Rational interpolation: II. Quadrature and convergence ⋆

Karl Deckers [*,1], Adhemar Bultheel

Department of Computer Science, K.U. Leuven, Heverlee, Belgium

Abstract

Consider an nth rational interpolatory quadrature rule $J_n^\sigma(f) = \sum_{j=1}^n \lambda_j f(x_j)$ to approximate integrals of the form $J_\sigma(f) = \int_{-1}^1 f(x)d\sigma(x)$, where σ is a (possibly complex) bounded measure with infinite support on the interval $[-1,1]$. First, we discuss the connection of $J_n^\sigma(f)$ with certain rational interpolatory quadratures on the complex unit circle to approximate integrals of the form $\int_{-\pi}^{\pi} \mathring{f}(e^{\mathbf{i}\theta})d\mathring{\sigma}(\theta)$. Next, we provide conditions to ensure the convergence of $J_n^\sigma(f)$ to $J_\sigma(f)$ for n tending to infinity. Finally, an upper bound for the error on the nth approximation and an estimate for the rate of convergence is provided.

Key words: Orthogonal rational functions, rational interpolation, rational quadrature rules, error bound, convergence rate.

1 Introduction

The central object of study in this paper is an integral of the form

$$J_\sigma(f) = \int_{-1}^1 f(x)d\sigma(x),$$

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* corresponding author.

Email addresses: `karl.deckers@cs.kuleuven.be` (Karl Deckers), `adhemar.bultheel@cs.kuleuven.be` (Adhemar Bultheel).

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where σ is a (possibly complex) bounded measure with infinite support on the interval $I = [-1, 1]$. Such integrals can be approximated by interpolatory quadrature rules with interpolation points that are zeros of orthogonal polynomials and are all in I. However, since σ need not be positive, one needs to introduce an auxiliary positive orthogonality measure μ. This leads to Gauss quadrature formulas with positive weights that approximate integrals of the form $J_\mu(f)$, and have a maximal (polynomial) domain of validity. If one of the endpoints (Radau) or both of them (Lobatto) are imposed as additional nodes, we get more general Gauss-type quadrature rules. In the ideal situation μ "resembles" σ as much as possible.

However, when f has singularities outside (but possibly close to) the interval I, it is often more appropriate to not consider a maximal polynomial domain of validity, but rather consider more general spaces of rational functions. In such a case the orthogonal polynomials are replaced by orthogonal rational functions with preassigned poles (to simulate the singularities of f).

A theory of orthogonal rational functions on the complex unit circle $\mathbb{T}$ has been studied intensively in [5]. Of course by a Joukowski Transform $x = \frac{1}{2}(z + z^{-1})$ one may map $x \in I$ to $z \in \mathbb{T}$ (see e.g. [3]), hence relating poles, nodes, weights, and measures on I and $\mathbb{T}$. In the classical situation, the poles for the circle situation are often taken in pairs $\{\beta_i, 1/\overline{\beta}_i\}$ with $|\beta_i| < 1$. This corresponds to taking real poles for the interval. We refer to this as the situation of "real poles"; see [20]. If, however, we want to consider arbitrary complex poles for I, then we need pairs $\{\beta_i, 1/\beta_i\}$ on $\mathbb{T}$; see [13].

In this paper we will investigate rational interpolatory quadrature rules to approximate the integrals of the form $J_\sigma(f)$, by making the connection with certain rational interpolatory quadrature rules for the approximation of integrals of the form $I_{\mathring{\sigma}}(\mathring{f}) = \int_{-\pi}^{\pi} \mathring{f}(e^{\mathbf{i}\theta}) d\mathring{\sigma}(\theta)$, where $\mathring{\sigma}$ is a (possibly complex) bounded measure with infinite support on $\mathbb{T}$. This connection then allows us to simultaneously obtain convergence results, error bounds and an estimate for the rate of convergence for both quadrature rules, on the interval as well as on the complex unit circle.

The outline of the paper is as follows. After giving the necessary theoretical background in Section 2, in Section 3 we discuss the connection of the rational interpolatory quadrature rules on the interval with certain rational interpolatory quadrature rules on the complex unit circle. Further, by considering the nodes from rational Gauss-type quadrature rules as interpolation points, the convergence result obtained in [10] will immediately induce convergence results for the rational interpolatory quadrature formulas themselves. Next, in Section 4 we provide error bounds and an estimate for the rate of convergence (root asymptotics of the error) for these quadrature rules. We conclude with some numerical experiments in Section 5.

2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$ and for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Further, the positive half line will be represented by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $a \in \mathbb{C}$, then $\Re\{a\}$ refers to the real part of a, while $\Im\{a\}$ refers to the imaginary part, and the imaginary unit will be denoted by $\mathbf{i}$. The unit circle and the open unit disk are denoted respectively by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Whenever the value zero is omitted in a set $X \subseteq \overline{\mathbb{C}}$, this will be represented by X_0. Similarly, the complement of a set $Y \subset \overline{\mathbb{C}}$ with respect to a set $X \subseteq \overline{\mathbb{C}}$ will be denoted by X_Y; i.e., $X_Y = \{t \in X : t \notin Y\}$. Further, if $b = \lceil a \rceil$ with $a \in \mathbb{R}$, then b is the smallest integer so that $b \geq a$. If, on the other hand, $b = \lfloor a \rfloor$ with $a \in \mathbb{R}$, then b is the largest integer so that $b \leq a$.

In this paper, we will consider quadrature formulas on the interval $I = [-1, 1]$ and on the complex unit circle $\mathbb{T}$. Although x and z are both complex variables, we reserve the notation x for the interval and z for the unit circle.

For any complex function $f(t)$, with $t = z$ or $t = x$, we define the involution operation or substar conjugate by $f_*(t) = \overline{f(1/\overline{t})}$. Next, we define the super-c conjugate by $f^c(t) = \overline{f(\overline{t})}$, and consequently f^c_* by $f^c_*(t) = f(1/t)$. Note that, if $f(t)$ has a pole at $t = p$, then $f_*(t)$ (respectively $f^c(t)$ and $f^c_*(t)$) has a pole at $t = 1/\overline{p}$ (respectively $t = \overline{p}$ and $t = 1/p$).

Let there be fixed a sequence of poles $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$, where the poles are arbitrary complex or infinite; hence, they do not have to appear in pairs of complex conjugates. We then define the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = \frac{x^k}{\prod_{j=1}^{k}(1 - x/\alpha_j)}, \quad k = 1, 2, \ldots . \tag{1}$$

These basis functions generate the nested spaces of rational functions with poles in $\mathcal{A}$ defined by $\mathcal{L}_{-1} = \{0\}$, $\mathcal{L}_0 = \mathbb{C}$ and $\mathcal{L}_k := \mathcal{L}\{\alpha_1, \ldots, \alpha_k\} = \text{span}\{b_0, \ldots, b_k\}$, $k = 1, 2, \ldots$. Further, with $\mathcal{L}$ we denote the closed linear span of all $\{b_k\}_{k=0}^{\infty}$. With the definition of the super-c conjugate we introduce $\mathcal{L}^c_k = \{f : f^c \in \mathcal{L}_k\} = \mathcal{L}\{\overline{\alpha}_1, \ldots, \overline{\alpha}_k\}$, while the product of two spaces of rational functions $\mathcal{L}_k$ and $\tilde{\mathcal{L}}_j = \mathcal{L}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j\}$ is defined by

$$\mathcal{L}_k \cdot \tilde{\mathcal{L}}_j = \{f \cdot g : f \in \mathcal{L}_k \text{ and } g \in \tilde{\mathcal{L}}_j\} = \mathcal{L}\{\alpha_1, \ldots, \alpha_k, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_j\}.$$

Note that $\mathcal{L}_k$ and $\mathcal{L}^c_k$ are rational generalizations of the space $\mathcal{P}_k$ of polynomials of degree less than or equal to k. Indeed, if $\alpha_j = \infty$ for every $j \geqslant 1$, the expression in (1) becomes $b_k(x) = x^k$.

Consider the integral

$$J_\sigma(f) := \int_{-1}^{1} f(x) d\sigma(x),$$

where σ is a (possibly complex) bounded measure with infinite support on I (in short, a complex measure on I). To approximate $J_\sigma(f)$, where f is a possibly complex function that can have singularities (possibly close to, but) outside the interval, rational interpolatory quadrature formulas (RIQs) are often preferred. An nth RIQ is obtained by integrating an interpolating rational function of degree $n-1$, and is of the form

$$J_n^\sigma(f) := \sum_{k=1}^{n} \lambda_{n,k}^\sigma f(x_{n,k}), \quad \{x_{n,k}\}_{k=1}^n \subset I, \quad x_{n,j} \neq x_{n,k} \text{ if } j \neq k, \quad \{\lambda_{n,k}^\sigma\}_{k=1}^n \subset \mathbb{C},$$

so that $J_\sigma(f) = J_n^\sigma(f)$ for at least every $f \in \mathcal{L}_{n-1}$. For reasons of notational simplicity, in the remainder we will write x_k and λ_k^σ, meaning $x_{m,k}$ and $\lambda_{m,k}^\sigma$ for a certain index m. At any time, the index m should be clear from the context.

Next, consider the inner product defined by

$$\langle f\,, g\rangle_\mu = J_\mu(f g^c), \quad f, g \in \mathcal{L}, \tag{2}$$

where μ is a positive bounded Borel measure with infinite support on I (in short, a positive measure on I), and let $\|f\|_{\mu,2} := \sqrt{\langle f\,, f\rangle_\mu}$. Orthogonalizing the basis functions $\{b_0, b_1, \ldots\}$ with respect to this inner product, we obtain a sequence of orthogonal rational functions (ORFs) $\{\varphi_0, \varphi_1, \ldots\}$, with $\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, so that $\varphi_k \perp_\mu \mathcal{L}_{k-1}$; i.e.; $\langle \varphi_k\,, \varphi_j\rangle_\mu = d_k \delta_{k,j}$, $d_k \in \mathbb{R}_0^+$ and $k, j = 0, 1, \ldots$, where $\delta_{k,j}$ is the Kronecker Delta.

