On (an error in) the Simultaneous Hall Condition

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Abstract

CoRR abs/1004.2626 states a necessary and sufficient condition, SIM-HC, for the existence of a perfect matching in an overlapping bipartite graph, a so-called simultaneous (perfect) matching, a generalization of Hall's marriage theorem. A suprisingly small counterexample shows that the condition is not sufficient. A short proof of the necessity of the condition is given here, and the culprit in the original proof is identified.

On (an error in) the Simultaneous Hall Condition*

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[1] states a necessary and sufficient condition, SIM-HC, for the existence of a perfect matching in an overlapping bipartite graph, a so-called simultaneous (perfect) matching, a generalization of Hall's marriage theorem. A suprisingly small counterexample shows that the condition is not sufficient. A short proof of the necessity of the condition is given here, and the culprit in the original proof is identified.

1 Introduction

Problems with *overlapping* all_different constraints occur all the time[1], and they have been studied extensively. A nice survey about the all_different constraint can be found in [5]. A necessary and sufficient condition (SIM-HC) for the existence of a solution for a set of overlapping all_different constraints would generalize the famous *Marriage Theorem* by P. Hall [3]. [1] proves such a SIM-HC for the existence of a perfect matching in an overlapping bipartite graph, a so-called simultaneous (perfect) matching, i.e. a solution for two overlapping all_different constraints. We refer to that paper often, in particular its **Theorem 2** and use freely its notation: the interested reader should have [1] handy - or the version at CoRR [2].

We report here on a counterexample to the main theorem in [1] **Theorem 2**. Only while writing down this report, we became aware of the PhD thesis [4] of one of the co-authors of [1] in which the author shows a different (larger) counterexample to that particular theorem: so the (negative) result was previously known, but we arrived at it independently.

We think it is important to expose the (negative) result better. It is indeed very tempting to trust [1]: it was written by respectable researchers, peer-reviewed, and impenetrable. Exposure of the inaccuracy of *Theorem 2* could encourage other researchers to search for and discover the *correct* version of

*This research was carried out in the context of a bachelor's thesis by the first two authors, in partial fulfilment of the requirements for the degree of Bachelor in Informatics, and under the supervision of the third author

[1] E.g. Sudoku, Latin Square, and most scheduling problems.

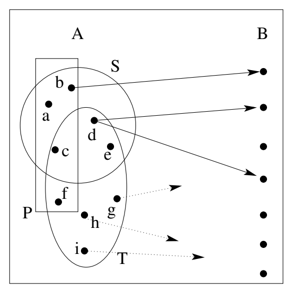
that theorem, and not try to prove new results starting from a flawed theorem. Moreover, we feel like exposing the (negative) result because two years after (at least a subset of) the authors of [1] themselves became aware of the flaw in their **Theorem 2**, they have not yet amended their CoRR version with a caveat about the theorem.

We also want to report on how we found the counterexample: exploratory programming played an important role. Exploratory programming is often defined as a technique within software engineering, but it is also a very valid and useful tool in the discovery and validation of (potential) theorems.

This report is structured as follows: Section 2 explains the theorem for which a counterexample was found. Section 3 shows the counterexample. Section 4 tries to pinpoint where the proof of **Theorem 2** goes wrong. Section 5 explains how the counterexample was found.

2 Theorem 2 in [1]

In the figure at the right, we have drawn part of the bipartite graph $(A \cup B, E)$ about which **Theorem 2** speaks: we have the arcs as an arrow in order to make the mapping N more apparent. The set of nodes A is the union of two non-empty overlapping sets S and T. [1] introduces the following notation:



- for $P \subseteq A$: $P^S = P \cap (S \setminus T)$ and $P^T = P \cap (T \setminus S)$
- for $P \subseteq A$: $N(P) = \{v | \exists u : (u, v) \in P\}$
- $|P|$ denotes the number of elements in a set P

In the figure, $P^S = \{a, b\}$ and $P^T = \{f, g, h, i\}$
In what follows, *SIM-BM* means *Simultaneous Bipartite Matching*. We now state

(Erroneous) Theorem 2 from [1]: *(Simultaneous Hall Condition (SIM-HC))*. Let $G =< A \cup B, E >$ and sets S, T be an overlapping bipartite graph. There exists a *SIM-BM*, *iff* $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P| \; \forall P \subseteq A$. ■

In Section 3 we give a counterexample to the *if* part of the above non-theorem. A quick proof of the *only if* part is given below - just to be on the safe side, it is formulated explicitly:

Theorem 2.1. *If G has a* SIM-BM*, then*

$$\forall P \subseteq A, |P| \leq |N(P)| + |N(P^S) \cap N(P^T)|.$$ [2]

<u>Proof</u> Let there be a SIM-BM M for G: M is a function that maps elements of A to elements of B. Then define $Same_T$ as follows:

$Same_T = \{t \in A^T | \exists s \in A^S, M(t) = M(s)\}$

Then M is a perfect match from $(A \setminus Same_T)$ to $N(A)$. So, by Hall's theorem we have $|A \setminus Same_T| \leq |N(A)$.

