Citation	P. Janssens, G. Pipeleers, M. Diehl, and J. Swevers, (2014), Energy-optimal time allocation of a series of point-to-point motions IEEE Transactions on Control Systems Technology, 22(6):2432-2435.
Archived version	Author manuscript: the content is identical to the content of the published paper, but without the final typesetting by the publisher
Published version	http://dx.doi.org/10.1109/TCST.2014.2301375
Journal homepage	http://www.ieeecss.org/publications/tcst
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IR	https://lirias.kuleuven.be/handle/123456789/357555

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Energy-optimal time allocation of a series of point-to-point motions

Pieter Janssens, Goele Pipeleers, Moritz Diehl and Jan Swevers

Abstract—When a system has to perform a series of point-to-point motions within a given time, the execution time for each individual motion is, in many cases, free to choose. This paper presents a dynamic programming algorithm to allocate the time to each motion in a series of point-to-point motions in an energy-optimal way. The algorithm is demonstrated on an XY-positioning stage and the results are compared with an equal time allocation scheme.

Index Terms—Motion control, Energy-optimal point-to-point motions, XY-positioning stage

I. INTRODUCTION

There is a growing interest in reducing the energy consumption of mechatronic systems such as industrial robots, pick-and-place machines, etc. [1], [2], [3], [4], [5], [6]. In [1], [4], [5] algorithms are presented to compute energy-optimal trajectories to perform a single point-to-point (PTP) motion within a given time. For linear time-invariant (LTI) systems, computing the energy-optimal PTP motion trajectory generally involves solving a convex optimization problem. The dissipated energy, most often modelled as the 2-norm of the system input [4], [6], is minimized subject to linear equality constraints, which impose the system dynamics and the system's initial and terminal conditions. System limitations such as maximum input current, maximum velocity, etc. can be incorporated in the optimization problem by formulating them as linear inequality constraints. Thanks to the convexity of the optimization problem, global optimality is guaranteed.

In case of a system performing a series of consecutive PTP motions with specified dwell time intervals, the energy-optimal system input can be obtained using the same methods as for the single PTP motion problem. In many practical situations, however, only the total execution time of the series of PTP motions is important, and hence the allocation of the total execution time over the different individual PTP motions is

free. As a result, the energy dissipation can be minimized by not only considering the system inputs, but also the time allocated to each motion as optimization variables.

In the field of multistage optimal control, many algorithms that optimize both the system inputs and the switching instants have been developed [7], [8], [9], [10], [11]. These algorithms allow different system dynamics in each stage, and the system states at the switching instants are not necessarily known in advance. [7], [8], [9] propose an algorithm that alternately updates the switching instants, and solves an optimal control problem with fixed switching instants. [11], [10] use time transformation to formulate the multistage optimal control problem as a nonlinear optimization problem. Either way, global optimality is not guaranteed [7], [8], [9], [10], [11] and computation times can be long.

The considered energy-optimal time allocation problem can be seen as a multistage optimal control problem where each PTP motion is a different stage, and the starting time of each motion but the first corresponds to a switching instant. For the considered application, however, (i) the system dynamics are the same for each stage, and (ii) the system states at the switching instants correspond to the given dwell specifications, and are known in advance. As a result, the minimal energy to perform a certain PTP motion, i.e. the stage cost, can be computed for each stage separately and depends on the motion time allocated to the stage only. To solve the optimal time allocation problem efficiently, we propose to precompute the stage costs for the different PTP motions as a function of the motion time. This allows formulating the optimal time allocation problem in terms of the motion times only (no longer involving the input and state trajectories): minimize the sum of the stage costs subject to an upper bound on the sum of the motion times. When formulated in discrete time, this problem amounts to a so-called separable nonlinear knapsack problem in which the variables are only coupled through one linear constraint [12], [13]. As dynamic programming (DP) is known to be very efficient in solving such problems [14], [15], we apply DP to solve the energy-optimal time allocation problem. Once the optimal time allocation is known, the energy-optimal system input for the complete series of PTP motions is computed by solving a single convex optimization problem.

