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Unified Analysis of Iterative Learning and Repetitive Controllers in Trial Domain

Goele Pipeleers, and Kevin L. Moore

Abstract—While iterative learning control (ILC) and repetitive control (RC) have much ground in common, they fundamentally differ in the initial conditions at each repetition. This difference has lead to distinct analysis techniques, hereby clouding the interrelations between both control strategies. To facilitate the transfer of results, this paper presents a unified approach to ILC and RC. Both control problems are formulated in the trial domain using so-called system lifting. For a given system, the corresponding ILC and RC trial-domain models differ, and a thorough system theoretic analysis and comparison of these models is performed. To illustrate the value of a unified formulation of ILC and RC, the analysis of the most commonly used ILC and RC structures is harmonized. This analysis reveals central differences and similarities between various stability, monotonic convergence and steady-state performance conditions.

I. INTRODUCTION

Repetitive controllers (RCs) are feedback controllers that improve the attenuation of periodic disturbances with the help of the error signal from previous periods [14], [26], [29], [51]. As such, RC is indispensable to meet the performance requirements in the control of disc drives [12], [17], active bearings [45], [60], electronic cam motion generation [30], satellite attitude control [58] etc. Iterative learning control (ILC) is an open-loop control strategy whereby the performance of a system executing the same task over and over again is improved by learning from previous executions [3], [9], [40]. ILC plays a crucial role in the control of industrial robots [42], [56], wafer stages [13], [27], chemical processes [22], [37] etc. Given the importance of these control strategies in numerous applications, the analysis and design of RCs and ILCs has received continuous research effort since the 80s.

In essence, RC and ILC are very similar as they both achieve superior performance for repeated input signals by means of the internal model principle [20], [49]. However, the control strategies fundamentally differ in the system's initial conditions at each repetition or trial. ILC operates in a discontinuous mode: prior to each trial, the system's initial conditions are assumed to be reset to the same value. RC, on the other hand, operates in a continuous mode: the trials/periods follow back to back without resetting the system state and hence, the initial conditions of one trial equal the system's final conditions of the previous trial. The discontinuous operation mode of ILC gives rise to two-dimensional system dynamics, since resetting

the system state decouples the time-domain dynamics during the trial from the trial-domain dynamics: the evolution of the signals over the trials. RC systems, on the other hand, feature one-dimensional dynamics that are fully described in the time domain.

Initially, RC and ILC designs relied on an approximate analysis that is based on the Fourier transform and inherently assumes an infinite period / trial length. Relying on this analysis, both controllers evolved to a structure involving a linear time-invariant learning filter L and robustness filter Q [2], [15], [33], [26], [11], [28], and the corresponding design guidelines for L and Q are similar as well [16], [35], [41], [55]. Although, thanks to their simplicity, these RC and ILC designs are still commonly used in practice, the literature witnessed a paradigm shift to RC and ILC analysis and design methods that do account for the finite period / trial length properly. Herein the different operation modes of RC and ILC have lead to the usage of different analysis techniques. The two-dimensional system dynamics of ILC systems demand specialized analysis tools, where the ones based on the lifted system representation [9], [39], obtained by grouping the signal samples during the trials into large vectors, called supervectors, have become most popular [1], [4], [21], [25], [52]. The one-dimensional dynamics of RC systems, on the other hand, comply with standard tools from linear system theory based on Laplace and Fourier transforms, and consequently, most RC designs are formulated in the time domain [31], [34], [44], [59].

The usage of different analysis techniques clouds the interrelation between ILC and RC, and to date, a clear and thorough system theoretic comparison between ILC and RC that accounts for the finite trial length is lacking. In addition, it complicates the transfer of advanced analysis and design methods between both control strategies. For instance, various lifted ILC algorithms have been used in RC applications, such as disc drives [50], raster scanning [5], [10], power electronics [32], [36], induction motors [48] etc, hereby ignoring the application's continuous operation mode.

To enable a thorough system theoretic comparison of ILC and RC and to enhance the transfer of results between both control strategies, this paper unifies the formulation and analysis of ILC and RC. To this end, both control problems are formulated in the trial domain using system lifting. The signal samples during the trials are grouped into supervectors, and reformulating the system dynamics in terms of the input and output supervectors yields a trial-domain system model. Section III details the RC and ILC system lifting and includes a thorough system theoretic analysis and comparison of resulting trial-domain models for the same underlying time-domain

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Fig. 1. Common structure of (a) an iterative learning controller and (b) a repetitive controller for a discrete-time SISO LTI system $P(q)$. In ILC the initial conditions are reset to the same value prior to each trial, while in RC the system state is never reset.

system. Although the RC system lifting has been elaborated for the control of linear periodically time-varying systems [6], [7], it is only scarcely used in an RC context [46]. Section III also carefully analyzes the relation between system properties in time and trial domain. In spite of the popularity of system lifting, this relation has not yet been discussed in detail in the ILC literature. Subsequently, Section IV streamlines the analysis of RC and ILC systems in the trial domain, and compares the resulting stability, monotonic convergence and performance conditions. Based on the relation between time-domain and trial-domain system properties, the results are linked to the approximate analyses based on the Fourier transform and assuming infinite trial length. To crystalize the developments, the results are applied to the $Q-L$ RC and ILC structures, presented in Section II. Section V illustrates the results for a numerical example and Section VI concludes the paper. For convenience, the symbols and notations used throughout the paper are summarized in Appendix A.

For the sake of clarity and conciseness, the paper focusses on stable and minimum-phase single-input, single-output (SISO) linear time-invariant (LTI) systems. The extension to non-minimum phase, unstable, or multiple-input, multiple-output systems is rather straightforward. In the case of linear time-varying systems of which the dynamics recur every trial/period, the system lifting is very similar to the LTI case [6], [7], [56], [57]. The system analysis approach presented in Section IV also remains largely valid, but the relation between time-domain and trial-domain system properties needs further investigation. While system lifting has been presented for ILC for nonlinear systems [54], the extension of our results to nonlinear systems requires additional research effort. Although only RC and ILC are considered in the paper, discrete repetitive processes [18], [19], [47] readily fit into the developed methodology.

II. COMMON RC AND ILC STRUCTURES

To clarify the elaborations, they will be applied to the most common ILC and RC structures, presented in Section II-A.

System lifting refers to representing the ILC and RC system dynamics in the trial domain. Prior to deriving the resulting trial-domain models in full mathematical detail (Section III), Section II-B already presents the trial-domain description of the ILC and RC systems introduced in Section II-A.

A. Time-domain Description

Figure 1(a) shows the common structure of an ILC for a discrete-time SISO LTI system $P(q)$ with input $u(k)$ and output $y(k)$. The index k labels the discrete time instants, while q denotes the one-sample advance operator. The system is subject to the exogenous input $w(k)$, being a reference command and/or a disturbance, over and over again, and the subsequent executions/trials are labeled by the index j . Each trial comprises a finite number N of time samples and prior to each trial, the system's initial conditions are reset to the same value (discontinuous operation mode). In order to reduce/eliminate the tracking error $e_j(k)$ as j increases, an ILC updates the control signal $u_j(k)$ with the help of signals from previous trials, invoking the need of a memory buffer "MEM". The update formula is most commonly of the following form:

$$u_{j+1}(k) = Q(q)(u_j(k) + L(q)e_j(k)) , \quad (1a)$$

with $Q(q)$ and $L(q)$ LTI filters.

Figure 1(b) shows the most common structure of an RC, where in this case the exogenous input $w(k)$ is periodic with an integer number N of time samples. To exploit the periodicity of the input signals, common RCs feature a one-period delay q^{-N} enclosed in a positive feedback loop. This way, the control signal $u(k)$ is adjusted based on signals from the previous period:

$$u(k+N) = Q(q)(u(k) + L(q)e(k)) . \quad (1b)$$

The similarity between Eq. (1b) and (1a) becomes clear by labeling the signal periods by the subscript $(\cdot)_j$. In addition, similar design guidelines apply to the filters $Q(q)$ and $L(q)$ in ILCs and RCs.

Fig. 2. Trial-domain description of (a) a common iterative learning control system and (b) a common repetitive control system.