Since $Same_T \subset A$, we have $|A \setminus Same_T| = |A| - |Same_T|$, and since also $|M(Same_T)| = |Same_T|$, we derive $|A| \leq |N(A)| + |M(Same_T)|$

Finally, since $M(Same_T) \subseteq (N(A^S) \cap N(A^T))$, we derive:

$$|A| \leq |N(A)| + |N(A^S) \cap N(A^T)|$$

Since A has a SIM-BM, every subset P of A has one as well, and satisfies the same inequality. In summary, we have

$$\forall P \subseteq A, |P| \leq |N(P)| + |N(P^S) \cap N(P^T)|$$ ■

3 The counterexample

The following is a CLP-like specification of the counterexample:

```
dom(U,[1,2]), dom(V,[1]), dom(X,[1,2,3]), dom(Y,[1,3]),
alldiff([U,V,X]), alldiff([V,X,Y])
```

In terms of [1], {U,V,X,Y} equals the set A, {1,2,3} equals the set B, S = {U,V,X} and T = {V,X,Y}. Figure 1 shows it at the left. One can check that it satisfies SIM-HC.

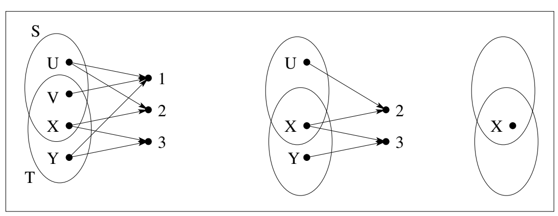


Figure 1: At the left the example; the other two show propagation at work

V has only one neighbour, so one can remove edge (V,1). That results in the graph in the middle. Then U and Y both have only one neighbour, and the result is at the right: there remain no possibilities for X.

To see that the initial graph satisfies SIM-HC, it sufficies to notice that

- for $P \subset A$ with $|P| = 1$, $N(P)$ contains at least one element

[2] We formulate it this way because we became acquainted with Hall's theorem by a formula containing $|P|$ at the left side. The formulation in [3] uses natural language expressing the same.

- for $P \subset A$ with $|P| = 2$, $N(P)$ contains at least two elements
- for $P \subset A$ with $|P| = 3$, $N(P)$ contains always three elements
- for $P = A$ we have $|P| = 4$, $|N(P)| = 3$ and $|N(P^S) \cap N(P^T)| = |\{1\}| = 1$

In all cases $|P| \leq |N(P)| + |N(P^S) \cap N(P^T)|$.

4 Where does the proof in [1] go wrong ?

The proof of the *if-part* of the theorem clearly does not work.

Take the example in 1: **Case 1** of [1] does not apply to it. Consider in **Case 2** $P = \{V\}$, then the resulting graph Q is exactly the graph in the middle of Figure 1. It clearly does not satisfy SIM-HC: it suffices to take (in this new graph) $P = \{U, X, Y\}$.

This means that the analysis in **Case 2** must be incorrect.

One problem with checking the proof in [1] is that it mixes the *if* and *only-if* part of the statement of the theorem. That results in confusion. However, consider the counterexample, and take in **Case 2** of the proof $P = \{V\}$: clearly $|P| = |N(P)| + |N(P^S) \cap N(P^T)| = 1$ as $P^S = P^T = \emptyset$. The graph Q subsequently constructed in the proof consists of the edges $\{(U, 2), (X, 2), (X, 3), (Y, 3)\}$, i.e. the graph in the middle of Figure 1. Then the proof says (we cite literally)

We claim that the SIM-HC holds also for Q

It is clear that SIM-HC does not hold for Q, so the error is somewhere in the proof of the claim.

In the proof, it says at some point:

Similarly $N(P^S \cup P^{S'}) = N(P^S) \cup N_Q(P^{S'})$

However, $P^S = \emptyset$, and $P^{S'} = \{U\}$ so the statement is equivalent to $N(\{U\}) = N_Q(\{U\})$; but $N(\{U\}) = \{1, 2\}$, while $N_Q(\{U\}) = \{1\}$, so, apparently, this line is in error.

5 How the counterexample was derivered

We tried to verify by *explorative programming* **Theorem 2** in [1]. We were not expecting to find a counterexample, even though some details of the proof escaped us: we trusted it to be true. We used the following approach:

For a given number of variables V, and domain size D, we first generate random set S and T of variables so that $S \cap T \neq \emptyset$ and $|S \cup T| = V$. We then associate each variable with domain $1..D$. That constitutes the initial constraint problem CSP_u and we continue only if this has a solution.

```
repeat
      copy CSP_u to CSP_s;
      randomly remove one value from a domain of a variable in CSP_u;
until not(has_a_solution(CSP_u))
```

At the end of this program, we know that CSP_s has a solution while CSP_u has not, and that CSP_s and CSP_u differ only by one domain element for one variable. So CSP_s and CSP_u are at the edge of having a SIM-BM or not, and they are good candidates for checking the necessary and sufficient SIM-HC of [1]. About 5% of the generated CSP_u satisfied the SIM-HC: the counterexample shown above is just one of them. The program was written in [6].

6 Future Work

We intend to work on establishing a sufficient condition for the existence of a SIM-BM, and to generalize the condition to more than two overlapping all_different constraints.

Acknowledgments

We thank M. Garcia de la Banda for verifying our counterexample, and for pointing at the induction argument as a possible culprit in the proof of **Theorem 2**.

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