The DP algorithm requires a lookup table, which can be precomputed, containing the minimally dissipated energy as a function of the motion time for each individual PTP motion. In case of an LTI system for which the energy dissipation is given by a convex expression, this energy lookup table can be obtained by solving a series of convex optimization problems.

Goele Pipeleers is Postdoctoral Fellow of the Research Foundation - Flanders. This work has been carried out within the framework of projects IWT-SBO 80032 (LECOPRO) of the Institute for the Promotion of Innovation through Science and Technology in Flanders (IWT-Vlaanderen), and G.0377.09 of the Research Foundation - Flanders (FWO - Flanders). This work also benefits from K.U.Leuven-BOF PFV/10/002 Center-of-Excellence Optimization in Engineering (OPTEC), from the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office, and from K.U.Leuven's Concerted Research Action GOA/10/11 "Global real-time optimal control of autonomous robots and mechatronic systems".

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As a result, the presented energy-optimal time allocation yields globally optimal results. In the case of nonlinear system dynamics or a nonconvex expression for the energy dissipation, the energy-optimal time allocation algorithm is still applicable but global optimality is no longer guaranteed.

In this paper, we only consider the class of systems for which such an energy lookup table can be computed by solving a series of convex optimization problems. This is for instance the case for DC permanent magnet servo motors and AC synchronous servo motors, which can be modelled as LTI systems, and for which the energy losses are typically modelled as a convex function of the system inputs [4], [5], [6], and possibly also the system states [1]. The proposed algorithm is validated for a series of 5 consecutive PTP motions carried out by an XY-positioning stage. The results of the proposed algorithm are compared with an equal time allocation scheme.

This paper is organised as follows. Section II discusses the computation of the energy lookup tables using convex optimization. The energy-optimal time allocation algorithm is detailed in section III. Section IV presents the application of the presented algorithm on an XY-positioning stage. Finally, section V summarizes the conclusions.

II. COMPUTING MINIMAL ENERGY LOOKUP TABLES

This section details the computation of the energy lookup tables in case of a discrete-time LTI system for which the energy consumption is given by a convex function of the state and input variables. We assume that the initial and terminal conditions of each PTP motion are given. For each PTP motion but the first, the initial conditions correspond with the terminal conditions of the previous PTP motion and meet the dwell specifications.

For each motion in the series of PTP motions, the energy lookup table $E(N)$ is constructed by solving an energy-optimal PTP motion problem for different values of execution time N . Consider a discrete-time LTI system with state space matrices A , B , C , D , input $u(k)$, output $y(k)$, and states $x(k)$. The energy-optimal system input to perform a single PTP motion with initial conditions x_{init} , u_{init} and terminal conditions x_{end} , u_{end} within N samples is obtained by solving the following optimization problem:

$$\underset{u, x, y}{\text{minimize}} \quad E_{\text{diss}}(u, x), \quad (1a)$$

$$\text{subject to} \quad x(1) = x_{\text{init}}, \quad u(1) = u_{\text{init}}, \quad (1b)$$

$$x(k+1) = Ax(k) + Bu(k), \quad (1c)$$

$$y(k) = Cx(k) + Du(k), \quad (1d)$$

$$x(N) = x_{\text{end}}, \quad u(N) = u_{\text{end}}, \quad (1e)$$

$$u_{\min} \leq u(k) \leq u_{\max}, \quad (1f)$$

$$\dot{y}_{\min} \leq \dot{y}(k) \leq \dot{y}_{\max}. \quad (1g)$$

In (1), the dissipated energy (1a), given by the convex function $E_{\text{diss}}(u, x)$, is minimized subject to a set of linear equality constraints containing the initial conditions (1b), the state-space equations (1c) for $k \in \{1, \dots, N-1\}$, the output equations (1d) for $k \in \{1, \dots, N\}$, and the terminal conditions

(1e). In addition, (1) also contains linear inequality constraints, (1f) and (1g) for $k \in \{1, \dots, N\}$, to ensure that the input $u(k)$ and the velocity $\dot{y}(k) := (y(k) - y(k-1))/T_s$ are within the system bounds.