B. Trial-domain Description

System lifting has become a standard technique to handle the two-dimensional dynamics of ILC systems, which originate from their discontinuous operation. The N time-domain samples of the signals during each trial are grouped into large vectors, commonly called supervectors [9], [40]. In case the system $P(q)$ has relative degree τ , these supervectors are given by:

$$\mathbf{u}_j = [u_j(0) \quad u_j(1) \quad \cdots \quad u_j(N-1)]^T, \quad (2a)$$

$$\mathbf{y}_j = [y_j(\tau) \quad y_j(\tau+1) \quad \cdots \quad y_j(\tau+N-1)]^T, \quad (2b)$$

$$\mathbf{w} = [w(\tau) \quad w(\tau+1) \quad \cdots \quad w(\tau+N-1)]^T, \quad (2c)$$

$$\mathbf{e}_j = \mathbf{w} - \mathbf{y}_j, \quad (2d)$$

where $(\cdot)^T$ denotes the transpose of a vector or matrix. Reformulating the input-output relation of the SISO LTI systems $P(q)$, $Q(q)$ and $L(q)$ in terms of the corresponding input and output supervectors yields the $N \times N$ trial-domain systems $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{L}$, respectively. While the mathematical model of these trial-domain systems will be derived in Section III, it can already be understood that they are static: In ILC, the system's initial conditions are reset to the same value at the beginning of each trial and hence, the system output of the current trial is decoupled from the input signals applied during previous trials and only depends on the system input of the current trial. Stated otherwise, on account of the discontinuous operation of an ILC, a dynamic time-domain system behaves statically in the trial domain. Lifting all the subsystems in Figure 1(a) yields the trial-domain representation of the ILC system shown in Figure 2(a). To distinguish between the time domain and the trial domain, plain characters are used in the former, while bold characters relate to latter. Consequently, the symbol $\mathbf{q}$ indicates the one-sample-advance operator in the trial domain¹, while the one-sample-advance operator in the time domain is denoted by q . Hence, $qx_j(k) = x_j(k+1)$, while $\mathbf{q}x_j(k) = x_{j+1}(k)$.

¹Notice that the boldfaced $\mathbf{q}$ notation is equivalent to the w -operator introduced in [38] and developed in [39].

Implied by its continuous operation, the dynamics of an RC system are one-dimensional and fully described in time domain. However, it is still possible to describe the system dynamics in a lifted form, where the corresponding supervectors are given by:

$$\mathbf{u}_j = [u(jN) \quad u(jN+1) \quad \cdots \quad u(jN+N-1)]^T, \quad (3a)$$

$$\mathbf{y}_j = [y(jN+\tau) \quad y(jN+\tau+1) \quad \cdots \quad y(jN+\tau+N-1)]^T, \quad (3b)$$

$$\mathbf{w} = [w(\tau) \quad w(\tau+1) \quad \cdots \quad w(\tau+N-1)]^T, \quad (3c)$$

$$\mathbf{e}_j = \mathbf{w} - \mathbf{y}_j. \quad (3d)$$

Although not essential, it is decided here to explicitly account for the system's relative degree τ in order to enhance the similarity with the ILC lifting. Since a repetitive control system operates in continuous mode (initial conditions are never reset), the system output not only depends on the input signal of the current period/trial, but of all preceding periods. Stated otherwise, in the repetitive control setting, a dynamic time-domain system also behaves dynamically in the trial domain. For this reason the argument $(\mathbf{q})$ is preserved in the trial-domain representation, shown in Figure 2(b), of the RC system of Figure 1(b).

At this point, it is worth emphasizing that the trial-domain systems shown in Figure 2 constitute equivalent descriptions of the corresponding ILC and RC systems shown in Figure 1. Also note that the notation is slightly abused to comply with the current practice in ILC lifting. Using the supervector definitions (2) and (3), $\mathbf{P}$ and $\mathbf{P}(\mathbf{q})$ in fact correspond to the trial-domain description of $q^\tau P(q)$, while $\mathbf{L}$ and $\mathbf{L}(\mathbf{q})$ correspond to $q^{-\tau} L(q)$.

III. SYSTEM LIFTING

The process of system lifting refers to representing the ILC and RC system dynamics in the trial domain. Transforming Figure 1 to Figure 2 requires lifting $q^\tau P(q)$, $q^{-\tau} L(q)$ and $Q(q)$. For the sake of generality, the ILC lifting (Section III-A) and the RC lifting (Section III-B) are elaborated for a general discrete-time SISO LTI system $G(q)$ with input $v(k)$ and

output $\eta(k)$. To obtain the models of the trial-domain systems shown in Figure 2, the results of this section should be applied with the substitutions $G(q) = q^\tau P(q)$, $G(q) = q^{-\tau} L(q)$ and $G(q) = Q(q)$. Since in ILC and RC systems the filter $Q(q)$ is generally noncausal, both causal and noncausal $G(q)$ are considered.

The order of $G(q)$ is denoted by n and its impulse response is indicated by $g(k)$. It is assumed that there exists an integer $m \geq 0$, such that $g(k) = 0$, for all $k < -m$. Hence, the general time-domain difference equation of $G(q)$ is given as follows

$$G(q) = \frac{b_{-m}q^m + \dots + b_{-1}q + b_0 + b_1q^{-1} + \dots + b_nq^{-n}}{a_0 + a_1q^{-1} + \dots + a_nq^{-n}} = \frac{b(q)}{a(q)}, \quad (4)$$

with $q^n b(q)$ and $q^n a(q)$ coprime polynomials. In case $G(q)$ is assumed to be causal ($m = 0$), matrices $A \in \mathbb{R}_{n \times n}$, $B \in \mathbb{R}_{n \times 1}$, $C \in \mathbb{R}_{1 \times n}$ and $D \in \mathbb{R}$ define a minimal state-space model of $G(q)$:

$$G(q) : \begin{cases} x(k+1) = Ax(k) + Bv(k) \\ \eta(k) = Cx(k) + Dv(k) \end{cases},$$

and in this case

$$\begin{cases} g(0) = D, \\ g(k) = CA^{k-1}B, \quad k > 0. \end{cases} \quad (5)$$

A. ILC System Lifting

In iterative learning control, the system's initial conditions are reset to the same value at the beginning of each trial. Without loss of generality, these initial conditions are assumed equal to zero [9]. Resetting the system state decouples the system output from the input signals applied during previous trials and consequently, the dynamic time-domain system $G(q)$ behaves statically in the trial domain. Below, the trial-domain system model is first derived under the assumption that $G(q)$ is causal. Then, the effect of noncausality is discussed, followed by the analysis of the trial-domain system's poles and zeros, and the trial-domain frequency response function (FRF).

1) *Trial-domain Model of a Causal System:* To obtain the trial-domain model of a causal system $G(q)$, the convolution relation

$$\eta_j(k) = \sum_{l=0}^k g(l)v_j(k-l), \quad k = 0, 1, \dots, N-1,$$

is reformulated in terms of the input supervector $\mathbf{v}_j$ and output supervector $\boldsymbol{\eta}_j$:

$$\underbrace{\begin{bmatrix} \eta_j(0) \\ \eta_j(1) \\ \vdots \\ \eta_j(N-1) \end{bmatrix}}_{\boldsymbol{\eta}_j} = \underbrace{\begin{bmatrix} g(0) & 0 & \cdots & 0 \\ g(1) & g(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ g(N-1) & \cdots & g(1) & g(0) \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} v_j(0) \\ v_j(1) \\ \vdots \\ v_j(N-1) \end{bmatrix}}_{\mathbf{v}_j} \quad (6)$$

The matrix $\mathbf{G}$ is Toeplitz since the underlying system $G(q)$ is assumed to be time-invariant, while the lower-triangularity of $\mathbf{G}$ follows from the causality of $G(q)$.

2) *Trial-domain Model of a Noncausal System:* In the case of a noncausal system $G(q)$, the standard practice in ILC is to disregard the noncausal response of $G(q)$, i.e. $\eta_j(k)$ for $k < 0$, and to use the same supervectors (6) as used when $G(q)$ is causal. This way, the trial-domain model of a noncausal system $G(q)$ amounts to

$$\mathbf{G} = \begin{bmatrix} g(0) & g(-1) & \cdots & g(-N+1) \\ g(1) & g(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & g(-1) \\ g(N-1) & \cdots & g(1) & g(0) \end{bmatrix}. \quad (7)$$

Disregarding the noncausal system response destroys the commutativity of linear systems. That is, for noncausal LTI systems $G_1(q)$ and $G_2(q)$, and with $\mathcal{C}$ denoting the operator that removes the noncausal part of a signal, the compositions $\mathcal{C} \circ G_1(q) \circ \mathcal{C} \circ G_2(q)$ and $\mathcal{C} \circ G_2(q) \circ \mathcal{C} \circ G_1(q)$ have a different outcome. In the trial-domain description this loss of commutativity reveals itself in the fact that the product of Toeplitz matrices is only commutative in case the matrices are lower-triangular.

3) *Poles and Zeros:* Being static, the trial-domain system $\mathbf{G}$ has neither poles nor zeros.

