34}X_{1,24}X_{1,14} . \end{aligned}$$

Because the slices of $\tilde{W}_N$ are dense in $\pi(N) = M_1 \otimes M_2$, we conclude that $(\Delta_1 \otimes \iota) m = m_{23} m_{13} (\Delta_1 \otimes \iota)$. Completely analogously, we verify that $(\iota \otimes \Delta_2) m = m_{13} m_{12} (\iota \otimes \Delta_2)$. Hence, m is a matching and above, we have seen already that $\pi : N \rightarrow M_m$ is an isomorphism of l.c. quantum groups. $\square$

8 Example: generalized quantum doubles

Let (M_1, Δ_1) and (M_2, Δ_2) be l.c. quantum groups. Suppose that $\mathcal{Z} \in M_1 \otimes M_2$ is a bicharacter in the sense that

$$(\Delta_1 \otimes \iota)(\mathcal{Z}) = \mathcal{Z}_{23} \mathcal{Z}_{13} \quad \text{and} \quad (\iota \otimes \Delta_2)(\mathcal{Z}) = \mathcal{Z}_{13} \mathcal{Z}_{12} .$$

(We see that, in fact, $\mathcal{Z}^*$ is the bicharacter.) Then, we can define a matching m as the inner isomorphism:

$$m(x) = \mathcal{Z} x \mathcal{Z}^* \quad \text{for all } x \in M_1 \otimes M_2 . \tag{8.1}$$

It is clear that m is a matching of (M_1, Δ_1) and (M_2, Δ_2) . Hence, we have an associated bicrossed product (M, Δ) and double crossed product (M_m, Δ_m) , as above.

A particular case is that of the *quantum double*: let (N, Δ_N) be a l.c. quantum group and put $(M_1, \Delta_1) = (N, \Delta_N^{\text{op}})$, $(M_2, \Delta_2) = (\hat{N}, \hat{\Delta}_N)$ and $\mathcal{Z} = W_N$. The associated double crossed products is the quantum double of (N, Δ_N) . Such a quantum double construction in the framework of l.c. quantum groups (or rather, Woronowicz algebras) has been done for the first time by T. Yamanouchi [23].

For this reason, we call the double crossed product associated with an inner matching as in Equation (8.1) a *generalized quantum double*.

Suppose for the rest of this section that we have a bicharacter $\mathcal{Z}$ as above and an inner matching m defined by Equation (8.1).

Proposition 8.1. *Through the isomorphism $\text{Ad } \mathcal{Z}^*$, the bicrossed product (M, Δ) is isomorphic with the tensor product $(\hat{M}_1 \otimes M_2, (\iota \otimes \sigma \otimes \iota)(\hat{\Delta}_1 \otimes \Delta_2))$.*

The ingredients of the double crossed product (i.e., the generalized quantum double) are given by Theorem 5.3, combined with the formulas:

- $J = \mathcal{Z}(\hat{J}_1 \otimes J_2)\mathcal{Z}^*$ and $\hat{J} = \mathcal{Z}(J_1 \otimes \hat{J}_2)\mathcal{Z}^*$;
- there exist group-like unitaries $k_1^{it} \in M_1$ and $k_2^{it} \in M_2$ such that

$$\mathcal{Z}^*(\hat{\delta}_1^{it} \otimes 1)\mathcal{Z} = \hat{\delta}_1^{it} \otimes k_2^{-it} \quad \text{and} \quad \mathcal{Z}^*(1 \otimes \hat{\delta}_2^{it})\mathcal{Z} = k_1^{-it} \otimes \hat{\delta}_2^{it}$$

and $\rho_1^{it} = k_1^{it}\delta_1^{it}$, $\rho_2^{it} = k_2^{it}\delta_2^{it}$ (observe that ρ_1 and ρ_2 are defined in Proposition 4.2, associated with a matching m);

- $Z = \nu_1^{i/4} \mathcal{Z}(J_1 \otimes J_2)\mathcal{Z}(J_1 \otimes J_2)$.

The double crossed product is unimodular if and only if $k_1 = \delta_1$ and $k_2 = \delta_2$. This is the case for the ordinary quantum double of a l.c. quantum group.

Proof. To determine the bicrossed product, we calculate:

$$\begin{aligned} (\mathcal{Z}^* \otimes \mathcal{Z}^*)W(\mathcal{Z} \otimes \mathcal{Z}) &= \mathcal{Z}_{34}^* W_{2,24} \mathcal{Z}_{12}^* \mathcal{Z}_{34} \hat{W}_{1,13} \mathcal{Z}_{12} \\ &= W_{2,24}(\iota \otimes \Delta_2)(\mathcal{Z}^*)_{324} \mathcal{Z}_{12}^* \mathcal{Z}_{34}(\Delta_1^{\text{op}} \otimes \iota)(\mathcal{Z})_{132} \hat{W}_{1,13} = W_{2,24} \hat{W}_{1,13}. \end{aligned}$$

This proves that $\text{Ad } \mathcal{Z}^*$ maps (M, Δ) onto the tensor product $(\hat{M}_1 \otimes M_2, (\iota \otimes \sigma \otimes \iota)(\hat{\Delta}_1 \otimes \Delta_2))$. From Proposition 4.2 in [13], the formulas for J and $\hat{J}$ follow.

Because $\mathcal{Z}^*$ is a corepresentation for (M_1, Δ_1) , we get that $(\iota \otimes \omega)(\mathcal{Z}^*) \in D(S_1)$ for all $\omega \in M_{2*}$ and $S_1(\iota \otimes \omega)(\mathcal{Z}^*) = (\iota \otimes \omega)(\mathcal{Z})$. An analogous statement holds for the left slices of $\mathcal{Z}$ and S_2 . Combining both, we conclude that $\mathcal{Z}$ is invariant under $\tau_{-t}^1 \otimes \tau_t^2$ and $R_1 \otimes R_2$. The formula for Z then follows.

Finally, $\eta_t^1(x) = \hat{\delta}_1^{it} x \hat{\delta}_1^{-it}$ defines a one-parameter group of automorphisms of M_1 satisfying $\Delta_1 \eta_t^1 = (\eta_t^1 \otimes \iota) \Delta_1$. It follows that $(\eta_t^1 \otimes \iota)(\mathcal{Z}^*)\mathcal{Z}$ is invariant under $\Delta_1 \otimes \iota$. Hence, there exist unitaries $k_2^{it} \in M_2$ such that $\mathcal{Z}^*(\hat{\delta}_1^{it} \otimes 1)\mathcal{Z} = \hat{\delta}_1^{it} \otimes k_2^{-it}$. Applying $\iota \otimes \Delta_2$ to this equality, we find that k_2^{it} is group-like. We analogously find k_1^{it} . Because the modular element of the tensor product $\hat{M}_1 \otimes M_2$ is given by $\hat{\delta}_1 \otimes \delta_2$, we get the following formula for the modular element δ of (M, Δ) :

$$\delta^{it} = \mathcal{Z}(\hat{\delta}_1^{it} \otimes \delta_2^{it})\mathcal{Z}^* = (\hat{\delta}_1^{it} \otimes 1)\alpha(k_2^{it}\delta_2^{it}).$$

So, by definition, $\rho_2^{it} = k_2^{it}\delta_2^{it}$.