For each value of N , the minimal energy is stored in the energy lookup table. In case of infeasibility, $+\infty$ is stored in the energy lookup table. It is clear that the computation of these energy lookup tables is computationally expensive. Fortunately, additional measures can be taken to reduce the computational cost. If the optimal solution of (1) for a given execution time N and a given travel distance does not hit the linear inequality constraints (1f) and (1g), this solution can be scaled down linearly to obtain the optimal solution of (1) for the same execution time N , but a smaller travel distance [16]. If (1) is infeasible for a given execution time N and travel distance, (1) will also be infeasible for problems with the same travel distance but shorter execution times, and for problems with the same execution time but larger travel distances. These observations already reduce the computational load significantly. Finally, the energy lookup tables can also be precomputed for a certain grid of travel distances such that the computational effort for every new problem reduces to solving the DP algorithm discussed in section III and solving a single convex optimization problem.

For some systems, the terminal conditions x_{end} , u_{end} of a PTP motion are not unambiguously related to the dwell specifications. This is for instance the case for an overactuated systems where a desired end-point position can be obtained in different ways. As a result, the minimal energy to perform a certain PTP motion not only depends on the allocated time, but also on the terminal conditions. In these situations, the DP algorithm presented in section III can only optimize the time allocation for fixed terminal states and inputs of each motion. In order to further optimize the obtained solution, the presented algorithm can be extended to optimize the terminal conditions of each PTP motion as well. However, this requires (i) the energy lookup tables to be computed for a finite set of feasible terminal conditions, and (ii) a DP algorithm that optimizes both the allocated time and the terminal conditions of each PTP motion. Due to the curse of dimensionality in dynamic programming, other optimization algorithms that don't assume fixed states at the switching instants, might be more appropriate in these situations [7], [8].

III. ENERGY-OPTIMAL TIME ALLOCATION ALGORITHM

This section discusses the DP algorithm, consisting of two phases, that uses the energy lookup table for each of the motions to compute the energy-optimal time allocation. For simplicity, we consider a single-input single-output system that needs to execute a series of Q PTP motions within N_{tot} samples. Let N_i for $i \in \{1, \dots, Q\}$, denote the execution time of each of the Q PTP motions. For each of the PTP motions, the minimally dissipated energy $E_i(N_i)$ as a function of the execution time N_i is assumed to be known. DP is applied to

solve the following separable nonlinear knapsack problem:

$$\begin{aligned} & \text{minimize}_{N_1, N_2, \dots, N_Q} E_1(N_1) + E_2(N_2) + \dots + E_Q(N_Q) \\ & \text{subject to } N_1 + N_2 + \dots + N_Q \leq N_{\text{tot}}. \end{aligned}$$

A. Phase 1: Forward computation of the minimal energy

The first phase of the energy-optimal time allocation algorithm starts by computing the minimal energy $E_{12}(N_{12})$ to perform just the first two PTP motions as a function of the execution time $N_{12} = N_1 + N_2$. For each value of N_{12} , the minimal energy $E_{12} = E_1 + E_2$ is computed by solving the following minimization problem:

$$E_{12}(N_{12}) = \text{minimize}_{N_1} E_1(N_1) + E_2(N_{12} - N_1). \quad (2)$$

This problem is solved by constructing the vector $E_1(N_1) + E_2(N_{12} - N_1)$ and simply finding its smallest element. For each of these minimization problems, the minimally dissipated energy $E_{12}(N_{12})$ and its minimizer $N_1^*(N_{12})$ are stored in a lookup table as a function of N_{12} .