4) *Frequency Response Function:* The FRF matrix of the trial-domain system $\mathbf{G}$ is independent of the trial-domain frequency $\mathbf{f}$ [1/trial]. However, for $N \rightarrow \infty$, the singular value decomposition of $\mathbf{G}$ allows reconstructing the FRF of the underlying time-domain system $G(q)$. This is a consequence of the asymptotic equivalence of Toeplitz matrices to circulant matrices [23]. On account of this equivalence, for a stable system $G(q)$, the columns of the inverse Fourier transformation matrix F_N

$$F_N[k+1, m+1] = \frac{1}{\sqrt{N}} e^{i2\pi km/N}, \quad \begin{matrix} k : 0, 1, \dots, N-1, \\ m : 0, 1, \dots, N-1, \end{matrix}$$

where i denotes the imaginary unit, constitute so-called asymptotic pseudomodes of $\mathbf{G}$ for $N \rightarrow \infty$ (Theorem 1.3 in [8]). These results mean that

$$\lim_{N \rightarrow \infty} \|\mathbf{G} - \boldsymbol{\Delta}\|_{\text{HS}} = 0, \quad (8a)$$

where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert-Schmidt matrix norm and

$$\boldsymbol{\Delta} = F_N \mathbf{diag}\left(G(e^{i2\pi m/N})\right) F_N^H, \quad m : 0, 1, \dots, N-1. \quad (8b)$$

Here, $(\cdot)^H$ denotes the complex conjugated transpose, and $\mathbf{diag}(G(e^{i2\pi m/N})) \in \mathbb{R}^{n \times N}$ is the diagonal matrix with $G(e^{i2\pi m/N})$ as its $(m+1)$ th diagonal entry. Hence, for increasing trial length N , $\hat{\mathbf{G}} = F_N^H \mathbf{G} F_N$ becomes more and more diagonal, and its diagonal elements get closer and closer to the FRF of $G(q)$ at frequencies $f_m = m/N$, $m = 0, 1, \dots, N-1$.

The asymptotic equivalence between Toeplitz matrices and circulant matrices is not restricted to lower-triangular Toeplitz matrices. As a result, for $N \rightarrow \infty$ the corresponding Toeplitz matrices become diagonalizable by F_N , irrespective of the causality of the underlying time-domain system, and hence the multiplication of these matrices is commutative. However,

in the case of a noncausal time-domain system this commutativity holds only asymptotically, and is compromised for finite N .

Let $\bar{\sigma}(\mathbf{G})$ denote the maximal singular value of the matrix $\mathbf{G}$ and $\|G(q)\|_\infty$ the $\mathcal{H}_\infty$ norm of $G(q)$. Then (8a) implies that

$$\lim_{N \rightarrow \infty} \bar{\sigma}(\mathbf{G}) = \|G(q)\|_\infty, \quad (9)$$

and this irrespective of the causality of $G(q)$. However, the causality determines how this limit is reached. In case $G(q)$ is causal, $\bar{\sigma}(\mathbf{G})$ constitutes a nondecreasing series as a function of N and hence, approaches $\|G(q)\|_\infty$ from below (Section 9.6 in [24]). This is not guaranteed for a noncausal system.

B. RC System Lifting

Since a repetitive control system operates in continuous mode (initial conditions are never reset), the system output not only depends on the input signal of the current period/trial, but of all preceding periods. Stated otherwise, in the repetitive control setting, a dynamic time-domain system $G(q)$ also behaves dynamically in the trial domain. Below, the trial-domain system model is first derived under the assumption that $G(q)$ is causal. Then, the effect of noncausality is discussed, followed by the analysis of the system's trial-domain poles, zeros, and FRF. Since RC systems are fully described in the time domain, the time-domain and trial-domain system model are equivalent in the sense that they describe the same dynamics (yet in different domains) and feature the same system properties.

1) *Trial-domain Model of a Causal System:* To derive a state-space model in trial domain, the time-domain state-space model of $G(q)$ is lifted as follows

$$\underbrace{x((j+1)N)}_{\mathbf{x}_{j+1}} = \underbrace{\mathbf{A}^N}_{\mathbf{A}} \underbrace{x(jN)}_{\mathbf{x}_j} + \underbrace{[A^{N-1}B \ A^{N-2}B \ \dots \ B]}_{\mathbf{B}} \underbrace{\begin{bmatrix} v(jN) \\ v(jN+1) \\ \vdots \\ v(jN+N-1) \end{bmatrix}}_{\mathbf{v}_j}, \quad (10a)$$

$$\underbrace{\begin{bmatrix} \eta(jN) \\ \eta(jN+1) \\ \vdots \\ \eta(jN+N-1) \end{bmatrix}}_{\boldsymbol{\eta}_j} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}}_{\mathbf{C}} \underbrace{x(jN)}_{\mathbf{x}_j} + \underbrace{\begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{N-2}B & \dots & CB & D \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} v(jN) \\ v(jN+1) \\ \vdots \\ v(jN+N-1) \end{bmatrix}}_{\mathbf{v}_j}. \quad (10b)$$

Hence, the trial-domain model $\mathbf{G}(\mathbf{q})$ has the same order n as the underlying time-domain system $G(q)$.

2) *Trial-domain Model of a Noncausal System:* To obtain a trial-domain model of a noncausal system $G(q)$, its time-domain difference equation (4) is lifted to the trial domain. For ease of exposition but without loss of generality, $n+m < N$ is assumed. Reformulating the time-domain difference equation in terms of the input and output supervector yields the following trial-domain description

$$\mathbf{G}(\mathbf{q}) = \underbrace{(\mathbf{a}_0 + \mathbf{a}_1 \mathbf{q}^{-1})^{-1}}_{\mathbf{a}(\mathbf{q})} \underbrace{(\mathbf{b}_{-1} \mathbf{q} + \mathbf{b}_0 + \mathbf{b}_1 \mathbf{q}^{-1})}_{\mathbf{b}(\mathbf{q})}, \quad (11a)$$

where

$$\mathbf{b}_0 = \begin{bmatrix} b_0 & b_{-1} & \dots & b_{-m} & 0 & \dots & 0 \\ b_1 & b_0 & b_{-1} & \dots & b_{-m} & \ddots & \vdots \\ \vdots & b_1 & \ddots & \ddots & & \ddots & 0 \\ b_n & \vdots & \ddots & \ddots & \ddots & & b_{-m} \\ 0 & b_n & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & b_0 & b_{-1} \\ 0 & \dots & 0 & b_n & \dots & b_1 & b_0 \end{bmatrix}, \quad (11b)$$

$$\mathbf{b}_1 = \begin{bmatrix} 0 & \dots & 0 & b_n & \dots & b_1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & b_n & \\ 0 & & & & 0 & \\ \vdots & & & & \vdots & \\ 0 & \dots & 0 & \dots & \dots & 0 \end{bmatrix}, \quad (11c)$$

$$\mathbf{b}_{-1} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ b_{-m} & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ b_{-1} & \dots & b_{-m} & 0 & \dots & 0 \end{bmatrix}, \quad (11d)$$

and the matrices $\mathbf{a}_0$ and $\mathbf{a}_1$ are obtained by replacing b_i by a_i in $\mathbf{b}_0$ and $\mathbf{b}_1$, respectively. Setting $m = 0$ yields the trial-domain difference equation of a causal system, which is equivalent to the trial-domain state-space description (10).

Irrespective of the causality of the underlying time-domain system, the trial-domain systems commute, i.e.: the product of their transfer function matrices is commutative. This comes at no surprise since the trial-domain model is an equivalent description of the time-domain dynamics, and in the time domain the product of the transfer functions commutes. In the trial domain, the commutative property is most easily observed in the difference equation description. For matrices $\mathbf{x}(\mathbf{q})$ and $\mathbf{y}(\mathbf{q})$ having a structure similar to $\mathbf{b}(\mathbf{q})$, straightforward calculations reveal that $\mathbf{x}(\mathbf{q})\mathbf{y}(\mathbf{q}) = \mathbf{y}(\mathbf{q})\mathbf{x}(\mathbf{q})$ and hence, $\mathbf{x}(\mathbf{q})^{-1}\mathbf{y}(\mathbf{q}) = \mathbf{y}(\mathbf{q})\mathbf{x}(\mathbf{q})^{-1}$. Consequently, given $\mathbf{G}_1(\mathbf{q}) = \mathbf{a}(\mathbf{q})^{-1}\mathbf{b}(\mathbf{q})$ and $\mathbf{G}_2(\mathbf{q}) = \mathbf{c}(\mathbf{q})^{-1}\mathbf{d}(\mathbf{q})$, then

$$\begin{aligned} \mathbf{G}_1(\mathbf{q})\mathbf{G}_2(\mathbf{q}) &= \mathbf{a}(\mathbf{q})^{-1}\mathbf{b}(\mathbf{q})\mathbf{c}(\mathbf{q})^{-1}\mathbf{d}(\mathbf{q}) \\ &= \mathbf{c}(\mathbf{q})^{-1}\mathbf{d}(\mathbf{q})\mathbf{a}(\mathbf{q})^{-1}\mathbf{b}(\mathbf{q}) = \mathbf{G}_2(\mathbf{q})\mathbf{G}_1(\mathbf{q}). \end{aligned}$$

