A 2.5D displacement based PML for elastodynamic wave propagation

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SUMMARY

This paper presents a perfectly matched layer (PML) technique for the numerical simulation of three-dimensional (3D) linear elastodynamic problems where the geometry is invariant in the longitudinal direction. Examples include transportation infrastructure, dams, lifelines, and alluvial valleys.

For longitudinally invariant geometries, a computationally efficient two-and-a-half-dimensional (2.5D) approach can be applied, where the Fourier transform from the longitudinal coordinate to the wavenumber domain allows for the representation of the 3D radiated wave field on a two-dimensional mesh. In this 2.5D framework, the equilibrium equations of a PML continuum are formulated in a weak form for an isotropic elastodynamic medium and discretized using a Galerkin approach.

The 2.5D PML methodology is validated by computing the Green's displacements of a homogeneous halfspace, demonstrating that the 2.5D PML absorbs all propagating waves for different angles of incidence. Furthermore, the dynamic stiffness of a rigid strip foundation and the efficiency of a vibration isolating screen are computed. The examples demonstrate that the PML methodology is computationally efficient, especially when only the response of the structure or the near field response is of interest.

KEY WORDS: Perfectly matched layer; Elastodynamics; 2.5D analysis

1. INTRODUCTION

The numerical solution of three-dimensional (3D) dynamic soil-structure interaction problems is computationally demanding, since the radiation of waves towards infinity has to be taken into account [1, 2]. A number of assumptions is often made in an attempt to reduce the computational effort: the structure can be assumed to be periodic or to have an invariant cross section, which is a valid assumption for roads, railway tracks, tunnels, dams, vibration isolation screens, lifelines, and alluvial valleys.

A computationally efficient two-and-a-half-dimensional (2.5D) approach can be applied for longitudinally invariant structures, where the Fourier transform of the longitudinal coordinate allows for the representation of the 3D response of the structure and the radiated wave field on a two-dimensional (2D) mesh [3, 4, 5, 6]. For the calculation of the response of structures to traveling seismic waves, the idea was first adopted by Hwang and Lysmer [8]. However, the term “2.5D” was coined by Yang and Hung [3], in a first application of the method to the simulation of wave propagation in a half-space induced by moving loads. The method was later

employed for the calculation of railway induced vibrations [9], wave barriers [10] and tunnels [11].

While absorbing boundaries may result in spurious reflections at the mesh truncation, a coupled 2.5D finite element - boundary element formulation presents a rigorous solution of the dynamic soil-structure interaction problem. Sheng et al. [7] employ this approach to the calculation of the response of railway tracks. Andersen and Nielsen apply a similar methodology to study the effect of vibration isolating screens along a railway track [12, 13]. A similar approach is followed by Lombaert and Degrande [14] and Lombaert et al. [15] to predict vibrations induced by road [14] and railway [15] traffic. The latter model has been validated by means of field measurements [15] and has been used to assess the vibration isolation efficiency of a floating slab track [16].

The boundary element method is commonly based on the 2.5D fundamental solution of a homogeneous full space for which a closed form analytical expression is available [17, 18]. A drawback of the homogeneous full space solution is that the free surface and the layer interfaces of the halfspace have to be discretized with boundary elements. As an alternative to the homogeneous full space solution, François et al. [6] propose a regularized 2.5D boundary element method that employs the fundamental solutions of a layered halfspace. The layer interfaces and the surface of the halfspace no longer have to be meshed, avoiding spurious reflections at mesh truncations. The boundary element mesh can be limited to the interface between the structure and the soil, significantly reducing the problem size.

The 2.5D coupled finite element - boundary element approach is less efficient in the case of embedded structures where the fundamental solution has to be evaluated for a large number of source/receiver depths. In this case, the use of a Perfectly Matched Layer (PML) technique may result in a significant gain in computational efficiency.

A PML is an absorbing layer that perfectly absorbs propagating waves with any angle of incidence. PMLs were first introduced by Bérenger in the framework of electromagnetic wave propagation [19], and can be written as a complex valued coordinate stretching of the governing equations [20]. Chew and Liu [21] first derived a PML for elastodynamic wave propagation where a finite difference time domain formulation is obtained through field splitting of the velocity and stress fields. Basu and Chopra [22] have formulated a (non split field) PML in the framework of displacement based time harmonic [22] and transient [23, 24, 25] elastodynamics.

Thus far, an elastodynamic PML in a 2.5D framework has not been developed. Therefore, the aim of this paper is to formulate a PML in the framework of the 2.5D finite element method to account for the radiation of waves, as an alternative to the 2.5D coupled finite element - boundary element or infinite/finite element approach. The 3D displacement based unsplit-field PML of Basu and Chopra [22] is used as a starting point for the formulation.

This paper is organized as follows. In section 2, the governing equations of the PML are formulated in a weak form and discretized in section 3, following a Galerkin approach through the introduction of 2.5D shape functions. In section 4, the choice of the attenuation functions in the PML is discussed by considering the analytical solution of the displacement inside a one-dimensional (1D) PML. Section 5 presents three applications of the PML. First, the Green's displacements of a homogeneous halfspace are compared to a reference solution, in order to verify the absorption of waves for all angles of incidence. Second, the dynamic stiffness of a rigid strip foundation is computed and compared to a 2.5D BE solution. In a third example, the efficiency of a vibration isolating screen in the soil is computed and the computational efficiency of the 2.5D PML method and the coupled 2.5D finite element - boundary element

method are compared.

2. FORMULATION OF THE PERFECTLY MATCHED LAYER

Consider a linear elastic domain Ω with an invariant geometry in the longitudinal direction y (figure 1). A PML Ω_{PML} is placed at the edge of the domain Ω in order to absorb waves travelling in the x - and z -direction. The characteristic property of this PML is that it is perfectly matched, as no reflections occur at the interface between the PML and the physical domain [22]. The PML is therefore a bounded representation of an unbounded extension of the domain Ω , where waves are strongly attenuated inside Ω_{PML} . The physical domain Ω_e of Ω is assumed to be an isotropic visco-elastic medium.

Figure 1. Domain Ω with longitudinal invariant cross-section A , subdivided into a PML Ω_{PML} and a physical domain Ω_e . The PML absorbs waves propagating in the x - and z -direction.