It follows immediately from Theorem 5.3 that (M_m, Δ_m) is unimodular if and only if $k_1 = \delta_1$ and $k_2 = \delta_2$. This is the case for the ordinary quantum double: we have $(M_1, \Delta_1) = (N, \Delta_N^{\text{op}})$ and hence, $\hat{\delta}_1^{it} = \hat{J}_N \hat{\delta}_N^{-it} \hat{J}_N$ satisfying

$$\mathcal{Z}^*(\hat{\delta}_1^{it} \otimes 1)\mathcal{Z} = (\hat{J}_N \otimes J_N)W_N(\hat{\delta}_N^{-it} \otimes 1)W_N^*(\hat{J}_N \otimes J_N) = \hat{J}_N \hat{\delta}_N^{-it} \hat{J}_N \otimes \hat{\delta}_N^{-it} = \hat{\delta}_1^{it} \otimes \delta_2^{-it}.$$

So, we are done. $\square$

9 C*-algebraic features of bicrossed and double crossed products

Throughout this section, we fix two l.c. quantum groups (M_1, Δ_1) , (M_2, Δ_2) and a matching m on them. We have an associated bicrossed product (M, Δ) and double crossed product (M_m, Δ_m) . We continue to use the notations fixed in Notation 2.4. So, the C*-algebras associated to (M_1, Δ_1) , (M_2, Δ_2) , (M, Δ) and (M_m, Δ_m) are denoted by A_1, A_2, A and A_m , respectively.

We denote by $\mathcal{K}$ the C*-algebra of compact operators on $H_1 \otimes H_2$ and by $\mathcal{K}_i$ the C*-algebra of compact operators on H_i , $i = 1, 2$.

The following result is rather easy to prove. We give formulas for the C*-algebras $A, \hat{A}$ associated with the bicrossed product and for the C*-algebra $\hat{A}_m$ associated with the double crossed product. Observe that it is much more delicate to describe the C*-algebra A_m , see Proposition 9.5 where we give a formula in the case of regular (M_1, Δ_1) and (M_2, Δ_2) .

Proposition 9.1. *We have*

$$A = [(\hat{A}_1 \otimes 1)\alpha(A_2)] \quad \text{and} \quad \hat{A} = [(1 \otimes \hat{A}_2)\beta(A_1)] .$$

Further, we have

$$\hat{A}_m = [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes 1) Z^*(1 \otimes \hat{A}_2) Z] .$$

Proof. We have

$$\begin{aligned} A &= [(\iota \otimes \iota \otimes \omega \otimes \mu)((\alpha \otimes \iota \otimes \iota)(W_{2,13} U_{\beta,23} \hat{W}_{1,13}))] \\ &= [(\iota \otimes \iota \otimes \omega \otimes \mu)((\alpha \otimes \iota \otimes \iota)(U_{\beta,23} W_{2,13} U_{\beta,21}^* \hat{W}_{1,13}))] \\ &= [\alpha(A_2)(\iota \otimes \iota \otimes \omega)((\alpha \otimes \iota)(U_{\beta,21}^* \hat{W}_{1,13}))] = [\alpha(A_2)(\hat{A}_1 \otimes 1)] , \end{aligned}$$

where we used that $(\iota \otimes \Delta_2)(U_\beta) = U_{\beta,13} U_{\beta,12}$ and the fact that $U_\beta \in M(\mathcal{K}_1 \otimes A_2)$.

We know that A is a C^* -algebra. This proves the formula for A and the formula for $\hat{A}$ is proven analogously. The formula for $\hat{A}_m$ is an immediate corollary of the formula for W_m in Theorem 5.3. $\square$

Next, we present a quantum version of Theorem 3.11 in [3]: we give a necessary and sufficient condition for (M, Δ) to be (semi-)regular.

Proposition 9.2. *We have that (M, Δ) is semi-regular if and only if both (M_1, Δ_1) and (M_2, Δ_2) are semi-regular and $A_1 \otimes A_2 \subset A_m$.*

We have that (M, Δ) is regular if and only if both (M_1, Δ_1) and (M_2, Δ_2) are regular and one of the following conditions holds true: $A_1 \otimes A_2 = A_m$ or $m(A_1 \otimes A_2) = A_1 \otimes A_2$.

Proof. First, observe that

$$[\mathcal{C}(W)] = [(\iota \otimes \iota \otimes \omega \otimes \mu)(\Sigma_{13} \Sigma_{24} W_{2,24} U_{\beta,34} U_{\alpha,12}^* \hat{W}_{1,13})] .$$

Because $(\iota \otimes \Delta_2)(U_\beta) = U_{\beta,13} U_{\beta,12}$ and $(\Delta_1 \otimes \iota)(U_\alpha) = U_{\alpha,23} U_{\alpha,13}$, we get

$$[\mathcal{C}(W)] = U_\beta [(\iota \otimes \iota \otimes \omega \otimes \mu)((\Sigma W_2)_{24} (U_\beta^* U_\alpha)_{12} (\Sigma \hat{W}_1)_{13})] U_\alpha^* = U_\beta [(1 \otimes \mathcal{C}(W_2)) U_\beta^* U_\alpha (\mathcal{C}(\hat{W}_1) \otimes 1)] U_\alpha^* .$$

Suppose that (M, Δ) is semi-regular. From Proposition 5.7 in [3], it follows that semi-regularity passes to closed quantum subgroups. So, (M_1, Δ_1) and (M_2, Δ_2) are semi-regular. From the calculation above, we conclude that

$$\mathcal{K} \subset [(1 \otimes \mathcal{C}(W_2)) U_\beta^* U_\alpha (\mathcal{C}(\hat{W}_1) \otimes 1)]$$

and hence,

$$\mathcal{K} = [(1 \otimes \mathcal{K}_2) \mathcal{K} (\mathcal{K}_1 \otimes 1)] \subset [(1 \otimes \mathcal{K}_2) U_\beta^* U_\alpha (\mathcal{K}_1 \otimes 1)] .$$

Hence, $\mathcal{K} \subset [(1 \otimes \mathcal{K}_2) Z^* (\mathcal{K}_1 \otimes 1)]$.

On the other hand, we have

$$\begin{aligned} A_m &= [(\iota \otimes \iota \otimes \omega \otimes \mu)((\Sigma V_1^* \Sigma)_{13} Z_{34}^* W_{2,24})] = [(\iota \otimes \iota \otimes \omega \otimes \mu)((\Sigma V_1^* \Sigma)_{13} ((1 \otimes \mathcal{K}_2) Z^* (\mathcal{K}_1 \otimes 1))_{34} W_{2,24})] \\ &\supset [(\iota \otimes \iota \otimes \omega \otimes \mu)((\Sigma V_1^* \Sigma)_{13} (\mathcal{K}_1 \otimes \mathcal{K}_2)_{34} W_{2,24})] = A_1 \otimes A_2 . \end{aligned}$$

Conversely, suppose that (M_1, Δ_1) and (M_2, Δ_2) are both semi-regular and $A_1 \otimes A_2 \subset A_m$. It is easy to check that $A_1 \otimes A_2 = [A_m(A_1 \otimes 1)]$. Hence,

$$m(A_1 \otimes A_2) = [(R_1 \otimes R_2) R_m(A_m) \beta(A_1)] \supset [(A_1 \otimes A_2) \beta(A_1)] .$$

But, using Proposition 9.1, we have

$$[A \hat{A}] = [(\hat{A}_1 \otimes 1) m(A_1 \otimes A_2) (1 \otimes \hat{A}_2)] \supset [(\hat{A}_1 A_1 \otimes A_2) \beta(A_1) (1 \otimes \hat{A}_2)] \supset [(\mathcal{K}_1 \otimes A_2) \beta(A_1) (1 \otimes \hat{A}_2)] .$$

Because $[A \hat{A}]$ is a C^* -algebra, we get

$$[A \hat{A}] \supset \mathcal{K}_1 \otimes [A_2 (\omega \otimes \iota) \beta(A_1) \hat{A}_2] = \mathcal{K}_1 \otimes [A_2 \hat{A}_2] \supset \mathcal{K}_1 \otimes \mathcal{K}_2 ,$$

where we used that, for $x \in A_1$, $\beta(x) = U_\beta(x \otimes 1)U_\beta^*$ with $U_\beta \in M(\mathcal{K}_1 \otimes A_2)$. Hence, (M, Δ) is semi-regular. This concludes the proof of the first statement of the proposition.