3) *Poles and Zeros*: The poles $\mathbf{p}_i$ and zeros $\mathbf{z}_i$ of $\mathbf{G}(\mathbf{q})$ correspond to the N th power of the poles p_i , respectively the zeros z_i , of $G(q)$:

$$\mathbf{p}_i = p_i^N, \quad i : 1, \dots, n, \quad (12a)$$

$$\mathbf{z}_i = z_i^N, \quad i : 1, \dots, n + m. \quad (12b)$$

For causal systems, relation (12a) readily follows from the state-space model lifting (10), since $\mathbf{A} = A^N$. In the more general case, relations (12) follow from the relation between the roots of $a(q)$ and $b(q)$ as defined in (4) and their trial-domain counterparts defined in (11). Consider, for instance, the numerator term $\mathbf{b}(\mathbf{q})$. Since $\mathbf{b}_1$ has rank n and $\mathbf{b}_{-1}$ has rank m , $\det(\mathbf{b}(\mathbf{q}))$ has $n + m$ roots, i.e. there exist $n + m$ values $\mathbf{z}_i$ for which $\det(\mathbf{b}(\mathbf{z}_i)) = 0$. These values must satisfy (12b) for some root z_i of $b(q)$, since

$$\mathbf{b}(q^N) \begin{bmatrix} 1 \\ q \\ \vdots \\ q^{N-1} \end{bmatrix} = b(q) \begin{bmatrix} 1 \\ q \\ \vdots \\ q^{N-1} \end{bmatrix}.$$

Hence, if $b(z_i) = 0$, $\mathbf{b}(z_i^N)$ is singular. The same arguments apply to the roots of $a(q)$ and $\mathbf{a}(\mathbf{q})$.

4) *Frequency Response Function*: On account of the continuous operation of repetitive controllers, the FRF of $\mathbf{G}(\mathbf{q})$ contains the same information as the FRF of $G(q)$. To investigate the relation between both FRFs, the following input signal is considered:

$$v(k) = e^{j2\pi f k} \rightarrow \mathbf{v}_j = \begin{bmatrix} 1 \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi(N-1)f} \end{bmatrix} e^{j2\pi N f j}.$$

Hence, all time-domain frequencies $f + m/N$ with m integer are mapped into the same trial-domain frequency $\mathbf{f} = fN$. Equivalently, the FRF of $\mathbf{G}(\mathbf{q})$ at frequency $\mathbf{f}$ contains the FRF of $G(q)$ at all frequencies $f = (\mathbf{f} + m)/N$ for $m = 0, \dots, N - 1$. It is readily verified that

$$\mathbf{G}(e^{j2\pi \mathbf{f}}) = U_N \Sigma U_N^H \quad (13a)$$

where

$$U_N[k + 1, m + 1] = \frac{1}{\sqrt{N}} e^{j2\pi k(\mathbf{f} + m)/N}, \quad k : 0, 1, \dots, N - 1, \\ m : 0, 1, \dots, N - 1, \quad (13b)$$

$$\Sigma = \text{diag} \left(G(e^{j2\pi(\mathbf{f} + m)/N}) \right). \quad (13c)$$

Consequently, for all systems $G(q)$ and any trial length N , the trial-domain FRF matrix $\mathbf{G}(e^{j2\pi \mathbf{f}})$ is diagonalizable by the system-independent matrix U_N . This common diagonalization confirms the commutativity of the product of the FRF matrices, as derived in Paragraph III-B2.

On account of relations (13), the eigenvalues $\lambda_m(\mathbf{G}(e^{j2\pi \mathbf{f}}))$ and singular values $\sigma_m(\mathbf{G}(e^{j2\pi \mathbf{f}}))$ for $m : 0, 1, \dots, N - 1$ equal

$$\lambda_m(\mathbf{G}(e^{j2\pi \mathbf{f}})) = G(e^{j2\pi(\mathbf{f} + m)/N}), \quad (14a)$$

$$\sigma_m(\mathbf{G}(e^{j2\pi \mathbf{f}})) = \left| G(e^{j2\pi(\mathbf{f} + m)/N}) \right|, \quad (14b)$$

and as a result:

$$\|\mathbf{G}(\mathbf{q})\|_\infty = \|G(q)\|_\infty. \quad (14c)$$

C. Relation Between ILC and RC Trial-domain Models

For the same time-domain system, the ILC and RC lifting yield different trial-domain models. However, by the construction of these models, the static ILC lifted model corresponds to the direct feed-through term of the dynamic RC lifted model. In case $G(q)$ is causal, this directly follows from substituting (5) in the $\mathbf{D}$ -matrix of the RC lifted system (10), which indeed yields the ILC lifted system $\mathbf{G}$ (6). To analyze this relationship for noncausal $G(q)$, the RC lifted difference equation (11) is reformulated as

$$\mathbf{G}(\mathbf{q}) = \mathbf{q} \mathbf{a}_0^{-1} \mathbf{b}_{-1} + \mathbf{a}_0^{-1} (\mathbf{b}_0 - \mathbf{a}_1 \mathbf{a}_0^{-1} \mathbf{b}_{-1}) + \\ \mathbf{q}^{-1} (\mathbf{a}_0 + \mathbf{a}_1 \mathbf{q}^{-1})^{-1} (\mathbf{b}_1 - \mathbf{a}_1 \mathbf{a}_0^{-1} \mathbf{b}_0 + \mathbf{a}_1 \mathbf{a}_0^{-1} \mathbf{a}_1 \mathbf{a}_0^{-1} \mathbf{b}_{-1}), \quad (15)$$

Hence, the direct feed-through term of this trial-domain model is $\mathbf{a}_0^{-1} (\mathbf{b}_0 - \mathbf{a}_1 \mathbf{a}_0^{-1} \mathbf{b}_{-1})$, and it is readily verified that this matrix equals the ILC lifted system (7).

In addition to this relationship between the ILC and RC lifted systems, both trial-domain models approach each other for $N \rightarrow \infty$. In case $G(q)$ is causal with zero relative degree, this follows from the fact that the RC lifted systems $\mathbf{G}(\mathbf{q})$ has n poles $\mathbf{p}_i$ and n zeros $\mathbf{z}_i$, which all migrate towards the origin for $N \rightarrow \infty$. This is a consequence of the relations $\mathbf{p}_i = p_i^N$ and $\mathbf{z}_i = z_i^N$, where p_i and z_i are respectively the stable poles and minimum-phase zeros of $G(q)$. Hence, for $N \rightarrow \infty$ the plant dynamics dissolve due to n pole-zero cancelations at the origin, and the static part $\mathbf{D}$ of the trial-domain model prevails. The fact that for $N \rightarrow \infty$ the ILC and RC trial-domain models approach each other can also be understood from (15). The static part of this difference equation is of full rank N , while the dynamic parts are rank deficient and of rank n or m . Hence, the latter parts become negligible for $N \rightarrow \infty$. To conclude, the asymptotic equivalence of the ILC and RC lifted systems also follows from their FRFs. In the RC case, the time-domain frequency axis is split up in intervals of width $1/N$ and each interval is reflected into a singular value curve of $\mathbf{G}(e^{j2\pi \mathbf{f}})$. Hence, for $N \rightarrow \infty$ the singular values of $\mathbf{G}(e^{j2\pi \mathbf{f}})$ become more and more static (i.e. independent of $\mathbf{f}$) and approach the FRF of $G(q)$. Hence, for all $\mathbf{f}$

$$\lim_{N \rightarrow \infty} \|\mathbf{G}(e^{j2\pi \mathbf{f}}) - \Delta\|_{\text{HS}} = 0,$$

with Δ defined in (8b), which also holds for the ILC case (8a).

IV. ANALYSIS OF ILC AND RC SYSTEMS

The process of system lifting transforms RC and ILC systems into particular multiple-input, multiple-output (MIMO) control systems. For instance, the MIMO systems shown in Figure 2 constitute an equivalent representation of the ILC and RC systems shown in Figure 1. Consequently, in the trial domain RC and ILC systems can be analyzed with standard tools from linear systems theory. The particular properties of RC and ILC lifted systems, as discussed in

Section III, allow interpreting the MIMO analysis results in the RC and ILC context. This section illustrates this analysis for the nominal stability (Section IV-A), monotonic convergence (Section IV-B) and steady-state performance (Section IV-C) of the RC and ILC systems shown in Figures 1 and 2.

