Non-intersecting squared Bessel paths with one positive starting and ending point

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Abstract

We consider a model of n non-intersecting squared Bessel processes with one starting point $a > 0$ at time $t = 0$ and one ending point $b > 0$ at time $t = T$. After proper scaling, the paths fill out a region in the tx -plane. Depending on the value of the product ab the region may come to the hard edge at 0, or not. We formulate a vector equilibrium problem for this model, which is defined for three measures, with upper constraints on the first and third measures and an external field on the second measure. It is shown that the limiting mean distribution of the paths at time t is given by the second component of the vector that minimizes this vector equilibrium problem. The proof is based on a steepest descent analysis for a 4×4 matrix valued Riemann-Hilbert problem which characterizes the correlation kernel of the paths at time t . We also discuss the precise locations of the phase transitions.

1 Introduction

Let $X(t)$ be a squared Bessel process with parameter $\alpha > -1$, i.e., it is a diffusion process (a strong Markov process with continuous sample paths) on $\mathbb{R}^+$ with transition probability density given by (cf. [11, 36])

$$p_t^\alpha(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\alpha/2} e^{-\frac{x+y}{2t}} I_\alpha\left(\frac{\sqrt{xy}}{t}\right), \quad x, y > 0, \quad (1.1)$$

$$p_t^\alpha(0, y) = \frac{y^\alpha}{(2t)^{\alpha+1} \Gamma(\alpha+1)} e^{-\frac{y}{2t}}, \quad y > 0, \quad (1.2)$$

where

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\alpha}}{k! \Gamma(k+\alpha+1)} \quad (1.3)$$

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is the modified Bessel function of the first kind of order α ; see [1, p.375]. If $d = 2(\alpha + 1)$ is an integer, the squared Bessel process can be viewed as the square of the distance to the origin of a d -dimensional Brownian motion. Some applications of this diffusion process can be found, for example, in [30].

We consider a model of n independent copies $X_j(t)$, $j = 1, \dots, n$, conditioned such that for some $T > 0$,

- $X_j(0) = a_j$, $X_j(T) = b_j$ for $j = 1, \dots, n$ for some given values $0 \leq a_1 < a_2 < \dots < a_n$, $0 \leq b_1 < b_2 < \dots < b_n$,
- the paths do not intersect each other for $0 < t < T$.

It follows from the seminal paper of Karlin and McGregor [31] that the positions of the non-intersecting squared Bessel paths at any given time $t \in (0, T)$ are a determinantal point process. This means that there exists a correlation kernel K_n such that for each positive integer m the m -point correlation function takes the determinantal form

$$\det[K_n(x_j, x_k)]_{j,k=1,\dots,m}. \quad (1.4)$$

Indeed, we have by [31] that the positions of the paths at time t have a joint probability density

$$\mathcal{P}_{n,t}(x_1, \dots, x_n) = \frac{1}{Z_{n,t}} \det[f_j(x_k)]_{j,k=1,\dots,n} \det[g_j(x_k)]_{j,k=1,\dots,n}, \quad (1.5)$$

where $f_j(x) = p_t^\alpha(a_j, x)$, $g_j(x) = p_{T-t}^\alpha(x, b_j)$, $j = 1, \dots, n$, and $Z_{n,t}$ is the normalization constant

$$Z_{n,t} = \int_{(0,\infty)^n} \mathcal{P}_{n,t}(x_1, \dots, x_n) dx_1 \dots dx_n. \quad (1.6)$$

Hence, (1.5) is a biorthogonal ensemble [9], which is a special case of a determinantal point process. The correlation kernel is given by

$$K_n(x, y) = \sum_{j=1}^n \phi_j(x) \psi_j(y), \quad (1.7)$$

where the functions ϕ_j , ψ_j , $j = 1, \dots, n$ are such that

$$\text{span}(\phi_1, \dots, \phi_n) = \text{span}(f_1, \dots, f_n), \quad \text{span}(\psi_1, \dots, \psi_n) = \text{span}(g_1, \dots, g_n),$$

and

$$\int_0^\infty \phi_i(x) \psi_k(x) dx = \delta_{i,k}, \quad i, k = 1, \dots, n. \quad (1.8)$$

We are interested in the situation where all paths start at the same position at $t = 0$ and end at the same position at $t = T$. Different types of initial and ending conditions are considered in [32, 34, 35], among others. In the confluent limit $a_j \rightarrow a \geq 0$ and $b_j \rightarrow b \geq 0$, the biorthogonal ensemble structure is preserved.

Figure 1: 50 rescaled non-intersecting squared Bessel paths with $a = b = 2$ (left) and $a = b = 1/4$ (right).

If $a = b = 0$, i.e., all paths start and end at the origin, this system is known as chiral or Laguerre ensemble in random matrix theory, cf. [20, 33, 36, 49]. The case where $a > 0$ and $b = 0$ was considered in [38] and [39]. In that situation, all paths, after proper scaling, initially stay away from the hard edge at $x = 0$, but at a certain critical time t^* the lowest paths hit the hard edge and are stuck to it from then on. The limiting mean density of the paths at time t is characterized by a non-real solution of an algebraic equation of order three; see also [38, Appendix] and [42] for an interpretation in terms of a vector equilibrium problem with two measures. If $t \neq t^*$, the local correlations obey the usual scaling limit from the random matrix theory, leading to the sine, Airy, and Bessel kernels [38]. A new kernel was established near the origin at the critical time t^* in [39]. The kernel admits a double integral representation which resembles the Pearcey kernel [8, 10, 12].

In this paper, we consider the case where both a and b are positive. To obtain interesting results, we make a time scaling as in [38] so that the time variable depends on the number of paths n . That is, we replace the variables t and T by

$$t \mapsto \frac{t}{2n}, \quad T \mapsto \frac{1}{2n}, \quad (1.9)$$

so that $0 < t < 1$. Now, letting $n \rightarrow \infty$, the paths fill out a region in the tx -plane that looks like one of the regions shown in Figures 1 and 2, depending on the product ab . Suppose a and b stay far away from the hard edge $x = 0$, such that $ab > 1/4$. Then we are in a situation as illustrated by the left picture in Figure 1. All paths remain positive for all time t . On the other hand, if $ab < 1/4$, there are two critical times t_1 and t_2 , as depicted in the right picture of Figure 1. All paths initially stay away from the hard edge (a wall) and the lowest paths hit the hard edge at the time t_1 . Since all paths are required to end at a positive position b , the lowest paths are stuck to the hard edge only for a while, and they leave the wall at a second critical time t_2 . The times t_1 and t_2 will come together if $ab = 1/4$, and we are then led to an intermediate situation shown in the left picture in Figure 2. The paths fill out a region bounded by the solid line, which

Figure 2: 50 rescaled non-intersecting squared Bessel paths with $a = b = 1/2$ and $t_c = 1/2$ (left) and around the point t_c (right).

is tangent to the hard edge at a multicritical time t_c ; see also the right picture in Figure 2 for a magnified view around t_c .

It is the aim of this paper to describe the region that is filled in the large n limit by the non-intersecting squared Bessel paths with one positive starting point and one positive ending point. Our results are stated in the next section.

2 Statement of results

2.1 A vector equilibrium problem

The main ingredient of our analysis is a vector equilibrium problem involving both an external field and two upper constraints. To this end, we define, as usual (see [48]), the logarithmic energy of a measure μ by

$$I(\mu) = \iint \log \frac{1}{|x - y|} d\mu(x) d\mu(y), \quad (2.1)$$

and the mutual energy of two measures μ, ν by

$$I(\mu, \nu) = \iint \log \frac{1}{|x - y|} d\mu(x) d\nu(y). \quad (2.2)$$

Consider the energy functional

$$E(\nu_1, \nu_2, \nu_3) = \sum_{j=1}^3 I(\nu_j) - \sum_{j=1}^2 I(\nu_j, \nu_{j+1}) + \int V(x) d\nu_2(x), \quad (2.3)$$

where

$$V(x) = \frac{x}{t(1-t)} - \frac{2\sqrt{ax}}{t} - \frac{2\sqrt{bx}}{1-t}. \quad (2.4)$$

The vector equilibrium problem is to minimize $E(\nu_1, \nu_2, \nu_3)$ among all measures ν_1, ν_2 and ν_3 satisfying the following conditions (E1)–(E3):

(E1) ν_1 is a measure on $\mathbb{R}^- := (-\infty, 0]$ with total mass $1/2$ that satisfies the constraint

$$\nu_1 \leq \rho_1, \quad (2.5)$$

where ρ_1 is the measure on $\mathbb{R}^-$ with density

$$\frac{d\rho_1}{dx} = \frac{\sqrt{a}}{\pi t} |x|^{-1/2}, \quad x \in \mathbb{R}^-. \quad (2.6)$$

(E2) ν_2 is a measure on $\mathbb{R}^+ := [0, \infty)$ with total mass 1.

(E3) ν_3 is a measure on $\mathbb{R}^-$ with total mass $1/2$ that satisfies the constraint

$$\nu_3 \leq \rho_3, \quad (2.7)$$

where ρ_3 is the measure on $\mathbb{R}^-$ with density

$$\frac{d\rho_3}{dx} = \frac{\sqrt{b}}{\pi(1-t)} |x|^{-1/2}, \quad x \in \mathbb{R}^-. \quad (2.8)$$

An electrostatic interpretation of the above equilibrium problem is the following: Consider three types of charged particles. The particles of the first and the third types are put on $\mathbb{R}^-$, both having total charge $\frac{1}{2}$. The particles of the second type are put on $\mathbb{R}^+$ with total charge 1. Particles of the same type repel each other. The particles of the first and the second types interact with each other with an attraction that is only half of the strength of the repulsion of particles of the same type. So do the particles of the second and the third types. The particles of the first and the third types do not interact directly. The particles of the second type are influenced by the external field V . The measures ρ_1 and ρ_3 in (2.6) and (2.8) play the role of upper constraints for the first and the third measures. Namely, we minimize the equilibrium problem with the extra condition that the particle densities of the first and the third types do not exceed the densities of ρ_1 and ρ_3 , respectively.

Vector equilibrium problems with mutual interaction can be traced back to [27, 28], see also [46]. The equilibrium problem with constraints appeared before mainly in the asymptotic analysis of discrete orthogonal polynomials, see e.g. [6, 21, 47].

Our first result concerns the structure of the minimizer of the vector equilibrium problem.

Theorem 2.1. *Let $a > 0$, $b > 0$ and $0 < t < 1$. Then there is a unique minimizer (μ_1, μ_2, μ_3) of the energy functional (2.3) under the conditions (E1), (E2), (E3). The minimizing measures satisfy:*

(a) *The support of μ_1 is the negative real axis and there exists $r_1 \geq 0$ such that*

$$\text{supp}(\rho_1 - \mu_1) = (-\infty, -r_1]. \quad (2.9)$$

The density of the measure $\rho_1 - \mu_1$ vanishes like a square root at $-r_1$ if $r_1 > 0$.

(b) The support of μ_2 is bounded and there exist $0 \leq p < q$ such that

$$\text{supp}(\mu_2) = [p, q], \quad (2.10)$$

and μ_2 has a density which is real analytic in the interior of the support and vanishes like a square root at the endpoint q and at p if $p > 0$.

(c) The support of μ_3 is the negative real axis and there exists $r_3 \geq 0$ such that

$$\text{supp}(\rho_3 - \mu_3) = (-\infty, -r_3]. \quad (2.11)$$

The density of the measure $\rho_3 - \mu_3$ vanishes as a square root at $-r_3$ if $r_3 > 0$.

Note that p, q, r_1 , and r_3 depend on a, b and t .

The proof of Theorem 2.1 will be given in Section 3. Throughout the whole paper, we will denote

$$\Delta_1 := (-\infty, -r_1), \quad \Delta_2 := (p, q), \quad \text{and} \quad \Delta_3 := (-\infty, -r_3), \quad (2.12)$$

i.e., Δ_j , $j = 1, 2, 3$ is the interior of the sets $\text{supp}(\rho_1 - \mu_1)$, $\text{supp}(\mu_2)$ and $\text{supp}(\rho_3 - \mu_3)$, respectively.

2.2 Classification into cases

It turns out that at least one of r_1 , p , r_3 is equal to 0, and generically no two consecutive ones are 0. Generically we also have a $O(|x|^{-1/2})$ behavior of the density at the hard edge 0. Then there are three generic cases.

Case I $r_1 = r_3 = 0$ and $p > 0$. The densities of $\rho_1 - \mu_1$ and $\rho_3 - \mu_3$ blow up like $O(|x|^{-1/2})$ as $x \rightarrow 0-$.

Case II $p > 0$ and one of r_1, r_3 is zero and the other is positive. Thus there are two subcases which play a symmetric role.

subcase IIa $r_1 > 0$, $p > 0$ and $r_3 = 0$.

subcase IIb $r_1 = 0$, $p > 0$ and $r_3 > 0$.

Case III $r_1 > 0$, $p = 0$ and $r_3 > 0$.

In Case I we have to add the condition that the densities of $\rho_1 - \mu_1$ and $\rho_3 - \mu_3$ blow up like an inverse square root at 0 in order to be in a generic case. There is also a non-generic case where one of these densities vanishes as a square root. This is exactly what happens in the transition from Case I to Case II. See Case V below.

The non-generic cases are:

Case IV Exactly two consecutive values from r_1, p, r_2 are zero. Thus $p = 0$ and one of r_1, r_3 is also zero, while the other is positive.

Case V $r_1 = r_3 = 0$ and $p > 0$. One of the densities of $\rho_j - \mu_j$ vanishes like a square root $O(|x|^{1/2})$ as $x \rightarrow 0-$, while the other blows up like $O(|x|^{-1/2})$ as $x \rightarrow 0-$.

Case VI All three are zero: $r_1 = p = r_3 = 0$.

The Case I corresponds to the situation as shown by the left picture in Figure 1. In this case $ab > 1/4$ and all paths stay away from the hard edge at $x = 0$ for all $t \in (0, 1)$. Both constraints are never active.

The right picture of Figure 1 which deals with $ab < 1/4$, contains the three Cases II, III, IV. If $t < t_1$ or $t > t_2$, then we are in Case II, where only one constraint is active. If $t_1 < t < t_2$, then we are then in Case III, where both constraints are active. The times t_1 and t_2 correspond to Case IV, which represents a phase transition between soft edge and hard edge.

The final Cases V and VI are contained in Figure 2 for the situation $ab = 1/4$ when the domain just touches the wall. We are in Case V if $t \neq t_c$, and in case VI if $t = t_c$.

The Cases IV–VI are non-generic critical cases. They correspond to certain phase transitions between the generic cases; see also Section 4.4 for a phase diagram in the situation where $ab < 1/4$.

We shall only consider the generic cases in this paper. For convenience (and without loss of generality), it is assumed that

$$r_3 \geq r_1 \tag{2.13}$$

in Cases II and III, which implies that $\Delta_3 \subset \Delta_1 \subset \mathbb{R}^-$ holds in all three cases.

2.3 Variational conditions

Denote by

$$U^\mu(x) = \int \log \frac{1}{|x - y|} d\mu(y) \tag{2.14}$$

the logarithmic potential of a measure μ . The minimizer (μ_1, μ_2, μ_3) is characterized by the following Euler-Lagrange variational conditions:

Proposition 2.2. *The measures μ_1 , μ_2 and μ_3 satisfy for some constant $l \in \mathbb{R}$,*

$$2U^{\mu_2}(x) = U^{\mu_1}(x) + U^{\mu_3}(x) - V(x) + l, \quad x \in \text{supp}(\mu_2), \tag{2.15}$$

$$2U^{\mu_2}(x) > U^{\mu_1}(x) + U^{\mu_3}(x) - V(x) + l, \quad x \in \mathbb{R}^+ \setminus \text{supp}(\mu_2), \tag{2.16}$$

where $V(x)$ is given in (2.4);

$$2U^{\mu_1}(x) = U^{\mu_2}(x), \quad x \in \text{supp}(\rho_1 - \mu_1), \tag{2.17}$$

$$2U^{\mu_1}(x) < U^{\mu_2}(x), \quad x \in \mathbb{C} \setminus \text{supp}(\rho_1 - \mu_1), \tag{2.18}$$

and

$$2U^{\mu_3}(x) = U^{\mu_2}(x), \quad x \in \text{supp}(\rho_3 - \mu_3), \tag{2.19}$$

$$2U^{\mu_3}(x) < U^{\mu_2}(x), \quad x \in \mathbb{C} \setminus \text{supp}(\rho_3 - \mu_3). \tag{2.20}$$

Note that the inequalities in (2.16), (2.18) and (2.20) are all strict. The strictness of the inequality in (2.16) follows from the strict convexity of the external fields acting the second measure; see Lemma 3.3 below.

The proof of Proposition 2.2 will be given in Section 3.

2.4 Limiting mean distribution

The following is the main theorem of this paper.