In order to prove the second statement, we can replace above every inclusion by an equality to obtain the equivalence between regularity of (M, Δ) and regularity of (M_1, Δ_1) , (M_2, Δ_2) and the equality $A_m = A_1 \otimes A_2$. The only different point is to prove that the regularity of (M, Δ) implies the regularity of (M_1, Δ_1) and (M_2, Δ_2) . But, suppose that (M, Δ) is regular. Then, $(\alpha \otimes \iota)(W_1)$ is a corepresentation of (M, Δ) and satisfies as such

$$[(\mathcal{K}_1 \otimes \mathcal{K}_2 \otimes 1)(\alpha \otimes \iota)(W_1)(1 \otimes 1 \otimes \mathcal{K}_1)] = \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathcal{K}_1 .$$

Because $\alpha(x) = U_\alpha(1 \otimes x)U_\alpha^*$, it follows immediately that (M_1, Δ_1) is regular. We get the regularity of (M_2, Δ_2) in an analogous way.

Finally, we suppose that (M_1, Δ_1) and (M_2, Δ_2) are regular and $m(A_1 \otimes A_2) = A_1 \otimes A_2$. Using Proposition 9.1, we immediately have

$$[A \hat{A}] = [(\hat{A}_1 \otimes 1)m(A_1 \otimes A_2)(1 \otimes \hat{A}_2)] = [\hat{A}_1 A_1 \otimes A_2 \hat{A}_2] = \mathcal{K}_1 \otimes \mathcal{K}_2 ,$$

yielding the regularity of (M, Δ) . $\square$

The (semi-)regularity of the double crossed product (M_m, Δ_m) is characterized more easily in the following proposition.

Proposition 9.3. *We have that (M_m, Δ_m) is semi-regular (resp., regular) if and only if both (M_1, Δ_1) and (M_2, Δ_2) are semi-regular (resp., regular).*

Proof. First observe that

$$[\mathcal{C}(W_m)] = [(\iota \otimes \iota \otimes \omega \otimes \mu)((\Sigma V_1)_{13}^* Z_{32}^* (\Sigma W_2)_{24})] = [(\iota \otimes \iota \otimes \omega)((\Sigma V_1)_{13}^* Z_{32}^* (1 \otimes \mathcal{C}(W_2)))] .$$

Suppose that (M_1, Δ_1) and (M_2, Δ_2) are semi-regular. Then, it follows that

$$[\mathcal{C}(W_m)] \supset [(\iota \otimes \iota \otimes \omega)((\Sigma V_1)_{13}^* Z_{32}^* (1 \otimes \mathcal{K}_2))] = [(\iota \otimes \iota \otimes \omega)((\Sigma V_1)_{13}^*) (1 \otimes \mathcal{K}_2)] \supset \mathcal{K}_1 \otimes \mathcal{K}_2 .$$

Hence, (M_m, Δ_m) is semi-regular.

Suppose, conversely, that (M_m, Δ_m) is semi-regular. Because $(\hat{M}'_1, \hat{\Delta}'_1)$ and $(\hat{M}'_2, \hat{\Delta}'_2)$ are closed quantum subgroups of $(\hat{M}_m, \hat{\Delta}_m)$, we get that they are semi-regular, using Proposition 5.7 in [3]. Hence, the same holds for (M_1, Δ_1) and (M_2, Δ_2) .

To prove the statement about regularity, we proceed exactly as in the final paragraph of the proof of the previous proposition. $\square$

Finally, we want to give some more detailed information about the C^* -algebras A_m and A , as well as the crossed product $A \rtimes_r A$ of the right action of (A, Δ) on itself. Our results will be quantum versions of the following results in [3]: Proposition 3.6, Lemma 3.9 and Proposition 3.10.

To obtain our results, we need actions of C^* -algebraic quantum groups on C^* -algebras. As shown in Section 5 of [3], the notion of continuity of such an action is problematic when the acting quantum group is non-regular. So, for the rest of this section, *we suppose that (M_1, Δ_1) and (M_2, Δ_2) are regular.*

Proposition 9.4. *Defining*

$$T_1 = [(\iota \otimes \mu)\beta(A_1) \mid \mu \in B(H_2)_*] \quad \text{and} \quad T_2 = [(\omega \otimes \iota)\alpha(A_2) \mid \omega \in B(H_1)_*] ,$$

we have that T_1 and T_2 are C^ -algebras, such that the restrictions of α and β :*

$$\alpha : T_2 \rightarrow M(A_1 \otimes T_2) , \quad \beta : T_1 \rightarrow M(T_1 \otimes A_2)$$

define C^* -algebraic actions of (A_1, Δ_1) on the left on T_2 and of $(A_2, \Delta_2^{\text{op}})$ on the right on T_1 , which are continuous in the strong sense:

$$[\alpha(T_2)(A_1 \otimes 1)] = A_1 \otimes T_2 \quad \text{and} \quad [\beta(T_1)(1 \otimes A_2)] = T_1 \otimes A_2 .$$

Moreover, we have

$$A = A_1 \rtimes T_2 \quad \text{and} \quad \hat{A} = T_1 \rtimes A_2 .$$

Proof. As in the proof of Proposition 5.7 in [3], we conclude that T_1 and T_2 are C^* -algebras. Further, we observe that

$$\begin{aligned} [\alpha(T_2)(A_1 \otimes 1)] &= [(\omega \otimes \iota \otimes \iota)(V_{1,12}\alpha(A_2)_{13}V_{1,12}^*(1 \otimes A_1 \otimes 1))] \\ &= [(\omega \otimes \iota \otimes \iota)((\mathcal{K}_1 \otimes 1)V_1(1 \otimes A_1))_{12}\alpha(A_2)_{13}] = A_1 \otimes T_2 , \end{aligned}$$

where we used that $V_1 \in M(\mathcal{K}_1 \otimes A_1)$ and $[(\mathcal{K}_1 \otimes 1)V_1(1 \otimes A_1)] = \mathcal{K}_1 \otimes A_1$ by regularity of (M_1, Δ_1) . It follows that $\alpha : T_2 \rightarrow M(A_1 \otimes T_2)$ is non-degenerate and that the restriction of α to T_2 is continuous in the strong sense. We analogously show the same kind of properties for T_1 and β .

By Definition 7.1 in [2], we have $A_1 \rtimes T_2 = [(\hat{A}_1 \otimes 1)\alpha(T_2)]$ and this a C^* -algebra. So,

$$\begin{aligned} A_1 \rtimes T_2 &= [(\hat{A}_1 \otimes 1)\alpha(T_2)(\hat{A}_1 \otimes 1)] = [(\omega \otimes \iota \otimes \iota)((1 \otimes \hat{A}_1 \otimes 1)W_{1,12}^*\alpha(A_2)_{23}W_{1,12}(1 \otimes \hat{A}_1 \otimes 1))] \\ &= [(\hat{A}_1 \otimes 1)\alpha(A_2)(\hat{A}_1 \otimes 1)] , \end{aligned}$$

because $W_1 \in M(\mathcal{K}_1 \otimes \hat{A}_1)$. But, from Proposition 9.1, we know that $A = [(\hat{A}_1 \otimes 1)\alpha(A_2)]$ and we know that A is a C^* -algebra. Hence, we are done. $\square$

Recall that comparing our work with [3], the double crossed product (M_m, Δ_m) should be considered as the ‘big’ group G in [3]. Hence, the following result is really the quantum version of [3], Lemma 3.9 and Proposition 3.10.