A. Stability

Let $\mathbf{H}(\mathbf{q})$ denote the closed loop of the feedback systems shown in Figure 2, and let its state matrix be indicated by $\mathbf{A}_\mathbf{H}$. Then, stability of the RC and ILC systems directly follows from the Schur stability condition:

Definition 1 (Schur stability). *A system $\mathbf{H}(\mathbf{q})$ with state matrix $\mathbf{A}_\mathbf{H}$ is Schur stable if and only if all eigenvalues $\lambda_i(\mathbf{A}_\mathbf{H})$ of $\mathbf{A}_\mathbf{H}$ lie in the open unit disc, that is:*

$$\rho(\mathbf{A}_\mathbf{H}) = \max_i \{|\lambda_i(\mathbf{A}_\mathbf{H})|\} < 1. \quad (16)$$

1) *ILC Stability:* For the ILC system shown in Figure 2(a) the closed-loop system $\mathbf{H}(\mathbf{q})$ is of order N . The vector $\mathbf{x}_{\mathbf{H},j} = \mathbf{u}_j$ constitutes a feasible state vector, yielding

$$\mathbf{A}_\mathbf{H} = \mathbf{Q}(I_N - \mathbf{L}\mathbf{P}). \quad (17)$$

Hence, this ILC system is stable if and only if (16) holds for (17). If the ILC lifted system matrices $\mathbf{Q}$, $\mathbf{L}$ and $\mathbf{P}$ are lower triangular, i.e. the time-domain systems $Q(q)$, $q^{-\tau}L(q)$ and $P(q)$ are causal, $\mathbf{A}_\mathbf{H}$ is lower triangular as well. In this case, stability condition (16) is equivalent to all diagonal elements of $\mathbf{A}_\mathbf{H}$ being smaller than 1 in absolute value. In case $Q(q)$ and/or $q^{-\tau}L(q)$ are noncausal, there is no clear relation between the eigenvalues of $\mathbf{A}_\mathbf{H}$ and system properties of $Q(q)$, $L(q)$ and $P(q)$. To the best of our knowledge, neither does there exist a sufficient stability condition with a clear time-domain interpretation, other than the monotonic convergence condition stated in Section IV-B.

2) *RC Stability:* Let's first assume that $P(q)$, $q^{-\tau}L(q)$ and $Q(q)$ are causal. The orders of these subsystems, and hence of $\mathbf{P}(\mathbf{q})$, $\mathbf{L}(\mathbf{q})$ and $\mathbf{Q}(\mathbf{q})$, are denoted by respectively n_G , n_L and n_Q , and the states of the corresponding trial-domain models are indicated by $\mathbf{x}_{\mathbf{P},j}$, $\mathbf{x}_{\mathbf{L},j}$ and $\mathbf{x}_{\mathbf{Q},j}$, respectively. The closed-loop system $\mathbf{H}(\mathbf{q})$ corresponding to Figure 2(b) is of order $n_G + n_L + n_Q + N$ and a feasible closed-loop state vector is given by

$$\mathbf{x}_{\mathbf{H},j} = \begin{bmatrix} \mathbf{x}_{\mathbf{P},j} \\ \mathbf{x}_{\mathbf{L},j} \\ \mathbf{x}_{\mathbf{Q},j} \\ \mathbf{u}_j \end{bmatrix}, \quad (18)$$

which yields the following state matrix $\mathbf{A}_\mathbf{H}$:

$$\mathbf{A}_\mathbf{H} = \begin{bmatrix} \mathbf{A}_\mathbf{P} & 0 & 0 & \mathbf{B}_\mathbf{P} \\ -\mathbf{B}_\mathbf{L}\mathbf{C}_\mathbf{P} & \mathbf{A}_\mathbf{L} & 0 & -\mathbf{B}_\mathbf{L}\mathbf{D}_\mathbf{P} \\ -\mathbf{B}_\mathbf{Q}\mathbf{D}_\mathbf{L}\mathbf{C}_\mathbf{P} & \mathbf{B}_\mathbf{Q}\mathbf{C}_\mathbf{L} & \mathbf{A}_\mathbf{Q} & \mathbf{B}_\mathbf{Q}(I_N - \mathbf{D}_\mathbf{L}\mathbf{D}_\mathbf{P}) \\ -\mathbf{D}_\mathbf{Q}\mathbf{D}_\mathbf{L}\mathbf{C}_\mathbf{P} & \mathbf{D}_\mathbf{Q}\mathbf{C}_\mathbf{L} & \mathbf{C}_\mathbf{Q} & \mathbf{D}_\mathbf{Q}(I_N - \mathbf{D}_\mathbf{L}\mathbf{D}_\mathbf{P}) \end{bmatrix}. \quad (19)$$

The matrices $\mathbf{D}_\mathbf{P}$, $\mathbf{D}_\mathbf{L}$, $\mathbf{D}_\mathbf{Q}$ of the RC lifted systems equal the ILC lifted systems $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{Q}$, respectively, and hence, (17) appears as the $(4,4)$ subblock of (19). However, little can be said in general about the relation between the stability of the

RC system and the ILC system involving the same systems $P(q)$, $Q(q)$ and $L(q)$.

The relation between ILC and RC stability is even more clouded in case $Q(q)$ and/or $q^{-\tau}L(q)$ are noncausal. To compute the corresponding closed-loop state matrix of the RC system, parts of the one period delay $\mathbf{q}^{-1}I_N$ need to be shifted around (see e.g. [45]), but other than that the calculations are straight-forward. As the result reveals no clear relation with the ILC closed-loop state matrix, it is omitted for reasons of conciseness.

Instead of checking the Schur stability condition (16), one can verify closed-loop stability using the following sufficient condition, which directly follows from the definition of the structured singular value [43]:

Lemma 1. *The RC system shown in Figure 2(b) is stable if*

$$\mu_\delta\{\mathbf{Q}(\mathbf{q})(I_N - \mathbf{L}(\mathbf{q})\mathbf{P}(\mathbf{q}))\} < 1,$$

where μ_δ denotes the structured singular value with respect to the uncertainty set

$$\delta = \{\delta(\mathbf{q})I_N \mid \delta(\mathbf{q}) = \text{stable, causal LTI system with } \|\delta(\mathbf{q})\|_\infty \leq 1\}.$$

This sufficient stability condition has a clear time-domain interpretation, since on account of relations (14):

$$\begin{aligned} \mu_\delta\{\mathbf{Q}(\mathbf{q})(I_N - \mathbf{L}(\mathbf{q})\mathbf{P}(\mathbf{q}))\} &= \|(\mathbf{Q}(\mathbf{q})(I_N - \mathbf{L}(\mathbf{q})\mathbf{P}(\mathbf{q})))\|_\infty \\ &= \|(Q(q)(1 - L(q)P(q)))\|_\infty. \end{aligned}$$

Combining this with lemma 1 yields the following corollary:

Corollary 1. *The RC system shown in Figure 2(b) is stable if*

$$\|(Q(q)(1 - L(q)P(q)))\|_\infty < 1.$$

Lemma 1 and corollary 1 hold irrespective of the causality of $Q(q)$ and $q^{-\tau}L(q)$.

B. Monotonic Convergence

Monotonic convergence has been introduced in the ILC literature to avoid large learning transients. It guarantees that $\mathbf{e}_j$ and/or $\mathbf{u}_j$ converge to their steady state values, denoted by respectively $\mathbf{e}_{ss}$ and $\mathbf{u}_{ss}$, in a monotonically decreasing way. Extending the common ILC monotonic convergence definition to general MIMO systems yields:

Definition 2 (Monotonic Convergence). *Consider a stable system $\mathbf{H}(\mathbf{q})$ to which a constant input $\mathbf{w}$ is applied. Let $\boldsymbol{\eta}_j$ denote the corresponding response and $\boldsymbol{\eta}_{ss}$ its steady-state value. Then, the system $\mathbf{H}(\mathbf{q})$ is called monotonic convergent in $\boldsymbol{\eta}_j$ under a given norm $\|\cdot\|$ if there exists a γ , $0 \leq \gamma < 1$, such that*

$$\|\boldsymbol{\eta}_{j+1} - \boldsymbol{\eta}_{ss}\| \leq \gamma \|\boldsymbol{\eta}_j - \boldsymbol{\eta}_{ss}\|, \quad \forall j : 0, 1, \dots \quad (20)$$

holds for all $\mathbf{w}$.