Whenever $\alpha_n \in \overline{\mathbb{R}}_I$, the the zeros x_k of $\varphi_n(x)$ are all distinct and in the open interval $(-1, 1)$, and hence, can be chosen as nodes for the quadrature formula $J_n^\sigma(f)$. For $\sigma = \mu$, we obtain in this way the n-point rational Gaussian quadrature formula, which has maximal domain of validity; i.e.; the approximation is exact for every function $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$. It is well known that the weights λ_k^μ in the rational Gaussian quadrature are all positive (see e.g. [9, Thm. 2.3.5]). Note, however, that the n-point rational Gaussian quadrature formula does not exist whenever the last pole $\alpha_n \notin \overline{\mathbb{R}}$.

For any other choice of nodes, the weights may be non-positive or even complex and the quadrature will only be exact in a smaller set of rational functions. For each node that is fixed in advance, the domain of validity will generally [2] decrease by one. A special case is obtained when one node in the n-point quadrature is fixed in advance, so that the weights are all positive and the

[2] For some specific choices for the nodes, the domain of validity may remain the same or may even decrease more (see also [2] for the polynomial case).

quadrature is exact for every $f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$, which corresponds to the n-point rational Gauss-Radau quadrature formula. However, the existence of this n-point rational Gauss-Radau quadrature depends on the choice of the node (e.g., it surely does not exist whenever the node is a zero of the ORF φ_{n-1}; see [12]), but it does not depend on whether $\alpha_n \in \overline{\mathbb{R}}_I$. Whenever two nodes in an $(n+1)$-point quadrature formula are fixed in advance, so that the weights are all positive and the quadrature is exact for every $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$, we obtain the $(n+1)$-point rational Gauss-Lobatto quadrature formula. The existence of this $(n+1)$-point rational Gauss-Lobatto quadrature not only depends now on the choice of the nodes, but also on the pole α_n (e.g., it is easily verified that it does not exist whenever $\alpha_n \notin \overline{\mathbb{R}}$; see also [11, Sect. 2]).

In the remainder we will refer to the nodes and weights of the rational Gauss-type quadratures as (μ, g)-nodes and (μ, g)-weights, where $g \in \{0, 1, 2\}$ refers to the Gauss-type: $g = 0$ for Gaussian, $g = 1$ for Gauss-Radau, and $g = 2$ for Gauss-Lobatto. Moreover, in the case of Gaussian and Gauss-Lobatto, we will assume that the last pole α_n, if not in $\overline{\mathbb{R}}_I$, is replaced with an arbitrary pole $\tilde{\alpha}_n \in \overline{\mathbb{R}}_I$, and that the space $\mathcal{L}_n$ is adapted accordingly (without changing the notation).

Another sequence of basis functions will be used for the unit circle. Given a sequence of complex numbers $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, we define the Blaschke products for $\mathcal{B}$ as

$$B_0(z) \equiv 1, \quad B_k(z) = \prod_{j=1}^{k} \frac{z - \beta_j}{1 - \overline{\beta}_j z}, \; k = 1, 2, \ldots \; . \tag{3}$$

These Blaschke products generate the nested spaces of rational functions $\mathring{\mathcal{L}}_{-1} = \{0\}$, $\mathring{\mathcal{L}}_0 = \mathbb{C}$ and $\mathring{\mathcal{L}}_k := \mathring{\mathcal{L}}\{\beta_1, \ldots, \beta_k\} = \text{span}\{B_0, \ldots, B_k\}$, $k = 1, 2, \ldots$. Similarly as before, we denote with $\mathring{\mathcal{L}}$ the closed linear span of all $\{B_k\}_{k=0}^{\infty}$. With the definition of the substar conjugate and the super-c conjugate we can define $\mathring{\mathcal{L}}_{k*} = \{f : f_* \in \mathring{\mathcal{L}}_k\}$, $\mathring{\mathcal{L}}_k^c = \{f : f^c \in \mathring{\mathcal{L}}_k\}$ and $\mathring{\mathcal{L}}_{k*}^c = \{f : f_*^c \in \mathring{\mathcal{L}}_k\}$. Note that $\mathring{\mathcal{L}}_k$ and $\mathring{\mathcal{L}}_k^c$ are rational generalizations of $\mathcal{P}_k$ too. Indeed, if all $\beta_j = 0$ (or equivalently, $1/\overline{\beta}_j = \infty$ for every $j \geqslant 1$), the expression in (3) becomes $B_k(z) = B_k^c(z) = z^k$.

Consider now the integral

$$I_{\mathring{\sigma}}(f) := \int_{-\pi}^{\pi} f(z) d\mathring{\sigma}(\theta), \; z = e^{\mathbf{i}\theta},$$

where $\mathring{\sigma}$ is a complex measure on $\mathbb{T}$ [3], and f is a (possibly complex) function

[3] The measure $\mathring{\sigma}$ on $\mathbb{T}$ induces a measure on $[-\pi, \pi]$ for which we shall use the same notation $\mathring{\sigma}$.

bounded on $\mathbb{T}$. The RIQs to approximate $I_{\mathring{\sigma}}(f)$ are then of the form

$$I_n^{\mathring{\sigma}}(f) := \sum_{k=1}^{n} \mathring{\lambda}_{n,k}^{\mathring{\sigma}} f(z_{n,k}), \quad \{z_{n,k}\}_{k=1}^{n} \subset \mathbb{T}, \quad z_{n,j} \neq z_{n,k} \text{ if } j \neq k, \quad \{\mathring{\lambda}_{n,k}^{\mathring{\sigma}}\}_{k=1}^{n} \subset \mathbb{C}, \tag{4}$$

so that $I_{\mathring{\sigma}}(f) = I_n^{\mathring{\sigma}}(f)$ for every $f \in \mathring{\mathcal{L}}_p \cdot \mathring{\mathcal{L}}_{q*}$, with $n-1 \leqslant p+q \leqslant 2n-2$ and $p, q \leqslant n-1$. From now on we will write z_k and $\mathring{\lambda}_k^{\mathring{\sigma}}$, meaning $z_{m,k}$ and $\mathring{\lambda}_{m,k}^{\mathring{\sigma}}$ for a certain index m, where the index m should again be clear at any time from the context.

Let $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$ denote an nth ORF with respect to the inner product

$$\langle f, g \rangle_{\mathring{\mu}} = I_{\mathring{\mu}}(f g_*), \quad f, g \in \mathring{\mathcal{L}},$$

where $\mathring{\mu}$ is a positive measure on $\mathbb{T}$, and let $\|f\|_{\mathring{\mu},2} := \sqrt{\langle f, f \rangle_{\mathring{\mu}}}$. We then define a para-orthogonal rational function

$$\mathring{Q}_{n,\tau}(z) = \phi_n(z) + \tau B_n(z) \phi_{n*}(z), \quad \tau \in \mathbb{T}. \tag{5}$$

The zeros z_k of $\mathring{Q}_{n,\tau}(z)$ are all distinct and on the unit circle $\mathbb{T}$, and hence, can be chosen as nodes for the quadrature formula $I_n^{\mathring{\sigma}}(f)$. In the special case in which $\mathring{\sigma} = \mathring{\mu}$, we obtain an n-point rational Szegő quadrature formula, which has maximal domain of validity ($p = q = n-1$). It is well known that in this case the weights $\mathring{\lambda}_k^{\mathring{\mu}}$ are all positive too. Due to the presence of the parameter τ in (5), however, the nodes and weights in an n-point rational Szegő quadrature formula are (unlike in the case of the interval) not unique. In the remainder we will refer to the nodes and weights of the rational Szegő quadratures as $\mathring{\mu}$-nodes and $\mathring{\mu}$-weights.

We denote the Joukowski Transformation $x = \frac{1}{2}(z+z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}_I$ and the unit circle $\mathbb{T}$ onto the interval I. When $z = e^{\mathbf{i}\theta}$, then $x = J(z) = \cos\theta$. In this paper we will assume that x and z are related by this transformation. The inverse mapping is denoted by $z = J^{inv}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}_I$. With the sequence $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$ we associate a sequence $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, so that $\beta_k = J^{inv}(\alpha_k)$ for every $k > 0$, and $\hat{\mathcal{B}} = \{\hat{\beta}_1, \hat{\beta}_2, \ldots\} \subset \mathbb{D}$ with $\hat{\beta}_{2k} = \overline{\hat{\beta}}_{2k-1} = \beta_k$, $k = 1, 2, \ldots$. Further, we denote the nested spaces of rational functions based on the sequence $\hat{\mathcal{B}}$ by $\hat{\mathring{\mathcal{L}}}_k := \mathring{\mathcal{L}}\{\hat{\beta}_1, \ldots, \hat{\beta}_k\}$, so that

$$\hat{\mathring{\mathcal{L}}}_{2k} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_k \quad \text{and} \quad \hat{\mathring{\mathcal{L}}}_{2k-1} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{k-1}.$$

A connection between quadrature formulas on the unit circle and the interval I is given in e.g. [3] and [4]. If σ is a complex measure on I, we obtain a complex measure on $\mathbb{T}$ by setting

$$\mathring{\sigma}(E) = \sigma\left(\{\cos\theta, \theta \in E \cap [0,\pi)\}\right) + \sigma\left(\{\cos\theta, \theta \in E \cap [-\pi, 0)\}\right). \qquad (6)$$

Clearly, this measure $\mathring{\sigma}$ is then symmetric (i.e.; $d\mathring{\sigma}(-\theta) = -d\mathring{\sigma}(\theta)$), so that $I_{\mathring{\sigma}}(f_*^c) = I_{\mathring{\sigma}}(f)$ for every function f on $\mathbb{T}$.

Note that by the Joukowski Transformation, a function $f(x)$ transforms into a function $\mathring{f}(z) = (f \circ J)(z)$, so that $\mathring{f}_*^c(z) = \mathring{f}(z)$ and $J_\sigma(f) = \frac{1}{2} I_{\mathring{\sigma}}(\mathring{f})$. Further, let $\mathring{\mathcal{S}}_n$ be defined by

$$\mathring{\mathcal{S}}_{2k-1} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{(k-1)*}\,, \quad \text{and} \quad \mathring{\mathcal{S}}_{2k} = \mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{k*},$$

and let $\mathring{\mathcal{S}} = \mathring{\mathcal{L}}^c \cdot \mathring{\mathcal{L}}_* = \mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$. From [14, Lem. 3.1] it then follows that every function $f \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ transforms into a function $\mathring{f} \in \mathring{\mathcal{S}}_{2k} \setminus \left(\mathring{\mathcal{S}}_{2k-1} \cup \mathring{\mathcal{S}}^c_{(2k-1)*}\right)$; see also [9, Chapt. 3.2].