Next, the third PTP motion is added to the first two in a similar way resulting in the minimal energy $E_{123}(N_{123})$ to perform the first three PTP motions within $N_{123} = N_{12} + N_3$ samples. For each value of N_{123} , the minimal energy E_{123} is computed by solving the following minimization problem:

$$E_{123}(N_{123}) = \text{minimize}_{N_{12}} E_{12}(N_{12}) + E_3(N_{123} - N_{12}). \quad (3)$$

Again, problem (3) is solved by constructing the vector $E_{12}(N_{12}) + E_3(N_{123} - N_{12})$ and finding its smallest element. The minimally dissipated energy $E_{123}(N_{123})$ and its minimizer $N_{12}^*(N_{123})$ are again stored in a lookup table as function of N_{123} .

This procedure is repeated until the last PTP motion is added and we obtained the final lookup tables: $E_{1\dots Q}(N_{1\dots Q})$ and $N_{1\dots Q-1}^*(N_{1\dots Q})$, where $N_{1\dots Q} \leq N_{\text{tot}}$.

B. Phase 2: Backward computation of the energy-optimal time allocation

The second phase of the algorithm starts by computing the energy-optimal execution time $N_{1\dots Q}$ of the complete series of Q PTP motions:

$$N_{1\dots Q} = \arg \text{minimize}_{N_{1\dots Q}} E_{1\dots Q}(N_{1\dots Q}). \quad (4)$$

Again, this problem is solved by simply finding the smallest element in a vector.

Next, the energy-optimal time allocation $\{N_1, N_2, \dots, N_Q\}$ is obtained by recursively going backwards through the lookup tables $N_{1\dots Q-1}^*(N_{1\dots Q})$, ..., $N_{12}^*(N_{123})$, $N_1^*(N_{12})$:

$$\begin{aligned} N_{1\dots Q-1} &= N_{1\dots Q-1}^*(N_{1\dots Q}) \\ \Rightarrow N_Q &= N_{1\dots Q} - N_{1\dots Q-1} \\ \Rightarrow N_{1\dots Q-2} &= N_{1\dots Q-2}^*(N_{1\dots Q-1}) \\ \Rightarrow N_{Q-1} &= N_{1\dots Q-1} - N_{1\dots Q-2} \\ &\vdots \\ \Rightarrow N_{12} &= N_{12}^*(N_{123}) \\ \Rightarrow N_3 &= N_{123} - N_{12} \\ \Rightarrow N_1 &= N_1^*(N_{12}) \\ \Rightarrow N_2 &= N_{12} - N_1. \end{aligned} \quad (5)$$

Fig. 1. Series of 5 PTP motions in the XY-plane

The obtained time allocation $\{N_1, N_2, \dots, N_Q\}$ satisfies the constraint $\sum_{i=1}^Q N_i \leq N_{\text{tot}}$.

IV. VALIDATION ON AN XY-POSITIONING STAGE

This section discusses the validation of the proposed algorithm on an XY-positioning stage consisting of two linear AC synchronous motors. At the end of each PTP motion, the system needs to stand still for 0.3 s to carry out a certain operation. The dynamics of both actuators are assumed to be identical and given by the following transfer function:

$$G(z) = \frac{1.147 \cdot 10^{-5}z + 1.146 \cdot 10^{-5}}{z^2 - 1.996z + 0.9958}, \quad T_s = 0.004 \text{ s}. \quad (6)$$

Since the majority of the energy losses scale with the squared 2-norm of the current [2], [17], this is usually chosen as the cost function in energy-optimal PTP motion control [6], [4], [18]. In [1] and in this paper, a more accurate model for the energy losses is used [2], [17]. Apart from the internal resistance losses, the losses due to viscous friction are also included in the objective function of optimization problem (1):

$$E_{\text{diss}} = R_x \|u_x\|_2^2 + c_x \|\dot{x}\|_2^2 + R_y \|u_y\|_2^2 + c_y \|\dot{y}\|_2^2. \quad (7)$$

In (7), R_x and R_y denote the motor resistances, c_x and c_y denote the viscous friction coefficients, u_x and u_y denote the input currents, and $\dot{x}$ and $\dot{y}$ denote the velocities of the x-motor and y-motor respectively.