In this paper the 2-norm is used. Suppose $\mathbf{H}(\mathbf{q})$ has the following state-space model:

$$\mathbf{H}(\mathbf{q}) : \begin{cases} \mathbf{x}_{\mathbf{H},j+1} = \mathbf{A}_\mathbf{H}\mathbf{x}_{\mathbf{H},j} + \mathbf{B}_\mathbf{H}\mathbf{w}_j \\ \boldsymbol{\eta}_j = \mathbf{C}_\mathbf{H}\mathbf{x}_{\mathbf{H},j} + \mathbf{D}_\mathbf{H}\mathbf{w}_j \end{cases}, \quad (21)$$

and initial state $\mathbf{x}_{\mathbf{H},0}$. To analyze the transient response of $\mathbf{H}(\mathbf{q})$ to a constant (trial-invariant) input $\mathbf{w}_j = \mathbf{w}$, it is transformed into the autonomous system:

$$\begin{cases} \tilde{\mathbf{x}}_{\mathbf{H},j+1} = \mathbf{A}_{\mathbf{H}}\tilde{\mathbf{x}}_{\mathbf{H},j} \\ \tilde{\boldsymbol{\eta}}_j = \mathbf{C}_{\mathbf{H}}\tilde{\mathbf{x}}_{\mathbf{H},j} \end{cases},$$

with initial state $\tilde{\mathbf{x}}_{\mathbf{H},0} = \mathbf{x}_{\mathbf{H},0} - \mathbf{x}_{\mathbf{H},ss}$, where

$$\tilde{\mathbf{x}}_{\mathbf{H},j} = \mathbf{x}_{\mathbf{H},j} - \mathbf{x}_{\mathbf{H},ss}, \quad \tilde{\boldsymbol{\eta}}_j = \boldsymbol{\eta}_j - \boldsymbol{\eta}_{ss}, \quad j = 0, 1, \dots$$

Monotonic convergence in $\boldsymbol{\eta}_j$ requires that for all $j : 0, 1, \dots$

$$\begin{aligned} \|\tilde{\boldsymbol{\eta}}_{j+1}\| &\leq \gamma \|\tilde{\boldsymbol{\eta}}_j\|, \\ \|\mathbf{C}_{\mathbf{H}}\mathbf{A}_{\mathbf{H}}\tilde{\mathbf{x}}_{\mathbf{H},j}\| &\leq \gamma \|\mathbf{C}_{\mathbf{H}}\tilde{\mathbf{x}}_{\mathbf{H},j}\|. \end{aligned}$$

The last inequality is guaranteed to hold for all vectors $\tilde{\mathbf{x}}_{\mathbf{H},j}$ if

$$\mathbf{A}_{\mathbf{H}}^T \mathbf{C}_{\mathbf{H}}^T \mathbf{C}_{\mathbf{H}} \mathbf{A}_{\mathbf{H}} \preceq \gamma^2 \mathbf{C}_{\mathbf{H}}^T \mathbf{C}_{\mathbf{H}}. \quad (22)$$

This yields the following sufficient monotonic convergence condition:

Lemma 2 (Monotonic Convergence). *The system $\mathbf{H}(\mathbf{q})$ given by state-space model (21) is monotonic convergent in $\boldsymbol{\eta}_j$ under the 2-norm if (22) holds for some γ , $0 \leq \gamma < 1$.*

1) *ILC Monotonic Convergence*: Here, lemma 2 is applied to the ILC system shown in Figure 2(a) in order to derive monotonic convergence conditions for the closed-loop system in the control effort $\mathbf{u}_j$ and the tracking error $\mathbf{e}_j$. As already pointed out in Section IV-A, the corresponding closed-loop system has order N , and $\mathbf{x}_{\mathbf{H},j} = \mathbf{u}_j$ constitutes a feasible state vector. The state matrix is given by (17), while $\boldsymbol{\eta}_j = \mathbf{u}_j$ and $\boldsymbol{\eta}_j = \mathbf{e}_j$ are obtained by using the following matrices $\mathbf{C}_{\mathbf{H}}$:

$$\begin{aligned} \mathbf{u}_j : \quad \mathbf{C}_{\mathbf{H}} &= \mathbf{I}_N, \\ \mathbf{e}_j : \quad \mathbf{C}_{\mathbf{H}} &= -\mathbf{P}. \end{aligned}$$

In both cases $\mathbf{C}_{\mathbf{H}}$ is invertible, whereby condition (22) is equivalent to

$$\bar{\sigma}(\mathbf{C}_{\mathbf{H}}\mathbf{A}_{\mathbf{H}}\mathbf{C}_{\mathbf{H}}^{-1}) \leq \gamma.$$

Lemma 3 (ILC Monotonic Convergence). *The ILC system shown in Figures 1(a) and 2(a) is monotonic convergent in $\mathbf{u}_j$ under the 2-norm if there exists a $\gamma_{\mathbf{u}}$, $0 \leq \gamma_{\mathbf{u}} < 1$, such that*

$$\bar{\sigma}(\mathbf{Q}(\mathbf{I}_N - \mathbf{L}\mathbf{P})) < \gamma_{\mathbf{u}}. \quad (23a)$$

and it is monotonic convergent in $\mathbf{e}_j$ under the 2-norm if there exists a $\gamma_{\mathbf{e}}$, $0 \leq \gamma_{\mathbf{e}} < 1$, such that

$$\bar{\sigma}(\mathbf{P}\mathbf{Q}(\mathbf{I}_N - \mathbf{L}\mathbf{P})\mathbf{P}^{-1}) < \gamma_{\mathbf{e}}. \quad (23b)$$

In case the time-domain systems $P(q)$, $Q(q)$ and $q^{-\tau}L(q)$ are causal, and hence, the matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{L}$ are lower triangular, (23a) and (23b) are equivalent thanks to the commutativity of the multiplication of lower triangular Toeplitz matrices. Moreover, as detailed in Section III-A, for a causal system $G(q)$, $\bar{\sigma}(\mathbf{G})$ constitutes a non-decreasing series as a function of N with $\lim_{N \rightarrow \infty} \bar{\sigma}(\mathbf{G}) = \|G(q)\|_{\infty}$ (9). Hence, under the causality assumption

$$\bar{\sigma}(\mathbf{Q}(\mathbf{I}_N - \mathbf{L}\mathbf{P})) \leq \|Q(q)(1 - L(q)P(q))\|_{\infty} \quad (24)$$

holds for all trial lengths N , which yields the following monotonic convergence criterion:

Lemma 4 (ILC Monotonic Convergence). *Consider the ILC system shown in Figures 1(a) and 2(a) where $P(q)$, $Q(q)$ and $q^{-\tau}L(q)$ are causal. This system is monotonic convergent in both $\mathbf{e}_j$ and $\mathbf{u}_j$ under the 2-norm if there exists a γ , $0 \leq \gamma < 1$, such that*

$$\|Q(q)(1 - L(q)P(q))\|_{\infty} < \gamma. \quad (25)$$

If the causality condition is violated, (24) is not guaranteed, especially not for short trial lengths. On the other hand, the asymptotic relation (9) still applies, such that for sufficiently large trial lengths (25) also implies monotonic convergence of the ILC system with noncausal $Q(q)$ and/or $q^{-\tau}L(q)$.

Since for any matrix X , $\rho(X) \leq \bar{\sigma}(X)$, lemmas 3 and 4 provide sufficient stability conditions for the stability of the ILC system.

2) *RC Monotonic Convergence*: For the RC system shown in Figure 2(b) the closed-loop order $n_{\mathbf{H}} > N$. Suppose $Q(q)$ and $q^{-\tau}L(q)$ are causal. Then $n_{\mathbf{H}} = n_G + n_Q + n_L + N$, and a feasible $\mathbf{x}_{\mathbf{H},j}$ is given by (18), which yields the state matrix $\mathbf{A}_{\mathbf{H}}$ given by (19). The outputs $\boldsymbol{\eta}_j = \mathbf{u}_j$ and $\boldsymbol{\eta}_j = \mathbf{e}_j$ are obtained by using

$$\begin{aligned} \mathbf{u}_j : \quad \mathbf{C}_{\mathbf{H}} &= \begin{bmatrix} 0 & 0 & 0 & \mathbf{I}_N \end{bmatrix}, \\ \mathbf{e}_j : \quad \mathbf{C}_{\mathbf{H}} &= \begin{bmatrix} -\mathbf{C}_{\mathbf{P}} & 0 & 0 & -\mathbf{D}_{\mathbf{P}} \end{bmatrix}. \end{aligned}$$

In both cases the pair $(\mathbf{A}_{\mathbf{H}}, \mathbf{C}_{\mathbf{H}})$ is readily verified to be observable. With $n_{\mathbf{H}} > N$ and $(\mathbf{A}_{\mathbf{H}}, \mathbf{C}_{\mathbf{H}})$ observable, monotonic convergence in the sense of definition 2 simply cannot be attained. Observability implies that a $\tilde{\mathbf{x}}_{\mathbf{H},0} \neq 0$ in the nullspace of $\mathbf{C}_{\mathbf{H}}$, i.e. $\tilde{\boldsymbol{\eta}}_0 = 0$, will at a certain j evolve to a $\tilde{\mathbf{x}}_{\mathbf{H},j}$ outside this nullspace: $\tilde{\boldsymbol{\eta}}_j \neq 0$. Consequently, (20) simply cannot hold for all trial-invariant inputs $\mathbf{w}$. Also, with $n_{\mathbf{H}} > N$ and $(\mathbf{A}_{\mathbf{H}}, \mathbf{C}_{\mathbf{H}})$ observable, the column spaces of $\mathbf{C}_{\mathbf{H}}^T$ and $\mathbf{A}_{\mathbf{H}}^T \mathbf{C}_{\mathbf{H}}^T$ do not coincide, whereby (22) is infeasible. Hence, in general monotonic convergence according to definition 2 cannot be guaranteed for RCs.