2.1. Equilibrium equations for a linear elastic medium

In the physical domain Ω_e , the stresses $\hat{\sigma}_{ij}$ and displacements $\hat{u}_i$ satisfy the frequency domain equilibrium equation:

$$\hat{\sigma}_{ij,j} + \omega^2 \rho \hat{u}_i = 0 \quad (1)$$

where ρ is the density and ω is the circular frequency. A hat above a variable denotes its representation in the frequency domain. The constitutive equation for the linear medium reads as:

$$\hat{\sigma}_{ij} = C_{ijkl} \hat{\epsilon}_{kl} \quad (2)$$

where C_{ijkl} is the constitutive tensor. The small strain tensor $\hat{\epsilon}_{ij}$ is defined as:

$$\hat{\epsilon}_{ij} = \frac{1}{2} (\hat{u}_{i,j} + \hat{u}_{j,i}) \quad (3)$$

2.2. Governing equations of the PML

In order to absorb waves inside the domain Ω_{PML} , the governing equations (1) and (3) are modified by considering stretched coordinates. Let s denote the coordinate in the direction

normal to the interface between the elastic domain and the PML (figure 2). The interface is located at s_o and the elastic domain extends from $s = 0$ to $s = s_o$ and the PML from $s = s_o$ to $s = s_t$. The stretched coordinate $\tilde{s}$ defined as [21, 26]:

$$\tilde{s} = \int_0^s \lambda_s(s) ds = s_o + \int_{s_o}^s \lambda_s(s) ds \quad (4)$$

where $\lambda_s(s)$ is a non-zero complex valued stretch function.

Figure 2. PML domain in the direction of coordinate s . The waves pass through the interface s_o without reflections, and decay with distance within the PML

In order to introduce the stretched coordinate into the governing equations, the partial derivatives with respect to $\tilde{s}$ is written as:

$$\frac{\partial}{\partial \tilde{s}} = \frac{1}{\lambda_s(s)} \frac{\partial}{\partial s} \quad (5)$$

For the 2.5D geometry under consideration, stretching is applied to the x and z -coordinates only (figure 1). Introducing the partial derivatives (equation (5)) of the stretched coordinates $\tilde{x}$ and $\tilde{z}$ into equation (1), yields the following modified equilibrium equations:

$$\underbrace{\begin{bmatrix} \frac{1}{\lambda_x} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{1}{\lambda_z} \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{1}{\lambda_x} \frac{\partial}{\partial x} & \frac{1}{\lambda_z} \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{1}{\lambda_z} \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{1}{\lambda_x} \frac{\partial}{\partial x} \end{bmatrix}}_{\mathbf{L}^T} \begin{Bmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\sigma}_{zz} \\ \hat{\sigma}_{xy} \\ \hat{\sigma}_{yz} \\ \hat{\sigma}_{zx} \end{Bmatrix} + \omega^2 \rho \begin{Bmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \end{Bmatrix} = 0 \quad (6)$$

where a vector notation is used for the symmetrical stress tensor $\hat{\sigma}_{ij}$ that collects stresses in the vector vector $\hat{\boldsymbol{\sigma}} = \{\hat{\sigma}_{xx}, \hat{\sigma}_{yy}, \hat{\sigma}_{zz}, \hat{\sigma}_{xy}, \hat{\sigma}_{yz}, \hat{\sigma}_{zx}\}^T$. Equation (6) is written briefly as:

$$\mathbf{L}^T \hat{\boldsymbol{\sigma}} + \omega^2 \rho \hat{\mathbf{u}} = 0 \quad (7)$$

Similarly, the small strain tensor in equation (3) is modified as:

$$\hat{\boldsymbol{\epsilon}} = \mathbf{L} \hat{\mathbf{u}} \quad (8)$$

where the vector $\hat{\boldsymbol{\epsilon}} = \{\hat{\epsilon}_{xx}, \hat{\epsilon}_{yy}, \hat{\epsilon}_{zz}, \hat{\gamma}_{xy}, \hat{\gamma}_{yz}, \hat{\gamma}_{zx}\}^T$ collects the normal and engineering strains.

In the case where the stretch functions $\lambda_x = \lambda_z = 1$, the governing equations inside Ω_{PML} are the same as equations (1) to (3) in the physical domain Ω_e .

It should be stressed that the governing equation (6) is defined independently of equations (1) to (3). Replacing the coordinates x and z by $\tilde{x}$ and by $\tilde{z}$ in equilibrium equations (1) to (3)

of the viscoelastic medium is only used as a motivation of equation (6). By simply considering equation (6) as the governing equation of the PML, issues of complex differentiability are avoided [22].

Since the stretching functions $\lambda_x(x)$ and $\lambda_z(z)$ are complex, they cause an artificially high attenuation of waves. Furthermore, it can be shown that the PML is non-reflective if the real part of the stretch functions $\lambda_x(x)$ and $\lambda_z(z)$ are continuous across the interface Ω [19, 21, 22], whereas the imaginary part may be discontinuous [27].

In order to attenuate both propagating and evanescent waves, the following stretching functions are classically used [19, 21, 22, 25, 27]:

$$\lambda_x(x) = 1 + f_x^e(x) - i \frac{f_x^p(x)}{a_0} \quad (9)$$

for waves propagating in the x -direction and:

$$\lambda_z(z) = 1 + f_z^e(z) - i \frac{f_z^p(z)}{a_0} \quad (10)$$

for waves propagating in the z -direction. The dimensionless frequency a_0 is defined as $a_0 = \omega L / C_s$ with L a characteristic length, commonly chosen as the thickness of the PML layer, and C_s the shear wave velocity. The attenuation functions $f_x^e(x)$ and $f_z^e(z)$ attenuate evanescent waves in the x -direction and z -direction, respectively. Similarly, the attenuation functions $f_x^p(x)$ and $f_z^p(z)$ attenuate propagating waves. The attenuation functions $f_x^e(x)$, $f_z^e(z)$, $f_x^p(x)$, and $f_z^p(z)$ are a function of the coordinate x or z only, and are independent of the frequency. The dimensionless frequency a_0 in equations 9 and 10 renders the stretching functions $\lambda_x(x)$ and $\lambda_z(z)$ frequency dependent, so that the dynamic element stiffness matrices of the PML layer have to be assembled for each frequency.

The attenuation functions $f_x^e(x)$, $f_z^e(z)$, $f_x^p(x)$ and $f_z^p(z)$ are chosen to be zero on the interface between the PML and the elastic domain Ω_e , so that the stretch functions $\lambda_x(x)$ and $\lambda_z(z)$ are continuous and the non-reflecting property applies. A linear or quadratic monomial is often chosen for the attenuation functions [22, 25, 28]. The slope of the attenuation functions is tuned so that sufficient attenuation is obtained inside the PML. The attenuation functions are further discussed in section 4.