Proposition 9.5. *We have*

$$A_m = [(\iota \otimes \iota \otimes \omega)(m_{13}(\iota \otimes \Delta_2^{\text{op}})(A_1 \otimes A_2)) \mid \omega \in B(H_2)_*] .$$

Further,

$$\Delta_1^{\text{op}} \otimes \Delta_2 : A_m \rightarrow M(A_1 \otimes A_m \otimes A_2)$$

defines a C^* -algebraic action of $(A_1, \Delta_1^{\text{op}})$ on the left and of (A_2, Δ_2) on the right on A_m . This action is continuous in the strong sense:

$$[(A_1 \otimes 1 \otimes 1 \otimes 1)(\Delta_1^{\text{op}} \otimes \Delta_2)(A_m)(1 \otimes 1 \otimes 1 \otimes A_2)] = A_1 \otimes A_m \otimes A_2 .$$

Moreover, the crossed product satisfies:

$$A \rtimes_r A \cong A_1 \rtimes A_m \rtimes_r A_2 .$$

Proof. One checks easily that $A_1 \otimes A_2 = [A_m(A_1 \otimes 1)]$. Hence, it follows that

$$\begin{aligned} [(\iota \otimes \iota \otimes \omega)(m_{13}(\iota \otimes \Delta_2^{\text{op}})(A_1 \otimes A_2))] &= [(\iota \otimes \iota \otimes \omega)(m_{13}(\iota \otimes \Delta_2^{\text{op}})(A_m) m(A_1 \otimes 1)_{13})] \\ &= [(\iota \otimes \iota \otimes \omega)(m_{12}^{-1}((1 \otimes 1 \otimes A_2)(\iota \otimes \Delta_2^{\text{op}})m(A_m)) m(A_1 \otimes 1)_{13})] . \end{aligned}$$

From Proposition 3.5, we get that

$$[(1 \otimes 1 \otimes A_2)(\iota \otimes \Delta_2^{\text{op}})m(A_m)] = [(\iota \otimes \iota \otimes \iota \otimes \omega \otimes \mu)((A_2 \otimes 1)W_2)_{35} W_{2,25} Z_{45} (\Sigma V_1^* \Sigma)_{14}] = m(A_m) \otimes A_2 ,$$

because $W_2 \in M(A_2 \otimes \mathcal{K}_2)$. Hence, we conclude that

$$[(\iota \otimes \iota \otimes \omega)(m_{13}(\iota \otimes \Delta_2^{\text{op}})(A_1 \otimes A_2))] = [A_m(T_1 \otimes 1)] . \quad (9.1)$$

Now, we use the second formula for V_m in Theorem 5.3 and we find that

$$[(T_1 \otimes 1)A_m] = [(\omega \otimes \iota \otimes \iota)((T_1 \otimes \mathcal{K}_2)\beta(A_1))_{21} U_{\alpha,21} V_{2,13}]. \quad (9.2)$$

Next, we claim that $[(T_1 \otimes \mathcal{K}_2)\beta(A_1)] = [(1 \otimes \mathcal{K}_2)\beta(A_1)]$. Because β is continuous in the strong sense on T_1 , we have $[(1 \otimes \mathcal{K}_2)\beta(T_1)] = T_1 \otimes \mathcal{K}_2$. To prove our claim, it suffices to show that $[T_1 A_1] = A_1$. This last statement can be proven as follows: from Theorem 5.3, we know that $(\iota \otimes m)(U_{\beta,13} V_{1,12}) = V_{1,12} U_{\beta,13}$, which gives

$$T_1 = [(\omega \otimes \iota \otimes \mu)((\iota \otimes m)(U_{\beta,13}^*) V_{1,12})]$$

and so, because $V_1 \in M(\mathcal{K}_1 \otimes A_1)$ and $U_\alpha \in M(A_1 \otimes \mathcal{K}_2)$,

$$[T_1 A_1] = [(\omega \otimes \iota \otimes \mu)(\iota \otimes \alpha)(U_\beta^* A_1)] = [(\omega \otimes \iota \otimes \mu)(U_{\alpha,23} U_{\beta,13}^* U_{\alpha,23}^* A_1)] = A_1.$$

Hence, our claim is proven. Combining this with Equation (9.2), we get

$$[(T_1 \otimes 1)A_m] = [(\omega \otimes \iota \otimes \iota)((1 \otimes \mathcal{K}_2)\beta(A_1))_{21} U_{\alpha,21} V_{2,13}] = A_m,$$

where we used again the second formula for V_m in Theorem 5.3. Combining this equation with Equation (9.1), we finally arrive at

$$[(\iota \otimes \iota \otimes \omega)(m_{13}(\iota \otimes \Delta_2^{\text{op}})(A_1 \otimes A_2))] = A_m.$$

It is easy to check that $\Delta_1^{\text{op}} \otimes \Delta_2$ defines a C^* -algebraic action of $(A_1, \Delta_1^{\text{op}})$ on the left and (A_2, Δ_2) on the right on A_m , which is continuous in the strong sense.

Because $R_m(A_m) = A_m$, we also get

$$A_m = [(\iota \otimes \iota \otimes \omega)(\iota \otimes \Delta_2)m^{-1}(A_1 \otimes A_2)].$$

Now, by Definition 7.1 in [2], $A \rtimes_r A = [A \hat{J} \hat{A} \hat{J}]$. Because $\hat{J} A \hat{J} = A$, we get, using $\text{Ad}((\hat{J}_1 \otimes \hat{J}_2)\hat{J})$,

$$A \rtimes_r A \cong (\hat{J}_1 \otimes \hat{J}_2)[A \hat{A}](\hat{J}_1 \otimes \hat{J}_2).$$

On the other hand, by definition,

$$A_1 \rtimes A_m \rtimes_r A_2 = [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes 1 \otimes 1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2)(\Delta_1^{\text{op}} \otimes \Delta_2)(A_m)],$$

which, by using the unitary $\hat{W}_1^* \otimes W_2$, is isomorphic with $[(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2)A_m]$.

Now, we observe that

$$(\hat{J}_1 \otimes \hat{J}_2)[A \hat{A}](\hat{J}_1 \otimes \hat{J}_2) = [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes 1) m^{-1}(A_1 \otimes A_2) (1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2)].$$

Because the left hand side is a C^* -algebra, we have

$$\begin{aligned} (\hat{J}_1 \otimes \hat{J}_2)[A \hat{A}](\hat{J}_1 \otimes \hat{J}_2) &= [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2) m^{-1}(A_1 \otimes A_2) (1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2)] \\ &= [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2) (\iota \otimes \iota \otimes \omega)(m^{-1}(A_1 \otimes A_2)_{12} V_{2,23}^*)] \\ &= [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2) (\iota \otimes \iota \otimes \omega)(V_{2,23}^* (\iota \otimes \Delta_2)m^{-1}(A_1 \otimes A_2))] \\ &= [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2) (\iota \otimes \iota \otimes \omega)(\iota \otimes \Delta_2)m^{-1}(A_1 \otimes A_2)] \\ &= [(\hat{J}_1 \hat{A}_1 \hat{J}_1 \otimes \hat{J}_2 \hat{A}_2 \hat{J}_2) A_m], \end{aligned}$$

where we used twice that $V_2 \in M(\hat{J}_2 \hat{A}_2 \hat{J}_2 \otimes \mathcal{K}_2)$. We have seen above that this last subspace is isomorphic with $A_1 \rtimes A_m \rtimes_r A_2$. So, we are done. $\square$

10 Appendix: the Radon-Nikodym derivative of a weight under a quantum group action

In this appendix, we generalize results of T. Yamanouchi [21, 22] and we give considerably simpler proofs.