A heuristic that often results in monotonic convergence is to make sure that

$$\|Q(q)(1 - L(q)P(q))\|_{\infty} = \|\mathbf{Q}(\mathbf{q})(\mathbf{I}_N - \mathbf{L}(\mathbf{q})\mathbf{P}(\mathbf{q}))\|_{\infty} < \gamma \quad (26)$$

holds for some γ , $0 \leq \gamma < 1$. According to corollary 1 this condition implies closed-loop stability of the RC system of Figure 2(b). The autonomous analysis for this system yields that

$$\tilde{\mathbf{v}}_j = \mathbf{Q}(\mathbf{q})(\mathbf{I}_N - \mathbf{L}(\mathbf{q})\mathbf{P}(\mathbf{q}))\tilde{\mathbf{u}}_j,$$

simulated from initial state

$$\begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{P},0} \\ \tilde{\mathbf{x}}_{\mathbf{L},0} \\ \tilde{\mathbf{x}}_{\mathbf{Q},0} \end{bmatrix}$$

satisfies $\tilde{\mathbf{v}}_j = \tilde{\mathbf{u}}_{j+1}$ for all $j : 0, 1, \dots$. Let $\tilde{\mathbf{v}}_j^{\text{zi}}$ and $\tilde{\mathbf{v}}_j^{\text{zs}}$ respectively the zero-input response and zero-state response parts of $\tilde{\mathbf{v}}_j$, then (26) implies

$$\|\tilde{\mathbf{v}}_j\|_{\mathcal{L}_2} \leq \|\tilde{\mathbf{v}}_j^{\text{zs}}\|_{\mathcal{L}_2} + \|\tilde{\mathbf{v}}_j^{\text{zi}}\|_{\mathcal{L}_2} \leq \gamma \|\tilde{\mathbf{u}}_j\|_{\mathcal{L}_2} + \|\tilde{\mathbf{v}}_j^{\text{zi}}\|_{\mathcal{L}_2}, \quad (27)$$

where the notation $\|\mathbf{x}_j\|_{\mathcal{L}_2}$ distinguishes the $\mathcal{L}_2$ signal norm from the 2-norm of the signal vector at trial j , denoted by $\|\mathbf{x}_j\|_2$. On the other hand, suppose the RC system is monotonically convergent in $\mathbf{u}_j$, then

$$\|\tilde{\mathbf{v}}_j\|_{\mathcal{L}_2} = \sqrt{\sum_{j=1}^{\infty} \|\tilde{\mathbf{u}}_{j+1}\|_2^2} \leq \gamma \sqrt{\sum_{j=1}^{\infty} \|\tilde{\mathbf{u}}_j\|_2^2} = \gamma \|\tilde{\mathbf{u}}_j\|_{\mathcal{L}_2}$$

Hence, monotonic convergence implies (27), with is also implied by (26). However, in general (26) cannot guarantee monotonic convergence, as it only accounts for the zero-state response part, and a reduction in the $\mathcal{L}_2$ signal norm does not guarantee a reduction of the vector 2-norm at every trial j .

C. Steady-state Performance

Performance of a control system relates to reference tracking and disturbance rejection. Hence, performance is determined by the closed-loop sensitivity function $\mathbf{S}(\mathbf{q})$, which relates $\mathbf{w}_j$, capturing both references and disturbances, to the tracking error $\mathbf{e}_j$. For the RC and ILC systems shown in Figure 2, $\mathbf{S}(\mathbf{q})$ is given by

$$\mathbf{S}(\mathbf{q}) = I_N - \mathbf{P}(\mathbf{q})(\mathbf{q}I_N - \mathbf{Q}(\mathbf{q})(I_N - \mathbf{L}(\mathbf{q})\mathbf{P}(\mathbf{q})))^{-1} \mathbf{Q}(\mathbf{q})\mathbf{L}(\mathbf{q}), \quad (28)$$

where the ILC result is obtained by replacing $\mathbf{P}(\mathbf{q})$, $\mathbf{L}(\mathbf{q})$ and $\mathbf{Q}(\mathbf{q})$ by $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{Q}$, respectively.

ILCs and RCs are feedback controllers specialized for trial-invariant reference and disturbance inputs, yielding good steady-state performance for such inputs. Mathematically this means that $\mathbf{S}(1)$ is small in terms of its singular values. Hereto both RC and ILC rely on the Internal Model Principle [20], [49], which states that in order to achieve perfect steady-state performance for a certain class of input signals, their signal generator should be included in a stable feedback loop. With $\mathbf{Q}(\mathbf{q}) = \mathbf{Q} = I_N$ both schemes in Figure 2 include a one-trial delay $\mathbf{q}^{-1}I_N$ enclosed in a positive feedback loop. Determined by its initial condition, such a system can autonomously generate any trial-invariant signal. The Internal Model Principle manifests itself in the closed-loop sensitivity, since with $\mathbf{Q}(\mathbf{q}) = I_N$, Eq. (28) yields $\mathbf{S}(1) = 0$.

For the RC case, application of decomposition (13) to $\mathbf{S}(1)$ yields

$$\mathbf{S}(1) = F_N \text{diag} \left\{ Q(e^{i\frac{2\pi m}{N}}) \left(1 - L(e^{i\frac{2\pi m}{N}}) G(e^{i\frac{2\pi m}{N}}) \right) \right\} F_N^H,$$

with $m : 0, 1, \dots, N-1$, and hence, the trial-invariant performance (i.e. the tracking/attenuation of periodic inputs), is fully determined by the time-domain FRFs $Q(e^{i2\pi f})$, $L(e^{i2\pi f})$ and $P(e^{i2\pi f})$ at the harmonic frequencies $f_m = m/N$, $m : 0, 1, \dots, N-1$, and this irrespective of the trial length N and the causality of the filters $Q(q)$ and $q^{-\tau}L(q)$. For ILCs on the other hand, the relation between the lifted system matrices and the time-domain FRFs, only holds asymptotically for $N \rightarrow \infty$.

D. Discussion

Throughout Sections IV-A and IV-B, the time-domain FRF condition

$$\|Q(q)(1 - L(q)P(q))\|_{\infty} \leq \gamma < 1. \quad (29)$$

appears frequently. In the RC case it provides a sufficient stability condition (corollary 1), irrespective of the causality of $Q(q)$ and $q^{-\tau}L(q)$, and the value of the trial length N . The conservatism involved may generally expected to be small, as it stems from the fact that

$$\mu_{\delta I_{n_H}}(\mathbf{A}_H) \leq \mu \left[\begin{array}{cc} \delta_1 I_{n_1} & 0 \\ 0 & \delta_2 I_{n_2} \end{array} \right] (\mathbf{A}_H), \quad (30)$$

for complex scalars δ , δ_1 and δ_2 and with $n_1 = n_H - N$ and $n_2 = N$. While stability requires the left-hand side to be < 1 , lemma 1 ensures the right-hand side to be < 1 . For $N \rightarrow \infty$, the difference between the two structured values in (30) dissolves since then, n_1 is negligible compared to n_2 . Although (29) often yields monotonic convergence of the RC system it cannot be guaranteed to do so.

In the ILC setting (29) provides a sufficient condition for monotonic convergence in the case of causal $Q(q)$ and $q^{-\tau}L(q)$. On account of $\rho(\mathbf{A}_H) \leq \bar{\sigma}(\mathbf{A}_H)$, condition (29) is also sufficient for stability of the ILC system with causal filters. In this case the conservatism of (29) with respect to the true stability condition is generally substantial. It stems from two sources: $\rho(\mathbf{A}_H) \leq \bar{\sigma}(\mathbf{A}_H)$ and $\bar{\sigma}(\mathbf{A}_H) \leq \|Q(q)(1 - L(q)P(q))\|_{\infty}$. The former source disappears for $N \rightarrow 1$, while the latter source vanishes for $N \rightarrow \infty$. If $Q(q)$ or $q^{-\tau}L(q)$ are noncausal, (29) only guarantees monotonic convergence and stability asymptotically for $N \rightarrow \infty$.

V. NUMERICAL ILLUSTRATION

This section illustrates the results of Sections II and IV numerically. Sections V-A and V-B respectively clarify the ILC and RC lifted system representation of

$$G(q) = q \frac{0.1833q + 0.1757}{q^2 - 1.523q + 0.8819}, \quad (31)$$

where the second factor corresponds to the zero-order-hold discretization of a second-order system with undamped natural frequency 1/10th of the sampling frequency and damping ratio 0.1. Subsequently, Section V-C illustrates some analysis results of Section IV for the ILC/RC system with $P(q) = G(q)$ given by (31), $Q(q) = Q = 0.65$ and $L(q) = L = 0.3$.