Consider an arbitrary set of m distinct nodes $\mathbf{x_m} := \{x_k\}_{k=1}^m \subset (-1,1)$, and let $\mathbf{z_{2m}} := \{z_k\}_{k=1}^{2m} \subset \mathbb{T} \setminus \{-1,1\}$ be the corresponding $2m$ distinct nodes on the unit circle, with $z_k = \overline{z}_{m+k}$ and $x_k = J(z_k) = J(z_{m+k})$ for $k = 1, \ldots, m$. We then distinguish the following three cases:

(0) $\mathbf{x_n^{[0]}} = \mathbf{x_n}$, and hence, $\mathbf{z_{2n}^{[0]}} = \mathbf{z_{2n}}$;
(1) $\mathbf{x_n^{[1]}} = \mathbf{x_{n-1}} \cup \{\pm 1\}$, and hence, $\mathbf{z_{2n-1}^{[1]}} = \mathbf{z_{2n-2}} \cup \{\pm 1\}$;
(2) $\mathbf{x_{n+1}^{[2]}} = \mathbf{x_{n-1}} \cup \{1,-1\}$, and hence, $\mathbf{z_{2n}^{[2]}} = \mathbf{z_{2n-2}} \cup \{1,-1\}$.

Let the parameter $s \in \{0,1,2\}$ refer to one of this three cases, and let $n(s)$ and $N(s)$ be defined by

$$n(s) = n + \lfloor s/2 \rfloor \qquad \text{and} \qquad N(s) = 2n - s \bmod 2.$$

This way we can briefly write that the set of $n(s)$ distinct nodes $\mathbf{x_{n(s)}^{[s]}} \subset I$ corresponds with the set of $N(s)$ distinct nodes $\mathbf{z_{N(s)}^{[s]}} \subset \mathbb{T}$, where s represents the number of nodes that are equal to 1 in absolute value. We then have the following theorem, which has been proved in [11, Sect. 3].

Theorem 1. *Suppose the positive measures μ on I and $\mathring{\mu}$ on $\mathbb{T}$ are related by* (6)*. Then the nodes $\mathbf{x_{n(s)}^{[s]}} \subset I$ are (μ, s)-nodes iff the nodes $\mathbf{z_{N(s)}^{[s]}} \subset \mathbb{T}$ are $\mathring{\mu}$-nodes.*

3 Rational interpolatory quadrature rules

In this section we will be concerned with the approximation of $J_\sigma(f)$ by means of an $n(s)$-point RIQ $J^\sigma_{n(s)}(f)$ based on a preassigned set of $n(s)$ distinct nodes

$\mathbf{x}_{n(s)}^{[s]}$ on I, where $s \in \{0,1,2\}$ and f is bounded on I. Note that it is always possible to determine the weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$ in such a way that the approximation is exact for every function $f \in \mathcal{L}_{n(s)-1}$. In this case, the weights are determined by requiring that $J_{n(s)}^\sigma(f) = J_\sigma(R_{n(s)-1})$, where $R_{n(s)-1}$ denotes the unique rational function in $\mathcal{L}_{n(s)-1}$ that interpolates the function f at the preassigned set of nodes. The success of such quadrature rules not only depends on the choice of nodes (see e.g. [18]), but also on the choice of poles. In this paper, however, we will assume that the poles are fixed in advance, and hence, we will only be concerned with the choice of nodes.

Let $\mathbf{z}_{N(s)}^{[s]}$ be the corresponding set of $N(s)$ distinct nodes on $\mathbb{T}$, with

$$
\begin{aligned}
&z_k = \overline{z}_{n-\lceil s/2\rceil+k} = J^{inv}(x_k) \in \mathbb{T} \setminus \{-1,1\}, \quad k = 1,\ldots,(n-\lceil s/2\rceil) \\
&z_{2n-1} = \pm 1 = x_n \quad \text{if} \quad s \neq 0, \quad \text{and} \quad z_{2n} = -z_{2n-1} = x_{n+1} \quad \text{if} \quad s = 2.
\end{aligned}
$$

Further, let the measure $\mathring{\sigma}$ on $\mathbb{T}$ be related to the measure σ on I by means of (6), and suppose $\mathring{f}$ is bounded on $\mathbb{T}$. We then can consider the approximation of $I_{\mathring{\sigma}}(\mathring{f})$ by means of the following $N(s)$-point quadrature rules:

$$
I_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = \sum_{k=1}^{N(s)} \mathring{\lambda}_k^{\mathring{\sigma}} \mathring{f}(z_k) = I_{\mathring{\sigma}}(\mathring{f}), \quad \forall \mathring{f} \in \mathring{\mathcal{S}}_{N(s)-1} \tag{7}
$$

and

$$
\tilde{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = \sum_{k=1}^{N(s)} \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} \mathring{f}(z_k) = I_{\mathring{\sigma}}(\mathring{f}), \quad \forall \mathring{f}_*^c \in \mathring{\mathcal{S}}_{N(s)-1}. \tag{8}
$$

Note that these quadrature rules – although a different domain of validity from the one in (4) – are of interpolatory type too, and are obtained by determining the weights $\mathring{\lambda}_k^{\mathring{\sigma}}$ and $\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}$ in such a way that $I_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = I_{\mathring{\sigma}}(S_{N(s)-1})$ and $\tilde{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = I_{\mathring{\sigma}}(\tilde{S}_{N(s)-1})$, where $S_{N(s)-1}$ and $\tilde{S}_{N(s)-1}$ denote the unique rational functions in respectively $\mathring{\mathcal{S}}_{N(s)-1}$ and $\mathring{\mathcal{S}}_{(N(s)-1)*}^c$ that interpolate the function $\mathring{f}$ at the preassigned set of nodes $\mathbf{z}_{N(s)}^{[s]}$. The following theorem now provides expressions for the weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$ in terms of the weights $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{N(s)}$ and $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{N(s)}$.

Theorem 2. *Consider the RIQ $J_{n(s)}^\sigma(f)$ for $J_\sigma(f)$, based on the set of nodes $\mathbf{x}_{n(s)}^{[s]}$, with corresponding set of weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$. Then for $s=0$ it holds that*

$$
\lambda_k^\sigma = \frac{\mathring{\lambda}_k^{\mathring{\sigma}} + \mathring{\lambda}_{n+k}^{\mathring{\sigma}}}{2} = \frac{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} + \tilde{\mathring{\lambda}}_{n+k}^{\mathring{\sigma}}}{2}, \quad k = 1,\ldots,n, \tag{9}
$$

while for $s \in \{1,2\}$ it holds that

$$\lambda_k^\sigma = \mathring{\lambda}_k^{\mathring{\sigma}} = \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n-1+k}^{\mathring{\sigma}} = \tilde{\mathring{\lambda}}_{n-1+k}^{\mathring{\sigma}}, \quad k = 1, \ldots, n-1$$
$$\lambda_n^\sigma = \frac{\mathring{\lambda}_{2n-1}^{\mathring{\sigma}}}{2} = \frac{\tilde{\mathring{\lambda}}_{2n-1}^{\mathring{\sigma}}}{2}, \quad \textit{and} \quad \lambda_{n+1}^\sigma = \frac{\mathring{\lambda}_{2n}^{\mathring{\sigma}}}{2} = \frac{\tilde{\mathring{\lambda}}_{2n}^{\mathring{\sigma}}}{2} \quad \textit{if} \quad s = 2,$$

where $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{N(s)}$ and $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{N(s)}$ are the set of weights in respectively the RIQs $I_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ for $I_{\mathring{\sigma}}(\mathring{f})$, based on the set of nodes $\mathbf{z}_{N(s)}^{[s]}$.

Proof. Let L_k, $k = 1, \ldots, n(s)$, denote the fundamental rational interpolating functions in $\mathcal{L}_{n(s)-1}$, so that $L_k(x_j) = \delta_{k,j}$ for $j = 1, \ldots, n(s)$, and define $g_k(z) := (L_k \circ J)(z) \in \mathring{\mathcal{S}}_{2[n(s)-1]}$. Then it holds that

$$g_k(z_j) = g_k(z_{n-\lceil s/2\rceil+j}) = \delta_{k,j}, \quad j = 1, \ldots, (n - \lceil s/2 \rceil)$$
$$g_k(z_{2n-1}) = \delta_{k,n} \quad \text{if} \quad s \neq 0, \quad \text{and} \quad g_k(z_{2n}) = \delta_{k,n+1} \quad \text{if} \quad s = 2,$$

so that $\lambda_k^\sigma = J_\sigma(L_k) = \frac{1}{2} I_{\mathring{\sigma}}(g_k)$. Since for $s = 0$ it holds that g_k is in $\mathring{\mathcal{S}}_{N(0)-1}$ as well as in $\mathring{\mathcal{S}}_{(N(0)-1)*}^c$, the equalities in (9) follow by applying the RIQs (7) and (8) respectively.