Let N be a von Neumann algebra with n.s.f. weight θ . If a l.c. group G acts continuously by automorphisms (α_g) on N , we can consider the Connes cocycles (Radon-Nikodym derivatives) $[D\theta \circ \alpha_g : D\theta]_t$ as a function $\mathbb{R} \rightarrow L^\infty(G) \otimes N$. In the case of an action of a l.c. quantum group (M, Δ) on N , Yamanouchi [21] defined the analogous Radon-Nikodym derivative $[D\theta \circ \alpha : D\theta]_t \in M \otimes N$. We repeat his definition.

We first introduce some needed preliminaries.

Fix an action $\alpha : N \rightarrow M \otimes N$ of the l.c. quantum group (M, Δ) on the von Neumann algebra N and fix an n.s.f. weight θ on N . Denote by $\tilde{J}_\theta$ and $\tilde{\nabla}_\theta$ the modular operators of the dual weight $\tilde{\theta}$, acting on the Hilbert space $H \otimes H_\theta$, with respect to the canonical GNS-construction for $\tilde{\theta}$ explained in Section 2.3.

On the crossed product $M_{\alpha \ltimes} N$, we have a dual action of $(\hat{M}, \hat{\Delta}^{\text{op}})$ and the second crossed product $\hat{M}_{\hat{\alpha} \ltimes} (M_{\alpha \ltimes} N)$ is canonically isomorphic with $B(H) \otimes N$ through the isomorphism $\Phi : B(H) \otimes N \rightarrow \hat{M}_{\hat{\alpha} \ltimes} (M_{\alpha \ltimes} N)$, see [13], Theorem 2.6. So, we can consider the bidual weight $\tilde{\tilde{\theta}}$ on $B(H) \otimes N$, with canonical GNS-map

$$(xy \otimes 1)\alpha(z) \mapsto \Lambda'(x) \otimes \hat{\Lambda}(y) \otimes \Lambda_\theta(z) \quad \text{for } x \in \mathcal{N}_{\varphi'}, y \in \mathcal{N}_{\tilde{\varphi}}, z \in \mathcal{N}_\theta$$

and where $\Lambda'(x) = J\Lambda(JxJ)$ is the canonical GNS-map for the left invariant weight on the commutant quantum group (M', Δ') , which is the dual of $(\hat{M}, \hat{\Delta}^{\text{op}})$. In the same sense as above, these elements $(xy \otimes 1)\alpha(z)$ span a core for the GNS-map of $\tilde{\tilde{\theta}}$.

From the results of [16], Section 2, it then follows that

$$\tilde{\tilde{\theta}}(\alpha(a^*)(x^*y \otimes 1)\alpha(b)) = \text{Tr}_{\tilde{\nabla}}(x^*y)\theta(a^*b)$$

for all $x, y \in \mathcal{N}_{\tilde{\nabla}}$, $a, b \in \mathcal{N}_\theta$, where we use the short-hand notation $\mathcal{N}_{\tilde{\nabla}} := \mathcal{N}_{\text{Tr}_{\tilde{\nabla}}}$ and where $\text{Tr}_{\tilde{\nabla}}$ is the n.s.f. weight on $B(H)$ with density $\tilde{\nabla}$. Also, the elements $(y \otimes 1)\alpha(b)$ span a core for the GNS-map of $\tilde{\tilde{\theta}}$.

On $B(H) \otimes N$, we have two natural weights: $\tilde{\tilde{\theta}}$ and $\text{Tr}_{\tilde{\nabla}} \otimes \theta$. The following result was first proven by Yamanouchi in [21] and called the Takesaki duality theorem for weights. We indicate how it follows from [13].

Lemma 10.1. *The following formula holds:*

$$[D\tilde{\tilde{\theta}} : D(\text{Tr}_{\tilde{\nabla}} \otimes \theta)]_t = \tilde{\nabla}_\theta^{it}(\hat{\nabla}^{-it} \otimes \nabla_\theta^{-it}).$$

Proof. Writing $T_{\hat{\alpha}}$ for the operator valued weight from $B(H) \otimes N$ to $M_{\alpha \ltimes} N$ associated with the integrable action $\hat{\alpha}$, we know from [13], Proposition 5.7 that $T_{\hat{\alpha}}(z) = \tilde{J}_\theta T_{\hat{\alpha}}^{-1}(\tilde{J}_\theta z \tilde{J}_\theta) \tilde{J}_\theta$. Here, $T_{\hat{\alpha}}$ is the operator valued weight from $M_{\alpha \ltimes} N$ to $\alpha(N)$ and $T_{\hat{\alpha}}^{-1}$ is the commutant operator valued weight from $\alpha(N)'$ to $(M_{\alpha \ltimes} N)'$. Using the language of spatial derivatives, this means that

$$\left(\frac{d\tilde{\tilde{\theta}}}{d(1 \otimes \theta(J_\theta \cdot J_\theta))} \right)^{it} = \left(\frac{d(\tilde{\tilde{\theta}}(\tilde{J}_\theta \cdot \tilde{J}_\theta) \circ T_{\hat{\alpha}}^{-1})}{d(\theta \circ \alpha^{-1})} \right)^{-it} = \left(\frac{d(\tilde{\tilde{\theta}}(\tilde{J}_\theta \cdot \tilde{J}_\theta))}{d(\theta \circ \alpha^{-1} \circ T_{\hat{\alpha}})} \right)^{-it} = \left(\frac{d(\tilde{\tilde{\theta}}(\tilde{J}_\theta \cdot \tilde{J}_\theta))}{d\tilde{\tilde{\theta}}} \right)^{-it} = \tilde{\nabla}_\theta^{it}.$$

Then, we get that

$$[D\tilde{\tilde{\theta}} : D(\text{Tr}_{\tilde{\nabla}} \otimes \theta)]_t = \left(\frac{d\tilde{\tilde{\theta}}}{d(1 \otimes \theta(J_\theta \cdot J_\theta))} \right)^{it} \left(\frac{d(\text{Tr}_{\tilde{\nabla}} \otimes \theta)}{d(1 \otimes \theta(J_\theta \cdot J_\theta))} \right)^{-it} = \tilde{\nabla}_\theta^{it}(\hat{\nabla}^{-it} \otimes \nabla_\theta^{-it}).$$

This concludes the proof. $\square$

Definition 10.2. Define $[D\theta \circ \alpha : D\theta]_t := \tilde{\nabla}_\theta^{it}(\hat{\nabla}^{-it} \otimes \nabla_\theta^{-it})$. We call $[D\theta \circ \alpha : D\theta]_t$ the Radon-Nikodym derivative of θ under the action α of (M, Δ) . $\blacktriangle$

We now prove the main properties of $[D\theta \circ \alpha : D\theta]_t$.

Theorem 10.3. *Write $D_t := [D\theta \circ \alpha : D\theta]_t$.*

- $D_t \in M \otimes N$.
- $(\Delta \otimes \iota)(D_t) = (\iota \otimes \alpha)(D_t)(1 \otimes D_t)$.