Fig. 1 depicts the series of 5 PTP motions in the XY-plane. The XY-table starts and ends at position $x = 0$ m, $y = 0$ m. The maximum current for the two systems is 5 A. The maximum velocity is 0.5 m/s. The complete series of point-to-point motions must be completed within 2.7 s.

The DP algorithm finds the energy-optimal time allocation in 0.15 s only. Fig. 2 shows the energy-optimal system input and corresponding velocity and position of the linear motors as a function of time. Note that the obtained solution is globally optimal and satisfies the constraints on the maximum current and maximum velocity. For the considered system, the minimal energy of each motion decreases as a function of motion time. As a result, the energy-optimal total execution time of the series of PTP motions is equal to the maximum available time 2.7 s. Table I shows the optimally allocated execution time, without the dwell time of 0.3 s, for each of the PTP motions.

The energy lookup tables can also be used to compute the minimal time required to perform the series of PTP motions.

Fig. 2. Energy-optimal system input and corresponding position and velocity of the linear motors as a function of time.

PTP motion	1	2	3	4	5
$N \cdot T_s$ (s)	0.376	0.2	0.136	0.316	0.172

TABLE I
OPTIMALLY ALLOCATED EXECUTION TIME $N \cdot T_s$ FOR A TOTAL
EXECUTION TIME OF 2.7 s.

For each of the PTP motions, the smallest feasible motion time is chosen such that one of the linear motors is performing a time-optimal PTP motion while the other is performing an energy-optimal PTP motion within the same execution time. This way, the XY-table performs a series of time-optimal PTP motions in an energy-optimal way. For this example, the minimal execution time is 2.612 s.

To show that the proposed algorithm yields significant energy savings, the results are compared with an “equal time allocation scheme”. The equal time allocation scheme allocates the same amount of time to each motion. Fig. 3 shows the required energy as a function of execution time in case of energy-optimal time allocation and equal time allocation. The energy is normalized with respect to the required energy for the time-optimal solution. The equal time allocation scheme already results in infeasibility for execution times smaller than 3.34 s. For the considered example, the energy savings of the proposed method over the equal time allocation scheme amount up to 43%.

Finally, to illustrate that the proposed algorithm yields a globally optimal solution of the energy-optimal time allocation problem, Fig. 4 shows the energy consumption E_{12} for the first two PTP motions as a function of the time $N_1 \cdot T_s$ allocated to the first motion, for a total execution time $N_{12} \cdot T_s$ of 1.4, 1.6, 1.8 and 2 s. The energy consumption E_{12} is scaled with respect to the optimal solution for $N_{12} \cdot T_s = 1.4$ s. In Fig. 4, the optimal time allocated to first motion as computed by the energy-optimal time allocation algorithm, is indicated by a black cross.

Fig. 3. Comparison of the time-energy trade-off curve with an equal time allocation scheme.

Fig. 4. Energy consumption E_{12} for the first two PTP motions as a function of the time allocated to the first motion, for a total execution time $N_{12} \cdot T_s$ of 1.4, 1.6, 1.8 and 2 s.

V. CONCLUSION

This paper presents a dynamic programming algorithm to compute the energy-optimal time allocation of a series of PTP motions that needs to be executed within a given time. By optimizing the dwell moments of the intermediate PTP motions, the energy consumption of mechatronic systems such as pick-and-place machines can be reduced significantly. The presented methodology was demonstrated on an XY-positioning stage and compared with a straightforward time allocation scheme.

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