Next, for $s \in \{1,2\}$ let $\mathring{l}_l$, $l = 1, \ldots, N(s)$, denote the fundamental rational interpolating functions in $\mathring{\mathcal{S}}_{N(s)-1}$, so that $\mathring{l}_l(z_j) = \delta_{l,j}$ for $j = 1, \ldots, N(s)$, and define the rational functions $h_k \in \mathring{\mathcal{S}}_{2[n(s)-1]}$ by

$$h_k(z) = \mathring{l}_k(z) + \mathring{l}_{k*}^c(z) = \mathring{l}_{n-1+k}(z) + \mathring{l}_{(n-1+k)*}^c(z), \quad k = 1, \ldots, n-1,$$
$$h_n(z) = \frac{\mathring{l}_{2n-1}(z) + \mathring{l}_{(2n-1)*}^c(z)}{2}, \quad \text{and} \quad h_{n+1}(z) = \frac{\mathring{l}_{2n}(z) + \mathring{l}_{(2n)*}^c(z)}{2} \quad \text{if} \quad s = 2.$$

Clearly, we then have that $h_k(z) = g_k(z)$ for $k = 1, \ldots, n(s)$, so that

$$\lambda_k^\sigma = \begin{cases} \frac{1}{2}\left\{I_{\mathring{\sigma}}(\mathring{l}_k) + I_{\mathring{\sigma}}(\mathring{l}_{k*}^c)\right\} = \frac{1}{2}\left\{I_{\mathring{\sigma}}(\mathring{l}_{n-1+k}) + I_{\mathring{\sigma}}(\mathring{l}_{(n-1+k)*}^c)\right\}, & k < n \\ \frac{1}{4}\left\{I_{\mathring{\sigma}}(\mathring{l}_{n-1+k}) + I_{\mathring{\sigma}}(\mathring{l}_{(n-1+k)*}^c)\right\}, & k \geqslant n. \end{cases}$$

Finally, since the measure $\mathring{\sigma}$ is symmetric, we have that

$$\mathring{\lambda}_k^{\mathring{\sigma}} = I_{\mathring{\sigma}}(\mathring{l}_k) = I_{\mathring{\sigma}}(\mathring{l}_{k*}^c), \quad k = 1, \ldots, N(s).$$

where the first equality follow by applying the RIQ (7). This concludes the proof. ∎

From the previous theorem it follows that, for $s = 0$, we need to compute $2n$ weights in the RIQs on $\mathbb{T}$ in order to obtain the n weights in the RIQ on I.

Under certain conditions on the nodes or on the measure σ and the poles, this number can be reduced to n, as shown in the following two theorems.

Theorem 3. *Consider the RIQ $J_n^\sigma(f)$ for $J_\sigma(f)$, based on the set of nodes $\mathbf{x}_{n(0)}^{[0]}$, with corresponding set of weights $\{\lambda_k^\sigma\}_{k=1}^n$. Define the rational function $\psi_n \in \mathcal{L}_n$ in such a way that $\psi_n(x_j) = 0$ for every $x_j \in \mathbf{x}_{n(0)}^{[0]}$, and let $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ and $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ be the set of weights in respectively the RIQs $I_{2n}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{I}_{2n}^{\mathring{\sigma}}(\mathring{f})$ for $I_{\mathring{\sigma}}(\mathring{f})$, based on the set of nodes $\mathbf{z}_{N(0)}^{[0]}$. Then it holds that*

$$\lambda_k^\sigma = \mathring{\lambda}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n+k}^{\mathring{\sigma}} \quad \text{and} \quad \lambda_k^\sigma = \tilde{\mathring{\lambda}}_k^{\mathring{\sigma}} = \tilde{\mathring{\lambda}}_{n+k}^{\mathring{\sigma}}, \quad k = 1, \ldots, n \tag{10}$$

iff

$$J_\sigma(\psi_n) = 0. \tag{11}$$

Proof. (We will only prove the first equality in (10); the second equality can be proved in a similar way). Let $\mathring{l}_k$, $k = 1, \ldots, 2n$, denote the fundamental rational interpolating functions in $\mathring{\mathcal{S}}_{2n-1}$, so that $\mathring{l}_k(z_j) = \delta_{k,j}$ for $j = 1, \ldots, 2n$. Then we have that

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \mathring{\lambda}_{n+k}^{\mathring{\sigma}} = I_{\mathring{\sigma}}(\mathring{l}_k) - I_{\mathring{\sigma}}(\mathring{l}_{n+k}) = I_{\mathring{\sigma}}(\mathring{l}_k) - I_{\mathring{\sigma}}(\mathring{l}_{(n+k)*}^c),$$

where the last equality follows from the fact that the measure $\mathring{\sigma}$ is symmetric. Note that, due to Theorem 2, it suffices to prove that $\mathring{\lambda}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n+k}^{\mathring{\sigma}}$ for $k = 1, \ldots, n$ iff $J_\sigma(\psi_n) = 0$. So, let us now define the rational function $\mathring{\psi}_{2n}(z) := (\psi_n \circ J)(z) = \mathring{\psi}_{(2n)*}^c(z) \in \mathring{\mathcal{S}}_{2n}$. Then it is easily verified that

$$\mathring{l}_k(z) = \frac{(z - \beta_n)\mathring{\psi}_{2n}(z)}{(z_k - \beta_n)(z - z_k)\mathring{\psi}_{2n}'(z_k)},$$

where the prime denotes the derivative with respect to z. Further, note that from $\mathring{\psi}_{2n}(z) = \mathring{\psi}_{2n}(1/z)$ it follows that $\mathring{\psi}_{2n}'(z) = -\frac{\mathring{\psi}_{2n}'(1/z)}{z^2}$, so that

$$\begin{aligned} I_{\mathring{\sigma}}(\mathring{l}_{(n+k)*}^c) &= \int_{-\pi}^{\pi} \frac{(1 - \beta_n z)\mathring{\psi}_{2n}(z)}{(z_{n+k} - \beta_n)(1 - z_{n+k}z)\mathring{\psi}_{2n}'(z_{n+k})} d\mathring{\sigma}(\theta) \\ &= \int_{-\pi}^{\pi} \frac{(1 - \beta_n z)\mathring{\psi}_{2n}(z)}{(1 - \beta_n z_k)(z - z_k)\left[-\frac{\mathring{\psi}_{2n}'(1/z_k)}{z_k^2}\right]} d\mathring{\sigma}(\theta) \\ &= \int_{-\pi}^{\pi} \frac{(1 - \beta_n z)\mathring{\psi}_{2n}(z)}{(1 - \beta_n z_k)(z - z_k)\mathring{\psi}_{2n}'(z_k)} d\mathring{\sigma}(\theta). \end{aligned}$$

Consequently,

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \mathring{\lambda}_{n+k}^{\mathring{\sigma}} = \int_{-\pi}^{\pi} \frac{\mathring{\psi}_{2n}(z)}{(z-z_k)\mathring{\psi}'_{2n}(z_k)} \left[\frac{z-\beta_n}{z_k-\beta_n} - \frac{1-\beta_n z}{1-\beta_n z_k} \right] d\mathring{\sigma}(\theta)$$
$$= \frac{(1-\beta_n^2) I_{\mathring{\sigma}}(\mathring{\psi}_{2n})}{(z_k-\beta_n)(1-\beta_n z_k)\mathring{\psi}'_{2n}(z_k)}.$$

As a result, $\mathring{\lambda}_k^{\mathring{\sigma}} = \mathring{\lambda}_{n+k}^{\mathring{\sigma}}$ iff $I_{\mathring{\sigma}}(\mathring{\psi}_{2n}) = 0$; i.e.; iff $J_\sigma(\psi_n) = 0$. This ends the proof.
■

Theorem 4. *Suppose σ is a real measure on I, and assume that $\alpha_n \in \overline{\mathbb{R}}_I$ and that the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$ are real and/or appear in complex conjugate pairs. Consider the RIQ $J_n^\sigma(f)$ for $J_\sigma(f)$, based on the set of nodes $\mathbf{x}_{n(0)}^{[0]}$, with corresponding set of weights $\{\lambda_k^\sigma\}_{k=1}^n$. Then it holds that*

$$\lambda_k^\sigma = \Re\{\mathring{\lambda}_k^{\mathring{\sigma}}\} \quad \textit{and} \quad \lambda_k^\sigma = \Re\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}, \quad k = 1, \ldots, n, \tag{12}$$

where $\{\mathring{\lambda}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ and $\{\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\}_{k=1}^{2n}$ are the set of weights in respectively the RIQs $I_{2n}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{I}_{2n}^{\mathring{\sigma}}(\mathring{f})$ for $I_{\mathring{\sigma}}(\mathring{f})$, based on the set of nodes $\mathbf{z}_{N(0)}^{[0]}$.

Proof. (We will only prove the first equality in (12); the second equality can be proved in a similar way). Let $\mathring{l}_k$, $k = 1, \ldots, 2n$, and $\mathring{\psi}_{2n}(z)$ be defined as before in the proof of the previous theorem. Note that, due to Theorem 2, it suffices to prove that $\mathring{\lambda}_k^{\mathring{\sigma}} = \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}}$ for $k = 1, \ldots, n$ under the conditions on the measure σ and the poles, given in the statement. We now have that

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}} = I_{\mathring{\sigma}}(\mathring{l}_k) - \overline{I_{\mathring{\sigma}}(\mathring{l}_{n+k})} = I_{\mathring{\sigma}}(\mathring{l}_k) - I_{\mathring{\sigma}}(\mathring{l}_{n+k}^c),$$

where the last equality follows from the fact that the measure $\mathring{\sigma}$ is real symmetric. Further, we have that

$$I_{\mathring{\sigma}}(\mathring{l}_{n+k}^c) = \int_{-\pi}^{\pi} \frac{(z-\overline{\beta}_n)\mathring{\psi}_{2n}^c(z)}{(\overline{z}_{n+k}-\overline{\beta}_n)(z-\overline{z}_{n+k})\overline{\mathring{\psi}'_{2n}(z_{n+k})}} d\mathring{\sigma}(\theta)$$
$$= \int_{-\pi}^{\pi} \frac{(z-\beta_n)\overline{\mathring{\psi}_{2n}(z)}}{(z_k-\beta_n)(z-z_k)\overline{\left[-z_k^2\mathring{\psi}'_{2n}(z_k)\right]}} d\mathring{\sigma}(\theta).$$

Consequently,

$$\mathring{\lambda}_k^{\mathring{\sigma}} - \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}} = \int_{-\pi}^{\pi} \frac{z_k(z-\beta_n)}{(z_k-\beta_n)(z-z_k)} 2\Re\left\{ \overline{z}_k \frac{\mathring{\psi}_{2n}(z)}{\mathring{\psi}'_{2n}(z_k)} \right\} d\mathring{\sigma}(\theta).$$