A. ILC System Lifting

To analyze relation (8) for $G(q)$ given by (31), Figure 3 shows $\|\mathbf{G} - \mathbf{\Delta}\|_{HS}$ as a function of N in the solid black line. The difference between $\mathbf{G}$ and $\mathbf{\Delta}$ has a two-fold origin: for finite N the matrix $\hat{\mathbf{G}} = F_N^H \mathbf{G} F_N$ is not perfectly diagonal, and its diagonal elements don't match the FRF of $G(q)$ perfectly. To analyze both contributions, Figure 3 also shows $\|\mathbf{G} - \mathbf{\Gamma}\|_{HS}$ and $\|\mathbf{\Gamma} - \mathbf{\Delta}\|_{HS}$ where

$$\mathbf{\Gamma} = F_N \text{diag}(\text{diag}(\hat{\mathbf{G}})) F_N^H.$$

Fig. 3. Evolution of $\|\mathbf{G} - \Delta\|_{\text{HS}}$, $\|\mathbf{G} - \Gamma\|_{\text{HS}}$ and $\|\Gamma - \Delta\|_{\text{HS}}$ as a function of N .

Hence, $\|\mathbf{G} - \Gamma\|_{\text{HS}}$ reflects the diagonal dominance of $\hat{\mathbf{G}}$, while $\|\Gamma - \Delta\|_{\text{HS}}$ reflects how well the diagonal elements of $\hat{\mathbf{G}}$ correspond to the FRF of $G(q)$ at frequencies $f_m = m/N$, $m = 0, 1, \dots, N-1$. Figure 3 reveals that the rate of convergence of $\mathbf{G}$ to Δ is proportional to $1/\sqrt{N}$ and predominantly determined by the rate at which $\hat{\mathbf{G}}$ becomes diagonal. On the other hand, the diagonal elements of $\hat{\mathbf{G}}$ approach the FRF of $G(q)$ at a $1/N$ rate.

Figure 4 confirms these findings by analyzing $\hat{\mathbf{G}}$ for $N = 20$ and $N = 200$. The left-hand side shows the magnitude of the elements of these matrices, and comparison of figures (a) and (b) reveals that for $N = 200$, $\hat{\mathbf{G}}$ is much more diagonal than for $N = 20$. However, even if $\hat{\mathbf{G}}$ is not very diagonal for $N = 20$, Figure 4(c) reveals that its diagonal elements already approximate the FRF of $G(q)$ quite closely. For $N = 200$, the diagonal elements of $\hat{\mathbf{G}}$ allow an accurate reconstruction of the FRF of $G(q)$.

To conclude, Figure 5 shows $\bar{\sigma}(\mathbf{G})$ as a function of N and confirms that this series converges to $\|G(q)\|_{\infty}$ for $N \rightarrow \infty$ in a nondecreasing way.

B. RC System Lifting

This section illustrates the relation between the time-domain FRF $G(e^{j2\pi f})$ and the trial-domain FRF $\mathbf{G}(e^{j2\pi f})$ for $G(q)$ given by (31) and $N = 8$. Figure 6(a) shows $|G(e^{j2\pi f})|$, while Figure 6(b) shows the singular value plot of $\mathbf{G}(e^{j2\pi f})$. As clarified by the colors, the time-domain frequency axis f is split up in intervals of width $1/N$, which corresponds to the fundamental frequency of signals periodic with N samples. The time-domain FRF on each of these intervals is mapped into one singular value line in the trial-domain FRF $\mathbf{G}(e^{j2\pi f})$. The circles indicate that all frequencies $f + m/N = 0.05 + m/8$ with $m = 0, 1, \dots, N-1$ are mapped into the same trial-domain frequency $\mathbf{f} = Nf = 0.4$.

C. Analysis

Figure 7 illustrates some ILC/RC analysis results for $P(q) = G(q)$ given by (31), $Q(q) = Q = 0.65$ and $L(q) = L = 0.3$. For these systems $\|Q(1 - LP(q))\|_{\infty} = 1.108$, as indicated in Figure 7. For the ILC system, $\mathbf{A}_{\mathbf{H}}$ is lower triangular with diagonal elements $Q(1 - Lp(1)) = 0.614$. Hence, $\rho(\mathbf{A}_{\mathbf{H}}) = 0.614 < 1$, irrespective of the trial length

Fig. 4. Magnitude of the elements of the matrix $\hat{\mathbf{G}} = F_N^H \mathbf{G} F_N$ for (a) $N = 20$ and (b) $N = 200$; and (c) comparison of its diagonal elements to the FRF of $G(q)$.

N , indicating that the ILC system is always stable. The singular values of $\mathbf{A}_{\mathbf{H}}$, on the other hand, do depend on the trial length N . By increasing N they evolve from $\rho(\mathbf{A}_{\mathbf{H}})$ for $N = 1$ to $\|Q(1 - LP(q))\|_{\infty}$ for $N \rightarrow \infty$. Hence, for large trial lengths the monotonic convergence condition (22) is no longer guaranteed. Figure 7 confirms that (29) constitutes a rather conservative stability condition for ILCs. For small N , this conservatism primarily stems from the fact that $\bar{\sigma}(\mathbf{A}_{\mathbf{H}}) < \|Q(1 - LP(q))\|_{\infty}$, while for large N the conservatism is dominated by $\rho(\mathbf{A}_{\mathbf{H}}) < \bar{\sigma}(\mathbf{A}_{\mathbf{H}})$.

Fig. 5. Evolution of $\bar{\sigma}(\mathbf{G})$ as a function of N .

Fig. 6. Singular value plot of (a) $G(e^{j2\pi f})$ and (b) $\mathbf{G}(e^{j2\pi f})$.

For the corresponding RC system, Figure 7 reveals that for large N values the closed-loop system is unstable, while there exist few N values ≤ 52 for which the closed-loop is stable. Figure 7 also confirms that for $N \rightarrow \infty$, $\rho(\mathbf{A}_H)$ converges to $\|Q(1 - LP(q))\|_\infty$ and hence, the conservatism of (29) as a stability condition dissolves.

VI. CONCLUSIONS AND FUTURE RESEARCH

This paper lays the foundation for a unified analysis and design of ILCs and RCs. To enhance the interaction between these control strategies they are both formulated in the trial domain using system lifting. While the discontinuous operation mode of ILCs translates in a static trial-domain model, the RC lifted model is dynamic and has the same order as the

Fig. 7. Evaluation of ILC and RC stability criteria as a function of N .

underlying time-domain system. Moreover, in the latter case, there is a close relationship between system's trial-domain properties, such as poles, zeros, FRF, and their time-domain counterparts. A thorough analysis and comparison of the ILC lifted model and the RC lifted model is performed, and while essential differences prevail for short trial lengths N , these differences fade as $N \rightarrow \infty$.

To illustrate the value of a unified formulation of ILC and RC, the analysis of the most commonly used ILC and RC structures is harmonized. This analysis reveals central differences and similarities between various stability, monotonic convergence and steady-state performance conditions. In addition to exposing these valuable insights, the analysis performed in this paper is intended as the preliminary for more extended analyses that, for instance, account for plant model uncertainty. Such analyses should unravel the relation between structured singular value based results such as [53], and the more commonly used robust stability conditions based on time-domain FRFs (see e.g. [9] and the references therein).

Formulated in the trial-domain, the ILC and RC problem are very similar. In both cases it involves designing a controller that yields an internally stable closed-loop system and superior performance for trial-invariant inputs. The latter specification is addressed by means of the Internal Model Principle [20], [49]. Hence, the ILCs and RCs face the same design specifications and only differ in the system they are designed for: either the static ILC lifted system or the dynamic RC lifted system. From this perspective, our paper allows extending advanced ILC design strategies to RCs, and vice versa.

APPENDIX A. NOTATION

k	: index labeling the discrete time samples
j	: index labeling the trials
N	: number of time samples per trial
q	: one sample advance operator in time domain: $qx_j(k) = x_j(k+1)$
$\mathbf{q}$	: one sample advance operator in trial domain: $\mathbf{q}x_j(k) = x_{j+1}(k)$
$\mathbf{G}(\mathbf{q})$	: lifted trial-domain model of the the time-domain system $G(q)$
i	: imaginary unit

I_n	: $n \times n$ identity matrix; the subscript is omitted if the dimension is clear from the context
$0_{n \times m}$	: $n \times m$ zero matrix; the subscript is omitted if the dimension is clear from the context
X^T	: transpose of matrix X
X^H	: Hermitian (complex conjugate) transpose of matrix X
$\lambda_i(X)$	: eigenvalues of matrix X
$\rho(X)$	: spectral radius of matrix X : $\rho(X) = \max_i \{ \lambda_i(X) \}$
$\sigma_i(X)$	: singular values of matrix X
$\bar{\sigma}(X)$	: maximum singular value of matrix X : $\bar{\sigma}(X) = \max_i \{\sigma_i(X)\}$
$\text{diag}(X)$	: vector containing the diagonal entries of the matrix X
$\text{diag}(x)$	: diagonal matrix with the elements of the vector x on its diagonal
$\ \cdot\ _{\text{HS}}$	: Hilbert-Schmidt matrix norm
$\ \cdot\ _{\mathcal{L}_2}$	: $\mathcal{L}_2$ signal norm
$\ \cdot\ _{\infty}$	: $\mathcal{H}_{\infty}$ system norm
$\mu(\cdot)_{\delta}$	: structured singular value with respect to uncertainty set δ

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