Proof. Denote by $(\tilde{\sigma})_t$ the modular automorphism group of the bidual weight $\tilde{\theta}$ on $B(H) \otimes N$. Because the dual weight $\tilde{\theta}$ is $\hat{\delta}$ -invariant (see [13], Proposition 2.5), we conclude from the proof of [13], Proposition 4.3, that $\tilde{\sigma}_t(x \otimes 1) = \hat{\delta}^{-it} \nabla^{-it} x \nabla^{it} \hat{\delta}^{it} \otimes 1$ for all $x \in M'$. From [16], Proposition 2.4, we know that $\hat{\delta}^{-it} \nabla^{-it} = \hat{\nabla}^{it} \delta^{it}$ and hence, $\tilde{\sigma}_t(x \otimes 1) = \hat{\nabla}^{it} x \hat{\nabla}^{-it} \otimes 1$ for all $x \in M'$. We easily conclude that $D_t \in M \otimes N$.

Next, we introduce the notation $\theta^\alpha := \tilde{\theta}$ to stress that we take a bidual weight with respect to the action α . Consider the action $\beta := (\sigma \otimes \iota)(\iota \otimes \alpha)$ of (M, Δ) on $B(H) \otimes N$. Because θ^α is an n.s.f. weight on $B(H) \otimes N$, we can define its bidual weight $(\theta^\alpha)^\beta$ on $B(H \otimes H) \otimes N$. From the discussion above, we know a canonical GNS-map for $(\theta^\alpha)^\beta$ given by

$$(y \otimes z \otimes 1)\beta(\alpha(x)) \mapsto \Lambda_{\hat{\nabla}}(y) \otimes \Lambda_{\hat{\nabla}}(z) \otimes \Lambda_\theta(x) \quad \text{for all } y, z \in \mathcal{N}_{\hat{\nabla}}, x \in \mathcal{N}_\theta,$$

where $\Lambda_{\hat{\nabla}}$ is a GNS-map for $\text{Tr}_{\hat{\nabla}}$. We also know that the elements $(y \otimes z \otimes 1)\beta(\alpha(x))$ span a core for the corresponding GNS-map. Observe that $(\Sigma V^* \Sigma)(\hat{\nabla} \otimes \hat{\nabla})(\Sigma V \Sigma) = Q \otimes \hat{\nabla}$, where Q is the closure of $\delta \hat{\nabla}$. Also observe that $\beta(\alpha(x)) = (\Delta^{\text{op}} \otimes \iota)\alpha(x)$ for all $x \in N$. Using the above core for $(\theta^\alpha)^\beta$, we conclude that

$$(\theta^\alpha)^\beta \circ \text{Ad}(\Sigma V \Sigma \otimes 1) = \text{Tr}_Q \otimes \theta^\alpha \quad \text{and} \quad (\text{Tr}_{\hat{\nabla}} \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta) \circ \text{Ad}(\Sigma V \Sigma \otimes 1) = \text{Tr}_Q \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta.$$

So, on the one hand, we have,

$$[D((\theta^\alpha)^\beta) : D(\text{Tr}_{\hat{\nabla}} \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta)]_t = (\Sigma V \Sigma \otimes 1)[D(\text{Tr}_Q \otimes \theta^\alpha) : D(\text{Tr}_Q \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta)]_t (\Sigma V^* \Sigma \otimes 1) = (\Delta^{\text{op}} \otimes \iota)(D_t).$$

On the other hand, we have

$$\begin{aligned} [D((\theta^\alpha)^\beta) : D(\text{Tr}_{\hat{\nabla}} \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta)]_t &= [D((\theta^\alpha)^\beta) : D((\text{Tr}_{\hat{\nabla}} \otimes \theta)^\beta)]_t [D((\text{Tr}_{\hat{\nabla}} \otimes \theta)^\beta) : D(\text{Tr}_{\hat{\nabla}} \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta)]_t \\ &= \beta([D\theta^\alpha : D(\text{Tr}_{\hat{\nabla}} \otimes \theta)]_t) [D((\text{Tr}_{\hat{\nabla}} \otimes \theta)^\beta) : D(\text{Tr}_{\hat{\nabla}} \otimes \text{Tr}_{\hat{\nabla}} \otimes \theta)]_t \\ &= (\sigma \otimes \iota)((\iota \otimes \alpha)(D_t)(1 \otimes D_t)), \end{aligned} \tag{10.1}$$

where we used the following well known fact from the theory of operator valued weights: $[D\tilde{\theta} : D\tilde{\mu}]_t = \beta([D\theta : D\mu]_t)$ where β is an action of a l.c. quantum group and θ, μ are n.s.f. weights with dual weights $\tilde{\theta}, \tilde{\mu}$. Combining both calculations above, the proof is finished. $\square$

The proof of the previous theorem can be adapted to prove the following useful and seemingly obvious formula.

Lemma 10.4. *Suppose that N is a von Neumann algebra with n.s.f. weight θ and let α and β be actions of (M_1, Δ_1) , resp. (M_2, Δ_2) on N . Suppose that there exists a $*$ -isomorphism $m : M_1 \otimes M_2 \rightarrow M_1 \otimes M_2$ ruling, in a sense, the commutation between α and β . More precisely,*

$$(\iota \otimes \beta)\alpha = (m\sigma \otimes \iota)(\iota \otimes \alpha)\beta.$$

If $m(\tau_t^1 \otimes \tau_t^2) = (\tau_t^1 \otimes \tau_t^2)m$, then

$$(\iota \otimes \beta)([D\theta \circ \alpha : D\theta]_t) (1 \otimes [D\theta \circ \beta : D\theta]_t) = (m\sigma \otimes \iota)((\iota \otimes \alpha)([D\theta \circ \beta : D\theta]_t) (1 \otimes [D\theta \circ \alpha : D\theta]_t)).$$

Proof. Consider the amplified action $\beta_a := (\sigma \otimes \iota)(\iota \otimes \beta)$ of (M_2, Δ_2) on $B(H_1) \otimes N$. First, we have the bidual weight θ^α on $B(H_1) \otimes N$, which has again a bidual weight $(\theta^\alpha)^{\beta_a}$ on $B(H_2 \otimes H_1) \otimes N$. The same calculation as in Equation (10.1) yields that

$$[D(\theta^\alpha)^{\beta_a} : D(\text{Tr}_{\hat{\nabla}_2} \otimes \text{Tr}_{\hat{\nabla}_1} \otimes \theta)]_t = (\sigma \otimes \iota) \left((\iota \otimes \beta) ([D\theta \circ \alpha : D\theta]_t) (1 \otimes [D\theta \circ \beta : D\theta]_t) \right). \quad (10.2)$$

By symmetry, we have an amplified action α_a and

$$[D(\theta^\beta)^{\alpha_a} : D(\text{Tr}_{\hat{\nabla}_1} \otimes \text{Tr}_{\hat{\nabla}_2} \otimes \theta)]_t = (\sigma \otimes \iota) \left((\iota \otimes \alpha) ([D\theta \circ \beta : D\theta]_t) (1 \otimes [D\theta \circ \alpha : D\theta]_t) \right). \quad (10.3)$$

Consider the unbounded operator $K := J_1 \delta_1 J_1 \otimes J_2 \delta_2 J_2$, affiliated with $M'_1 \otimes M'_2$. Then, $K \otimes 1$ is invariant under the modular automorphism group of $\text{Tr}_{\hat{\nabla}_1} \otimes \text{Tr}_{\hat{\nabla}_2} \otimes \theta$ and it commutes with

$$[D(\theta^\beta)^{\alpha_a} : D(\text{Tr}_{\hat{\nabla}_1} \otimes \text{Tr}_{\hat{\nabla}_2} \otimes \theta)]_t \in M_1 \otimes M_2 \otimes N.$$