Theorem 2.3. *Let $a, b > 0$, $0 < t < 1$, and let K_n denote the correlation kernel for the positions of the non-intersecting squared Bessel paths at time t in the rescaled time variables (1.9). Let (μ_1, μ_2, μ_3) be the minimizer of the vector equilibrium problem stated in Section 2.1. Suppose that we are in one of the generic cases I, II, or III, described above, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = \frac{d\mu_2}{dx}(x), \quad x \in \mathbb{R}^+. \quad (2.21)$$

The limit (2.21) expresses that the density of the measure μ_2 is the limiting mean density of the non-intersecting squared Bessel paths at time t as $n \rightarrow \infty$.

It is an unusual feature that it is the second measure which plays the main role here. In recent papers [23, 24] concerning eigenvalue distribution of a two-matrix model in random matrix theory, a vector equilibrium problem with three measures appears as well. In that situation, however, the first component of the minimizer is relevant to the limiting mean distribution.

We prove Theorem 2.3 by means of a steepest descent analysis of a relevant Riemann-Hilbert problem. From this analysis, we also obtain the universality results for the local path correlations. For Cases I, II, III, we obtain the sine kernel (in the bulk), Airy kernel (at the soft edges) and Bessel kernel of order α (at the hard edge 0), which are the typical limiting kernels in random matrix theory, see e.g. [15, 26, 44]. We will not discuss this any further but instead refer to the papers [4, 7, 8, 23] for a more detailed analysis in similar situations.

We expect the conclusion of Theorem 2.3 to remain valid in the Cases IV–VI. The local path correlations at the origin in Case IV should be the same as those derived in [39], since they both correspond to phase transitions between a soft and a hard edge. The Cases V–VI represent new critical phenomena. It is conjectured that they are related to the inhomogeneous Painlevé II equation, which bears some connections with the ‘critical separation’ case for non-intersecting Brownian motion with two starting points and two ending points [2, 18, 19, 22]. These results will be reported in a future publication.

Theorem 2.3 will be proved in Section 6.

2.5 About the proof of Theorem 2.3

The proof of Theorem 2.3 relies on the Riemann-Hilbert (RH) problem for multiple orthogonal polynomials of mixed type and its connection to the correlation kernel (1.7). To see this, we start with a proposition which gives us the explicit structure of the biorthogonal ensemble in the confluent case.

Proposition 2.4. *In confluent case $a_j \rightarrow a > 0$, $b_j \rightarrow b > 0$, $j = 1, \dots, n$, the positions of the non-intersecting squared Bessel paths at time $t \in (0, T)$ are a biorthogonal ensemble (1.5) with functions*

$$f_{2j-1}(x) = x^{j-1} p_t^\alpha(a, x), \quad j = 1, \dots, n_1 := \lceil n/2 \rceil, \quad (2.22)$$

$$f_{2j}(x) = x^{j-1} p_t^{\alpha+1}(a, x), \quad j = 1, \dots, n_2 := n - n_1, \quad (2.23)$$

$$g_{2j-1}(x) = x^{j-1} p_{T-t}^\alpha(x, b), \quad j = 1, \dots, n_1, \quad (2.24)$$

$$g_{2j}(x) = x^{j-1} p_{T-t}^{\alpha-1}(x, b), \quad j = 1, \dots, n_2. \quad (2.25)$$

Proof. The functions (2.22) and (2.23) follow as in the proof of Proposition 2.1 in [38].

In the confluent limit $b_j \rightarrow b$, the linear space spanned by the functions $x \mapsto p_{T-t}^\alpha(x, b_j)$, $j = 1, \dots, n$, tends to the linear space spanned by

$$x \mapsto \frac{\partial^{j-1}}{\partial y^{j-1}} p_{T-t}^\alpha(x, b), \quad j = 1, \dots, n. \quad (2.26)$$

Using the differential relations satisfied by the modified Bessel functions of the first kind (see [1]), we can prove that

$$\begin{aligned} \frac{\partial}{\partial y} p_t^\alpha(x, y) &= \frac{1}{2t} (p_t^{\alpha-1}(x, y) - p_t^\alpha(x, y)), \\ y \frac{\partial}{\partial y} p_t^{\alpha-1}(x, y) &= \frac{x}{2t} p_t^\alpha(x, y) - \left(\frac{y}{2t} - (\alpha - 1) \right) p_t^{\alpha-1}(x, y). \end{aligned}$$

Now it is easy to show inductively that the linear span of (2.26) is the same as the linear space spanned by the functions (2.24) and (2.25). $\square$

Next, we introduce two sets of weight functions $w_{1,1}, w_{1,2}$ and $w_{2,1}, w_{2,2}$,

$$\begin{aligned} w_{1,1}(x) &= x^{\alpha/2} e^{-\frac{nx}{t}} I_\alpha \left(\frac{2n\sqrt{ax}}{t} \right), & w_{1,2}(x) &= x^{(\alpha+1)/2} e^{-\frac{nx}{t}} I_{\alpha+1} \left(\frac{2n\sqrt{ax}}{t} \right), \\ w_{2,1}(x) &= x^{-\alpha/2} e^{-\frac{nx}{1-t}} I_\alpha \left(\frac{2n\sqrt{bx}}{1-t} \right), & w_{2,2}(x) &= x^{-(\alpha-1)/2} e^{-\frac{nx}{1-t}} I_{\alpha-1} \left(\frac{2n\sqrt{bx}}{1-t} \right). \end{aligned}$$

which are collected in two row vectors

$$\mathbf{w}_1 = (w_{1,1}, w_{1,2}), \quad \mathbf{w}_2 = (w_{2,1}, w_{2,2}). \quad (2.27)$$

Note that by (1.1) and (1.9)

$$\begin{aligned} f_{2j-1}(x) &\propto x^{j-1} w_{1,1}(x), & f_{2j}(x) &\propto x^{j-1} w_{1,2}(x), \\ g_{2j-1}(x) &\propto x^{j-1} w_{2,1}(x), & g_{2j}(x) &\propto x^{j-1} w_{2,2}(x), \end{aligned} \quad (2.28)$$

where f_j, g_j , $j = 1, \dots, n$ are given in (2.22)–(2.25). The proportionality constants are irrelevant for us.

We then look for a 4×4 matrix valued function Y satisfying the following RH problem:

- (1) Y is defined and analytic in $\mathbb{C} \setminus \mathbb{R}^+$.
- (2) For $x > 0$, Y possesses continuous boundary values $Y_+(x)$ (from the upper half plane) and $Y_-(x)$ (from the lower half plane), which satisfy

$$Y_+(x) = Y_-(x) \begin{pmatrix} I_2 & W(x) \\ 0 & I_2 \end{pmatrix}, \quad (2.29)$$

where I_k denotes the $k \times k$ identity matrix and $W(x)$ is the rank-one matrix

$$\begin{aligned} W(x) &= \mathbf{w}_1^T(x) \mathbf{w}_2(x) \\ &= e^{-\frac{nx}{(1-t)t}} \begin{pmatrix} I_\alpha \left(\frac{2n\sqrt{ax}}{t} \right) I_\alpha \left(\frac{2n\sqrt{bx}}{1-t} \right) & \sqrt{x} I_\alpha \left(\frac{2n\sqrt{ax}}{t} \right) I_{\alpha-1} \left(\frac{2n\sqrt{bx}}{1-t} \right) \\ \sqrt{x} I_{\alpha+1} \left(\frac{2n\sqrt{ax}}{t} \right) I_\alpha \left(\frac{2n\sqrt{bx}}{1-t} \right) & x I_{\alpha+1} \left(\frac{2n\sqrt{ax}}{t} \right) I_{\alpha-1} \left(\frac{2n\sqrt{bx}}{1-t} \right) \end{pmatrix}. \end{aligned} \quad (2.30)$$

- (3) As $z \rightarrow \infty$, $z \in \mathbb{C} \setminus \mathbb{R}^+$, we have

$$Y(z) = (I_4 + O(1/z)) \operatorname{diag}(z^{n_1}, z^{n_2}, z^{-n_1}, z^{-n_2}). \quad (2.31)$$

- (4) $Y(z)$ has the following behavior near the origin:

$$Y(z) = O \begin{pmatrix} 1 & 1 & h(z) & h(z) \\ 1 & 1 & h(z) & h(z) \\ 1 & 1 & h(z) & h(z) \\ 1 & 1 & h(z) & h(z) \end{pmatrix}, \quad Y(z)^{-1} = O \begin{pmatrix} h(z) & h(z) & h(z) & h(z) \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad (2.32)$$

as $z \rightarrow 0$, $z \in \mathbb{C} \setminus \mathbb{R}^+$, where

$$h(z) = \begin{cases} |z|^\alpha, & \text{if } -1 < \alpha < 0, \\ \log |z|, & \text{if } \alpha = 0, \\ 1, & \text{if } \alpha > 0, \end{cases} \quad (2.33)$$

and the O condition in (2.32) is taken to be entrywise.

There exists a unique solution Y to the above RH problem which involves multiple orthogonal polynomials of mixed type associated with the modified Bessel weights $\mathbf{w}_1(x)$, $\mathbf{w}_2(x)$ and their Cauchy transforms; see [13] for details. Due to the singularity of the weight matrix near the origin, we need the condition (2.32) to ensure the uniqueness of the solution.

It also follows from the results in [13] that the correlation kernel (1.7) admits the following representation in terms of the solution of the RH problem:

$$K_n(x, y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 0 & w_{2,1}(y) & w_{2,2}(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} w_{1,1}(x) \\ w_{1,2}(x) \\ 0 \\ 0 \end{pmatrix}. \quad (2.34)$$

The representation (2.34) is based on the Christoffel-Darboux formula for multiple orthogonal polynomials of mixed type, see also [3].

We will obtain the limit (2.21) from a Deift/Zhou steepest descent analysis [15, 16, 17] for the RH problem for Y . It consists of a series of explicit and invertible transformations

$$Y \mapsto X \mapsto U \mapsto T \mapsto R, \quad (2.35)$$

which leads to a RH problem for a matrix valued function R which is such that R tends to the identity matrix as $n \rightarrow \infty$. Analyzing the effect of the transformations (2.35) we obtain the limit (2.21).

The rest of this paper is organized as follows. In Section 3 we analyze the equilibrium problem and establish Theorem 2.1 and Proposition 2.2. In Section 4 we introduce the Riemann surface built from the solution to the vector equilibrium problem and define some auxiliary functions. Section 5 contains the steepest descent analysis of the RH problem for $Y(z)$ and Section 6 gives the proof of the main Theorem 2.3. Throughout this paper, it is also assumed that n is an even number so that

$$n_1 = n_2 = n/2. \quad (2.36)$$

This assumption is not essential and is only made to simplify the presentation.

3 Analysis of the vector equilibrium problem

The main purpose of this section is to prove Theorem 2.1 and Proposition 2.2, which yields the existence and main properties of the minimizers μ_1 , μ_2 and μ_3 . Our basic tool is the balayage (sweeping out) of a measure.

For any finite measure ν we denote with $\|\nu\|$ the total mass of ν . A positive measure $\hat{\nu}$ on a closed set $K \subset \mathbb{C}$ with positive capacity is called the balayage of ν onto K if $\|\nu\| = \|\hat{\nu}\|$ and

$$U^\nu(x) = U^{\hat{\nu}}(x) + C, \quad \text{q.e. } x \in K, \quad (3.1)$$

for some constant C , where “q.e.” means quasi-every, i.e., with the exception of a set with zero capacity, and U^ν is the logarithmic potential (2.14), see [48] for background on potential theory. To emphasize the dependence on K , we also write

$$\hat{\nu} = \text{Bal}(\nu, K). \quad (3.2)$$

A relation between ν and $\text{Bal}(\nu, K)$ is established as follows:

$$\text{Bal}(\nu, K) = \int \text{Bal}(\delta_s, K) d\nu(s), \quad (3.3)$$

where δ_z denotes the Dirac delta measure at the point z .

For the purpose of this paper, we are interested in the case where

$$K = K_c = (-\infty, -c], \quad \text{with } c \geq 0.$$

Then $C = 0$ in (3.1), since K_c is unbounded. By using simple contour integration, it can be shown that, if $s \in \mathbb{R}^+$, the balayage of δ_s onto K_c has the density

$$\frac{d\text{Bal}(\delta_s, K_c)}{dx} = \frac{1}{\pi} \frac{\sqrt{s+c}}{\sqrt{|x+c|}(s-x)}, \quad x \in K_c. \quad (3.4)$$

In particular, for $c = 0$,

$$\frac{d\text{Bal}(\delta_s, \mathbb{R}^-)}{dx} = \lim_{c \rightarrow 0^+} \frac{d\text{Bal}(\delta_s, K_c)}{dx} = \frac{1}{\pi} \frac{\sqrt{s}}{\sqrt{|x|}(s-x)}, \quad x \in \mathbb{R}^-. \quad (3.5)$$

Observe that the density (3.4) is decreasing as $|x|$ increases, with a decay rate $O(|x|^{-3/2})$ as $|x| \rightarrow \infty$. It will become clear in what follows how the concept of balayage measure is linked to the equilibrium problem.

3.1 Equilibrium problems for ν_1 and ν_3

If ν_2 and ν_3 are given, the equilibrium problem for ν_1 is to minimize

$$I(\nu) - \int U^{\nu_2}(x) d\nu(x), \quad (3.6)$$

among all measures ν on $\mathbb{R}^-$ with total mass $1/2$ and satisfying the constraint $\nu \leq \rho_1$, where ρ_1 is the measure on $\mathbb{R}^-$ with density given by (2.6). Clearly, this problem only depends on ν_2 . The properties of ν_1 are given in the following lemma.

Lemma 3.1. *(The measure ν_1) Suppose ν_2 and ν_3 are fixed so that conditions (E2) and (E3) of the equilibrium problem are satisfied. Then the measure ν_1 that minimizes $E(\nu_1, \nu_2, \nu_3)$ subject to the condition (E1) exists. It has the following properties:*

(i) *If the constraint ρ_1 is not active, then the measure ν_1 is given by*

$$\nu_1 = \frac{1}{2} \text{Bal}(\nu_2, \mathbb{R}^-), \quad (3.7)$$

or, equivalently, in view of (3.3) and (3.5),

$$\frac{d\nu_1(x)}{dx} = \frac{1}{2\pi\sqrt{|x|}} \int \frac{\sqrt{s}}{s-x} d\nu_2(s), \quad x \in \mathbb{R}^-. \quad (3.8)$$

(ii) If the constraint ρ_1 is active, then there exists a constant $r_1 > 0$, such that

$$\text{supp}(\rho_1 - \nu_1) = (-\infty, -r_1]. \quad (3.9)$$

Moreover, we have

$$\begin{cases} 2U^{\nu_1}(x) = U^{\nu_2}(x), & x \in \text{supp}(\rho_1 - \nu_1), \\ 2U^{\nu_1}(x) < U^{\nu_2}(x), & x \in \mathbb{R}^- \setminus \text{supp}(\rho_1 - \nu_1), \end{cases} \quad (3.10)$$

and $\rho_1 - \nu_1$ vanishes like a square root at $-r_1$.

(iii) Suppose ν_2 has a compact support. Then here exists a constant $\tilde{K} > 0$, depending only on $\max \text{supp}(\nu_2)$ such that

$$\frac{d\nu_1(x)}{dx} \leq \frac{\tilde{K}}{|x|^{3/2}}, \quad x \in \mathbb{R}^-. \quad (3.11)$$

Proof. (i) Suppose that $\nu_1 = \text{Bal}(\nu_2, \mathbb{R}^-)/2$. Then $2U^{\nu_1}(x) = U^{\nu_2}(x)$ for $x \in \mathbb{R}^-$, which is the variational condition for (3.6) without constraint. This proves part (i).

(ii) Part (ii) is proved by using iterated balayage algorithm of [37]. The main idea is to construct a sequence of real numbers $\{r_{1,k}(t)\}$ and a sequence of measures $\{\nu_{1,k}\}$ with limit $r_1(t)$ and ν_1 , respectively. Since this argument is similar to those given in [23], we omit the proof here.