3. 2.5D FINITE ELEMENT IMPLEMENTATION

In the following, the equilibrium equations of the domain Ω , including both the visco-elastic medium Ω_e and the PML Ω_{PML} (figure 1), are formulated in a weak form. The equilibrium equations (6) of the PML are used throughout the domain Ω , where unit stretch functions $\lambda_x = \lambda_z = 1$ are selected inside Ω_e .

3.1. Virtual work equation

In order to avoid integration by parts of the stretch functions λ_x and λ_z in the weak form of the virtual work equation, the equilibrium equations (6) are multiplied with a factor $\lambda_x \lambda_z$:

$$\lambda_x \lambda_z \mathbf{L}^T \hat{\boldsymbol{\sigma}} + \omega^2 \lambda_x \lambda_z \rho \hat{\mathbf{u}} = 0 \quad (11)$$

Furthermore, tractions $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n} = \hat{\mathbf{t}}^n = \bar{\hat{\mathbf{t}}}^n$ are imposed on the Neumann boundary Γ_t with unit outward normal vector $\mathbf{n}$. The equilibrium equation (11) and the Neumann boundary conditions are imposed in a weak form by considering a virtual displacement field $\hat{\mathbf{v}}$ on the domain Ω :

$$\int_{\Omega} \hat{\mathbf{v}}^T \lambda_x \lambda_z \mathbf{L}^T \hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}}) d\Omega + \omega^2 \int_{\Omega} \hat{\mathbf{v}}^T \lambda_x \lambda_z \rho \hat{\mathbf{u}} d\Omega - \int_{\Gamma_t} \hat{\mathbf{v}}^T \lambda_x \lambda_z (\hat{\mathbf{t}}^n - \bar{\hat{\mathbf{t}}}^n) d\Gamma = 0 \quad (12)$$

Integration by parts of equation (12) and application of the divergence theorem yields:

$$\begin{aligned} \int_{\Omega} \lambda_x \lambda_z \hat{\boldsymbol{\epsilon}}^T(\hat{\mathbf{v}}) \hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}}) d\Omega - \omega^2 \int_{\Omega} \hat{\mathbf{v}}^T \lambda_x \lambda_z \rho \hat{\mathbf{u}} d\Omega &= \int_{\Gamma} \hat{\mathbf{v}}^T \lambda_x \lambda_z \hat{\mathbf{t}}^{n'} d\Gamma \\ &\quad - \int_{\Gamma_t} \hat{\mathbf{v}}^T \lambda_x \lambda_z (\hat{\mathbf{t}}^n - \bar{\hat{\mathbf{t}}}^n) d\Gamma \end{aligned} \quad (13)$$

where $\hat{\boldsymbol{\epsilon}}(\hat{\mathbf{v}}) = \mathbf{L} \hat{\mathbf{v}}$ is the virtual strain tensor. The stress vector $\hat{\mathbf{t}}^{n'}$ is computed as $\hat{\mathbf{t}}^{n'} = \hat{\boldsymbol{\sigma}} \cdot \mathbf{n}'$, where the stretched normal vector $\mathbf{n}' = \{n_x/\lambda_x, n_y, n_z/\lambda_z\}^T$ is used instead of the (unstretched) normal vector $\mathbf{n} = \{n_x, n_y, n_z\}^T$. In the following, it is assumed that the Neumann boundary Γ_t is restricted to the part of the boundary Γ where the following condition applies:

$$\hat{\mathbf{t}}^{n'} = \hat{\mathbf{t}}^n \quad (14)$$

Equation (14) holds (a) on the boundary of the physical domain Ω_e ($\lambda_x = \lambda_z = 1$) and (b) on parts of the boundary of Ω_{PML} with a unit outward vector $\mathbf{n} = \{1, 0, 0\}^T$ or $\mathbf{n} = \{0, 0, 1\}^T$ where $\lambda_x = 1$ or $\lambda_z = 1$, respectively. For example, the condition (14) applies on surface ① (figure 1) if $\lambda_z = 1$. With the restriction (14) on the Neumann boundary, equation (13) is further elaborated as:

$$\int_{\Omega} \lambda_x \lambda_z \hat{\boldsymbol{\epsilon}}^T(\hat{\mathbf{v}}) \hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}}) d\Omega - \omega^2 \int_{\Omega} \hat{\mathbf{v}}^T \lambda_x \lambda_z \rho \hat{\mathbf{u}} d\Omega = \int_{\Gamma_u} \hat{\mathbf{v}}^T \lambda_x \lambda_z \hat{\mathbf{t}}^{n'} d\Gamma + \int_{\Gamma_t} \hat{\mathbf{v}}^T \lambda_x \lambda_z \hat{\mathbf{t}}^n d\Gamma \quad (15)$$

where Γ_u is the part of Γ on which homogeneous Dirichlet conditions are applied. As the virtual displacement field $\hat{\mathbf{v}}$ is chosen to satisfy the Dirichlet conditions on the boundary Γ_u , the first integral on the right hand side of equation (15) vanishes:

$$\int_{\Omega} \lambda_x \lambda_z \hat{\boldsymbol{\epsilon}}^T(\hat{\mathbf{v}}) \hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}}) d\Omega - \omega^2 \int_{\Omega} \hat{\mathbf{v}}^T \lambda_x \lambda_z \rho \hat{\mathbf{u}} d\Omega = \int_{\Gamma_t} \hat{\mathbf{v}}^T \lambda_x \lambda_z \hat{\mathbf{t}}^n d\Gamma \quad (16)$$

3.2. Discretization

Next, the finite element discretization is introduced into the virtual work equation (16). As the domain Ω is longitudinally invariant and has an invariant cross section A , the displacement vector $\hat{\mathbf{u}}(\mathbf{x}, \omega)$ is discretized as:

$$\hat{\mathbf{u}}(\mathbf{x}, \omega) \simeq \mathbf{N}(x, z) \hat{\underline{\mathbf{u}}}(y, \omega) \quad (17)$$

where $\mathbf{N}(x, z)$ are the globally defined shape functions defined over the section A and $\hat{\underline{\mathbf{u}}}(y, \omega)$ is the vector with the degrees of freedom at all nodes of the 2D mesh. The vector $\hat{\underline{\mathbf{u}}}(y, \omega)$ is a function of the longitudinal coordinate y and the frequency ω .