Hence, $K \otimes 1$ is invariant under the modular automorphism group of $(\theta^\beta)^{\alpha_a}$. So, we can define the n.s.f. weight $\mu := ((\theta^\beta)^{\alpha_a})_{K \otimes 1}$. We know a GNS-map for $(\theta^\beta)^{\alpha_a}$ and because the closure of $(\hat{\nabla}_1 \otimes \hat{\nabla}_2)K$ is $P_1 \otimes P_2$, we have a canonical GNS-map for μ given by

$$(z \otimes 1) \alpha_a(\beta(x)) \mapsto \Lambda_{P_1 \otimes P_2}(z) \otimes \Lambda_\theta(x) \quad \text{for all } z \in \mathcal{N}_{P_1 \otimes P_2}, x \in \mathcal{N}_\theta.$$

The elements $(z \otimes 1) \alpha_a(\beta(x))$ span a core for this GNS-map. Denote by $\mathcal{Z}$ the canonical implementation of the automorphism m , which is a unitary on $H_1 \otimes H_2$. Because m commutes with $\tau_t^1 \otimes \tau_t^2$ and because $P_1^{it} \otimes P_2^{it}$ is the canonical implementation of $\tau_t^1 \otimes \tau_t^2$, we get that $\mathcal{Z}$ commutes with $P_1 \otimes P_2$. Using the core for the GNS-map of $(\theta^\beta)^{\alpha_a}$ and using the given formula

$$(\iota \otimes \beta) \alpha = (m \sigma \otimes \iota)(\iota \otimes \alpha) \beta,$$

we conclude that

$$\mu = \rho \circ \text{Ad}(\Sigma \mathcal{Z} \otimes 1),$$

where $\rho := ((\theta^\alpha)^{\beta_a})_{\Sigma K \Sigma \otimes 1}$.

Finally, it follows from Equation (10.3) that

$$[D\mu : D(\text{Tr}_{P_1 \otimes P_2} \otimes \theta)]_t = (\sigma \otimes \iota) \left((\iota \otimes \alpha) ([D\theta \circ \beta : D\theta]_t) (1 \otimes [D\theta \circ \alpha : D\theta]_t) \right),$$

while it follows from Equation (10.2) that

$$[D\rho : D(\text{Tr}_{P_2 \otimes P_1} \otimes \theta)]_t = (\sigma \otimes \iota) \left((\iota \otimes \beta) ([D\theta \circ \alpha : D\theta]_t) (1 \otimes [D\theta \circ \beta : D\theta]_t) \right).$$

Applying $\text{Ad}(\Sigma \mathcal{Z} \otimes 1)$ and using that $\mu = \rho \circ \text{Ad}(\Sigma \mathcal{Z} \otimes 1)$ and that $\mathcal{Z}$ commutes with $P_1 \otimes P_2$, we arrive at the statement of the lemma. $\square$

As a corollary, we prove the following result on closed quantum subgroups.

Proposition 10.5. *Let (M, Δ) be a l.c. quantum group. Suppose that $N \subset M$ is a von Neumann subalgebra such that $\Delta(N) \subset N \otimes N$.*

Then, $(N, \Delta|_N)$ is a l.c. quantum group if and only if $R(N) = N$ and $\tau_t(N) = N$ for all $t \in \mathbb{R}$.

Proof. If $(N, \Delta|_N)$ is a l.c. quantum group, the inclusion map is a morphism and we know that $R(N) = N$ and $\tau_t(N) = N$ for all $t \in \mathbb{R}$ by Proposition 5.45 in [8]. So, we only have to prove the converse implication.

Define $\alpha := \Delta|_N$, which we consider as a left action of (M, Δ) on N and $\beta := \Delta^{\text{op}}|_N$, which we consider as a left action of (M, Δ^{op}) on N . Choose an n.s.f. weight θ on N and define

$$u_t := [D\theta \circ \alpha : D\theta]_t \in M \otimes N, \quad w_t := [D\theta \circ \beta : D\theta]_t \in M \otimes N.$$

We will show that there exists a (σ_t^θ) -cocycle (v_t) in N such that $u_t = \alpha(v_t^*)(1 \otimes v_t)$ for all $t \in \mathbb{R}$. If we then correct the weight θ with the cocycle (v_t) , we find a left invariant weight on $(N, \Delta|_N)$.

Because $\alpha(N) \subset N \otimes N$, we can use the argument of the proof of Proposition 3.12 in [13] to conclude that $u_t \in N \otimes N$ and analogously, $w_t \in N \otimes N$ for all $t \in \mathbb{R}$. Remark that at this point, we use in a crucial way that R and τ_t leave N globally invariant.

By Theorem 10.3, we know that $(\Delta \otimes \iota)(u_t) = (\iota \otimes \Delta)(u_t)(1 \otimes u_t)$, which we can rewrite as

$$(\Delta \otimes \iota)(\tilde{u}_t W) = (\tilde{u}_t W)_{13}(\tilde{u}_t W)_{23} \quad \text{with} \quad \tilde{u}_t = (\hat{J} \otimes J)u_t^*(\hat{J} \otimes J) .$$

Because $\tilde{u}_t \in M \otimes M'$, the corepresentation $\tilde{u}_t W$ is covariant with respect to the standard representation of M on H in the sense that $\Delta(x) = (\tilde{u}_t W)^*(1 \otimes x)(\tilde{u}_t W)$ for all $x \in M$. As we remarked after Definition 2.4 in [3], this gives the existence of faithful, normal $*$ -homomorphisms $\hat{\pi}_t : \hat{M} \rightarrow B(H)$ satisfying $(\iota \otimes \hat{\pi}_t)(W) = \tilde{u}_t W$.

On the other hand, we have analogously,

$$(\Delta^{\text{op}} \otimes \iota)(\tilde{w}_t \Sigma V^* \Sigma) = (\tilde{w}_t \Sigma V^* \Sigma)_{13}(\tilde{w}_t \Sigma V^* \Sigma)_{23} \quad \text{where} \quad \tilde{w}_t = (\hat{J} \otimes J)w_t^*(\hat{J} \otimes J) .$$

So, $\tilde{w}_t \Sigma V^* \Sigma$ is a corepresentation of (M, Δ^{op}) which is covariant with respect to the standard representation of $(M, \Delta^{\text{op}})^\wedge = \hat{M}'$. For the same reason as above, we find faithful, normal $*$ -homomorphisms $\hat{\pi}'_t : \hat{M}' \rightarrow B(H)$ satisfying $(\iota \otimes \hat{\pi}'_t)(\Sigma V^* \Sigma) = \tilde{w}_t \Sigma V^* \Sigma$.

Applying Lemma 10.4 to the actions α and β (with $m = \iota$), we find that

$$(\iota \otimes \Delta)(w_t)(1 \otimes u_t) = (\sigma \otimes \iota)((\iota \otimes \Delta^{\text{op}})(u_t)(1 \otimes w_t)) ,$$

which can be rewritten as

$$(\Sigma V^* \Sigma w_t)_{13} \quad \text{and} \quad (W u_t)_{23} \quad \text{commute} .$$

This means that the ranges of $\hat{\pi}_t$ and $\hat{\pi}'_t$ commute.