(iii) Suppose ν_2 has a compact support and let

$$b = \max \text{supp}(\nu_2).$$

By part (ii), the constraint is active on some interval $[-r_1, 0)$ (which could be empty). Since $\|\nu_1\| = 1/2$, we then have that $r_1 \leq c_1$ where $c_1 > 0$ is the value for which $\rho_1([-c_1, 0]) = 1/2$. By properties of balayage we then have that the restriction of ν_1 to K_{c_1} is less than the balayage of $\nu_2/2$ onto K_{c_1} . Hence by (3.3) and (3.4)

$$\frac{d\nu_1(x)}{dx} \leq \frac{1}{2\pi\sqrt{|x+c_1|}} \int \frac{\sqrt{s+c_1}}{s-x} d\nu_2(s), \quad x \in (-\infty, -c_1]. \quad (3.12)$$

The function $s \mapsto \frac{\sqrt{s+c_1}}{s-x}$ is increasing for $s \in [-c_1, -x-2c_1]$. Thus if $x \leq -2c_1 - b$, then the maximum on $\text{supp}(\nu_2)$ is taken at its right end point $s = b$, and therefore

$$\frac{d\nu_1(x)}{dx} \leq \frac{1}{2\pi\sqrt{|x+c_1|}} \frac{\sqrt{b+c_1}}{b-x}, \quad x \in (-\infty, -2c_1 - b]. \quad (3.13)$$

Then there is a constant $\tilde{K}$, which only depends on b , such that (3.11) holds for $x \leq -2c_1 - b$. Since $\nu_1 \leq \rho_1$, we can then adjust the constant if necessary, such that (3.11) holds for every $x < 0$. □

If ν_1 and ν_2 are fixed, the equilibrium problem for ν_3 is to minimize (3.6) among all measures ν on $\mathbb{R}^-$ with total mass $1/2$ and satisfying the constraint $\nu \leq \rho_3$, where ρ_3 is given in (2.8). Hence, the properties of ν_3 can be derived in a manner similar to the analysis given above. We omit the proof and state the results in the following lemma:

Lemma 3.2. *(The measure ν_3) Suppose ν_1 and ν_2 are fixed so that conditions (E1) and (E2) of the equilibrium problem are satisfied. Then the measure ν_3 that minimizes $E(\nu_1, \nu_2, \nu_3)$ subject to the condition (E3) exists. It has the following properties:*

(i) *If the constraint ρ_3 is not active, the measure ν_3 is given by*

$$\frac{d\nu_3(x)}{dx} = \frac{1}{2\pi\sqrt{|x|}} \int \frac{\sqrt{z}}{z-x} d\nu_2(z), \quad x \in \mathbb{R}^-. \quad (3.14)$$

(ii) *If the constraint ρ_3 is active, there exists a constant $r_3(t) > 0$, such that*

$$\text{supp}(\rho_3 - \nu_3) = (-\infty, -r_3(t)]. \quad (3.15)$$

Moreover, we have

$$\begin{cases} 2U^{\nu_3}(x) = U^{\nu_2}(x), & x \in \text{supp}(\rho_3 - \nu_3), \\ 2U^{\nu_3}(x) < U^{\nu_2}(x), & x \in \mathbb{R}^- \setminus \text{supp}(\rho_3 - \nu_3), \end{cases} \quad (3.16)$$

and $\rho_3 - \nu_3$ vanishes like a square root at $-r_3(t)$.

(iii) *Suppose ν_2 has a compact support. Then there exists a constant $\tilde{K} > 0$, depending only on $\max \text{supp}(\nu_2)$ such that*

$$\frac{d\nu_3(x)}{dx} \leq \frac{\tilde{K}}{|x|^{3/2}}, \quad x \in \mathbb{R}^-. \quad (3.17)$$

3.2 Equilibrium problem for ν_2

If ν_1 and ν_3 satisfying conditions (E1) and (E3) are fixed, the equilibrium problem for ν_2 is to minimize

$$I(\nu) + \int (V(x) - U^{\nu_1}(x) - U^{\nu_3}(x)) d\nu(x) \quad (3.18)$$

among all probability measures ν on $\mathbb{R}^+$, where V is given by (2.4). This is a usual equilibrium problem with external field

$$V(x) - U^{\nu_1}(x) - U^{\nu_3}(x) \quad (3.19)$$

see e.g. [15, 48].

Lemma 3.3. *(The measure ν_2) Suppose ν_1 and ν_3 are fixed so that conditions (E1) and (E3) in the equilibrium problem are satisfied, in particular we have $\nu_j \leq \rho_j$ for $j = 1, 3$. Then the measure ν_2 that minimizes $E(\nu_1, \nu_2, \nu_3)$ subject to condition (E2) exists and has the following properties:*

- (i) *The external field (3.19) is strictly convex on $\mathbb{R}^+$, and there exist two constants $0 \leq p < q$ such that*

$$\text{supp}(\nu_2) = [p, q]. \quad (3.20)$$

- (ii) *If $0 \notin \text{supp}(\rho_1 - \nu_1) \cup \text{supp}(\rho_3 - \nu_3)$ (so that both constraints are active in a neighborhood of 0), then (3.19) is real analytic on $\mathbb{R}^+$ (including 0).*

Proof. (i) The second derivative of (3.19) is, for $x > 0$,

$$\begin{aligned} V''(x) - (U^{\nu_1})''(x) - (U^{\nu_3})''(x) \\ = \frac{\sqrt{a}}{2t} x^{-3/2} + \frac{\sqrt{b}}{2(1-t)} x^{-3/2} - \int_{-\infty}^0 \frac{d\nu_1(s)}{(x-s)^2} - \int_{-\infty}^0 \frac{d\nu_3(s)}{(x-s)^2}. \end{aligned} \quad (3.21)$$

Since ν_1 satisfies the constraint $\nu_1 \leq \rho_1$ we have for $x > 0$,

$$\int_{-\infty}^0 \frac{d\nu_1(s)}{(x-s)^2} < \int_{-\infty}^0 \frac{d\rho_1(s)}{(x-s)^2} = \frac{\sqrt{a}}{\pi t} \int_{-\infty}^0 \frac{ds}{\sqrt{|s|}(x-s)^2} = \frac{\sqrt{a}}{2t} x^{-3/2}, \quad (3.22)$$

where we used the explicit expression (2.6) for ρ_1 . The above inequality is strict since ν_1 cannot be equal to ρ_1 on the whole negative real axis. Similarly, we find

$$\int_{-\infty}^0 \frac{d\nu_3(s)}{(x-s)^2} < \frac{\sqrt{b}}{2(1-t)} x^{-3/2}, \quad x > 0. \quad (3.23)$$

Combining (3.21)–(3.23), we find that (3.21) is positive, and therefore (3.19) is strictly convex. It then follows from [48, Theorem IV.1.10] that the support of ν_2 consists of one interval, say, $[p, q]$ with $0 \leq p < q$.

- (ii) Suppose that $s > 0$ is such that $\nu_1 = \rho_1$ and $\nu_3 = \rho_3$ on $(-s, 0)$. Thus, if $x \geq 0$,

$$-U^{\nu_1}(x) = \int_{-s}^0 \log|x-y| d\rho_1(y) + \int_{-\infty}^{-s} \log|x-y| d\rho_1(y) \quad (3.24)$$

where the second term is real analytic on $\mathbb{R}^+$, and for the first term we use the explicit expression (2.6) for ρ_1 . Then we obtain for the derivative of (3.24) (where “real analytic” means a real analytic function on $\mathbb{R}^+$)

$$\begin{aligned} (-U^{\nu_1})'(x) &= \frac{\sqrt{a}}{\pi t} \int_{-s}^0 \frac{dy}{(x-y)\sqrt{-y}} + \text{“real analytic”} \\ &= \frac{\sqrt{a}}{\pi t} \int_{-\infty}^0 \frac{dy}{(x-y)\sqrt{-y}} + \text{“real analytic”} \\ &= \frac{\sqrt{a}}{t} x^{-1/2} + \text{“real analytic”}. \end{aligned} \quad (3.25)$$

Then by integrating (3.25) we find that $U^{\nu_1}(x) + \frac{2\sqrt{ax}}{t}$ is real analytic for $x \in \mathbb{R}^+$ (including at $x = 0$).

Similarly, we find that $U^{\nu_3}(x) + \frac{2\sqrt{bx}}{1-t}$ is real analytic on $\mathbb{R}^+$ as well. Then V is real analytic because of (2.4) and (3.19). □

3.3 Proofs of Theorem 2.1 and Proposition 2.2

Proof of uniqueness. We rewrite the energy functional $E(\nu_1, \nu_2, \nu_3)$ as

$$E(\nu_1, \nu_2, \nu_3) = \frac{1}{2}I(\nu_2) + \frac{1}{4}I(\nu_2 - 2\nu_1) + \frac{1}{4}I(\nu_2 - 2\nu_3) + \int V(x)d\nu_2(x). \quad (3.26)$$

It is well-known that if two distinct measures ν and μ have finite logarithmic energy and $\int d\nu = \int d\mu$, then

$$I(\nu - \mu) \geq 0. \quad (3.27)$$

Hence, it follows that the energy functional is strictly convex. The minimizer is therefore uniquely determined by the Euler-Lagrange variational conditions listed in Proposition 2.2. □

Proof of existence. It is easily seen from (3.26)–(3.27) that

$$E(\nu_1, \nu_2, \nu_3) \geq \frac{1}{2}I(\nu_2) + \int V(x)d\nu_2(x). \quad (3.28)$$

Thus the energy functional is bounded from below. As a consequence, we can find a sequence $(\nu_{1,n}, \nu_{2,n}, \nu_{3,n})$ such that

$$E(\nu_{1,n}, \nu_{2,n}, \nu_{3,n}) \leq \frac{1}{n} + \inf E(\nu_1, \nu_2, \nu_3), \quad (3.29)$$

where both the sequence and the infimum are taken subject to conditions (E1)–(E3) of the equilibrium problem. In addition, we may assume that

$$\text{supp}(\nu_{2,n}) \subset [p, q],$$

and, in view of (3.11), (3.17),

$$\frac{d\nu_{i,n}(x)}{dx} \leq \frac{\tilde{K}}{|x|^{3/2}}, \quad i = 1, 3$$

with $0 \leq p < q$ and $\tilde{K} > 0$ independent of n . Therefore, there exists a weakly convergent subsequence of $(\nu_{1,n}, \nu_{2,n}, \nu_{3,n})$, with a certain weak limit (ν_1, ν_2, ν_3) . By the fact that the energy functional E is lower semi-continuous [29], the vector of measures (ν_1, ν_2, ν_3) is then the required minimizer of the energy minimization problem. □

Proofs of Theorem 2.1 and Proposition 2.2. We already showed that there exists a unique minimizer (μ_1, μ_2, μ_3) for the equilibrium problem. The parts (a)–(c) of Theorem 2.1 and Proposition 2.2 are now immediate by Lemmas 3.1–3.3. $\square$

4 Riemann surface and auxiliary functions

In this section, we shall construct a Riemann surface $\mathcal{R}$ with the minimizer (μ_1, μ_2, μ_3) of the vector equilibrium problem and introduce some auxiliary functions for further use.

4.1 A four-sheeted Riemann surface $\mathcal{R}$

The Riemann surface $\mathcal{R}$ has four sheets $\mathcal{R}_j$, $j = 1, \dots, 4$, which are defined as

$$\begin{aligned}\mathcal{R}_1 &= \mathbb{C} \setminus \text{supp}(\rho_1 - \mu_1), & \mathcal{R}_2 &= \mathbb{C} \setminus (\text{supp}(\rho_1 - \mu_1) \cup \text{supp}(\mu_2)), \\ \mathcal{R}_3 &= \mathbb{C} \setminus (\text{supp}(\mu_2) \cup \text{supp}(\rho_3 - \mu_3)), & \mathcal{R}_4 &= \mathbb{C} \setminus \text{supp}(\rho_3 - \mu_3).\end{aligned}\quad (4.1)$$

These four sheets are connected as follows: $\mathcal{R}_1$ is connected to $\mathcal{R}_2$ via $\text{supp}(\rho_1 - \mu_1)$, $\mathcal{R}_2$ is connected to $\mathcal{R}_3$ via $\text{supp}(\mu_2)$, and $\mathcal{R}_3$ is connected to $\mathcal{R}_4$ via $\text{supp}(\rho_3 - \mu_3)$. Each connection is in the usual crosswise manner; see Figure 3.

The Riemann surface is compactified by adding two points at infinity: one is common to the sheets $\mathcal{R}_1$ and $\mathcal{R}_2$, and the other one is common to the sheets $\mathcal{R}_3$ and $\mathcal{R}_4$. Since the Riemann surface $\mathcal{R}$ has 4 sheets and 6 simple branch points, it then follows from Hurwitz's theorem (see e.g. [45]) that it has genus 0. This fact will be helpful in our future construction of the global parametrix.

4.2 The ξ -functions

We use F_j to denote the Cauchy transform of the measure μ_j , i.e.,

$$F_j(z) = \int \frac{1}{z-x} d\mu_j(x), \quad z \in \mathbb{C} \setminus \text{supp}(\mu_j), \quad j = 1, 2, 3. \quad (4.2)$$

With the above notation, the Sokhotski-Plemelj formulas now read

$$F_{j,+}(z) + F_{j,-}(z) = 2 \text{PV} \int \frac{1}{z-x} d\mu_j(x), \quad (4.3)$$

$$F_{j,+}(z) - F_{j,-}(z) = -2\pi i \frac{d\mu_j}{dz}, \quad (4.4)$$

for $z \in \text{supp}(\mu_j)$, where PV denotes the Cauchy principal value. The ξ -functions are then defined as follows.

Theorem 4.1. *The function $\xi : \bigcup_{j=1}^4 \mathcal{R}_j \rightarrow \mathbb{C}$ defined by*

$$\xi(z) = \begin{cases} -F_1(z) + \frac{\sqrt{a}}{t\sqrt{z}} + \frac{1}{t(1-t)}, & z \in \mathcal{R}_1, \\ F_1(z) - F_2(z) - \frac{\sqrt{a}}{t\sqrt{z}} + \frac{1}{t(1-t)}, & z \in \mathcal{R}_2, \\ F_2(z) - F_3(z) + \frac{\sqrt{b}}{(1-t)\sqrt{z}}, & z \in \mathcal{R}_3, \\ F_3(z) - \frac{\sqrt{b}}{(1-t)\sqrt{z}}, & z \in \mathcal{R}_4, \end{cases} \quad (4.5)$$

Figure 3: The Riemann surface $\mathcal{R}$.

has an extension to a meromorphic function (also denoted by ξ) on $\mathcal{R}$. In (4.5), the principal branch of the square root function $\sqrt{z}$ is used, i.e., with a branch cut along $(-\infty, 0]$.

Proof. We first need to show the analyticity of ξ on each individual sheet $\mathcal{R}_j$. To see this, we note that $\mu_1 = \rho_1$ on $\mathbb{R}^- \setminus \text{supp}(\rho_1 - \mu_1) = (-r_1, 0)$. It then follows from (4.4) and the definition (2.6) of ρ_1 that

$$\begin{aligned} F_{1,+}(x) - F_{1,-}(x) &= -2\pi i \frac{d\mu_1(x)}{dx} = -2\pi i \frac{d\rho_1(x)}{dx} \\ &= -2i \frac{\sqrt{a}}{t|x|^{1/2}} = \frac{\sqrt{a}}{t} \left(\frac{1}{\sqrt{x_+}} - \frac{1}{\sqrt{x_-}} \right), \end{aligned} \quad (4.6)$$

for $x \in (-r_1, 0)$. By the definition of ξ , this implies that ξ has an analytic continuation to $(-r_1, 0)$ on both $\mathcal{R}_1$ and $\mathcal{R}_2$. Similarly, we have by (4.4) and (2.8)

$$F_{3,+}(x) - F_{3,-}(x) = \frac{\sqrt{b}}{1-t} \left(\frac{1}{\sqrt{x_+}} - \frac{1}{\sqrt{x_-}} \right), \quad (4.7)$$

for $x \in (-r_2, 0)$, which gives the analyticity of ξ on the sheets $\mathcal{R}_3$ and $\mathcal{R}_4$.

It remains to check the analyticity of ξ if one moves from one sheet to another via a specific cut. This is always the case in view of the variational conditions for the equilibrium problem. For convenience, we will only show that ξ is analytic when one crosses $\text{supp}(\rho_1 - \mu_1)$, since the other cases can be verified similarly.

We start with differentiating U^{μ_1} on $\text{supp}(\rho_1 - \mu_1)$ and obtain

$$2 \frac{d}{dx} U^{\mu_1}(x) = -2 \text{PV} \int \frac{1}{x-s} d\mu_1(s) = -F_{1,+}(x) - F_{1,-}(x), \quad (4.8)$$

where the second equality follows from Sokhotski-Plemelj formula (4.3). On the other hand, differentiating the right-hand side of the variational condition (2.17), we have

$$2 \frac{d}{dx} U^{\mu_1}(x) = \frac{d}{dx} U^{\mu_2}(x) = - \int \frac{1}{x-s} d\mu_2(s) = -F_2(x). \quad (4.9)$$

Combining (4.8) and (4.9) leads to

$$F_{1,+}(x) + F_{1,-}(x) = F_2(x), \quad x \in \text{supp}(\rho_1 - \mu_1). \quad (4.10)$$

This, together with the fact that $\sqrt{x_-} = -\sqrt{x_+}$ for $x \in \mathbb{R}^-$, gives the analyticity of ξ when one crosses $\text{supp}(\rho_1 - \mu_1)$ and moves from $\mathcal{R}_1$ to $\mathcal{R}_2$. $\square$

As a consequence of the above theorem, the following result about the asymptotic behavior of $F_j(z)$ at infinity is easy to prove.

Lemma 4.2. *There exist two constants c_1 and c_3 such that*

$$F_1(z) = \frac{1}{2z} - \frac{c_1}{z^{3/2}} + O\left(\frac{1}{z^2}\right), \quad (4.11)$$

$$F_2(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad (4.12)$$

$$F_3(z) = \frac{1}{2z} - \frac{c_3}{z^{3/2}} + O\left(\frac{1}{z^2}\right), \quad (4.13)$$

as $z \rightarrow \infty$.