In order to apply the constitutive equation $\hat{\boldsymbol{\sigma}} = \mathbf{C}\hat{\boldsymbol{\epsilon}}$, the strain vector $\hat{\boldsymbol{\epsilon}}$ is first derived from the displacement vector $\hat{\mathbf{u}}(y, \omega)$ as:

$$\hat{\boldsymbol{\epsilon}} = \mathbf{L}_1 \mathbf{N} \hat{\mathbf{u}} + \mathbf{L}_2 \mathbf{N} \frac{\partial \hat{\mathbf{u}}}{\partial y} \quad (18)$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ read as:

$$\mathbf{L}_1 = \begin{bmatrix} \frac{1}{\lambda_x} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda_z} \frac{\partial}{\partial z} \\ 0 & \frac{1}{\lambda_x} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{\lambda_z} \frac{\partial}{\partial z} & 0 \\ \frac{1}{\lambda_z} \frac{\partial}{\partial z} & 0 & \frac{1}{\lambda_x} \frac{\partial}{\partial x} \end{bmatrix} \quad \text{and} \quad \mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (19)$$

so that $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \partial/\partial y$. The strain vector $\hat{\boldsymbol{\epsilon}}$ is related to both the displacement vector $\hat{\mathbf{u}}$ and its derivative $\partial \hat{\mathbf{u}}/\partial y$ with respect to the longitudinal coordinate y . Equation (18) is alternatively written as:

$$\hat{\boldsymbol{\epsilon}} = \mathbf{B}_1 \hat{\mathbf{u}} + \mathbf{B}_2 \frac{\partial \hat{\mathbf{u}}}{\partial y} \quad (20)$$

where $\mathbf{B}_1 = \mathbf{L}_1 \mathbf{N}$ and $\mathbf{B}_2 = \mathbf{L}_2 \mathbf{N}$.

The stress vector $\hat{\boldsymbol{\sigma}}$ is related to the strain vector $\hat{\boldsymbol{\epsilon}}$ through the constitutive relation $\hat{\boldsymbol{\sigma}} = \mathbf{C}\hat{\boldsymbol{\epsilon}}$, resulting in:

$$\hat{\boldsymbol{\sigma}} = \mathbf{C} \mathbf{B}_1 \hat{\mathbf{u}} + \mathbf{C} \mathbf{B}_2 \frac{\partial \hat{\mathbf{u}}}{\partial y} \quad (21)$$

3.3. Galerkin procedure

A standard Galerkin procedure is followed, where the same discretization is used for the virtual displacement vector $\hat{\mathbf{v}}(\mathbf{x}, \omega) = \mathbf{N}(x, z) \hat{\mathbf{v}}(y, \omega)$ as for the displacement vector $\hat{\mathbf{u}}(\mathbf{x}, \omega)$. The virtual strain vector $\hat{\boldsymbol{\epsilon}}(\hat{\mathbf{v}})$ is derived from the virtual displacement vector $\hat{\mathbf{v}}(y, \omega)$ as:

$$\hat{\boldsymbol{\epsilon}}(\hat{\mathbf{v}}) = \mathbf{B}_1 \hat{\mathbf{v}} + \mathbf{B}_2 \frac{\partial \hat{\mathbf{v}}}{\partial y} \quad (22)$$

Introducing the constitutive equation (21) and the strain-displacement relation (22) in the virtual work equation (16) yields:

$$\begin{aligned} -\omega^2 \int_{\Omega} \hat{\mathbf{v}}^T \lambda_x \lambda_z \rho \mathbf{N}^T \mathbf{N} \hat{\mathbf{u}} d\Omega + \int_{\Omega} \lambda_x \lambda_z \left(\hat{\mathbf{v}}^T \mathbf{B}_1^T + \left(\frac{\partial \hat{\mathbf{v}}}{\partial y} \right)^T \mathbf{B}_2^T \right) \mathbf{C} \left(\mathbf{B}_1 \hat{\mathbf{u}} + \mathbf{B}_2 \frac{\partial \hat{\mathbf{u}}}{\partial y} \right) d\Omega \\ = \int_{\Gamma_t} \hat{\mathbf{v}}^T \lambda_x \lambda_z \mathbf{N}^T \hat{\mathbf{t}}^n d\Gamma \end{aligned} \quad (23)$$

Equation (23) is further elaborated, rewriting the volume integrals as an integral over the

longitudinal coordinate y and the cross section A :

$$\begin{aligned} & -\omega^2 \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left(\int_A \lambda_x \lambda_z \rho \mathbf{N}^T \mathbf{N} d\Gamma \right) \hat{\mathbf{u}} dy + \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left(\int_A \lambda_x \lambda_z \mathbf{B}_1^T \mathbf{C} \mathbf{B}_1 d\Gamma \right) \hat{\mathbf{u}} dy \\ & + \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left(\int_A \lambda_x \lambda_z \mathbf{B}_1^T \mathbf{C} \mathbf{B}_2 d\Gamma \right) \frac{\partial \hat{\mathbf{u}}}{\partial y} dy + \int_{-\infty}^{\infty} \left(\frac{\partial \hat{\mathbf{v}}}{\partial y} \right)^T \left(\int_A \lambda_x \lambda_z \mathbf{B}_2^T \mathbf{C} \mathbf{B}_1 d\Gamma \right) \hat{\mathbf{u}} dy \\ & + \int_{-\infty}^{\infty} \left(\frac{\partial \hat{\mathbf{v}}}{\partial y} \right)^T \left(\int_A \lambda_x \lambda_z \mathbf{B}_2^T \mathbf{C} \mathbf{B}_2 d\Gamma \right) \frac{\partial \hat{\mathbf{u}}}{\partial y} dy = \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left(\int_{G_t} \lambda_x \lambda_z \mathbf{N}^T \hat{\mathbf{t}}^n ds \right) dy \quad (24) \end{aligned}$$

where G_t is the intersection of the surface Γ_t with the plane $y = 0$. The discretized equation is obtained through elimination of the virtual displacement vector $\hat{\mathbf{v}}(y, \omega)$. This requires integration by parts on the terms containing derivatives $\partial \hat{\mathbf{v}}(y, \omega) / \partial y$ in equation (24):

$$\begin{aligned} & -\omega^2 \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left[\int_A \lambda_x \lambda_z \rho \mathbf{N}^T \mathbf{N} d\Gamma \right] \hat{\mathbf{u}} dy + \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left[\int_A \lambda_x \lambda_z \mathbf{B}_1^T \mathbf{C} \mathbf{B}_1 d\Gamma \right] \hat{\mathbf{u}} dy \\ & + \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left[\int_A \lambda_x \lambda_z \mathbf{B}_1^T \mathbf{C} \mathbf{B}_2 d\Gamma - \int_A \lambda_x \lambda_z \mathbf{B}_2^T \mathbf{C} \mathbf{B}_1 d\Gamma \right] \frac{\partial \hat{\mathbf{u}}}{\partial y} dy \\ & - \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left[\int_A \lambda_x \lambda_z \mathbf{B}_2^T \mathbf{C} \mathbf{B}_2 d\Gamma \right] \frac{\partial^2 \hat{\mathbf{u}}}{\partial y^2} dy = \int_{-\infty}^{\infty} \hat{\mathbf{v}}^T \left[\int_{G_t} \lambda_x \lambda_z \mathbf{N}^T \hat{\mathbf{t}}^n ds \right] dy \quad (25) \end{aligned}$$