Next, note that $\mathring{\psi}_{2n}(z)$ is of the form

$$\mathring{\psi}_{2n}(z) = \frac{c_n \prod_{j=1}^{2n}(z-z_j)}{\prod_{j=1}^{n}(1-\beta_j z)(z-\beta_j)}, \quad c_n \in \mathbb{C}_0,$$

and that $\overline{\mathring{\psi}_{2n}(z)} = \frac{\overline{c}_n}{c_n}\mathring{\psi}_{2n}(z)$ due to the assumption on the poles. Without loss of generality, we may as well assume that $c_n = 1$, so that

$$\begin{aligned}\mathring{\lambda}_k^{\mathring{\sigma}} - \overline{\mathring{\lambda}_{n+k}^{\mathring{\sigma}}} &= 2\Re\left\{\frac{\overline{z}_k}{\mathring{\psi}'_{2n}(z_k)}\right\} z_k \int_{-\pi}^{\pi} \frac{(z-\beta_n)\mathring{\psi}_{2n}(z)}{(z_k-\beta_n)(z-z_k)} d\mathring{\sigma}(\theta) \\ &= 2\Re\left\{\frac{\overline{z}_k}{\mathring{\psi}'_{2n}(z_k)}\right\} z_k \mathring{\psi}'_{2n}(z_k) I_{\mathring{\sigma}}(\mathring{l}_k).\end{aligned}$$

Finally, it holds that

$$\Re\left\{\frac{\overline{z}_k}{\mathring{\psi}'_{2n}(z_k)}\right\} = \frac{\prod_{j=1}^n |z_k-\beta_j|^2}{\prod_{j=1,j\neq k}^n |z_k-z_j|^2}\Re\left\{\frac{1}{z_k-z_{n+k}}\right\} = 0.$$

■

Remark 5. Note that, if assumption (11) in Theorem 3 is satisfied, it holds that $J_\sigma(f) = J_n^\sigma(f)$ for every $f \in \mathcal{L}_n$; see also [8, Rem. 4.7] for the polynomial case.

Remark 6. The condition on the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$ in Theorem 4 is sufficient but not necessary. A necessary but not sufficient condition is that the RIQ $J_\sigma(f) \approx J_n^\sigma(f)$ is exact for every $f \in \tilde{\mathcal{L}}_m$ but not for $f \in \tilde{\mathcal{L}}_{m+1} \setminus \tilde{\mathcal{L}}_m$, where $\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_m \subseteq \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$ and $\tilde{\mathcal{L}}_m^c = \tilde{\mathcal{L}}_m$.

So far, we have been mainly concerned with the algebraic aspects of the quadrature rules for $J_\sigma(f)$ and $I_{\mathring{\sigma}}(f)$, but nothing has been said yet about the "goodness" of these quadrature rules with respect to the numerical aspects. For this purpose, we will consider de case in which the interpolation points $\mathbf{x}_{n(s)}^{[s]}$ on I are (μ, s)-nodes. In the remainder of this section, we will restrict ourselves to the case of the interval. Similar results are easily proved with the aid of [6, Sect. 3 and 4] and [13, Sect. 3] for the quadrature rules (7) and (8), based on an arbitrary sequence of $\mathring{\mu}$-nodes on $\mathbb{T}$ (and hence, for a complex measure $\mathring{\sigma}$ on $\mathbb{T}$ which is not necessarily symmetric). In fact, more general subspaces of $\mathring{\mathcal{S}}$, and of the form $\mathring{\mathcal{L}}_{p(n-1)}^c \cdot \mathring{\mathcal{L}}_{q(n-1)*}$, with n the number of nodes in the RIQ, can then be considered.

In Theorem 9 we will prove – under certain conditions – the convergence of the RIQs $J_{n(s)}^\sigma(f)$ for n tending to infinity. For this, we first need the following two lemmas.

Lemma 7. *Suppose μ is a positive measure on I, and consider the sequence of orthonormal rational functions $\{\varphi_k\}_{k=0}^n$, with $\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, so that $\varphi_k \perp_\mu \mathcal{L}_{k-1}$ and $\|\varphi_k\|_{\mu,2} = 1$. Further, suppose $g(x)$ is a function on I such that*

$\|g\|_{\mu,2} < \infty$, and let P_n^g be its orthogonal projection in $L_2^\mu(I)$ onto $\mathcal{L}_n^c$; i.e.;

$$P_n^g(x) = \sum_{k=0}^{n} c_k \varphi_k^c(x), \quad c_k = J_\mu\left(\varphi_k g\right).$$

Then it holds that $J_\mu\left(fP_n^g\right) = J_\mu\left(fg\right)$ for every $f \in \mathcal{L}_n$.

Proof. Since the sequence $\{\varphi_k\}_{k=0}^n$ forms an orthonormal basis for $\mathcal{L}_n$, it suffices to prove that

$$J_\mu\left(\varphi_j P_n^g\right) = J_\mu\left(\varphi_j g\right), \quad j = 0, \ldots, n.$$

We now have for $j = 0, \ldots, n$ that

$$J_\mu\left(\varphi_j P_n^g\right) = \sum_{k=0}^{n} c_k J_\mu\left(\varphi_j \varphi_k^c\right) = c_j = J_\mu\left(\varphi_j g\right).$$

This concludes the proof. ■

Lemma 8. *Suppose μ is a positive measure on I, and let σ be a complex measure on I, such that σ is absolutely continuous with respect to μ ($\sigma \ll \mu$) and*

$$\|g\|_{\mu,2} < \infty, \quad g(x) = \frac{d\sigma(x)}{d\mu(x)}. \tag{13}$$

Consider the set of (μ, s)-nodes $\mathbf{x}_{n(s)}^{[s]} \subset I$, and let $\{\lambda_k^\mu\}_{k=1}^{n(s)}$ denote the corresponding (μ, s)-weights. Let $a_{n,s}$ be given by

$$a_{n,s} = \begin{cases} 0, & s < 2 \\ \frac{J_\mu(\varphi_n g)}{J_{n+1}^\mu\left(|\varphi_n|^2\right)}, & s = 2. \end{cases}$$

Define the weights

$$\lambda_k^\sigma = \lambda_k^\mu\left[a_{n,s}\varphi_n^c(x_k) + P_{n-1}^g(x_k)\right], \quad k = 1, \ldots, n(s),$$

where P_{n-1}^g is the orthogonal projection in $L_2^\mu(I)$ of $g(x)$ onto $\mathcal{L}_{n-1}^c$. Then the RIQ $J_\sigma(f) \approx J_{n(s)}^\sigma(f)$, based on the set of nodes $\mathbf{x}_{n(s)}^{[s]}$ and weights $\{\lambda_k^\sigma\}_{k=1}^{n(s)}$ is exact for every $f \in \mathcal{L}_{n(s)-1}$.

Proof. First, note that for $\sigma \ll \mu$, we have with $g(x) = \frac{d\sigma(x)}{d\mu(x)}$ that $J_\mu(fg) = J_\sigma(f)$ for every function f. So, consider now the case in which $f \in \mathcal{L}_{n-1}$. We then have that

$$\begin{aligned} J_{n(s)}^\sigma(f) &= a_{n,s} J_{n(s)}^\mu(f\varphi_n^c) + J_{n(s)}^\mu(fP_{n-1}^g) \\ &= a_{n,s} J_{n(s)}^\mu(f\varphi_n^c) + J_\mu\left(fP_{n-1}^g\right) = a_{n,s} J_{n(s)}^\mu(f\varphi_n^c) + J_\sigma(f), \end{aligned}$$

where the last equality follows from the previous lemma. Thus, we find that $J^{\sigma}_{n(s)}(f) = J_{\sigma}(f)$ iff $a_{n,s}J^{\mu}_{n(s)}(f\varphi^c_n) = 0$. Recall that the last pole α_n, if not real, was assumed to be replaced with $\tilde{\alpha}_n \in \overline{\mathbb{R}}_I$ for $s \in \{0, 2\}$, so that $a_{n,s}J^{\mu}_{n(s)}(f\varphi^c_n) = 0$ iff $s \neq 1$ or $a_{n,1} = 0$. Hence, we have proved the statement for $s < 2$.

Consider now the case in which $s \in \{0, 2\}$ and $f = \varphi_n$. It then holds that

$$
\begin{aligned}
J^{\sigma}_{n(s)}(\varphi_n) &= a_{n,s}J^{\mu}_{n(s)}(|\varphi_n|^2) + J^{\mu}_{n(s)}(\varphi_n P^g_{n-1}) \\
&= a_{n,s}J^{\mu}_{n(s)}\left(|\varphi_n|^2\right) + J_{\mu}\left(\varphi_n P^g_{n-1}\right) = a_{n,s}J^{\mu}_{n(s)}\left(|\varphi_n|^2\right).
\end{aligned}
$$

Note that $J^{\mu}_{n(s)}\left(|\varphi_n|^2\right) = 0$ for $s = 0$, due to the fact that $\varphi_n(x_k) = 0$ for every $(\mu, 0)$-node x_k. Consequently, the quadrature is exact for every $f \in \mathcal{L}_n$ iff $J_{\mu}(\varphi_n g) = J_{\sigma}(\varphi_n) = 0$ (see also Remark 5). Thus, if P^g_n denotes the orthogonal projection in $L^{\mu}_2(I)$ of $g(x)$ onto $\mathcal{L}^c_n$, then we deduce form the previous lemma that $P^g_n(x) \equiv P^g_{n-1}(x)$ whenever $J_{\mu}(\varphi_n g) = 0$. As a result, the statement remains valid for this special situation, and we may as well put $a_{n,0} = 0$ too.

Finally, for the case in which $s = 2$, it holds that $J^{\mu}_{n+1}\left(|\varphi_n|^2\right) \in \mathbb{R}^+_0$. Thus, setting $a_{n,2} = J_{\mu}(\varphi_n g)/J^{\mu}_{n+1}\left(|\varphi_n|^2\right)$, we get that

$$
J^{\sigma}_{n+1}(\varphi_n) = \frac{J_{\mu}(\varphi_n g)}{J^{\mu}_{n+1}\left(|\varphi_n|^2\right)} J^{\mu}_{n+1}\left(|\varphi_n|^2\right) = J_{\mu}(\varphi_n g) = J_{\sigma}(\varphi_n).
$$

This concludes the proof. ∎

The following theorem now provides a convergence result for the RIQs $J^{\sigma}_{n(s)}(f)$ for n tending to infinity.