We use $\xi_j(z)$ to denote the restriction of ξ to $\mathcal{R}_j$. The relations between ξ_j and the solution to the equilibrium problem can be established in the following way.

Proposition 4.3. *We have that*

$$d\mu_1(x) = \frac{1}{2\pi i} (\xi_{1,+}(x) - \xi_{1,-}(x)) dx + d\rho_1(x), \quad x \in \mathbb{R}^-, \quad (4.14)$$

$$d\mu_2(x) = \frac{1}{2\pi i} (\xi_{2,+}(x) - \xi_{2,-}(x)) dx, \quad x \in \mathbb{R}^+, \quad (4.15)$$

$$d\mu_3(x) = \frac{1}{2\pi i} (\xi_{3,+}(x) - \xi_{3,-}(x)) dx + d\rho_3(x), \quad x \in \mathbb{R}^-. \quad (4.16)$$

Proof. This follows by a combination of (4.5), the Sokhotski-Plemelj formula (4.4), and the definitions (2.6) and (2.8) of ρ_1 and ρ_2 . $\square$

The Riemann surface has two points at infinity and the meromorphic function (4.5) takes on two distinct finite values at these points (namely $\frac{1}{t(1-t)}$ and 0). Thus ξ is not a constant, and being a meromorphic function on a compact Riemann surface it must have at least one pole. From (4.5) it is clear that the only possible poles are at the values of $z = 0$ that correspond to branch points. It thus follows that at least one of r_1, p , and r_3 is zero, and we can identify the various cases in terms of the structure of $\mathcal{R}$.

In Case I we have $r_1 = r_3 = 0$ and $p > 0$. It is only in this case that ξ has two poles. In all other cases, ξ has exactly one pole, and then ξ is actually a conformal map from $\mathcal{R}$ onto the Riemann sphere.

4.3 The spectral curve

Next we calculate the spectral curve associated to the vector equilibrium problem; see also [5] and the references therein for similar calculations. From Theorem 4.1, it follows that the four function $\xi_j(z)$, $j = 1, 2, 3, 4$ are solutions to an algebraic equation of degree four

$$\begin{aligned} 0 = P(\xi, z) &= (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4) \\ &= \xi^4 + A(z)\xi^3 + B(z)\xi^2 + C(z)\xi + D(z), \end{aligned} \quad (4.17)$$

with coefficients $A(z)$, $B(z)$, $C(z)$, $D(z)$ that can only have a pole in $z = 0$. In fact, because of the behavior of (4.5) at $z = 0$, we have that $A(z)$ has no pole at $z = 0$ (and therefore is a constant), $B(z)$ and $C(z)$ have at most a simple pole at $z = 0$. Also $D(z)$ has at most a simple pole at $z = 0$, unless we are in Case I, in which case $D(z)$ has a double pole at $z = 0$. Indeed, in this last case we have that all functions ξ_j behave like $z^{-1/2}$ as $z \rightarrow 0$, and then $D(z) = \xi_1(z)\xi_2(z)\xi_3(z)\xi_4(z)$ behaves like z^{-2} as $z \rightarrow 0$.

Straightforward calculations using (4.5) and Lemma 4.2 yield in all cases

$$A(z) = -\xi_1 - \xi_2 - \xi_3 - \xi_4 = -\frac{2}{t(1-t)}, \quad (4.18)$$

$$\begin{aligned} B(z) &= \xi_1\xi_2 + \xi_1\xi_3 + \xi_1\xi_4 + \xi_2\xi_3 + \xi_2\xi_4 + \xi_3\xi_4 \\ &= \frac{1}{t^2(1-t)^2} - \frac{b}{z(1-t)^2} - \frac{a}{zt^2} + \frac{1}{zt(1-t)}, \end{aligned} \quad (4.19)$$

$$C(z) = \frac{2b}{zt(1-t)^3} - \frac{1}{zt^2(1-t)^2}, \quad (4.20)$$

$$D(z) = -\frac{b}{zt^2(1-t)^4} + \frac{c}{z^2t^2(1-t)^2}, \quad (4.21)$$

where $c = 0$ in all cases but in Case I.

Lemma 4.4. *The spectral curve is given by (4.17) with coefficients given by (4.18)–(4.21). The constant c that appears in (4.21) is equal to*

$$c = \begin{cases} \left(\sqrt{ab} - \frac{1}{2}\right)^2 & \text{in Case I,} \\ 0 & \text{in all other cases.} \end{cases} \quad (4.22)$$

In Case I, we have

$$\begin{aligned} \sqrt{p} &= (1-t)\sqrt{a} + t\sqrt{b} - \sqrt{2t(1-t)} > 0, \\ \sqrt{q} &= (1-t)\sqrt{a} + t\sqrt{b} + \sqrt{2t(1-t)}. \end{aligned} \quad (4.23)$$

Proof. We already saw that the spectral curve is given by (4.17), (4.18)–(4.21) and that $c = 0$ in all cases, except in Case I.

Now suppose we are in Case I, so that $r_1 = r_3 = 0$ and $p > 0$. Consider the following functions

$$\eta_1(z) = \begin{cases} z\xi_2(z^2), & \operatorname{Re} z > 0, \\ z\xi_1(z^2), & \operatorname{Re} z < 0, \end{cases} \quad \eta_2(z) = \begin{cases} z\xi_1(z^2), & \operatorname{Re} z > 0, \\ z\xi_2(z^2), & \operatorname{Re} z < 0, \end{cases} \quad (4.24)$$

$$\eta_3(z) = \begin{cases} z\xi_3(z^2), & \operatorname{Re} z > 0, \\ z\xi_4(z^2), & \operatorname{Re} z < 0, \end{cases} \quad \eta_4(z) = \begin{cases} z\xi_4(z^2), & \operatorname{Re} z > 0, \\ z\xi_3(z^2), & \operatorname{Re} z < 0. \end{cases} \quad (4.25)$$

Since, we are in Case I, we have that all functions η_j are analytic across the imaginary axis. Also the singularities at $z = 0$ are removable because of the extra factors z in (4.24) and (4.25). It follows that η_1 and η_3 are analytic in $\mathbb{C} \setminus [\sqrt{p}, \sqrt{q}]$, while η_2 and η_4 are analytic in $\mathbb{C} \setminus [-\sqrt{q}, -\sqrt{p}]$.

In fact the two functions η_1 and η_3 give a meromorphic function on the two-sheeted Riemann surface with cut along $[\sqrt{p}, \sqrt{q}]$, whose only pole is at infinity on the first sheet. Since

$$\begin{aligned} \eta_1(z) &= \frac{z}{t(1-t)} - \frac{\sqrt{a}}{t} - \frac{1}{2z} + O(z^{-2}) \\ \eta_3(z) &= \frac{\sqrt{b}}{1-t} + \frac{1}{2z} + O(z^{-2}) \end{aligned}$$

as $z \rightarrow \infty$, it then follows that $\eta_1(z)$ and $\eta_3(z)$ are the two solutions of the quadratic equation

$$\eta^2 - \left(\frac{z}{t(1-t)} - \frac{\sqrt{a}}{t} + \frac{\sqrt{b}}{1-t} \right) \eta + \left(\frac{\sqrt{b}z}{t(1-t)^2} - \frac{\sqrt{ab}}{t(1-t)} + \frac{1}{2t(1-t)} \right) = 0. \quad (4.26)$$

Similarly, $\eta_2(z)$ and $\eta_4(z)$ are the two solutions of

$$\eta^2 - \left(\frac{z}{t(1-t)} + \frac{\sqrt{a}}{t} - \frac{\sqrt{b}}{1-t} \right) \eta + \left(-\frac{\sqrt{b}z}{t(1-t)^2} - \frac{\sqrt{ab}}{t(1-t)} + \frac{1}{2t(1-t)} \right) = 0.$$

In particular, it follows that

$$\eta_1(z)\eta_2(z)\eta_3(z)\eta_4(z) = -\frac{bz^2}{t^2(1-t)^4} + \frac{(\sqrt{ab} - \frac{1}{2})^2}{t^2(1-t)^2}.$$

Then in view of (4.24) and (4.25) we obtain

$$z^4 D(z^2) = z^4 \xi_1(z^2)\xi_2(z^2)\xi_3(z^2)\xi_4(z^2) = -\frac{bz^2}{t^2(1-t)^4} + \frac{(\sqrt{ab} - \frac{1}{2})^2}{t^2(1-t)^2}.$$

Comparing this with (4.21) we indeed find that c is given by (4.22) in Case I.

The branch points of (4.26) are equal to $\sqrt{p}$ and $\sqrt{q}$. Simple calculations then lead to (4.23). $\square$

Remark 1. *We notice that the spectral curve in Case I is equivalent with the one in [18], while in Cases II and III it is equivalent with the one in [14] (with $\sqrt{a}, \sqrt{b}$ instead of a, b). This implies that the limiting distribution of the non-intersecting squared Bessel paths is obtained from the one of the corresponding non-intersecting Brownian motions by a simple squaring transform.*

One could extend the above ideas even further by also lifting the vector equilibrium problem into its square-root form. The lifted measures are then supported on the real or imaginary axis. This approach can be used to obtain an equilibrium problem interpretation for the spectral curve in [14], which allows in turn to obtain an alternative derivation of the results in that paper. We will not go into the details.

Proposition 4.5. *We are in Case I if and only if $ab > 1/4$.*

Proof. The spectral curve depends continuously on the parameters a , b and t . Therefore the change from Case I to one of the other cases can only occur when $ab = 1/4$.

If we are in Case I, then $\sqrt{p}$ is given by (4.23) which in particular implies that $(1-t)\sqrt{a} + t\sqrt{b} - \sqrt{2t(1-t)} > 0$. For $t = 1/2$ and $a = b < 1/2$ this inequality is not satisfied and so for these values of parameters we are not in Case I. Then it follows that for all a and b with $ab \leq 1/4$ we are not in Case I.

If we are not in Case I, then $c = 0$ and the spectral curve (4.17) has four real branch points $-r_1$, $-r_2$, p and q . For the specific values $t = 1/2$, $a = 1$ and $b = 1/2$, we calculated with Maple that there are two non-real branch points. Thus for those values of parameters we cannot have $c = 0$, and so are in Case I. By continuity we are in Case I for all a and b with $ab > 1/4$. This completes the proof of the proposition. $\square$

4.4 Phase transitions

Let us assume that the ending point b is fixed. In figure 4 we give a phase diagram divided into three different regions according with the limiting distribution of the non-intersecting squared Bessel paths as $n \rightarrow \infty$. The horizontal axis of the figure denotes the time $t \in (0, 1)$ and the vertical axis denotes the starting point a .

Figure 4: The left figure shows the phase diagram for the non-intersecting squared Bessel paths for $b = 1/4$. The solid curve is given in (4.27). The right figure shows a simulation of 50 non-intersecting paths for $a = 1/3$ which corresponds to the dashed line in the left figure.

In the case $ab < 1/4$, for t small, we are in Case II. The paths start at the positive value a at $t = 0$ and they initially remain positive. The limiting mean density is supported in the interval $[p, q]$ with $p > 0$ (see Theorems 2.1 and 2.3). For this case there is only one active constraint.

We have computed the discriminant $Disc(z)$ of (4.17) with coefficients given by (4.18)–(4.21) and $c = 0$ in (4.21) with respect to ξ using Maple and we obtain

$$Disc(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + 4(4ab - 1)(a - 2at + at^2 + bt^2 + t^2 - t)^3 z$$

for some constants a_4 , a_3 and a_2 . We do not include the explicit expressions of a_4 , a_3 and a_2 because they are too long and not too interesting. Therefore there are exactly four zeros $Disc(z)$ which correspond to the branch points of the Riemann surface.

For each $ab < 1/4$ there are two critical times $t_{1,2}$ where $Disc(z)$ has a double zero at $z = 0$, namely

$$t_{1,2} = \frac{2a + 1 \pm \sqrt{1 - 4ab}}{2(a + b + 1)}. \quad (4.27)$$

The critical times t_1 and t_2 are the only two values of t for which the coefficient of z in $Disc(z)$ is equal to zero. The time t_1 corresponds to the time where the first paths touch the wall. If $t_1 < t < t_2$ then we are in Case III. Here the limiting mean density of the paths is supported on the interval $[0, q]$ (see Theorems 2.1 and 2.3). Both constraints are active in this case. If $t > t_2$ then we are back to the Case II.

The figure on the right of Figure 4 shows the transitions between Case II and Case III for the choice of the parameters $a = 1/3$ and $b = 1/4$. This choice corresponds to the dashed line in the left figure. The critical times t_1 and t_2 where the transition occurs are given by (4.27).

On the other hand, if $ab > 1/4$ then the paths start at the positive value a at $t = 0$ and they remain positive. In this case the limiting mean density is supported in the interval $[p, q]$ with $p > 0$. No constraint is active in this case.

4.5 The λ -functions

For the measures μ_j , $j = 1, 2, 3$ that minimize the energy functional (2.3), we first define the associated g -functions as

$$g_j(z) = \int \log(z - s) d\mu_j(s), \quad (4.28)$$

where we take the principal branch of the logarithm, i.e.,

$$\log(z - s) = \log|z - s| + i \arg(z - s), \quad \arg(z - s) \in (-\pi, \pi). \quad (4.29)$$

On account of Theorem 2.1, it is easily seen that g_1 and g_3 are analytic in $\mathbb{C} \setminus \mathbb{R}^-$, while g_2 is analytic in $\mathbb{C} \setminus (-\infty, q]$.

Noting that

$$g'_j(z) = F_j(z), \quad (4.30)$$

we find the asymptotic behavior of $g_j(z)$ at infinity as given in the following lemma.

Lemma 4.6. *With the constants c_1 and c_3 given in Lemma 4.2, we have that*

$$g_1(z) = \frac{1}{2} \log z + \frac{2c_1}{\sqrt{z}} + O\left(\frac{1}{z}\right), \quad (4.31)$$

$$g_2(z) = \log z + O\left(\frac{1}{z}\right), \quad (4.32)$$

$$g_3(z) = \frac{1}{2} \log z + \frac{2c_3}{\sqrt{z}} + O\left(\frac{1}{z}\right), \quad (4.33)$$

as $z \rightarrow \infty$.

Proof. The asymptotics of g_2 follows from the fact that μ_2 is a probability measure with compact support.

The formulas (4.31) and (4.33) for g_1 and g_3 can be derived by integrating the asymptotic formulas for F_1 and F_3 given in Lemma 4.2. The integration constant is zero, due to the fact that

$$\int \log(z - s) d\mu_j(s) = \log(z) \|\mu_j\| + \int \log\left(1 - \frac{s}{z}\right) d\mu_j(s), \quad j = 1, 3,$$

and $\int \log(1 - s/z) d\mu_j(s) \rightarrow 0$ as $z \rightarrow \infty$. $\square$

The λ -functions are then defined as the following anti-derivatives of the ξ -functions:

$$\lambda_1(z) = -g_1(z) + \frac{2\sqrt{az}}{t} + \frac{z}{t(1-t)} - l, \quad (4.34)$$

$$\lambda_2(z) = g_1(z) - g_2(z) - \frac{2\sqrt{az}}{t} + \frac{z}{t(1-t)} - l, \quad (4.35)$$

$$\lambda_3(z) = g_2(z) - g_3(z) + \frac{2\sqrt{bz}}{1-t}, \quad (4.36)$$

$$\lambda_4(z) = g_3(z) - \frac{2\sqrt{bz}}{1-t}, \quad (4.37)$$

where $l \in \mathbb{R}$ is the variational constant in Proposition 2.2. Note that λ_1 and λ_4 are analytic in $\mathbb{C} \setminus (-\infty, 0]$ while λ_2 and λ_3 are analytic in $\mathbb{C} \setminus (-\infty, q]$. Some properties of λ -functions are shown in the following two lemmas.