Since equation (25) should hold for any virtual displacement $\hat{\mathbf{v}}(y, \omega)$, it is equivalent with:

$$-\omega^2 \mathbf{M} \hat{\mathbf{u}}(y, \omega) + \mathbf{K}^0 \hat{\mathbf{u}}(y, \omega) + \mathbf{K}^1 \frac{\partial \hat{\mathbf{u}}(y, \omega)}{\partial y} + \mathbf{K}^2 \frac{\partial^2 \hat{\mathbf{u}}(y, \omega)}{\partial y^2} = \hat{\mathbf{f}}(y, \omega) \quad (26)$$

where the mass matrix $\mathbf{M}$ is defined as:

$$\mathbf{M} = \int_A \lambda_x \lambda_z \rho \mathbf{N}^T \mathbf{N} d\Gamma \quad (27)$$

The stiffness matrices $\mathbf{K}^0$, $\mathbf{K}^1$, and $\mathbf{K}^2$ are defined as:

$$\mathbf{K}^0 = \int_A \lambda_x \lambda_z \mathbf{B}_1^T \mathbf{C} \mathbf{B}_1 d\Gamma \quad (28)$$

$$\mathbf{K}^1 = \int_A \lambda_x \lambda_z (\mathbf{B}_1^T \mathbf{C} \mathbf{B}_2 - \mathbf{B}_2^T \mathbf{C} \mathbf{B}_1) d\Gamma \quad (29)$$

$$\mathbf{K}^2 = \int_A \lambda_x \lambda_z \mathbf{B}_2^T \mathbf{C} \mathbf{B}_2 d\Gamma \quad (30)$$

The external load vector $\hat{\mathbf{f}}(y, \omega)$ contains contributions of surface tractions and is evaluated for every point y on the longitudinal axis:

$$\hat{\mathbf{f}}(y, \omega) = \int_{G_t} \lambda_x \lambda_z \mathbf{N}^T \hat{\mathbf{t}}^n(x, y, z, \omega) d\Gamma \quad (31)$$

3.4. Solution

The integrals in equations (27) to (30) are evaluated by means of 2D Gaussian quadrature, using reduced integration. The matrices $\mathbf{M}$ and $\mathbf{K}^0$ correspond to an uncoupled combination

of the classical 2D in-plane and out-of-plane finite element mass and stiffness matrices, respectively. The stiffness matrices $\mathbf{K}^1$ and $\mathbf{K}^2$ account for 3D wave propagation in the structure, coupling the in-plane and out-of-plane degrees of freedom.

The differential equation (26) is solved by a Fourier transform of the longitudinal coordinate y to the horizontal wavenumber k_y , where the Fourier transform is defined as $\mathcal{F}[f(y), k_y] = \int_{-\infty}^{\infty} \exp(+ik_y y) f(y) dy$. This results in the wavenumber transform of equation (26):

$$[-\omega^2 \mathbf{M} + \mathbf{K}^0 - ik_y \mathbf{K}^1 - k_y^2 \mathbf{K}^2] \tilde{\mathbf{u}}(k_y, \omega) = \tilde{\mathbf{f}}(k_y, \omega) \quad (32)$$

where a tilde above a variable denotes its representation in the frequency-wavenumber domain.

Equation (32) is solved for a range of frequencies and wavenumbers. Inverse Fourier transforms from the wavenumber k_y to the coordinate y and from the frequency ω to the time t yield the solution in the time-space domain.

4. ATTENUATION FUNCTIONS

In this section, the choice of the attenuation functions $f_x^e(x)$ and $f_x^p(x)$ in the stretching functions (9) and (10) is discussed in detail. The propagation of shear waves with a velocity $C_s = 100$ m/s in a semi-infinite 1D PML (figure 3) is studied analytically. The PML is subjected to a unit harmonic transverse displacement $u_z(t) = \exp(i\omega t)$ at the left end $x = 0$ at 20 Hz. As the bar is semi-infinite, radiation conditions apply for $x \rightarrow \infty$.

Figure 3. 1D PML modelled as (a) a continuum and (b) after discretization with finite elements, where the PML is truncated to a length L .

If zero attenuation $f_x^e(x) = f_x^p(x) = 0$ functions are considered in the PML, the displacements are found as the solution of equations (1) to (3), corresponding to the displacements in a linear elastic semi-infinite bar (figure 4):

$$\hat{u}_z(x, \omega) = \exp(-ik_s x) \quad (33)$$

where the wavenumber k_s is related to the shear wavelength λ_s as $k_s = \omega/C_s = 2\pi/\lambda_s$. Only propagative waves are present in the displacement solution (33). Since both evanescent and propagative waves can occur in a general wave propagation problem, it is commonly advised

Figure 4. (a) Real and (b) imaginary part of the transverse displacement $\hat{u}_z(x, \omega)$ along a linear elastic semi-infinite bar.

to attenuate both wave types [25, 29]. In the present example, the effect of the attenuation function $f_x^e(x)$ for evanescent waves on the propagative solution (33) is demonstrated.