Write $\pi(x) = x$ for $x \in M$. Define $\mathcal{U}_t := (\hat{\pi}'_t \otimes \iota)(V^*) (\pi \otimes \iota)(W^*) (\hat{\pi}_t \otimes \iota)(\tilde{V})$ with $\tilde{V} = (J \otimes J)\hat{W}^*(J \otimes J)$. Checking it on a slice of W , we get that $(\hat{\pi}_t \otimes \iota)\hat{\Delta}^{\text{op}}(x) = (\pi \otimes \iota)(W)(\hat{\pi}_t(x) \otimes 1)(\pi \otimes \iota)(W^*)$ for all $x \in \hat{M}$. So,

$$(\pi \otimes \iota)(W^*) (\hat{\pi}_t \otimes \iota)(\tilde{V}) (1 \otimes a) = (\hat{\pi}_t(a) \otimes 1)(\pi \otimes \iota)(W^*) (\hat{\pi}_t \otimes \iota)(\tilde{V}) \quad \text{for all} \quad a \in \hat{M} .$$

So, we arrive at

$$\mathcal{U}_t(1 \otimes a)\mathcal{U}_t^* = \hat{\pi}_t(a) \otimes 1 \quad \text{for} \quad a \in \hat{M} , \quad \mathcal{U}_t(1 \otimes b)\mathcal{U}_t^* = \pi(b) \otimes 1 \quad \text{for} \quad b \in M ,$$

where the second formula is checked in an analogous way as the first one. Because $\hat{M}M$ is strongly dense in $B(H)$, it follows that there exist normal, faithful, $*$ -homomorphisms $\Psi_t : B(H) \rightarrow B(H)$ such that $\mathcal{U}_t(1 \otimes x)\mathcal{U}_t^* = \Psi_t(x) \otimes 1$ for all $x \in B(H)$. It is also clear that the map $t \mapsto \Psi_t(x)$ is strong $*$ -continuous for all $x \in B(H)$.

We prove that $\Psi_t(B(H)) = B(H)$. So, suppose that $x \in \Psi_t(B(H))'$. Because $\Psi_t(a) = a$ for $a \in M$, it follows that $x \in M'$ and we write $y = JxJ \in M$. Further, $x \in \hat{\pi}_t(\hat{M})'$, which means that $1 \otimes x$ commutes with $\tilde{u}_t W$. Hence, $1 \otimes y$ commutes with $W u_t$. So,

$$\Delta(y) = u_t(1 \otimes y)u_t^* .$$

From this, we conclude that $\Delta^{\text{op}}(y) \in M \otimes N$ and obviously $(\Delta^{\text{op}} \otimes \iota)\Delta^{\text{op}}(y) = (\iota \otimes \beta)\Delta^{\text{op}}(y)$. This gives us that $\Delta^{\text{op}}(y) \in \beta(N)$ and hence, $y \in N$. So,

$$\alpha(y) = u_t(1 \otimes y)u_t^* = \tilde{\nabla}^{it}(1 \otimes \sigma_{-t}^\theta(y))\tilde{\nabla}^{-it} ,$$

from which we conclude that $\Delta(\sigma_{-t}^\theta(y)) = 1 \otimes \sigma_{-t}^\theta(y)$. We finally find that $\sigma_{-t}^\theta(y) \in \mathbb{C}$ and so, $y \in \mathbb{C}$. This proves that $\Psi_t(B(H)) = B(H)$.

So, Ψ_t is a family of isomorphisms of $B(H)$ which is pointwise strong $*$ -continuous. It is then easy to find a strongly continuous family $(\tilde{v}_t)$ of unitaries in $B(H)$ such that $\Psi_t = \text{Ad } \tilde{v}_t$ for all $t \in \mathbb{R}$. Because $\Psi_t(a) = a$

for all $a \in M$, we get $\tilde{v}_t \in M'$. We define $v_t := J\tilde{v}_t^*J$ and conclude from the equation $(\iota \otimes \hat{\pi}_t)(W) = \tilde{u}_t W$ that

$$u_t^* W^* = (1 \otimes v_t^*) W^* (1 \otimes v_t) \quad \text{and hence,} \quad u_t = \Delta(v_t^*)(1 \otimes v_t).$$

But then, we observe that $\Delta(v_t) \in N \otimes M$, because $u_t \in N \otimes N$, and the same reasoning as above yields $v_t \in N$ for all $t \in \mathbb{R}$. So, we write $u_t = \alpha(v_t^*)(1 \otimes v_t)$.

Because $u_t = [D\tilde{\theta} : D(\text{Tr}_{\hat{\nabla}} \otimes \theta)]_t$, we know that $u_{t+s} = u_t (\tau_t \otimes \sigma_t^\theta)(u_s)$. So, we can calculate

$$\begin{aligned} \alpha(v_{t+s}^*)(1 \otimes v_{t+s}) &= u_{t+s} = u_t (\tau_t \otimes \sigma_t^\theta)(\alpha(v_s^*)(1 \otimes v_s)) = \tilde{\nabla}^{it}(\hat{\nabla}^{-it} \otimes \nabla_\theta^{-it}) (\tau_t \otimes \sigma_t^\theta)(\alpha(v_s^*)(1 \otimes v_s)) \\ &= \tilde{\nabla}^{it} \alpha(v_s^*)(1 \otimes v_s)(\hat{\nabla}^{-it} \otimes \nabla_\theta^{-it}) = \alpha(\sigma_t^\theta(v_s^*)) u_t (1 \otimes \sigma_t^\theta(v_s)) \\ &= \alpha(\sigma_t^\theta(v_s^*)v_t^*)(1 \otimes v_t \sigma_t^\theta(v_s)). \end{aligned}$$

It follows that $v_t \sigma_t^\theta(v_s) v_{t+s}^*$ is invariant under Δ and hence, a scalar. We find a continuous function $\lambda : \mathbb{R}^2 \rightarrow \mathbb{T}$ such that $v_{t+s} = \lambda(t, s) v_t \sigma_t^\theta(v_s)$. It is then easy to check that λ is a 2-cocycle and so, necessarily, a 2-coboundary. This means that there exists a continuous function $\mu : \mathbb{R} \rightarrow \mathbb{T}$ such that $\lambda(t, s) = \mu(t)\mu(s)\mu(t+s)$. If we replace v_t by $\mu(t)v_t$, we still have $u_t = \alpha(v_t^*)(1 \otimes v_t)$ and we get moreover that $v_{t+s} = v_t \sigma_t^\theta(v_s)$ for all $s, t \in \mathbb{R}$. Because the map $t \mapsto v_t$ is strong*-continuous, it follows that (v_t) is a (σ_t^θ) -cocycle.

So, there exists a unique n.s.f. weight φ_N on N such that $[D\varphi_N : D\theta]_t = v_t$. Then,

$$\begin{aligned} [D\tilde{\varphi}_N : D(\text{Tr}_{\hat{\nabla}} \otimes \varphi_N)]_t &= [D\tilde{\varphi}_N : D\tilde{\theta}]_t [D\tilde{\theta} : D(\text{Tr}_{\hat{\nabla}} \otimes \theta)]_t [D(\text{Tr}_{\hat{\nabla}} \otimes \theta) : D(\text{Tr}_{\hat{\nabla}} \otimes \varphi_N)]_t \\ &= \alpha(v_t) u_t (1 \otimes v_t^*) = 1. \end{aligned}$$

So, $[D\varphi_N \circ \alpha : D\varphi_N]_t = 1$, which means that $(\iota \otimes \varphi_N)\alpha(x) = \varphi_N(x) 1$ for all $x \in \mathcal{M}_{\varphi_N}^+$. Hence, φ_N is a left invariant weight on $(N, \Delta|_N)$.

Because $R(N) = N$, we can restrict R to N and we find an anti-automorphism of N that anti-commutes with $\Delta|_N$. So, we also get a right invariant weight $\varphi|_N R$ and $(N, \Delta|_N)$ is a l.c. quantum group. $\square$

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