Lemma 4.7. *We have*

$$\begin{aligned}\lambda_{1,+}(x) &= \lambda_{1,-}(x), & x &\in (-r_1, 0), \\ \lambda_{2,+}(x) &= \lambda_{2,-}(x) - 2\pi i, & x &\in (-r_1, p), \\ \lambda_{3,+}(x) &= \lambda_{3,-}(x) + 2\pi i, & x &\in (-r_3, p), \\ \lambda_{4,+}(x) &= \lambda_{4,-}(x), & x &\in (-r_3, 0),\end{aligned}$$

$$\begin{aligned}\lambda_{1,\pm}(x) &= \lambda_{2,\mp}(x) \mp \pi i, & x &\in \Delta_1, \\ \lambda_{2,\pm}(x) &= \lambda_{3,\mp}(x), & x &\in \Delta_2, \\ \lambda_{3,\pm}(x) &= \lambda_{4,\mp}(x) \mp \pi i, & x &\in \Delta_3,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}(\lambda_1(x) - \lambda_2(x)) &< 0, & x &\in (-r_1, 0), \\ \operatorname{Re}(\lambda_2(x) - \lambda_3(x)) &> 0, & x &\in \mathbb{R}^+ \setminus \Delta_2, \\ \operatorname{Re}(\lambda_3(x) - \lambda_4(x)) &< 0, & x &\in (-r_3, 0).\end{aligned}$$

Proof. These are reformulations of the variational conditions of the equilibrium problem in Proposition 2.2. $\square$

Lemma 4.8. *The λ -functions have the following asymptotics for $z \rightarrow \infty$:*

$$\begin{aligned}\lambda_1(z) &= \frac{z}{t(1-t)} + \frac{2\sqrt{az}}{t} - l - \frac{1}{2} \log z - \frac{2c_1}{\sqrt{z}} + O(z^{-1}), \\ \lambda_2(z) &= \frac{z}{t(1-t)} - \frac{2\sqrt{az}}{t} - l - \frac{1}{2} \log z + \frac{2c_1}{\sqrt{z}} + O(z^{-1}), \\ \lambda_3(z) &= \frac{2\sqrt{bz}}{1-t} + \frac{1}{2} \log z - \frac{2c_3}{\sqrt{z}} + O(z^{-1}), \\ \lambda_4(z) &= -\frac{2\sqrt{bz}}{1-t} + \frac{2c_3}{\sqrt{z}} + \frac{1}{2} \log z + O(z^{-1}).\end{aligned}$$

Proof. Obvious from the definitions (4.34)–(4.37) and from (4.31)–(4.33). $\square$

4.6 The ϕ -functions

We end this section with the introduction of the ϕ -functions, which will be used to simplify the jump matrices during our steepest descent analysis. Recall that ξ_j is the

restriction of ξ to the sheet $\mathcal{R}_j$ of the Riemann surface $\mathcal{R}$. We define

$$\phi_1(z) = \frac{1}{2} \int_{-r_1}^z (\xi_1(y) - \xi_2(y)) dy, \quad z \in \mathbb{C} \setminus ((-\infty, -r_1] \cup [p, \infty)), \quad (4.38)$$

$$\phi_2(z) = \frac{1}{2} \int_q^z (\xi_2(y) - \xi_3(y)) dy, \quad z \in \mathbb{C} \setminus (-\infty, q], \quad (4.39)$$

$$\phi_3(z) = \frac{1}{2} \int_{-r_3}^z (\xi_3(y) - \xi_4(y)) dy, \quad z \in \mathbb{C} \setminus ((-\infty, -r_3] \cup [p, \infty)). \quad (4.40)$$

5 Steepest descent analysis for $Y(z)$

In this section, we will perform the steepest descent analysis of the RH problem for Y . It consists of a series of explicit and invertible transformations. In Section 6, these transformations will be used to prove Theorem 2.3.

To this end, we mainly follow the theme laid out in [38], although the argument is somehow more involved since our RH problem is of size 4×4 whereas the dimension treated in [38] is 3×3 . Furthermore, we have more generic cases to consider here.

5.1 First transformation $Y \mapsto X$

It is the aim of this transformation to simplify the block matrix (2.30) appearing in the jump condition (2.29) for Y . The idea is to use the special properties of modified Bessel functions, in a very similar way as in [38]. To start with, we set four functions:

$$\begin{aligned} y_1(z) &= z^{(\alpha+1)/2} I_{\alpha+1}(2\sqrt{z}), & y_2(z) &= z^{(\alpha+1)/2} K_{\alpha+1}(2\sqrt{z}), \\ y_3(z) &= z^{-(\alpha-1)/2} I_{\alpha-1}(2\sqrt{z}), & y_4(z) &= z^{-(\alpha-1)/2} K_{\alpha-1}(2\sqrt{z}), \end{aligned} \quad (5.1)$$

where K_ν is the modified Bessel function of the second kind; see [1, Section 9.6] for its main properties. It is readily seen that $y_3(z)$ is well defined in the whole complex plane, while the function $y_j(z), j = 1, 2, 4$ is analytic in the complex plane with a branch cut along the negative real axis. Indeed, it follows from formulas 9.6.30–9.6.31 in [1] that their jumps on $\mathbb{R}^-$ are given in the following ways

$$\begin{aligned} y_{1,+}(x) &= e^{2i\alpha\pi} y_{1,-}(x), \\ y_{2,+}(x) &= y_{2,-}(x) + i\pi e^{i\alpha\pi} y_{1,-}(x), \\ y_{4,+}(x) &= e^{-2i\alpha\pi} y_{4,-}(x) + i\pi e^{-i\alpha\pi} y_3(x), \end{aligned} \quad (5.2)$$

for $x < 0$. In view of the derivative properties of the modified Bessel functions [1, 9.6.26], we also have

$$\begin{aligned} y_1'(z) &= z^{\alpha/2} I_\alpha(2\sqrt{z}), & y_2'(z) &= -z^{\alpha/2} K_\alpha(2\sqrt{z}), \\ y_3'(z) &= z^{-\alpha/2} I_\alpha(2\sqrt{z}), & y_4'(z) &= -z^{-\alpha/2} K_\alpha(2\sqrt{z}). \end{aligned} \quad (5.3)$$

A combination of (5.1), (5.3) and (2.30) implies that we can represent the weight matrix $W(x)$ in terms of the functions y_1 , y_3 and their derivatives as

$$\begin{aligned} W(x) &= e^{-\frac{nx}{t(1-t)}} \begin{pmatrix} x^{\alpha/2} I_{\alpha} \left(\frac{2n\sqrt{ax}}{t} \right) \\ x^{(\alpha+1)/2} I_{\alpha+1} \left(\frac{2n\sqrt{ax}}{t} \right) \end{pmatrix} \begin{pmatrix} x^{-\alpha/2} I_{\alpha} \left(\frac{2n\sqrt{bx}}{1-t} \right) & x^{-(\alpha-1)/2} I_{\alpha-1} \left(\frac{2n\sqrt{bx}}{1-t} \right) \end{pmatrix} \\ &= e^{-\frac{nx}{t(1-t)}} \begin{pmatrix} \tau_1^{-\alpha} y_1'(\tau_1^2 x) \\ \tau_1^{-\alpha-1} y_1(\tau_1^2 x) \end{pmatrix} \begin{pmatrix} \tau_3^{\alpha} y_3'(\tau_3^2 x) & \tau_3^{\alpha-1} y_3(\tau_3^2 x) \end{pmatrix}, \end{aligned} \quad (5.4)$$

where

$$\tau_1 = \tau_{1,n} = \frac{n\sqrt{a}}{t}, \quad \tau_3 = \tau_{3,n} = \frac{n\sqrt{b}}{1-t}. \quad (5.5)$$

For later use, we also need the following Wronskian relations:

$$y_1'(z)y_2(z) - y_1(z)y_2'(z) = z^{\alpha}/2, \quad y_3'(z)y_4(z) - y_3(z)y_4'(z) = z^{-\alpha}/2, \quad (5.6)$$

for $z \in \mathbb{C} \setminus \mathbb{R}^-$; see formula 9.6.15 of [1].

The first transformation $Y \mapsto X$ is then defined by (compare with [38])

$$X(z) = C_1 Y(z) \operatorname{diag}(A_1(z), A_2(z)), \quad (5.7)$$

where

$$A_1(z) = \tau_1^{-\alpha} z^{-\alpha/2} \begin{pmatrix} -\frac{1}{\pi i} y_2'(\tau_1^2 z) & y_1'(\tau_1^2 z) \\ -\frac{1}{\pi i} \tau_1^{-1} y_2(\tau_1^2 z) & \tau_1^{-1} y_1(\tau_1^2 z) \end{pmatrix}, \quad (5.8)$$

$$A_2(z) = 2\tau_3^{\alpha} z^{\alpha/2} \begin{pmatrix} y_4(\tau_3^2 z) & -\pi i y_3(\tau_3^2 z) \\ -\tau_3 y_4'(\tau_3^2 z) & \pi i \tau_3 y_3'(\tau_3^2 z) \end{pmatrix}, \quad (5.9)$$

and with C_1 the constant 4×4 matrix

$$C_1 = \operatorname{diag} \left(\sqrt{2\pi\tau_1} \begin{pmatrix} i & 0 \\ \frac{4(\alpha+1)^2-1}{16\tau_1} & 1 \end{pmatrix}, \frac{1}{\sqrt{2\pi\tau_3}} \begin{pmatrix} 0 & 1 \\ i & -i\frac{4(\alpha-1)^2-1}{16\tau_3} \end{pmatrix} \right).$$

Note that, on account of (5.6), we have

$$\det A_1 = -\frac{1}{2\pi\tau_1} i, \quad \det A_2 = 2\pi\tau_3 i. \quad (5.10)$$

Hence, it is easily seen that $\det X = 1$. Now, we have

Proposition 5.1. *The matrix valued function $X(z)$ defined by (5.7) is the unique solution of the following RH problem.*

(1) $X(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(2) For $x \in \mathbb{R}$, $X(x)$ satisfies the following jump condition

$$X_+(x) = X_-(x) \begin{cases} I + e^{-\frac{nx}{t(1-t)}} E_{23}, & \text{if } x > 0, \\ \text{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}) - E_{21} - E_{43}, & \text{if } x < 0, \end{cases} \quad (5.11)$$

where we denote by E_{ij} the 4×4 elementary matrix whose entries are all 0, except for the (i, j) -th entry, which is 1.

(3) As $z \rightarrow \infty$, $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$X(z) = \left(I + O\left(\frac{1}{z}\right) \right) \text{diag}\left(z^{-1/4}, z^{1/4}, z^{-1/4}, z^{1/4}\right) \frac{1}{\sqrt{2}} \text{diag}\left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right) \\ \times \text{diag}\left(z^{n/2} e^{-2n\sqrt{az}/t}, z^{n/2} e^{2n\sqrt{az}/t}, z^{-n/2} e^{-2n\sqrt{bz}/(1-t)}, z^{-n/2} e^{2n\sqrt{bz}/(1-t)}\right),$$

uniformly valid for z bounded away from the negative real axis.

(4) $X(z)$ has the following behavior near the origin:

$$\begin{aligned} X(z) \text{diag}(|z|^{\alpha/2}, |z|^{-\alpha/2}, |z|^{\alpha/2}, |z|^{-\alpha/2}) &= O(1), & \text{if } \alpha > 0, \\ X(z) \text{diag}((\log |z|)^{-1}, 1, (\log |z|)^{-1}, 1) &= O(1), & \text{if } \alpha = 0, \\ X(z) &= O(z^{\alpha/2}), \quad X^{-1}(z) = O(z^{\alpha/2}), & \text{if } -1 < \alpha < 0. \end{aligned} \quad (5.12)$$

Proof. The proof follows [38]. It is easily seen from (5.7)–(5.9) and (2.29) that

$$\begin{aligned} X_-^{-1}(x) X_+(x) &= \begin{pmatrix} A_{1,-}^{-1}(x) & 0 \\ 0 & A_{2,-}^{-1}(x) \end{pmatrix} Y_-^{-1}(x) Y_+(x) \begin{pmatrix} A_{1,+}(x) & 0 \\ 0 & A_{2,+}(x) \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} I_2 & A_{1,-}^{-1}(x) W(x) A_{2,+}(x) \\ 0 & I_2 \end{pmatrix}, & \text{if } x > 0, \\ \begin{pmatrix} A_{1,-}^{-1}(x) A_{1,+}(x) & 0 \\ 0 & A_{2,-}^{-1}(x) A_{2,+}(x) \end{pmatrix}, & \text{if } x < 0. \end{cases} \end{aligned} \quad (5.13)$$

In view of (5.4) and (5.6), we obtain from a direct calculation that

$$\begin{aligned} &A_{1,-}^{-1}(x) W(x) A_{2,+}(x) \\ &= e^{-\frac{nx}{(1-t)t}} A_{1,-}^{-1}(x) \begin{pmatrix} \tau_1^{-\alpha} y_1'(\tau_1^2 x) \\ \tau_1^{-\alpha-1} y_1(\tau_1^2 x) \end{pmatrix} \begin{pmatrix} \tau_3^\alpha y_3'(\tau_3^2 x) & \tau_3^{\alpha-1} y_3(\tau_3^2 x) \end{pmatrix} A_{2,+}(x) \\ &= e^{-\frac{nx}{(1-t)t}} \begin{pmatrix} 0 \\ x^{\alpha/2} \end{pmatrix} \begin{pmatrix} x^{-\alpha/2} & 0 \end{pmatrix} = e^{-\frac{nx}{(1-t)t}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (5.14)$$

for $x > 0$. Also, by (5.2) and (5.3), it can be shown that

$$A_{1,-}^{-1}(x)A_{1,+}(x) = \begin{pmatrix} e^{-i\pi\alpha} & 0 \\ -1 & e^{i\pi\alpha} \end{pmatrix} \quad \text{and} \quad A_{2,-}^{-1}(x)A_{2,+}(x) = \begin{pmatrix} e^{-i\pi\alpha} & 0 \\ -1 & e^{i\pi\alpha} \end{pmatrix}, \quad (5.15)$$

for $x < 0$. The jump condition (5.11) now follows from a combination of (5.13)–(5.15).

The large z behavior of $X(z)$ shown in item (3), follows very similarly as in [38]. The relevant results are

$$A_1(z) = \frac{1}{\sqrt{2\pi\tau_1}} \left[\begin{pmatrix} -i & 0 \\ i\frac{4(\alpha+1)^2-1}{16\tau_1} & 1 \end{pmatrix} + O(z^{-1}) \right] \frac{z^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-2\tau_1\sqrt{z}\sigma_3}, \quad (5.16)$$

and

$$A_2(z) = \sqrt{2\pi\tau_3} \left[\begin{pmatrix} \frac{4(\alpha-1)^2-1}{16\tau_3} & -i \\ 1 & 0 \end{pmatrix} + O(z^{-1}) \right] \frac{z^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-2\tau_3\sqrt{z}\sigma_3}, \quad (5.17)$$

as $z \rightarrow \infty$, uniformly for z bounded away from the negative real axis. Substituting the asymptotic formulas (5.16) and (5.17) of $A_1(z)$ and $A_2(z)$ into (5.7), the asymptotic behavior of $X(z)$ at infinity is immediate.

Finally, the known behavior of the modified Bessel functions near zero given by formulas 9.6.7–9.6.9 in [1] yields

$$\begin{aligned} y_1(z) &\sim \frac{1}{\Gamma(\alpha+2)} z^{\alpha+1}, & y_1'(z) &\sim \frac{1}{\Gamma(\alpha+1)} z^\alpha, \\ y_2(z) &\sim \frac{1}{2}\Gamma(\alpha+1), & y_2'(z) &\sim \begin{cases} -\frac{1}{2}\Gamma(\alpha), & \alpha > 0, \\ \frac{1}{2}\log(z), & \alpha = 0, \\ -\frac{1}{2}\Gamma(-\alpha)z^\alpha, & \alpha < 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} y_3(z) &\sim \frac{1}{\Gamma(\alpha)}, & y_3'(z) &\sim \frac{1}{\Gamma(\alpha+1)}, \\ y_4(z) &\sim \begin{cases} \frac{1}{2}\Gamma(\alpha-1)z^{1-\alpha}, & \alpha > 1, \\ -\frac{1}{2}\log(z), & \alpha = 1, \\ \frac{1}{2}\Gamma(1-\alpha), & \alpha < 1, \end{cases} & y_4'(z) &\sim \begin{cases} -\frac{1}{2}\Gamma(\alpha)z^{-\alpha}, & \alpha > 0, \\ \frac{1}{2}\log(z), & \alpha = 0, \\ -\frac{1}{2}\Gamma(-\alpha), & \alpha < 0. \end{cases} \end{aligned}$$

The behavior at zero of $X(z)$ in item (4) then follows from a straightforward calculation.

This completes the proof of Proposition 5.1. $\square$

5.2 Second transformation $X \mapsto U$

The second transformation $X \mapsto U$ is defined by

$$U(z) = C_2 X(z) \operatorname{diag} \left(e^{n \left(\lambda_1(z) - \frac{z}{i(1-i)} \right)}, e^{n \left(\lambda_2(z) - \frac{z}{i(1-i)} \right)}, e^{n \lambda_3(z)}, e^{n \lambda_4(z)} \right), \quad (5.18)$$

where C_2 is the constant matrix

$$C_2 = (I + 2inc_1 E_{21} + 2inc_3 E_{34}) \operatorname{diag}(e^{nl}, e^{nl}, 1, 1), \quad (5.19)$$

with c_1 and c_3 as in Lemma 4.2 and with l the variational constant in Proposition 2.2.