Theorem 9. *Suppose μ is a positive measure on I, and let σ be a complex measure on I, such that $\sigma \ll \mu$ and $\|g\|_{\mu,2} < \infty$, where g is defined as before in* (13). *Assume $\sum_{j=1}^{\infty}(1 - |J^{inv}(\alpha_j)|) = \infty$, and consider the RIQs $J_{\sigma}(f) \approx J^{\sigma}_{n(s)}(f)$, $n = 1, 2, \ldots$, based on the sets of (μ, s)-nodes $\mathbf{x}^{[s]}_{n(s)} \subset I$. Then it holds that $\lim_{n\to\infty} J^{\sigma}_{n(s)}(f) = J_{\sigma}(f)$ for all functions f bounded on I, for which the Riemann–Stieltjes integral $J_{\mu}(f)$ exists.*

Proof. For every $n > 0$, let $R_{n(s)-1} \in \mathcal{L}_{n(s)-1}$ denote the interpolating rational function for f at the sets of nodes $\mathbf{x}^{[s]}_{n(s)}$. Since $J^{\sigma}_{n(s)}(f) = J^{\sigma}_{n(s)}(R_{n(s)-1}) = J_{\sigma}(R_{n(s)-1})$, we have that

$$
\begin{aligned}
\left|J_{\sigma}(f) - J^{\sigma}_{n(s)}(f)\right| &= \left|J_{\sigma}(f - R_{n(s)-1})\right| \\
&= \left|J_{\mu}\left([f - R_{n(s)-1}]g\right)\right| \leqslant J_{\mu}\left(\left|f - R_{n(s)-1}\right| |g|\right).
\end{aligned}
$$

Making use of the Cauchy–Schwarz inequality, and setting $\|g\|_{\mu,2} = M < \infty$, we obtain that

$$\left|J_\sigma(f) - J^\sigma_{n(s)}(f)\right| \leqslant M \cdot \left\|f - R_{n(s)-1}\right\|_{\mu,2},$$

where it follows from [10, Thms. 8, 10 and 12] that $\lim_{n\to\infty} \left\|f - R_{n(s)-1}\right\|_{\mu,2} = 0$. ■

Under the same assumptions on the measure σ as in the previous theorem, it follows from the Banach-Steinhaus Theorem (see e.g. [17, Thm. 2.5]) that there exists a positive constant M_s so that for every $n > 0$, $S_n = \sum_{k=1}^{n(s)} |\lambda^\sigma_k| \leqslant M_s$. The next theorem implies as a special case (i.e., with $f(x) \equiv 1$) that the sequence S_n converges; namely that $\lim_{n\to\infty} S_n = \int_{-1}^{1} |d\sigma(x)|$. First, we need the following lemma.

Lemma 10. *Suppose μ is a positive measure on I, and assume $\sum_{j=1}^{\infty}(1 - |J^{inv}(\alpha_j)|) = \infty$. For $n = 1, 2, \ldots$, let $J^\mu_{n(s)}(f)$ denote the rational Gauss-type quadrature based on the (μ, s)-nodes. Then it holds that $\lim_{n\to\infty} J^\mu_{n(s)}(f) = J_\mu(f)$ for all μ-integrable functions f.*

Proof. The statement directly follows from [10, Lems. 2 and 4], together with Theorem 1. ■

Theorem 11. *Suppose μ is a positive measure on I, and let σ be a complex measure on I, such that $\sigma \ll \mu$ and $\|g\|_{\mu,2} < \infty$, where g is defined as before in (13), and assume that $\sum_{j=1}^{\infty}(1 - |J^{inv}(\alpha_j)|) = \infty$. Consider the sets of (μ, s)-nodes $\mathbf{x}^{[s]}_{n(s)} \subset I$, $n = 1, 2, \ldots$, and let $\{\lambda^\sigma_k\}_{k=1}^{n(s)}$ be the sets of weights in the corresponding RIQ $J_\sigma(f) \approx J^\sigma_{n(s)}(f)$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n(s)} |\lambda^\sigma_k|\, f(x_k) = J_\mu(f \cdot |g|), \tag{14}$$

for all functions f bounded on I, for which the function $f \cdot |g|$ is μ-integrable.

Proof. For ease of notation, we will denote the sum on the left hand side in (14) by $J^\mu_{n(s)}(f \cdot \left|g^\perp_n\right|)$, where $g^\perp_n(x) := a_{n,s}\varphi^c_n(x) + P^g_{n-1}(x) \in \mathcal{L}^c_{n(s)-1}$; see Lemma 8.

Choose an arbitrary $\hat{g} \in L^\mu_2(I)$ (i.e.; $\|\hat{g}\|_{\mu,2} < \infty$). By the triangle inequality and because the weights λ^σ_k depend linearly on σ, it follows that

$$\begin{aligned} T_n &:= \left|J^\mu_{n(s)}(f \cdot \left|g^\perp_n\right|) - J_\mu(f \cdot |g|)\right| \\ &\leqslant J^\mu_{n(s)}\left(|f| \cdot \left|g^\perp_n - \hat{g}^\perp_n\right|\right) + J_\mu\left(|f| \cdot |g - \hat{g}|\right) + \left|J^\mu_{n(s)}(f \cdot \left|\hat{g}^\perp_n\right|) - J_\mu(f \cdot |\hat{g}|)\right|. \end{aligned}$$

Since f is bounded on I, there exists a number M_f such that $|f(x)| \leqslant M_f < \infty$ for every $x \in I$. Hence,

$$T_n \leqslant M_f J^{\mu}_{n(s)} \left(\left|g^{\perp}_n - \hat{g}^{\perp}_n\right|\right) + M_f J_{\mu} \left(|g - \hat{g}|\right) + \left|J^{\mu}_{n(s)}(f \cdot \left|\hat{g}^{\perp}_n\right|) - J_{\mu}(f \cdot |\hat{g}|)\right| \\ = T_{n,1} + T_{n,2} + T_{n,3},$$

where $T_{n,1} = \left|J^{\mu}_{n(s)}(f \cdot \left|\hat{g}^{\perp}_n\right|) - J_{\mu}(f \cdot |\hat{g}|)\right|$, $T_{n,2} = M_f J^{\mu}_{n(s)} \left(\left|g^{\perp}_n - \hat{g}^{\perp}_n\right|\right)$, and $T_{n,3} = M_f J_{\mu} \left(|g - \hat{g}|\right)$. To estimate $T_{n,3}$, we use the Cauchy–Schwarz inequality to get

$$T_{n,3} \leqslant M_f \left\|g - \hat{g}\right\|_{\mu,2} \cdot K_{\mu}, \quad K_{\mu} := \|1\|_{\mu,2} < \infty.$$

For $T_{n,2}$ we note that for every function f_n of the form $f_n(x) = c_n \varphi^c_n(x) + f_{n-1}(x)$, with $f_{n-1} \in \mathcal{L}^c_{n-1}$, it holds that

$$\begin{aligned} J^{\mu}_{n(s)} \left(|f_n|\right) &\leqslant |c_n| \, J^{\mu}_{n(s)} \left(|\varphi^c_n|\right) + J^{\mu}_{n(s)} \left(|f_{n-1}|\right) \\ &\leqslant \sqrt{\sum_{k=1}^{n(s)} \lambda^{\mu}_k} \cdot \left\{ |c_n| \sqrt{J^{\mu}_{n(s)} \left(|\varphi_n|^2\right)} + \sqrt{J^{\mu}_{n(s)} \left(|f_{n-1}|^2\right)} \right\} \\ &= K_{\mu} \cdot \left\{ |c_n| \sqrt{J^{\mu}_{n(s)} \left(|\varphi_n|^2\right)} + \left\|f_{n-1}\right\|_{\mu,2} \right\}, \end{aligned}$$

where the last equality follows from the fact that the quadratures are all exact in $\mathcal{L}_{n-1} \cdot \mathcal{L}^c_{n-1}$.

For our purpose, c_n equals $a_{n,s} - \hat{a}_{n,s}$. So, let us first consider the case in which $s < 2$. We then have that

$$T_{n,2} \leqslant M_f \left\|g^{\perp}_n - \hat{g}^{\perp}_n\right\|_{\mu,2} \cdot K_{\mu} \leqslant M_f \left\|g - \hat{g}\right\|_{\mu,2} \cdot K_{\mu},$$

so that

$$T_n \leqslant T_{n,1} + 2M_f \left\|g - \hat{g}\right\|_{\mu,2} \cdot K_{\mu}.$$

Consider now the case in which $\hat{g} = P^g_N \in \mathcal{L}^c_N$. From [10, Cor. 15] it then follows that for every $\epsilon' > 0$ there exists an integer l, so that for every $N \geqslant l$:

$$2M_f \left\|g - P^g_N\right\|_{\mu,2} \cdot K_{\mu} < \epsilon'/2,$$

Thus, for $n > N \geqslant l$ we get that

$$T_n < \epsilon'/2 + \left|J^{\mu}_{n(s)}(f \cdot |P^g_N|) - J_{\mu}(f \cdot |P^g_N|)\right|.$$

Further, since

$$\begin{aligned} \left|J_{\mu} \left(f \cdot |g|\right) - J_{\mu} \left(f \cdot |P^g_N|\right)\right| &\leqslant J_{\mu} \left(|f| \cdot |g - P^g_N|\right) \\ &\leqslant \|f\|_{\mu,2} \cdot \left\|g - P^g_N\right\|_{\mu,2} \leqslant M_f \left\|g - P^g_N\right\|_{\mu,2} \cdot K_{\mu} < \epsilon'/4, \end{aligned}$$

and $f \cdot |g|$ is assumed to be μ-integrable, it follows that $f \cdot |P_N^g|$ is μ-integrable too. Consequently, from Lemma 10 we deduce that for every $\epsilon' > 0$ there exists an integer k, so that for every $n \geqslant k$:

$$\left| J^{\mu}_{n(s)}(f \cdot |P_N^g|) - J_\mu(f \cdot |P_N^g|) \right| < \epsilon'/2.$$

Taking $n > \max\{N, k\}$ and setting $\epsilon = \epsilon'$, we obtain in this way that $T_n < \epsilon$. This proves the statement for $s < 2$.