In the case where the attenuation functions are non-zero, the solution of the equilibrium equation (6) is found by replacing x by $\tilde{x}$ in equation (33):

$$\hat{u}_z(\tilde{x}, \omega) = \exp(-ik_s \tilde{x}) \quad (34)$$

Equation (4) for the stretched coordinate $\tilde{x}$ is elaborated by introducing the stretch function (9):

$$\tilde{x} = \int_0^x \left(1 + f_x^e(x) - i \frac{f_x^p(x)}{a_0} \right) dx \quad (35)$$

Equation (35) is rewritten as:

$$\tilde{x} = x + \int_0^x f_x^e(x) dx - \frac{i}{a_0} \int_0^x f_x^p(x) dx \quad (36)$$

The integrals in equation (36) are denoted as $F_x^e(x)$ and $F_x^p(x)$, respectively:

$$\tilde{x} = x + F_x^e(x) - \frac{i}{a_0} F_x^p(x) \quad (37)$$

Equation (37) allows for writing the displacement solution (34) as:

$$\hat{u}_z(x, \omega) = \exp(-ik_s x) \exp(-ik_s F_x^e(x)) \exp\left(-\frac{F_x^p(x)}{L}\right) \quad (38)$$

The following linear attenuation functions $f_x^e(x)$ and $f_x^p(x)$ are selected [25, 29]:

$$f_x^e(x) = f_{x0}^e \frac{x}{L} \quad (39)$$

$$f_x^p(x) = f_{x0}^p \frac{x}{L} \quad (40)$$

where the parameters f_{x0}^e and f_{x0}^p are used to adjust the amount of attenuation. For this particular choice, the integrals $F_x^e(x)$ and $F_x^p(x)$ read as:

$$F_x^e(x) = \frac{f_{x0}^e}{2} \frac{x^2}{L} \quad (41)$$

$$F_x^p(x) = \frac{f_{x0}^p x^2}{2L} \quad (42)$$

Substituting equations (41) and (42) into the displacement solution (38) yields:

$$\hat{u}_z(x, \omega) = \exp(-ik_s x) \exp\left(-ik_s \frac{f_{x0}^e x^2}{2L}\right) \exp\left(-\frac{f_{x0}^p x^2}{2L^2}\right) \quad (43)$$

Figure 5 shows the displacement solution (38) in the PML for $f_{x0}^e = f_{x0}^p = 10$ and $L = 20$ m. The attenuation function $f_x^e(x)$ results in a gradual stretch of the coordinate x . Correspondingly, the (apparent) wavelength λ'_s of the shear waves gradually decreases with x . The apparent wavelength λ'_s of the shear waves in the PML is found as:

$$\lambda'_s(x) = \frac{\lambda_s}{\text{Re}(\lambda_x(x))} = \frac{\lambda_s}{1 + f_x^e(x)} \quad (44)$$

where $\lambda_s = C_s/f = 5$ m is the actual shear wavelength. In the present example, the ratio between the actual shear wavelength λ_s and the apparent shear wavelength λ'_s at the truncation $x = L$ is equal to $1 + f_x^e(L) = 11$. In contrast, the attenuation function $f_x^p(x)$ strongly attenuates

Figure 5. (a) Real and (b) imaginary part of the transverse displacement $\hat{u}_z(x, \omega)$ along the 1D PML for $f_{x0}^e = f_{x0}^p = 10$. The analytical solution (solid line) is compared with the finite element solution using 264 finite elements (crosses).

the solution but does not modify the wavelength. This is demonstrated in figure 6 that shows the displacement solution (38) in the PML selecting $f_{x0}^e = 0$ and $f_{x0}^p = 10$.

In practical applications the PML is truncated and discretized with finite elements. The finite element mesh should therefore be sufficiently fine to represent the displacement solution (38). As a rule of thumb, a minimum of six elements per wavelength is necessary for an accurate finite element solution. Figures 5 and 6 compare the analytical solution (38) to a finite element solution using 264 8-node quadratic finite elements with an element length $l_e = 0.075$ m, corresponding to 66 elements per wavelength at $x = 0$, and to $66/11 = 6$ elements per wavelength at the truncation $x = L$. This is sufficient for an accurate finite element solution. The difference between the finite element solution and the analytical solution is mainly due to the domain truncation.

This demonstrates that the use of a non-zero attenuation function $f_x^e(x)$ for the absorption of (possible) evanescent waves poses stringent demands on the mesh size, since the wavelength of propagating waves is shortened for large values of f_{x0}^e . Therefore, zero stretch functions $f_{x0}^e = f_{z0}^e = 0$ for evanescent waves are applied in the following. As will be demonstrated in

Figure 6. (a) Real and (b) imaginary part of the transverse displacement $\hat{u}_z(x, \omega)$ along the 1D PML for $f_{x0}^e = 0$ and $f_{x0}^p = 10$. The analytical solution (solid line) is compared with the finite element solution using 264 finite elements (crosses).

the examples in section 5, evanescent waves can be attenuated by selecting a sufficiently large domain size.

On the other hand, the attenuation function $f_x^p(x)$ for propagating waves should be chosen such that the displacements are sufficiently small at $x = L$, limiting the truncation error. From this point of view, a very high value of f_{x0}^p is optimal. However, a high value of f_{x0}^p leads to a strong attenuation, characterized by the term $\exp(-F_x^p(x))$ in equation (38), that requires a dense mesh for an accurate finite element solution. Therefore, a compromise should be sought for the parameter f_{x0}^p that allows for a sufficiently coarse mesh but where the truncation error is sufficiently small. A constant value $f_{x0}^p = f_{z0}^p = 20$, as suggested by Basu and Chopra [22], is used in the following applications.

5. APPLICATIONS

5.1. Green's functions of a homogeneous halfspace

The 2.5D PML methodology is further validated by computing the displacements of a homogeneous halfspace with a shear wave velocity $C_s = 150$ m/s, a compressional wave velocity $C_p = 300$ m/s, and a density $\rho = 1800$ m/s. Material damping is modeled through the use of complex Lamé coefficients $\mu^* = \mu(1 + 2i\beta_s)$ and $(\lambda + 2\mu)^* = (\lambda + 2\mu)(1 + 2i\beta_p)$, where $\beta_s = 0.02$ and $\beta_p = 0.02$ represent the hysteretic material damping ratios for shear and dilatational waves, respectively. The halfspace is excited by a vertical line load on the surface that is harmonic with respect to the longitudinal coordinate y . In the frequency-space domain, the load $\hat{p}_z(x, y, z, \omega)$ is given by:

$$\hat{p}_z(x, y, z, \omega) = \delta(x)\delta(z) \cos(k_y y) \quad (45)$$

where $k_y = 2\pi/\lambda_y$ is the wavenumber corresponding to the wavelength λ_y . The line load is applied along the line $(x = 0, z = 0)$ in the direction $\mathbf{e}_z$. Since the load is harmonic with respect to the y -coordinate, the resulting displacements $\hat{u}_i(x, y, z, \omega)$ are also harmonic:

$$\hat{u}_i(x, y, z, \omega) = \tilde{u}_i(x, k_y, z, \omega) \cos(k_y y) \quad (46)$$

The corresponding displacements $\tilde{u}_i(x, k_y, z, \omega)$ are referred to as the Green's displacements in the frequency-wavenumber domain.