In view of the jump relations (5.11) for X and the definition (5.18), we see that the jump relations for U are given by

$$U_+ = U_- \begin{cases} \Lambda_-^{-1} (I + E_{23}) \Lambda_+, & \text{on } \mathbb{R}^+, \\ \Lambda_-^{-1} (\operatorname{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}) - E_{21} - E_{43}) \Lambda_+, & \text{on } \mathbb{R}^-, \end{cases} \quad (5.20)$$

where we define the diagonal matrix

$$\Lambda(z) = \operatorname{diag}(e^{n\lambda_1(z)}, e^{n\lambda_2(z)}, e^{n\lambda_3(z)}, e^{n\lambda_4(z)}). \quad (5.21)$$

The jump matrix for U can be reformulated with the help of the ϕ -functions in (4.38)–(4.40) as stated in the following proposition.

Proposition 5.2. *The matrix valued function $U(z)$ defined by (5.18) is the unique solution of the following RH problem.*

(1) $U(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(2) For $x \in \mathbb{R}$, we have that $U_+ = U_- J_U$, where

$$J_U = \operatorname{diag} \left(1, \begin{pmatrix} e^{2n\phi_{2,+}} & 1 \\ 0 & e^{2n\phi_{2,-}} \end{pmatrix}, 1 \right) \quad \text{on } \Delta_2, \quad (5.22)$$

$$J_U = I + e^{-2n\phi_{2,+}} E_{23} \quad \text{on } \mathbb{R}^+ \setminus (0 \cup \overline{\Delta_2}), \quad (5.23)$$

$$J_U = \operatorname{diag} \left(\begin{pmatrix} e^{-i\pi\alpha} e^{2n\phi_{1,+}} & 0 \\ -1 & e^{i\pi\alpha} e^{2n\phi_{1,-}} \end{pmatrix}, \begin{pmatrix} e^{-i\pi\alpha} e^{2n\phi_{3,+}} & 0 \\ -1 & e^{i\pi\alpha} e^{2n\phi_{3,-}} \end{pmatrix} \right) \quad \text{on } \Delta_3, \quad (5.24)$$

$$J_U = \left(\begin{pmatrix} e^{-i\pi\alpha} e^{2n\phi_{1,+}} & 0 \\ -1 & e^{i\pi\alpha} e^{2n\phi_{1,-}} \end{pmatrix} \begin{pmatrix} e^{-i\pi\alpha} & 0 \\ -e^{2n\phi_3} & e^{i\pi\alpha} \end{pmatrix} \right) \quad \text{on } \Delta_1 \setminus \overline{\Delta_3}, \quad (5.25)$$

$$J_U = \operatorname{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}) - e^{2n\phi_1} E_{21} - e^{2n\phi_3} E_{43} \quad \text{on } \mathbb{R}^- \setminus \overline{\Delta_1}, \quad (5.26)$$

with Δ_j , $j = 1, 2, 3$ being defined in (2.12) and $\overline{\Delta_j}$ denotes the closure of Δ_j .

(3) As $z \rightarrow \infty$, $z \in \mathbb{C} \setminus \mathbb{R}$, we have that

$$U(z) = \left(I + O\left(\frac{1}{z}\right) \right) \text{diag} \left(z^{-1/4}, z^{1/4}, z^{-1/4}, z^{1/4} \right) \frac{1}{\sqrt{2}} \text{diag} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \quad (5.27)$$

uniformly valid for z bounded away from the negative real axis.

(4) $U(z)$ has the same behavior as $X(z)$ near the origin; see (5.12).

Before proceeding to the proof of Proposition 5.2, we remind the reader that, due to our assumption (2.13), the jump condition (5.25) only appears in Cases II and III, while (5.26) only appears in Case III.

Proof. The jump conditions in item (2) are obtained in a straightforward way from Lemma 4.7 and the definitions of ϕ -functions in (4.38)–(4.40). For convenience, we give a proof of (5.22), the other jumps can be proved in similar ways.

If $x \in \Delta_2$, it follows from (5.20) and Lemma 4.7 that

$$J_U(x) = \text{diag} \left(1, \begin{pmatrix} e^{n(\lambda_{2,+} - \lambda_{2,-})(x)} & 1 \\ 0 & e^{n(\lambda_{3,+} - \lambda_{3,-})(x)} \end{pmatrix}, 1 \right). \quad (5.28)$$

On account of our definition of $\lambda_{2,3}$ in (4.35) and (4.36), it suffices to show that

$$g_{2,+}(x) - g_{2,-}(x) = -2\phi_{2,+}(x) = 2\phi_{2,-}(x), \quad (5.29)$$

for $x \in \Delta_2$. To see this, we obtain from (4.28) and (4.15) that

$$g_{2,+}(x) - g_{2,-}(x) = 2\pi i \int_x^q d\mu_2(s) = \int_q^x (\xi_{2,-}(s) - \xi_{2,+}(s)) ds. \quad (5.30)$$

The relation (5.29) now is immediate, with the aid of (5.30) and the fact that $\xi_{2,-}(s) = \xi_{3,+}(s)$ provided $s \in \Delta_2$.

To prove the asymptotic behavior of U near infinity, we note that the asymptotics of X and the large z behavior of the λ -functions given in Lemma 4.8 yield

$$\begin{aligned} X(z) & \text{diag} \left(e^{n(\lambda_1(z) - \frac{z}{i(1-i)})}, e^{n(\lambda_2(z) - \frac{z}{i(1-i)})}, e^{n\lambda_3(z)}, e^{n\lambda_4(z)} \right) \\ & = \left(I + O\left(\frac{1}{z}\right) \right) \text{diag} \left(z^{-1/4}, z^{1/4}, z^{-1/4}, z^{1/4} \right) \frac{1}{\sqrt{2}} \text{diag} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) C(z), \end{aligned} \quad (5.31)$$

as $z \rightarrow \infty$, where

$$C(z) = \text{diag} \left(e^{-nl} \left(1 - \frac{2nc_1}{z^{1/2}} \right), e^{-nl} \left(1 + \frac{2nc_1}{z^{1/2}} \right), 1 - \frac{2nc_3}{z^{1/2}}, 1 + \frac{2nc_3}{z^{1/2}} \right) + O\left(\frac{1}{z}\right),$$

where the matrix denoted with $O\left(\frac{1}{z}\right)$ is diagonal as well.

Now we can move $C(z)$ in (5.31) to the left as in the proof of Proposition 5.1. The result is that (5.31) is equal to

$$C_2^{-1} \left(I + O\left(\frac{1}{z}\right) \right) \text{diag} \left(z^{-1/4}, z^{1/4}, z^{-1/4}, z^{1/4} \right) \frac{1}{\sqrt{2}} \text{diag} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right).$$

Then (5.27) is immediate by the definition of U in (5.18).

The behavior of U near the origin follows from the corresponding behavior of X , and the fact that the λ -functions are bounded at the origin. $\square$

5.3 Third transformation $U \mapsto T$

Our next transformation involves factorizations of jump matrices and the so-called lens opening, which is a key ingredient of the steepest descent analysis. The main goal of this step is to convert the highly oscillatory jump matrices into a more convenient form (e.g., constant matrices) on the original contour, thereby introducing extra jump matrices with exponentially small off-diagonal entries on the new contours.

The existence of lenses is guaranteed by the next lemma.

Lemma 5.3. (*Lenses around Δ_j .*) *There exist Jordan curves Δ_2^+ , Δ_2^- connecting p to q in the upper and lower half plane, respectively, such that $\text{Re } \phi_2(z) < 0$ for every non-real z in the region enclosed by Δ_2^+ , Δ_2^- . This region is called the lens around Δ_2 .*

Similarly, for $j = 1, 3$ there exist Jordan curves Δ_j^+ , Δ_j^- connecting $-\infty$ to $-r_j$ in the upper and lower half plane, respectively, such that $\text{Re } \phi_j(z) > 0$ for every non-real z in the region enclosed by Δ_j^+ , Δ_j^- , $j = 1, 3$. This region is called the lens around Δ_j . If we are in Case I or $r_3 = r_1$ in Case III, then we may (and do) assume that $\Delta_3^\pm = \Delta_1^\pm$.

Proof. We first prove the claim for Δ_2 . Recalling that $\xi_{2,+}(x) = \xi_{3,-}(x)$ for $x \in \Delta_2$, it follows from (4.15) and the definition (4.39) of $\phi_2(x)$ that

$$\phi_{2,+}(x) = -\phi_{2,-}(x) = \pi i \int_q^x d\mu_2, \quad x \in \Delta_2. \quad (5.32)$$

Hence, $\phi_{2,\pm}$ is purely imaginary on Δ_2 . Furthermore, by differentiating the above formula with respect to x on both sides, we obtain

$$\frac{d}{dx} \text{Im } \phi_{2,+}(x) > 0 \quad \text{and} \quad \frac{d}{dx} \text{Im } \phi_{2,-}(x) < 0. \quad (5.33)$$

Therefore, an appeal to the Cauchy-Riemann equations yields the existence of a region around Δ_2 , such that $\text{Re } \phi_2(z) < 0$ for every $z \notin \{p, q\}$ in that region.

To prove the claim for Δ_1 , note that $\xi_{1,+}(x) = \xi_{2,-}(x)$ for $x \in \Delta_1$. Then we obtain from (4.38) and (4.14) that

$$\phi_{1,+}(x) = -\phi_{1,-}(x) = \pi i \int_{-r_1}^x d(\mu_1 - \rho_1), \quad x \in \Delta_1. \quad (5.34)$$

Figure 5: Lenses around Δ_j , $j = 1, 2, 3$. In Case I or $r_3 = r_1$ in Case III, we have $\Delta_1^\pm = \Delta_3^\pm$.

In view of the constraint condition $\mu_1 \leq \rho_1$, it is immediate that

$$\frac{d}{dx} \operatorname{Im} \phi_{1,+}(x) < 0 \quad \text{and} \quad \frac{d}{dx} \operatorname{Im} \phi_{1,-}(x) > 0. \quad (5.35)$$

Again, by the Cauchy-Riemann equations, our claim holds. The claim for Δ_3 can be proved similarly. $\square$

An illustration of Lemma 5.3 is shown in Figure 5.

The 2×2 middle block of the jump matrix (5.22) on Δ_2 admits the factorization

$$\begin{pmatrix} e^{2n\phi_{2,+}} & 1 \\ 0 & e^{2n\phi_{2,-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{2,-}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{2,+}} & 1 \end{pmatrix}. \quad (5.36)$$

Similarly, for the 2×2 upper left and lower right blocks of the jump matrix (5.24) on Δ_1 and Δ_3 we have the factorizations

$$\begin{pmatrix} e^{-i\pi\alpha} e^{2n\phi_{j,+}} & 0 \\ -1 & e^{i\pi\alpha} e^{2n\phi_{j,-}} \end{pmatrix} = \begin{pmatrix} 1 & -e^{-i\pi\alpha} e^{-2n\phi_{j,-}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \times \begin{pmatrix} 1 & -e^{i\pi\alpha} e^{-2n\phi_{j,+}} \\ 0 & 1 \end{pmatrix}, \quad j = 1, 3. \quad (5.37)$$

Based on the factorizations (5.36)–(5.37), we define

$$T(z) = U(z) \left(I \mp e^{2n\phi_2(z)} E_{32} \right), \quad (5.38)$$

for z in the domain bounded by $\Delta_2^\pm$ and Δ_2 ;

$$T(z) = U(z) \left(I \pm e^{\pm i\pi\alpha} e^{-2n\phi_1(z)} E_{12} \right), \quad (5.39)$$

for z in the domain bounded by $\Delta_1^\pm$, $\Delta_3^\pm$ and Δ_1 ;

$$T(z) = U(z) \left(I \pm e^{\pm i\pi\alpha} e^{-2n\phi_1(z)} E_{12} \pm e^{\pm i\pi\alpha} e^{-2n\phi_3(z)} E_{34} \right), \quad (5.40)$$

for z in the domain bounded by $\Delta_3^\pm$ and Δ_3 , and we let

$$T(z) = U(z), \quad (5.41)$$

for z outside of the lenses.

It then follows from straightforward calculations that $T(z)$ is characterized by the following RH problem:

Proposition 5.4. *The matrix valued function $T(z)$ is the unique solution of the following RH problem.*

- (1) T is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup \Delta_1^\pm \cup \Delta_2^\pm \cup \Delta_3^\pm)$.
- (2) For each of the oriented contours shown in Figure 5, T has a jump $T_+ = T_- J_T$, where

$$\begin{aligned} J_T &= \text{diag} \left(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right), \text{ on } \Delta_2, \\ J_T &= I + e^{-2n\phi_2} E_{23}, \text{ on } \mathbb{R}^+ \setminus (0 \cup \overline{\Delta_2}), \\ J_T &= \text{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}) - e^{2n\phi_1} E_{21} - e^{2n\phi_3} E_{43}, \text{ on } (-r_1, 0) \text{ (Case III)}, \\ J_T &= \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} e^{-i\pi\alpha} & 0 \\ -e^{2n\phi_3} & e^{i\pi\alpha} \end{pmatrix} \right), \text{ on } \Delta_1 \setminus \overline{\Delta_3} \text{ (Cases II and III)}, \\ J_T &= \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right), \text{ on } \Delta_3, \\ J_T &= I + e^{2n\phi_2} E_{32}, \text{ on } \Delta_2^\pm, \\ J_T &= I - e^{\pm i\alpha\pi} e^{-2n\phi_1} E_{12}, \text{ on } \Delta_1^\pm, \\ J_T &= I - e^{\pm i\alpha\pi} e^{-2n\phi_3} E_{34}, \text{ on } \Delta_3^\pm, \\ J_T &= I - e^{\pm i\alpha\pi} e^{-2n\phi_1} E_{12} - e^{\pm i\alpha\pi} e^{-2n\phi_3} E_{34}, \text{ on } \Delta_1^\pm = \Delta_3^\pm \text{ (Case I or } r_3 = r_1 \text{ in Case III)}. \end{aligned}$$

- (3) As $z \rightarrow \infty$, we have

$$T(z) = \left(I + O\left(\frac{1}{z}\right) \right) \text{diag} \left(z^{-1/4}, z^{1/4}, z^{-1/4}, z^{1/4} \right) \frac{1}{\sqrt{2}} \text{diag} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right),$$

uniformly for $z \in \mathbb{C} \setminus \mathbb{R}$.

(4) $T(z)$ has the same behavior near the origin as $U(z)$ (and $X(z)$), see (5.12), provided that $z \rightarrow 0$ outside the lenses that end in 0. [The behavior inside the lenses is different but will not be needed for us.]

(5) T is bounded at the finite endpoints of Δ_j , $j = 1, 2, 3$ other than the origin.

Proof. All the properties follow from straightforward calculations.

To show that the asymptotic behavior of T given in item (3) holds uniformly up to the negative real axis, one needs to trace back the transformations $Y \mapsto X \mapsto U \mapsto T$. It turns out the entries of T are actually linear combinations of the modified Bessel functions, whose asymptotic expansions are uniformly valid up to the negative real axis. We omit the details here. $\square$

5.4 Global parametrix

By the above constructions, it is easily seen from Lemmas 4.8 and 5.3 that the jump matrices J_T in the RH problem for $T(z)$ all tend to the identity matrix exponentially fast as $n \rightarrow \infty$, except for the jump matrices on Δ_j , $j = 1, 2, 3$. Hence, we may approximate T by a solution N_α of the following 4×4 model RH problem:

(1) N_α is analytic in $\mathbb{C} \setminus (\overline{\Delta_1} \cup \overline{\Delta_2} \cup \overline{\Delta_3})$.

(2) N_α has continuous boundary values on Δ_j , $j = 1, 2, 3$ and satisfies $N_{\alpha,+} = N_{\alpha,-} J_{N_\alpha}$, where

$$J_{N_\alpha} = \begin{cases} \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right), & \text{on } \Delta_3, \\ \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e^{-i\pi\alpha}, e^{i\pi\alpha} \right), & \text{on } \Delta_1 \setminus \overline{\Delta_3} \text{ (Cases II and III),} \\ \text{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}), & \text{on } (-r_1, 0) \text{ (Case III),} \\ \text{diag} \left(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right), & \text{on } \Delta_2. \end{cases}$$

(3) As $z \rightarrow \infty$, $z \in \mathbb{C} \setminus \mathbb{R}^-$,

$$N_\alpha(z) = \left(I + O\left(\frac{1}{z}\right) \right) \text{diag} \left(z^{-1/4}, z^{1/4}, z^{-1/4}, z^{1/4} \right) \frac{1}{\sqrt{2}} \text{diag} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right).$$

We construct the solution to this RH problem with the help of meromorphic differentials, following the idea in [41]. We will give the construction for Case III; the modifications for Cases I and II will be briefly commented at the end of this section.

Consider the Riemann surface in Figure 3. Denote with p, q (sheets 2 and 3), $-r_1$ (sheets 1 and 2), $-r_3$ (sheets 3 and 4) the finite branch points. Also denote with $\infty_{1,2}$ the point at infinity that is common to the first and second sheet, with $\infty_{3,4}$ the point at infinity that is common to the third and fourth sheet, and with 0_1 and 0_4 the origin on the first and fourth sheet respectively. Recall that we are working in Case III, so $p = 0$ and $\min(r_1, r_3) > 0$.