Next, for $s = 2$ we have that

$$|c_n| \sqrt{J^{\mu}_{n+1}\left(|\varphi_n|^2\right)} = \frac{|J_\mu(\varphi_n(g - \hat{g}))|}{\sqrt{J^{\mu}_{n+1}\left(|\varphi_n|^2\right)}} \leqslant \frac{\|g - \hat{g}\|_{\mu,2}}{\sqrt{J^{\mu}_{n+1}\left(|\varphi_n|^2\right)}},$$

so that

$$T_n \leqslant T_{n,1} + 2M_f \|g - \hat{g}\|_{\mu,2} \cdot K_\mu \left(1 + \frac{1}{2}\left[J^{\mu}_{n+1}\left(|\varphi_n|^2\right)\right]^{-1/2}\right).$$

Note that $J_\mu\left(|\varphi_n|^2\right) = 1$, and hence, from Lemma 10 we now deduce that that for every $\epsilon'' > 0$ there exists an integer m, so that for every $n \geqslant m$

$$1 - \epsilon'' < \left[J^{\mu}_{n+1}\left(|\varphi_n|^2\right)\right]^{-1/2} < 1 + \epsilon''.$$

Proceeding as before, with $\hat{g} = P_N^g$, and taking $n > \max\{N, k, m\}$, we obtain in this way that $T_n < \frac{\epsilon'}{4}(5 + \epsilon'') = \epsilon$. This concludes the proof. ■

4 Error bound and rate of convergence

In this section we will provide an error bound for the RIQs considered in the previous section. For this, we will again pass from the interval to the unit circle by means of the Joukowski Transformation $x = J(z) \in \mathbb{C}$. Setting $x = \Re\{x\} + \mathbf{i}\Im\{x\}$ and $z = \rho e^{\mathbf{i}\theta}$, we obtain that $\Re\{x\} + \mathbf{i}\Im\{x\} = J(\rho e^{\mathbf{i}\theta}) = \frac{1}{2}(\rho + \rho^{-1})\cos\theta + \mathbf{i}\frac{1}{2}(\rho - \rho^{-1})\sin\theta$. Thus, for $0 < \rho < 1$, the circles

$$\mathcal{C}_\rho := \{z \in \mathbb{C} : |z| = \rho\} \quad \text{and} \quad \mathcal{C}_{\frac{1}{\rho}} := \{z \in \mathbb{C} : |z| = \rho^{-1}\} \tag{15}$$

map onto the ellipse

$$\mathcal{E}_\rho := \left\{ x \in \mathbb{C} : \left(\frac{2\Re\{x\}}{\rho + \rho^{-1}}\right)^2 + \left(\frac{2\Im\{x\}}{\rho - \rho^{-1}}\right)^2 = 1 \right\}. \tag{16}$$

Let f be an analytic function on some open neighborhood of I, and suppose that μ is a positive measure on I. Consider an n-point RIQ $J_n^\mu(f) =$

$\sum_{j=1}^{n} \lambda_k^\mu f(x_k)$ which is exact in the space $\tilde{\mathcal{L}}_m := \mathcal{L}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\}$, with $\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_m \subseteq \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$, and let $E_n^\mu(f) := J_\mu(f) - J_n^\mu(f)$ denote the error on the nth approximation. Further, suppose that U is the domain of analyticity of $\pi_m f$, where $\pi_m(x) = \prod_{j=1}^m (1 - x/\tilde{\alpha}_j)$, and let $0 < \rho < 1$ be such that $\mathcal{E}_\rho \subset U$. Then the following upper bound has been proved in [15, Eq. (5)]:

$$|E_n^\mu(f)| \leqslant \frac{\|\pi_m f\|_{\mathcal{E}_\rho}}{2\pi} \left(\int_{-1}^1 d\mu(x) + \sum_{k=1}^n |\lambda_k^\mu| \right) \int_{\mathcal{E}_\rho} e_{m+1} \left(\frac{1}{\pi_m(\cdot)(v - \cdot)} \right) |dv| , \tag{17}$$

where

$$e_{m+1} \left(\frac{1}{\pi_m(\cdot)(v - \cdot)} \right) = \min_{p \in \mathcal{P}_{m+1},\, p(v)=1} \left\| \frac{p(\cdot)}{\pi_m(\cdot)(v - \cdot)} \right\|_I$$

and $\|h\|_K$ stands for the maximum value of the continuous function $|h|$ on the compact set K. In [15, Sect. 3] the poles $\tilde{\alpha}_j$ where assumed to be real and/or appearing in complex conjugate pairs. However, following the steps in the proof of (17), it is easy to see that the expression for the upper bound remains valid without this assumption on the poles. Furthermore, the result is easily extended to the case of a complex measure σ just by replacing $d\mu(x)$ and λ_k^μ with $|d\sigma(x)|$ and λ_k^σ respectively; thus,

$$|E_n^\sigma(f)| \leqslant \frac{\|\pi_m f\|_{\mathcal{E}_\rho}}{2\pi} \left(\int_{-1}^1 |d\sigma(x)| + \sum_{k=1}^n |\lambda_k^\sigma| \right) \int_{\mathcal{E}_\rho} e_{m+1} \left(\frac{1}{\pi_m(\cdot)(v - \cdot)} \right) |dv| .$$

We now have the following lemma, which is a generalization of [15, Lem. 1] to the case of arbitrary complex poles outside I.

Lemma 12. *Suppose $x = J(z) \in I$, $v = J(w) \in \mathbb{C}$ and $\tilde{\alpha}_j = J(\tilde{\beta}_j) \in \overline{\mathbb{C}}_I$. Let us denote*

$$V_{m+1} = (1 + w^2) \prod_{j=1}^m (1 + \tilde{\beta}_j^2), \quad \mathring{\pi}_m(z) = \prod_{j=1}^m (z - \tilde{\beta}_j),$$

and set

$$\frac{\Phi_{m+1}(x)}{V_{m+1}} = \frac{\pi_m(x)(v - x)}{2^{m+1} v} \left\{ \frac{(z - \overline{w})\mathring{\pi}_m^c(z)}{(1 - zw) z^m \mathring{\pi}_{m*}^c(z)} + \frac{(1 - z\overline{w}) z^m \mathring{\pi}_{m*}(z)}{(z - w)\mathring{\pi}_m(z)} \right\}.$$

Then, $\Phi_{m+1}(x)$ is a polynomial in the variable x of degree $m + 1$.

Proof. First, note that

$$V_{m+1} \pi_m(x)(1 - x/v) = \mathring{\pi}_m(z) \mathring{\pi}_{m*}^c(z) \frac{(z - w)(1 - zw)}{z},$$

so that

$$2^{m+1} \Phi_{m+1}(x) = \frac{(z - w)(z - \overline{w})}{z^{m+1}} \mathring{\pi}_m(z) \mathring{\pi}_m^c(z) + z^{m-1} (1 - zw)(1 - z\overline{w}) \mathring{\pi}_{m*}^c(z) \mathring{\pi}_{m*}(z).$$

Next, with $z = e^{\mathbf{i}\theta}$ we have

$$\frac{(z-w)(z-\overline{w})}{z^{m+1}}\mathring{\pi}_m(z)\mathring{\pi}_m^c(z) = \frac{(e^{\mathbf{i}\theta}-w)(e^{\mathbf{i}\theta}-\overline{w})}{e^{\mathbf{i}(m+1)\theta}}\prod_{j=1}^{m}(e^{\mathbf{i}\theta}-\tilde{\beta}_j)(e^{\mathbf{i}\theta}-\overline{\tilde{\beta}_j})$$

and

$$z^{m-1}(1-zw)(1-z\overline{w})\mathring{\pi}_{m*}^c(z)\mathring{\pi}_{m*}(z) \\ = e^{\mathbf{i}(m+1)\theta}\left(\frac{1}{e^{\mathbf{i}\theta}}-w\right)\left(\frac{1}{e^{\mathbf{i}\theta}}-\overline{w}\right)\prod_{j=1}^{m}\left(\frac{1}{e^{\mathbf{i}\theta}}-\tilde{\beta}_j\right)\left(\frac{1}{e^{\mathbf{i}\theta}}-\overline{\tilde{\beta}_j}\right).$$

Therefore, $\Phi_{m+1}(x)$ is a trigonometrical polynomial of degree $m+1$ at most, and hence, can be expressed as a linear combination of $\cos(k\theta)$, $k = 0, \ldots, m+1$. An easy calculation shows that the coefficient of $\cos((m+1)\theta)$ is $(1 + |w|^2\prod_{j=1}^{m}\left|\tilde{\beta}_j\right|^2)/2^m \neq 0$. This concludes the proof. ■

Note that for $|z| = 1$, it holds that $|(z-\overline{w})/(1-zw)| = |(1-z\overline{w})/(z-w)| = 1$, and that

$$\frac{\mathring{\pi}_m^c(z)}{z^m\mathring{\pi}_{m*}^c(z)} = \tilde{B}_m^c(z) \quad \text{and} \quad \frac{z^m\mathring{\pi}_{m*}(z)}{\mathring{\pi}_m(z)} = \tilde{B}_{m*}(z),$$

where $B_m(z)$ is the Blaschke product, given by (3). Hence, from the previous lemma it follows that

$$\left\|\frac{\Phi_{m+1}(\cdot)}{\pi_m(\cdot)(v-\cdot)}\right\|_I \leqslant \frac{|V_{m+1}|}{2^m|v|} = \frac{2|w|}{2^m}\prod_{j=1}^{m}\left|1+\tilde{\beta}_j^2\right|.$$

We are now in a position to prove the following generalization of [15, Thm. 1].