The Green's displacements are computed using the 2.5D PML method. The homogeneous halfspace is meshed with a grid of 40×20 equally sized square 8-node serendipity elements from $x = -5$ m to $x = 5$ m up to a depth of 5 m. The element size $l_e = 0.25$ m corresponds to 11.4 elements per Rayleigh wavelength $\lambda_R = C_R/f = 2.80$ m which is sufficient for an accurate finite element solution. The thickness of the PML layer is 0.5 m, comprising two 2.5D finite elements. The PML is clamped at its edges.

Figure 7 shows the Green's displacements for different dimensionless wavenumbers $\bar{k}_y$ defined as $\bar{k}_y = k_y C_s / \omega$. On the surface of the halfspace, the wave field is characterized by 2D Rayleigh waves with plane wave fronts. The velocity $C_R = 139.8$ m/s of the Rayleigh waves corresponds to a wavelength $\lambda_R = C_R/f = 2.80$ m.

The direction θ of the wave fronts depends on the wavenumber k_y . At $\bar{k}_y = 0$, the wave fronts are parallel to the y -axis, corresponding to 2D wave propagation. In the case where $0 < k_y < k_R$, the angle θ between the wave fronts and the (x, z) -plane is computed as:

$$\theta = \arccos \frac{k_y}{k_R} \quad (47)$$

The angle varies from $\theta = 0$ at $k_y = 0$ to $\theta = \pi/2$ at $k_y = k_R$, as can be observed in figure 7. In the (x, z) -plane, the wave fronts are characterized by an apparent wavelength $\lambda_{Rx} = 2\pi/k_{Rx}$, corresponding to a wavenumber k_{Rx} :

$$k_{Rx} = \sqrt{k_R^2 - k_y^2} \quad (48)$$

In the case where $k_y > k_R$, the wavenumber k_{Rx} in equation (48) becomes imaginary, and the corresponding displacement solution is composed of evanescent waves. This is illustrated in figure 7d, where the displacement solution exponentially decreases in the x -direction.

A reference solution for the Green's displacements of the homogeneous halfspace is obtained by means of the direct stiffness method [30, 31] as implemented in the ElastoDynamics Toolbox (EDT) for MATLAB [32]. Figure 7 compares the PML solution with the reference solution. Figure 8 further compares the vertical displacements along the x -axis with the reference solution, which demonstrates the good accuracy of the 2.5D PML solution. In the case $0 < k_y < k_R$, the PML absorbs the propagative waves as expected. In the case $k_y > k_R$, the computational domain is sufficiently extended, so that the evanescent wave solution is attenuated in a regular way. This justifies the use of zero attenuation functions $f_x^e(x) = 0$ and $f_z^e(z) = 0$.

5.2. Dynamic stiffness of a strip foundation

In this example, the dynamic stiffness of a strip foundation is computed. The rigid strip foundation has a width $b = 3$ m and is resting on a homogeneous halfspace with a shear wave velocity $C_s = 150$ m/s, a compressional wave velocity $C_p = 300$ m/s, a density $\rho = 1800$ kg/m³ and a material damping ratio $\beta_s = \beta_p = 0.02$ in both deviatoric and volumetric deformation.

The vertical dynamic stiffness of the strip foundation is first computed using the PML approach. The finite element mesh is shown in figure 9, where symmetry is exploited to reduce the computational effort. A square domain of 3×3 m is meshed with equally sized finite elements with an element size $l_e = 0.25$ m. A PML with a thickness of 1 m is considered at the edge of the mesh, where linear attenuation functions with $f_{x0}^p = f_{z0}^p = 20$ are used to attenuate

Figure 7. Real part of the vertical displacement $\hat{u}_z(x, y, z, \omega)$ in a homogeneous halfspace excited at the surface by a vertical harmonic load at 50 Hz for a dimensionless wavenumber (a) $\bar{k}_y = 0$, (b) $\bar{k}_y = 0.5$, (c) $\bar{k}_y = 1.0$, and (d) $\bar{k}_y = 1.5$. The PML solution (left) is compared with the reference solution computed with EDT (right).

Figure 8. Real part of the vertical displacement $\hat{u}_z(x, y, z, \omega)$ in a homogeneous halfspace excited at the surface by a vertical harmonic load at 50 Hz for a dimensionless wavenumber (a) $\bar{k}_y = 0$, (b) $\bar{k}_y = 0.5$, (c) $\bar{k}_y = 1$, and (d) $\bar{k}_y = 1.5$. The PML solution (solid line) is compared with the reference solution computed with EDT (dashed line).

propagative waves. The PML is clamped at its edges, while symmetry conditions are applied at $x = 0$ m.

Figure 9. Finite element mesh for the computation of the dynamic stiffness of the rigid strip foundation.

The strip foundation is subjected to a unit vertical displacement. The vertical dynamic stiffness is found as the sum of the vertical reaction forces on the foundation-soil interface. The real and imaginary part of the vertical dynamic stiffness are plotted in figure 10 up to 100 Hz for the wavenumber $k_y = 0$, corresponding to a 2D situation.

A reference solution is computed using 100 and 400 equally sized boundary elements over the interface. The symmetry conditions are not exploited in the boundary element method. The boundary elements have constant shape functions, so that the tractions are constant within the element. The Green's functions of the homogeneous halfspace, computed with EDT, are used as the fundamental solution in the boundary element method. Figure 10 shows a good agreement between the PML solution and the boundary element solution. This demonstrates that the PML solution results in an accurate computation of the dynamic stiffness of the foundation, although the computational domain is small with respect to the wavelength of shear and dilatational waves, especially at lower frequencies. For example, the wavelength at 10 Hz equals $\lambda_s = C_s/f = 15$ m, which is 5 times larger than the domain size. While the attenuation functions $f_x^e(x)$ and $f_z^e(z)$ have zero value, the evanescent waves are well attenuated in the finite element mesh in a regular way.

The computation time for the PML solution is 10.6 s (for 50 frequencies), while the boundary element solution takes respectively 21.7 s and 250 s on the same processor for a solution using 100 and 400 boundary elements. It is difficult to generally compare both methods, however, as the computation time is biased by the specific implementation of both methods. Since the assembly of the stiffness matrices does not have to be repeated for every wavenumber, the efficiency of the PML method will be even larger if the solution is computed for a large number of wavenumbers. It can therefore be concluded that the PML solution is computationally more attractive than the boundary element solution for the present example.

Figure 10. Real (left) and imaginary (right) part of the vertical dynamic stiffness of the strip foundation as a function of frequency for the wavenumber (a) $\bar{k}_y = 0$, (b) $\bar{k}_y = 0.5$, and (c) $\bar{k}_y = 1.0$. The PML solution (solid line) is compared with a boundary element solution using 100 (dotted line) and 400 (dashed line) boundary elements along the interface.