Let ω_i , $i = 1, 2, 3, 4$ be four meromorphic differentials (of the third kind) on the Riemann surface with simple poles at the above listed points (and nowhere else), and with residues at these points given by the following table:

x	p	q	$-r_1$	$-r_3$	$\infty_{1,2}$	$\infty_{3,4}$	0_1	0_4
$\text{Res}(\omega_1, x)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$1/2$	$3/2$	$-\alpha/2$	$\alpha/2$
$\text{Res}(\omega_2, x)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$5/2$	$-\alpha/2$	$\alpha/2$
$\text{Res}(\omega_3, x)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$3/2$	$1/2$	$-\alpha/2$	$\alpha/2$
$\text{Res}(\omega_4, x)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$5/2$	$-1/2$	$-\alpha/2$	$\alpha/2$

(5.42)

The differences among ω_i are that their residues at $\infty_{1,2}$ and $\infty_{3,4}$ are all different. Such meromorphic differentials exist because the sum of the residues is zero. The fact that our Riemann surface has genus 0 implies that each meromorphic differential is unique; cf. [25, 45].

We then build the first row of N_α with the aid of meromorphic differential ω_1 introduced in (5.42). Let p_1 be an arbitrary reference point on the first sheet of the Riemann surface, chosen on the positive real line such that $p_1 > q$. Define four functions

$$u_j(z) = \int_{p_1}^{z_j} \omega_1, \quad j = 1, \dots, 4,$$

where we denote with z_j the point z lying on the j th sheet. The integration path from p_1 to z_j obeys the following rules: It can only move from one sheet to another by going from the lower side of Δ_j on sheet j to the upper side of Δ_j on sheet $j + 1$. Moreover, the path is not allowed to go through a pole of ω_1 , and it must not intersect the intervals $[-r_1, 0]$ on the first and second sheets, nor the intervals $[-r_3, 0]$ on the third and fourth sheets, except possibly at the endpoint of the path.

An illustration of the integration path for $u_3(z)$ is shown in Figure 6.

With the above conventions, the functions $u_j(z)$ satisfy the following jump properties

$$\begin{aligned}
u_{1,+} &= u_{1,-} + 2\pi i \text{Res}(\omega_1, 0_1) = u_{1,-} - \alpha\pi i, & \text{on } (-r_1, 0), \\
u_{2,+} &= u_{2,-} - 2\pi i (\text{Res}(\omega_1, 0_1) + \text{Res}(\omega_1, \infty_{1,2}) + \text{Res}(\omega_1, -r_1)) \\
&= u_{2,-} + \alpha\pi i, & \text{on } (-r_1, 0), \\
u_{3,+} &= u_{3,-} - 2\pi i (\text{Res}(\omega_1, 0_4) + \text{Res}(\omega_1, \infty_{3,4}) + \text{Res}(\omega_1, -r_3)) \\
&= u_{3,-} - 2\pi i (1 + \alpha/2), & \text{on } (-r_3, 0), \\
u_{4,+} &= u_{4,-} + 2\pi i \text{Res}(\omega_1, 0_4) = u_{4,-} + \alpha\pi i, & \text{on } (-r_3, 0).
\end{aligned}
\tag{5.43}$$

Figure 6: Integration path for $u_3(z)$.

Each of these jumps can be easily verified by deforming the integration path of $u_{j,+} - u_{j,-}$ to a closed loop and checking which poles are inside (or outside) the loop. Similarly, we have

$$\begin{aligned}
u_{2,+} &= u_{1,-}, & u_{2,-} &= u_{1,+} + 2\pi i \operatorname{Res}(\omega_1, \infty_{1,2}) = u_{1,+} + \pi i, & \text{on } \Delta_1, \\
u_{3,+} &= u_{2,-}, & u_{3,-} &= u_{2,+} - 2\pi i \operatorname{Res}(\omega_1, q) = u_{2,+} + \pi i, & \text{on } \Delta_2, \\
u_{4,+} &= u_{3,-}, & u_{4,-} &= u_{3,+} + 2\pi i \operatorname{Res}(\omega_1, \infty_{3,4}) = u_{3,+} + 3\pi i, & \text{on } \Delta_3.
\end{aligned} \tag{5.44}$$

Now set

$$v_j(z) = \exp(u_j(z)), \quad j = 1, \dots, 4, \tag{5.45}$$

and consider the row vector $\mathbf{v}(z) := (v_1(z), v_2(z), v_3(z), v_4(z))$. By (5.42)–(5.44) we find that

$$\mathbf{v}_+ = \mathbf{v}_- J_{N_\alpha}, \quad \text{on } \Delta_1 \cup \Delta_2 \cup \Delta_3, \tag{5.46}$$

in Case III. Thus $\mathbf{v}(z)$ satisfies the required jumps for the row vectors of N_α .

Next we discuss the asymptotics of $\mathbf{v}(z)$ for $z \rightarrow \infty$. Taking into account that $\infty_{1,2}$ is a branch point on the Riemann surface, its local coordinate on the Riemann surface can be chosen as $w(z) = 1/\sqrt{z}$. Therefore

$$\begin{aligned}
u_j(z) &= \operatorname{Res}(\omega_1, \infty_{1,2}) \log w(z) + \gamma_j + O(w(z)) \\
&= -\frac{1}{4} \log z + \gamma_j + O(z^{-1/2}), \quad z \rightarrow \infty,
\end{aligned}$$

for $j = 1, 2$, where γ_1 and γ_2 are certain constants. From the relation $u_{2,+} = u_{1,-}$ on $(-\infty, -r_1)$ we find that $\gamma_2 = \gamma_1 + \text{Res}(\omega_1, \infty_{1,2})\pi i = \gamma_1 + \pi i/2$. Similarly we also have

$$u_j(z) = -\frac{3}{4} \log z + \gamma_j + O(z^{-1/2}), \quad z \rightarrow \infty,$$

for $j = 3, 4$, where γ_3 and γ_4 are certain constants. Combining all of this and using (5.42) and (5.45), we find that

$$\mathbf{v}(z) = \left(e^{\gamma_1} z^{-1/4}, i e^{\gamma_1} z^{-1/4}, 0, 0 \right) + O(z^{-3/4}), \quad z \rightarrow \infty.$$

The behavior of $\mathbf{v}(z)$ at the origin can be found from similar arguments as above and is given by

$$\mathbf{v}(z) = O\left(z^{-\alpha/2}, z^{-1/4}, z^{-1/4}, z^{\alpha/2}\right), \quad z \rightarrow 0.$$

Finally, we also find that

$$\mathbf{v}(z) = O(|z - \kappa|^{-1/4}), \quad \text{as } z \rightarrow \kappa, \quad (5.47)$$

if κ is a branch point other than the origin.

Summarizing, we see that $\frac{e^{-\gamma_1}}{\sqrt{2}} \mathbf{v}(z)$ satisfies all the constraints for the first row of N_α . The other meromorphic differentials ω_i , $i = 2, 3, 4$ in Table 5.42 can be used to build the i -th row of N_α in a similar way, we omit the details here.

Our construction of N_α leads to

$$N_\alpha(z) = O(|z - \kappa|^{-1/4}), \quad \text{as } z \rightarrow \kappa, \quad (5.48)$$

if κ is a branch point other than the origin, and

$$N_\alpha(z) \text{diag}(z^{\alpha/2}, z^{1/4}, z^{1/4}, z^{-\alpha/2}) = O(1), \quad \text{as } z \rightarrow 0 \quad (\text{Case III}). \quad (5.49)$$

Similar constructions can be given in Cases I and II. For instance, we may build the first row of N_α in terms of a meromorphic differential ω with prescribed simple poles and residues as shown in the following tables:

x	p	q	$0_{1,2}$	$0_{3,4}$	$\infty_{1,2}$	$\infty_{3,4}$
$\text{Res}(\omega, x)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$1/2$	$3/2$

(5.50)

for Case I, and

x	p	q	$0_{1,2}$	$-r_3$	$\infty_{1,2}$	$\infty_{3,4}$	0_3	0_4
$\text{Res}(\omega, x)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$1/2$	$3/2$	$-\alpha/2$	$\alpha/2$

(5.51)

for Case II. Such constructions will again give us (5.48), but the behavior near the origin is different:

$$N_\alpha(z) = O(z^{-1/4}), \quad \text{as } z \rightarrow 0 \quad (\text{Case I}), \quad (5.52)$$

$$N_\alpha(z) \text{diag}(z^{1/4}, z^{1/4}, z^{\alpha/2}, z^{-\alpha/2}) = O(1), \quad \text{as } z \rightarrow 0 \quad (\text{Case II}). \quad (5.53)$$

In what follows, we will construct local parametrices near each branch point.

5.5 Parametrix near the nonzero branch points

Near each of the branch points in $\{p, q, -r_1, -r_3\} \setminus \{0\}$, we need to build a local parametrix P^{Airy} using the 2×2 RH problem for Airy functions. This construction is very well-known and we omit the details.

5.6 Parametrix near the branch point 0 (hard edge)

The local parametrix at the origin is built by means of modified Bessel functions. Such a construction was given before for a 2×2 RH problem in [40] and in a 3×3 setting in [38, 43]. Our construction will be similar to the one in [38]. For $-1 < \alpha < 0$ there are some extra complications that were solved in an ad hoc way in [38]. We will be facing similar problems and solve them in a more conceptual way.

5.6.1 RH problem for modified Bessel functions

First we recall the construction of [40]. Define the 2×2 matrix valued function $\Psi^{\text{Bessel}}(\zeta)$ by

$$\Psi^{\text{Bessel}}(\zeta) = \begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi} K_\alpha(2\zeta^{1/2}) \\ 2\pi i \zeta^{1/2} I'_\alpha(2\zeta^{1/2}) & -2\zeta^{1/2} K'_\alpha(2\zeta^{1/2}) \end{pmatrix},$$

for $|\arg \zeta| < 2\pi/3$,

$$\Psi^{\text{Bessel}}(\zeta) = \begin{pmatrix} \frac{1}{2} H_\alpha^{(1)}(2(-\zeta)^{1/2}) & \frac{1}{2} H_\alpha^{(2)}(2(-\zeta)^{1/2}) \\ \pi \zeta^{1/2} (H_\alpha^{(1)})'(2(-\zeta)^{1/2}) & \pi \zeta^{1/2} (H_\alpha^{(2)})'(2(-\zeta)^{1/2}) \end{pmatrix} e^{\frac{i\pi\alpha}{2}\sigma_3},$$

for $2\pi/3 < \arg \zeta < \pi$, and

$$\Psi^{\text{Bessel}}(\zeta) = \begin{pmatrix} \frac{1}{2} H_\alpha^{(2)}(2(-\zeta)^{1/2}) & -\frac{1}{2} H_\alpha^{(1)}(2(-\zeta)^{1/2}) \\ -\pi \zeta^{1/2} (H_\alpha^{(2)})'(2(-\zeta)^{1/2}) & \pi \zeta^{1/2} (H_\alpha^{(1)})'(2(-\zeta)^{1/2}) \end{pmatrix} e^{-\frac{i\pi\alpha}{2}\sigma_3},$$

for $-\pi < \arg \zeta < -2\pi/3$, where I_α and K_α are modified Bessel functions and $H_\alpha^{(i)}$, $i = 1, 2$ is the Hankel function; cf. [1, Chapter 9].

Denote with γ_j , $j = 1, 2, 3$ the complex rays $\{\zeta \in \mathbb{C} \mid \arg \zeta = (j+1)\pi/3\}$. According to [40], Ψ^{Bessel} satisfies the following RH problem:

- (1) Ψ^{Bessel} is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^3 \gamma_j$.

(2) With the rays γ_j , $j = 1, 2, 3$ all oriented towards the origin, Ψ^{Bessel} has the jumps

$$\Psi_+^{\text{Bessel}} = \Psi_-^{\text{Bessel}} \times \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha} & 1 \end{pmatrix}, & \text{on } \gamma_1, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } \gamma_2, \\ \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha} & 1 \end{pmatrix}, & \text{on } \gamma_3. \end{cases}$$

(3) Uniformly for $\zeta \rightarrow \infty$ we have

$$\Psi^{\text{Bessel}}(\zeta) = (2\pi\zeta^{1/2})^{-\sigma_3/2} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + O(\zeta^{-1/2}) \right) e^{2\zeta^{1/2}\sigma_3}. \quad (5.54)$$

(4) As $\zeta \rightarrow 0$ in $|\arg \zeta| < 2\pi/3$ we have

$$\Psi^{\text{Bessel}}(\zeta) = \begin{cases} O \begin{pmatrix} \zeta^{\alpha/2} & \zeta^{-\alpha/2} \\ \zeta^{\alpha/2} & \zeta^{-\alpha/2} \end{pmatrix}, & \text{if } \alpha > 0, \\ O \begin{pmatrix} 1 & \log |\zeta| \\ 1 & \log |\zeta| \end{pmatrix}, & \text{if } \alpha = 0, \\ O(\zeta^{\alpha/2}), & \text{if } \alpha < 0. \end{cases} \quad (5.55)$$

5.6.2 Construction for $\alpha \geq 0$

In the construction of the local parametrix at the origin we will restrict ourselves to Case II, which is in some sense the most general and difficult of the three cases. The modifications for Cases I and III will be briefly commented at the end of this section.

We denote by B_δ a small fixed disk with radius $\delta > 0$ centered at the origin, such that it does not contain any other branch points and consider all the jump matrices J_T of T restricted in B_δ ; cf. item (2) in Proposition 5.4. We note that the $(2, 3)$ entry of J_T on $(0, \delta)$ is $e^{-2n\phi_{2,+}}$. Since $\text{Re } \phi_{2,+} = \text{Re } (\lambda_2 - \lambda_3) > c > 0$ on $(0, \delta)$ (cf. Lemma 4.7), this entry is exponentially small as $n \rightarrow \infty$. Similarly the $(4, 3)$ entry of J_T on $(-\delta, 0)$ is $-e^{2n\phi_3}$ which is also exponentially small.

By neglecting the exponentially small entries in the jump matrices for $T(z)$ in B_δ , we are led to the following RH problem for a 4×4 matrix valued function $Q(z)$:

(1) Q is analytic in $B_\delta \setminus (\Delta_1 \cup \Delta_1^\pm)$.

Figure 7: Contours for the local parametrix around 0 in Case II, for $\alpha \geq 0$ (left picture) and for $-1 < \alpha < 0$ (right picture).

- (2) On each of the oriented contours in the left picture of Figure 7, Q has a jump $Q_+ = Q_- J_Q$, where

$$J_Q = \begin{cases} \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e^{-i\pi\alpha}, e^{i\pi\alpha} \right), & \text{on } \Delta_1 \cap B_\delta = (-\delta, 0), \\ I - e^{\pm\alpha\pi i} e^{-2n\phi_1} E_{12}, & \text{on } \Delta_1^\pm \cap B_\delta. \end{cases}$$

- (3) As $z \rightarrow 0$ outside the lens enclosed by $\Delta_1^\pm$ we have

$$\begin{aligned} Q(z) \text{diag}(|z|^{\alpha/2}, |z|^{-\alpha/2}, |z|^{\alpha/2}, |z|^{-\alpha/2}) &= O(1), & \text{if } \alpha > 0, \\ Q(z) \text{diag}((\log |z|)^{-1}, 1, (\log |z|)^{-1}, 1) &= O(1), & \text{if } \alpha = 0. \end{aligned} \quad (5.56)$$

- (4) As $n \rightarrow \infty$,

$$Q(z) = N_\alpha(z) \left(I + O\left(\frac{1}{n}\right) \right), \quad (5.57)$$

uniformly for $z \in \partial B_\delta \setminus (\Delta_1 \cup \Delta_1^\pm)$, where N_α is the global parametrix built in Section 5.4.

Before solving this RH problem, let us consider the function

$$f_1(z) := (\phi_1(z))^2. \quad (5.58)$$

Since the density of $\rho_1 - \mu_1$ blows up like $O(|x|^{-1/2})$ as $x \rightarrow 0$, we obtain from (4.14) and the definition of $\phi_1(z)$ in (4.38) that $f_1(z)$ is analytic in B_δ and gives a conformal mapping from a neighborhood of the origin onto itself, such that $f_1(x)$ is real and positive

for $x \in (0, \delta)$. For our purpose, we may deform the contours $\Delta_1^\pm$ near 0 such that f_1 maps $\Delta_1^\pm \cap B_\delta$ to the rays with angles $2\pi/3$ and $-2\pi/3$, respectively.

We now define

$$Q(z) = E(z) \operatorname{diag} \left(\sigma_1 \Psi^{\text{Bessel}}(n^2 f_1(z)) \sigma_1, z^{-\alpha/2}, z^{\alpha/2} \right) \operatorname{diag}(e^{n\phi_1(z)}, -e^{-n\phi_1(z)}, 1, 1), \quad (5.59)$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and where the prefactor $E(z)$ is analytic in B_δ and is chosen to

satisfy the matching condition on ∂B_δ , see below. With this definition, the items (1), (2) and (3) in the RH problem for $Q(z)$ are satisfied, by virtue of items (1), (2) and (4) in the RH problem for $\Psi^{\text{Bessel}}(z)$.