5.3. A vibration isolating screen in the soil

In the third example, the efficiency of a vibration isolating screen in the soil is examined. A vibration isolating screen is a soft or stiff wave barrier. Due to the impedance contrast between the isolating screen and the soil, waves are reflected, so reducing vibration levels behind the screen.

Figure 1 shows a vibration isolating screen embedded in a homogeneous halfspace with a shear wave velocity $C_s = 150$ m/s, a compressional wave velocity $C_p = 300$ m/s, a density $\rho = 1800$ kg/m³ and a material damping ratio $\beta_s = \beta_p = 0.05$ in both deviatoric and volumetric deformation. The incident wave field is generated by a unit vertical point load acting on the surface of the halfspace at a distance of 5 m from the vibration isolating screen.

First, the reference case of a unit point load on the surface of a halfspace is considered, corresponding to the 3D Green's displacements. The Green's displacements are computed with the 2.5D PML method, on the domain from $x = -10$ m to $x = 10$ m up to a depth of 12 m. A mesh of square 8-node finite elements is used with an element size $l_e = 0.5$ m. On the edge of the domain, a PML with a thickness of 1 m and attenuation coefficients $f_{x0}^e = f_{z0}^e = 0$ and $f_{x0}^p = f_{z0}^p = 20$ is considered. The results in the spatial domain are obtained by using an FFT for the inverse wavenumber transform, considering 150 wavenumbers in the range from $\bar{k}_y = 0$ to $\bar{k}_y = 3$.

Figure 11. Real part of the vertical displacement for a unit vertical point load at the surface of a homogeneous halfspace at (a) 20 Hz and (b) 40 Hz.

Figure 11 shows the real part of the vertical displacement due to a vertical point source on the surface of the halfspace at 20 Hz and 40 Hz. The displacement in the soil is dominated by Rayleigh waves with cylindrical wave fronts. The Rayleigh waves have a velocity $C_R = 139.8$ m/s which corresponds to a wavelength $\lambda_R = C_R/f = 6.95$ m at the frequency $f = 20$ Hz and $\lambda_R = 3.48$ m at a frequency $f = 40$ Hz. The penetration depth of the Rayleigh waves is proportional to the wavelength: most of the wave energy is located above a depth of one wavelength. The design of the vibration isolating screen is based on this penetration depth at the lowest frequency for which isolation is desired. A concrete vibration isolating screen with a depth of 8 m and a width of 0.6 m is considered, which is expected to isolate vibrations above a frequency of 20 Hz. The concrete has a Young's modulus $E_c = 30$ GPa, a Poisson's ratio $\nu_c = 0.2$ and a density $\rho_c = 2600$ kg/m³, and is therefore stiffer than the soil.

A vertical point source is considered at the surface of the halfspace at a distance of 5 m from the centerline of the screen. Figure 12 shows the real part of the vertical displacement in the free field at frequencies of 20 Hz and 40 Hz. The incident waves are reflected by the vibration isolating screen, reducing vibration levels behind the screen. At 40 Hz, two lines of destructive interference between direct and reflected Rayleigh waves are observed. As the concrete is much stiffer than the soil, it acts as a rigid wave barrier and a small displacement amplitude is observed along the soil-screen interface.

Figure 12. Real part of the vertical displacement for a unit harmonic point load at a distance of 5 m from the center of the concrete vibration isolating screen at (a) 20 Hz and (b) 40 Hz.

The efficiency of the vibration isolating screen can be quantified by the insertion loss IL_z of the vertical displacement, defined as the ratio of the vertical displacement amplitudes $|u_z^{\text{uniso}}(\omega)|$ and $|u_z^{\text{iso}}(\omega)|$ in the case without and with vibration isolating screen, as expressed in dB:

$$IL_z = 20 \log_{10} \left(\frac{|u_z^{\text{uniso}}(\omega)|}{|u_z^{\text{iso}}(\omega)|} \right) \quad (49)$$

Figure 13 shows the vertical insertion loss at 20 Hz and 40 Hz as expected. The screening efficiency is larger at 40 Hz than at 20 Hz. This indicates that the vibration isolating screen effectively reduces the vibration levels behind the screen.

The results are in correspondence with results obtained with a 2.5D coupled finite element - boundary element formulation [6]. The computation time of the 2.5D PML method is 180 s per frequency, while the same computation using the 2.5D coupled finite element - boundary element approach takes 2 hours on the same processor. This large computation time is due to the calculation of the Green's functions of the halfspace for a large number of source and receiver depths [6], for each wavenumber and frequency.

6. CONCLUSION

In this paper, a 2.5D displacement based PML has been derived. The equilibrium equations of a PML continuum have been formulated in a weak form and discretized using a Galerkin approach through the introduction of 2.5D shape functions.

The choice of the attenuation function has been discussed by considering the analytical solution of the displacement inside a 1D PML. It has been demonstrated that the use of attenuation coefficients for evanescent waves poses stringent conditions on the mesh size, since it compresses the wavelength of propagating waves. It is therefore suggested to use

Figure 13. Vertical insertion loss IL_z for a unit harmonic point load at a distance $d = 5$ m from the concrete vibration isolating screen at (a) 20 Hz and (b) 40 Hz.

zero attenuation coefficients for evanescent waves and to extend the domain size.

The 2.5D PML methodology has been validated by three examples.

In the first example, the Green's displacements of a homogeneous halfspace are compared to a reference solution, demonstrating that the 2.5D PML absorbs all propagating waves for different angles of incidence.

In the second example, the dynamic stiffness of a rigid strip foundation has been computed, where a 2.5D PML solution is compared to a boundary element solution. This demonstrates that the 2.5D PML solution results in an accurate computation of the dynamic stiffness, though the computational domain is small with respect to the wavelength of shear and dilatational waves, especially at lower frequencies. A comparison of the computation time shows that the PML solution is computationally more attractive than a boundary element solution.

The same conclusion is drawn for the third example, where the efficiency of a vibration isolating screen is computed. The computation time for the boundary element method is large due to the calculation of the Green's functions of the halfspace for a large number of source and receiver depths.

The main advantage of the 2.5D approach is that the system matrices of the PML have to be assembled only once for each frequency. The total computation time is dominated by the solution time of the 2.5D finite element equations in the frequency-wavenumber domain. As a result, the 2.5D PML method is a computationally efficient approach for the solution of 3D dynamic soil-structure interaction problems where the geometry is invariant in the longitudinal direction.

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