To achieve the matching condition in item (4) of the RH problem for $Q(z)$, we take $E(z)$ in (5.59) as

$$E(z) = N_\alpha(z) \operatorname{diag}(1, -1, 1, 1) \operatorname{diag} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, z^{\alpha/2}, z^{-\alpha/2} \right) \\ \times \operatorname{diag}((2\pi n)^{-1/2} f_1(z)^{-1/4}, (2\pi n)^{1/2} f_1(z)^{1/4}, 1, 1). \quad (5.60)$$

Obviously $E(z)$ is analytic for $z \in B_\delta \setminus \mathbb{R}^-$. Moreover, one checks that $E(z)$ is also analytic across $(-\delta, 0)$. Finally, since $f_1(z)^{1/4} = O(z^{1/4})$ as $z \rightarrow 0$ and using (5.49) we find that

$$E(z) = O(z^{-1/2}), \quad z \rightarrow 0, \quad (5.61)$$

so $E(z)$ cannot have a pole at zero. We conclude that $E(z)$ is analytic in the disk B_δ . By virtue of (5.54), the matching condition (5.57) in the RH problem for Q is satisfied.

In summary, we have established the following proposition.

Proposition 5.5. *The matrix valued function Q defined by (5.59) and (5.60) satisfies the conditions (1)–(4) of the RH problem for Q .*

In view of the fact that $\det N_\alpha(z) = 1$ and $\det \Psi^{\text{Bessel}} = 1$ (see [40]), we see that

$$\det Q(z) \equiv 1, \quad z \in B_\delta. \quad (5.62)$$

If we take Q as the local parametrix for T , the final transformation would be defined as

$$R(z) = T(z)Q(z)^{-1}, \quad z \in B_\delta. \quad (5.63)$$

Then R would be analytic in $B_\delta \setminus (0, \infty)$ with the following jump on $(0, \delta)$,

$$R_-(x)^{-1} R_+(x) = Q(x) T_-(x)^{-1} T_+(x) Q(x)^{-1} \\ = Q(x) (I + e^{-2n\phi_{2,+}(x)} E_{23}) Q(x)^{-1} \\ = I + e^{-2n\phi_{2,+}(x)} Q(x) E_{23} Q(x)^{-1}. \quad (5.64)$$

Similarly, the jump on $(-\delta, 0)$ would be

$$R_-(x)^{-1} R_+(x) = I - e^{-i\pi\alpha} e^{2n\phi_3(x)} Q_+(x) E_{43} Q_+(x)^{-1}. \quad (5.65)$$

Lemma 5.6. *For $\alpha \geq 0$, the matrix $Q(x)E_{23}Q(x)^{-1}$ is bounded as $x \rightarrow 0$, $x > 0$. Similarly, $Q_+(x)E_{43}Q_+(x)^{-1}$ is bounded as $x \rightarrow 0$, $x < 0$.*

Proof. We prove the first statement only; the second follows similarly. Observe that

$$Q(x)E_{23}Q(x)^{-1} = Q(x) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} Q(x)^{-1}. \quad (5.66)$$

If $\alpha > 0$, it follows from (5.56) and $\det Q(z) \equiv 1$ that both $Q(x) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} Q(x)^{-1}$ behave like $O(z^{\alpha/2})$, and the lemma follows.

If $\alpha = 0$, we use (5.55) and $\det \Psi^{\text{Bessel}}(\zeta) = 1$ to get

$$\left(\Psi^{\text{Bessel}} \right)^{-1}(\zeta) = \begin{pmatrix} O(1) & O(1) \\ O(\log |\zeta|) & O(\log |\zeta|) \end{pmatrix} \quad \text{as } \zeta \rightarrow 0.$$

Inserting this formula in (5.59), we have

$$Q^{-1}(x) = \text{diag} \left(\begin{pmatrix} O(1) & O(1) \\ O(\log x) & O(\log x) \end{pmatrix}, O(1), O(1) \right) E^{-1}(x), \quad (5.67)$$

as $x \rightarrow 0$ and $x > 0$, where $E^{-1}(x)$ is bounded near the origin. Since $Q(x) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T$ is bounded by (5.56) and $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} Q(x)^{-1}$ is bounded by (5.67), the lemma follows for $\alpha = 0$ as well. $\square$

The above lemma, together with (5.64) and the fact that $\text{Re } \phi_{2,+}(x) > c > 0$ for some constant c on $(0, \delta)$, implies that, if $\alpha \geq 0$, the jump matrix for R is exponentially close to the identity matrix as $n \rightarrow \infty$, uniformly for $x \in (0, \delta)$. A similar statement holds for the jump on $(-\delta, 0)$. We therefore take the parametrix $P = Q$ in case $\alpha \geq 0$.

5.6.3 Construction for $-1 < \alpha < 0$

If $-1 < \alpha < 0$, then the $(2, 3)$ entry of the jump matrix J_T on $(0, \delta)$ and the $(4, 3)$ entry of J_T on $(-\delta, 0)$ cannot be simply neglected. Hence, we need to build a 4×4 matrix valued function P such that

- (1) P is analytic in $B_\delta \setminus (\mathbb{R} \cup \Delta_1^\pm)$.
- (2) For each of the oriented contours shown in the right picture of Figure 7, P has a jump $P_+ = P_- J_P$, where

$$J_P = \begin{cases} \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} e^{-i\pi\alpha} & 0 \\ -e^{2n\phi_3} & e^{i\pi\alpha} \end{pmatrix} \right), & \text{on } \Delta_1 \cap B_\delta = (-\delta, 0), \\ I - e^{\pm\alpha\pi i} e^{-2n\phi_1} E_{12}, & \text{on } \Delta_1^\pm \cap B_\delta, \\ I + e^{-2n\phi_{2,+}} E_{23}, & \text{on } (0, \delta). \end{cases}$$

(3) $P(z)$ behaves near the origin like:

$$P(z) = O(z^{\alpha/2}), \quad P^{-1}(z) = O(z^{\alpha/2}).$$

(4) As $n \rightarrow \infty$,

$$P(z) = N_\alpha(z) \left(I + O\left(\frac{1}{n}\right) \right), \quad (5.68)$$

uniformly for $z \in \partial B_\delta \setminus (\Delta_1 \cup \Delta_1^\pm)$, where N_α is the global parametrix built in Section 5.4.

We use the matrix valued function Q given by (5.59) and (5.60), which works as the parametrix for the case $\alpha \geq 0$, to construct P . More precisely, recall the sectors Ω_j , $j = 1, \dots, 4$, in the right picture of Figure 7. By setting

$$S(z) = \Lambda^{-1}(z) S_j(z) \Lambda(z), \quad z \in \Omega_j, \quad (5.69)$$

where Λ is defined in (5.21) and S_j , $j = 1, \dots, 4$ is a constant matrix that depends on the sector Ω_j , we construct P in the form

$$P(z) = Q(z) S(z). \quad (5.70)$$

The construction of these four matrices S_j is different in the three Cases I–III; we will again explain it for Case II.

Let us give names to the jump matrix J_P on the four jump contours around zero as follows. $J_{P,1}$ is the jump matrix of P on $(0, \delta)$, $J_{P,2}$ is the jump matrix on Δ_1^+ , $J_{P,3}$ on $(-\delta, 0)$ and $J_{P,4}$ on Δ_1^- . Similarly we define the jump matrices $J_{Q,k}$, $k = 1, \dots, 4$. Note that $J_{Q,2} = J_{P,2}$, $J_{Q,4} = J_{P,4}$, $J_{Q,1}$ differs from $J_{P,1}$ in the $(2, 3)$ entry, and $J_{Q,3}$ differs from $J_{P,3}$ in the $(4, 3)$ entry,

To obtain the correct jumps for P , the matrices S_j in (5.70) should be such that

$$\begin{aligned} \Lambda J_{P,1}^{-1} \Lambda^{-1} &= S_1^{-1} (\Lambda J_{Q,1}^{-1} \Lambda^{-1}) S_4, \\ \Lambda J_{P,k} \Lambda^{-1} &= S_k^{-1} (\Lambda J_{Q,k} \Lambda^{-1}) S_{k-1}, \quad \text{for } k = 2, 3, 4. \end{aligned} \quad (5.71)$$

Multiplying these equations into a telescoping product form yields the following relation for S_1 :

$$\Lambda (J_{P,1}^{-1} J_{P,4} J_{P,3} J_{P,2}) \Lambda^{-1} = S_1^{-1} \Lambda \left(J_{Q,1}^{-1} J_{Q,4} J_{Q,3} J_{Q,2} \right) \Lambda^{-1} S_1. \quad (5.72)$$

Now $\Lambda (J_{P,1}^{-1} J_{P,4} J_{P,3} J_{P,2}) \Lambda^{-1}$ and $\Lambda (J_{Q,1}^{-1} J_{Q,4} J_{Q,3} J_{Q,2}) \Lambda^{-1}$ are both constant matrices. A calculation shows that they have the spectral decompositions

$$\Lambda (J_{P,1}^{-1} J_{P,4} J_{P,3} J_{P,2}) \Lambda^{-1} = \begin{pmatrix} e^{-i\pi\alpha} & 0 & 0 & 0 \\ -1 & e^{i\pi\alpha} & -e^{-i\pi\alpha} & 0 \\ 0 & 0 & e^{-i\pi\alpha} & 0 \\ 0 & 0 & -1 & e^{i\pi\alpha} \end{pmatrix} =: V_P D V_P^{-1}, \quad (5.73)$$

$$\Lambda (J_{Q,1}^{-1} J_{Q,4} J_{Q,3} J_{Q,2}) \Lambda^{-1} = \text{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}) - E_{21} =: V_Q D V_Q^{-1}, \quad (5.74)$$

with common eigenvalue matrix

$$D = \text{diag}(e^{-i\pi\alpha}, e^{i\pi\alpha}, e^{-i\pi\alpha}, e^{i\pi\alpha}), \quad (5.75)$$

and where the eigenvector matrices V_P and V_Q can be chosen as

$$V_P = I + \frac{1}{2i \sin(\pi\alpha)} (E_{21} + e^{-i\pi\alpha} E_{23} + E_{43}), \quad V_Q = I + \frac{1}{2i \sin(\pi\alpha)} E_{21}.$$

On account of the spectral decompositions (5.73)–(5.74), the equation (5.72) can be rewritten as $D(V_Q^{-1} S_1 V_P) = (V_Q^{-1} S_1 V_P) D$, which allows the solution $S_1 = V_Q V_P^{-1}$. So we can take

$$S_1 = V_Q V_P^{-1} = I - \frac{1}{2i \sin(\pi\alpha)} (e^{-i\pi\alpha} E_{23} + E_{43}). \quad (5.76)$$

The constant matrices S_2 , S_3 and S_4 are then obtained recursively from S_1 and (5.71), which now read

$$S_2 = I + \frac{1}{2i \sin(\pi\alpha)} (E_{13} - e^{-i\alpha\pi} E_{23} - E_{43}), \quad (5.77)$$

$$S_3 = I - \frac{1}{2i \sin(\pi\alpha)} (E_{13} + e^{i\pi\alpha} E_{23} + E_{43}), \quad (5.78)$$

$$S_4 = I - \frac{1}{2i \sin(\pi\alpha)} (e^{i\pi\alpha} E_{23} + E_{43}). \quad (5.79)$$

With S_k , $k = 1, \dots, 4$, given in (5.76)–(5.79), we conclude that P defined in (5.70) indeed satisfies the items (1)–(4) in the RH problem for $P(z)$ stated above.

The above description was for Case II. For Cases I and III similar constructions can be given. The key point is that in each case, the cyclic products of jump matrices in the left hand sides of (5.73) and (5.74), respectively, allow spectral decompositions with common eigenvalue matrix (5.75). We leave these constructions to interested readers.

Combining the results in this subsection and Section 5.6.2, we take $P = QS$ as the parametrix in B_δ for Case II, where $S = I$ if $\alpha \geq 0$, and S is given by (5.69), if $-1 < \alpha < 0$.

5.7 Final transformation

Using the local parametrices P , P^{Airy} and the global parametrix N_α , we define the final transformation as follows

$$R(z) = \begin{cases} T(z)(P^{\text{Airy}}(z))^{-1}, & \text{in the disks around } \{p, q, -r_1, -r_3\} \setminus \{0\}, \\ T(z)(P(z))^{-1}, & \text{in the disk } B_\delta \text{ around the origin,} \\ T(z)(N_\alpha(z))^{-1}, & \text{elsewhere.} \end{cases} \quad (5.80)$$

From our construction of the parametrices, it follows that R satisfies the following RH problem:

Figure 8: Jump contours Σ_R for the RH problem for R in Case I, for $\alpha \geq 0$ (top) and for $-1 < \alpha < 0$ (bottom).

- (1) R is analytic in $\mathbb{C} \setminus \Sigma_R$, where Σ_R depends on α and is different for all three cases. For Cases I and III, an illustration is shown in Figures 8 and 9, respectively.
- (2) R has jumps $R_+ = R_- J_R$ on Σ_R that satisfy

$$J_R(z) = I + O(1/n), \quad (5.81)$$

uniformly for z on the boundaries of the disks;

$$J_R(x) = I + O(x^\alpha e^{-cn}), \quad (5.82)$$

on $(0, \delta)$ (Case I), or on $(-\delta, 0) \cup (0, \delta)$ (Case II), or on $(-\delta, 0)$ (Case III), for some constant $c > 0$, if $\alpha \geq 0$;

$$J_R(z) = I + O(e^{-cn|z|}), \quad (5.83)$$

on the other parts of Σ_R , for some constant $c > 0$.

- (3) $R(z) = I + O(1/z)$ as $z \rightarrow \infty$.

Then, as in [16, 17, 38], we may conclude that

$$R(z) = I + O\left(\frac{1}{n(|z| + 1)}\right), \quad (5.84)$$

as $n \rightarrow \infty$, uniformly for z in the complex plane outside of Σ_R . The convergence holds in fact uniformly up to Σ_R , as long as we stay away from the points of self-intersection of Σ_R (including the origin).

Figure 9: Jump contours Σ_R for the RH problem for R in Case III, for $\alpha \geq 0$ (top) and for $-1 < \alpha < 0$ (bottom).

6 Proof of Theorem 2.3

With the above preparations, we are ready to establish Theorem 2.3. For this purpose, we need to represent the correlation kernel $K_n(x, y)$ in terms of T by following the sequence of transformations $Y \mapsto X \mapsto U \mapsto T$ in the RH analysis.

Let $x, y \in \Delta_2$. Recalling the representation of $K_n(x, y)$ in (2.34), we first find from (5.7) that

$$K_n(x, y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 0 & y^{-\alpha/2} e^{-\frac{ny}{1-t}} & 0 \end{pmatrix} X_+^{-1}(y) X_+(x) \begin{pmatrix} 0 & x^{\alpha/2} e^{-\frac{nx}{t}} & 0 & 0 \end{pmatrix}^T.$$

In view of (5.18), this becomes

$$K_n(x, y) = \frac{x^{\alpha/2} y^{-\alpha/2}}{2\pi i(x-y)} \begin{pmatrix} 0 & 0 & e^{n\lambda_{3,-}(y)} & 0 \end{pmatrix} U_+^{-1}(y) U_+(x) \begin{pmatrix} 0 & e^{-n\lambda_{2,+}(x)} & 0 & 0 \end{pmatrix}^T.$$

By (5.38), it follows that

$$K_n(x, y) = \frac{x^{\alpha/2} y^{-\alpha/2}}{2\pi i(x-y)} \begin{pmatrix} 0 & -e^{n\lambda_{2,-}(y)} & e^{n\lambda_{3,-}(y)} & 0 \end{pmatrix} T_+^{-1}(y) T_+(x) \times \begin{pmatrix} 0 & e^{-n\lambda_{2,+}(x)} & e^{-n\lambda_{3,+}(x)} & 0 \end{pmatrix}^T. \quad (6.1)$$

Now, we obtain from (5.84) and standard arguments (e.g. [7, Section 9]) that

$$T_+^{-1}(y) T_+(x) = I + O(x-y), \quad \text{as } x \rightarrow y, \quad (6.2)$$

uniformly in n . Also note that $\lambda_{3,\pm} = \lambda_{2,\mp}$. Thus, by taking $y \rightarrow x$, it follows from (6.2) and L'Hôpital's rule that

$$K_n(x, x) = \frac{n}{2\pi i}(\xi_{2,+}(x) - \xi_{2,-}(x)) + O(1) = n \frac{d\mu_2}{dx}(x) + O(1), \quad (6.3)$$

where the last equality follows from (4.15), and so

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = \frac{d\mu_2}{dx}(x), \quad x \in \Delta_2. \quad (6.4)$$

If $x \in \mathbb{R}^+ \setminus \Delta_2$, a similar argument as given above yields

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = 0. \quad (6.5)$$

This completes the proof of Theorem 2.3.

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