Theorem 13. *Suppose σ is a complex measure on I. Let f be an analytic function on a neighborhood of the interval I, and suppose $J_n^\sigma(f) = \sum_{k=1}^{n}\lambda_k^\sigma f(x_k)$ is an nth RIQ for $J_\sigma(f)$ that is exact in the space $\tilde{\mathcal{L}}_m := \mathcal{L}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\}$, with $\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_m \subseteq \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$. Let $E_n^\sigma(f) := J_\sigma(f) - J_n^\sigma(f)$, and define Υ_m and F_m by*

$$\Upsilon_m = \prod_{j=1}^{m}\left\{1+\left[J^{inv}(\tilde{\alpha}_j)\right]^2\right\} \quad \textit{and} \quad F_m(x) = f(x)\prod_{j=1}^{m}(1-x/\tilde{\alpha}_j).$$

Let U be the domain of analyticity of F_m. Further, let $\rho < 1$ be such that $\mathcal{E}_\rho \subset U$, where the ellipse $\mathcal{E}_\rho$ is defined as before in (16). Then

$$|E_n^\sigma(f)| \leqslant \left(\frac{2\rho^{m+1}}{1-\rho^2}\right)|\Upsilon_m|\,\|F_m\|_{\mathcal{E}_\rho}\left(\int_{-1}^{1}|d\sigma(x)| + \sum_{k=1}^{n}|\lambda_k^\sigma|\right)G_m(\rho), \qquad (18)$$

where $\|F_m\|_{\mathcal{E}_\rho} := \max\{|F_m(x)| : x \in \mathcal{E}_\rho\}$, *and*

$$G_m(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^{m} \frac{1}{|1-\rho J^{inv}(\tilde{\alpha}_j)e^{\mathrm{i}t}|^2}dt, \tag{19}$$

with the product in the integrand equal to 1 *when* $m = 0$.

Proof. Proceeding as in [15, Sect. 3], if we choose

$$p(x) = \frac{2^m\overline{w}^m}{1-|w|^2}\frac{1}{\sqrt{v^2-1}}\prod_{j=1}^{m}\left|1-w\tilde{\beta}_j\right|^{-2}\Phi_{m+1}(x),$$

then p is a polynomial of degree $m+1$ such that $p(v) = 1$ and

$$\left\|\frac{p(\cdot)}{\pi_m(\cdot)(v-\cdot)}\right\|_I \leqslant \frac{2\,|w|^{m+1}}{\left|\sqrt{v^2-1}\right|(1-|w|^2)}\prod_{j=1}^{m}\frac{\left|1+\tilde{\beta}_j^2\right|}{\left|1-w\tilde{\beta}_j\right|^2},$$

so that

$$e_{m+1}\left(\frac{1}{\pi_m(\cdot)(v-\cdot)}\right) \leqslant \frac{2\,|w|^{m+1}}{\left|\sqrt{v^2-1}\right|(1-|w|^2)}\prod_{j=1}^{m}\frac{\left|1+\tilde{\beta}_j^2\right|}{\left|1-w\tilde{\beta}_j\right|^2}.$$

Note that we here obtain the same result as in [15], just before the end of the proof of [15, Thm. 1]. The statement now follows from the last step in that proof.

■

The main advantage of the upper bound given in the previous theorem, is that the integral (19) is always computable by the Residue Theorem; see [15, Lem. 2]. We can also obtain an error bound for the RIQs on the unit circle, considered in the previous section. In this respect, we have the following theorem, which is a generalization of [16, Thm. 1].

Theorem 14. *Suppose* $\mathring{\sigma}$ *is a complex measure on* $\mathbb{T}$. *Let* $\mathring{f}$ *be an analytic function on a neighborhood of the unit circle* $\mathbb{T}$, *and suppose* $I_n^{\mathring{\sigma}}(\mathring{f}) = \sum_{k=1}^{n}\mathring{\lambda}_k^{\mathring{\sigma}}\mathring{f}(z_k)$ *is an nth RIQ* $I_{\mathring{\sigma}}(\mathring{f})$ *that is exact in the space* $\mathring{\tilde{\mathcal{L}}}_p^c\cdot\mathring{\tilde{\mathcal{L}}}_{q*}$, *with* $\mathring{\tilde{\mathcal{L}}}_k := \mathring{\mathcal{L}}\{\tilde{\beta}_1,\ldots,\tilde{\beta}_k\}$ *and* $p, q < n$. *Let* $\mathring{E}_n^{\mathring{\sigma}}(\mathring{f}) := I_{\mathring{\sigma}}(f) - I_n^{\mathring{\sigma}}(\mathring{f})$, *and define* $\mathring{F}_{p,q}$ *by*

$$\mathring{F}_{p,q}(z) = \mathring{\pi}_p^c(z)\mathring{\pi}_{q*}(z)\mathring{f}(z), \quad \textit{where} \quad \mathring{\pi}_m(z) := \prod_{j=1}^{m}(1-\overline{\tilde{\beta}_j}z).$$

Let $\mathring{U}$ *be the domain of analyticity of* $\mathring{F}_{p,q}$. *Further, let* $r < 1 < R$ *be such that*

$\mathcal{C}_r \cup \mathcal{C}_R \subset \mathring{U}$, where the circles $\mathcal{C}_\rho$ are defined as before in (15). Then

$$\left|\mathring{E}_n(\mathring{f})\right| \leqslant \left\|\mathring{F}_{p,q}\right\|_{\mathcal{C}_r\cup\mathcal{C}_R} \left(\int_{-\pi}^{\pi} |d\mathring{\sigma}(\theta)| + \sum_{k=1}^{n} \left|\mathring{\lambda}_k^{\mathring{\sigma}}\right|\right) \times \left(\frac{r^{q+1}}{1-r^2}\sqrt{\mathring{G}_p(r)\mathring{G}_q(r)} + \frac{R^{1-p}}{R^2-1}\sqrt{\mathring{G}_p\left(\frac{1}{R}\right)\mathring{G}_q\left(\frac{1}{R}\right)}\right), \quad (20)$$

where $\left\|\mathring{F}_{p,q}\right\|_{\mathcal{C}_r\cup\mathcal{C}_R} := \max\left\{\left|\mathring{F}_{p,q}(z)\right| : z \in \mathcal{C}_r \cup \mathcal{C}_R\right\}$, *and*

$$\mathring{G}_m(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{j=1}^{m}\frac{1}{\left|1-\rho\tilde{\beta}_j e^{\mathbf{i}t}\right|^2}dt, \quad |\rho| < 1,$$

with the product in the integrand equal to 1 when $m = 0$.

Proof. The proof is exactly the same as the one of [16, Thm. 1] when replacing 'π_q' and 'ω_q' (i.e.; those with index 'q') with 'π_q^c' and 'ω_q^c' respectively. ∎

Note that for $\mathring{f}(z) = (f \circ J)(z)$, and σ and $\mathring{\sigma}$ related by (6), it holds that

$$J_\sigma(f) = \frac{1}{2}I_{\mathring{\sigma}}(\mathring{f}) \quad \text{and} \quad \int_{-1}^{1} |d\sigma(x)| = \frac{1}{2}\int_{-\pi}^{\pi} |d\mathring{\sigma}(\theta)|,$$

while it follows from Theorem 2 that $J_{n(s)}^{\sigma}(f) = \frac{1}{2}\mathring{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = \frac{1}{2}\tilde{\mathring{I}}_{N(s)}^{\mathring{\sigma}}(\mathring{f})$, so that

$$E_{n(s)}^{\sigma}(f) = \frac{1}{2}\mathring{E}_{N(s)}^{\mathring{\sigma}}(\mathring{f}) = \frac{1}{2}\tilde{\mathring{E}}_{N(s)}^{\mathring{\sigma}}(\mathring{f}). \quad (21)$$

Furthermore, whenever either $s \in \{1, 2\}$, or $s = 0$ and condition (11) is satisfied, it also holds that

$$\sum_{k=1}^{n(s)} |\lambda_k^{\sigma}| = \frac{1}{2}\sum_{k=1}^{N(s)}\left|\mathring{\lambda}_k^{\mathring{\sigma}}\right| = \frac{1}{2}\sum_{k=1}^{N(s)}\left|\tilde{\mathring{\lambda}}_k^{\mathring{\sigma}}\right|,$$

while we can deduce from Theorems 2–3 that the quadrature rules $\mathring{I}_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ and $\tilde{\mathring{I}}_{N(s)}^{\mathring{\sigma}}(\mathring{f})$ have the same domain of validity; they are exact for every $\mathring{f} \in \mathring{\mathcal{S}}_{2(n-s \bmod 2)}$ (note that $2(n - s \bmod 2) = N(s) - 1$ for $s = 1$, but that $2(n - s \bmod 2) = N(s)$ for $s \in \{0, 2\}$). As a result, the error bound (18) could also be obtained with the aid of (20) by setting $p = q = m$ and $r = 1/R = \rho$, so that

$$\left(\frac{r^{q+1}}{1-r^2}\sqrt{\mathring{G}_p(r)\mathring{G}_q(r)} + \frac{R^{1-p}}{R^2-1}\sqrt{\mathring{G}_p\left(\frac{1}{R}\right)\mathring{G}_q\left(\frac{1}{R}\right)}\right) = \frac{2\rho^{m+1}}{1-\rho^2}\mathring{G}_m(\rho),$$

and noticing that for $\tilde{\beta}_j = J^{inv}(\tilde{\alpha}_j)$, it holds that $\mathring{G}_m(\rho) = G_m(\rho)$ and $\left\|\mathring{F}_{m,m}\right\|_{\mathcal{C}_\rho \cup \mathcal{C}_{\frac{1}{\rho}}} = |\Upsilon_m| \, \|F_m\|_{\mathcal{E}_\rho}$. If, however, condition (11) is not satisfied for $s = 0$, we need to consider the auxiliary quadrature rule

$$\hat{I}^{\mathring{\sigma}}_{N(0)}(\mathring{f}) = \frac{1}{2}\left[I^{\mathring{\sigma}}_{N(0)}(\mathring{f}) + \tilde{I}^{\mathring{\sigma}}_{N(0)}(\mathring{f})\right],$$

with weights

$$\hat{\mathring{\lambda}}^{\mathring{\sigma}}_k = \frac{\mathring{\lambda}^{\mathring{\sigma}}_k + \tilde{\mathring{\lambda}}^{\mathring{\sigma}}_k}{2} = \frac{\mathring{\lambda}^{\mathring{\sigma}}_k + \mathring{\lambda}^{\mathring{\sigma}}_{n+k}}{2} = \frac{\tilde{\mathring{\lambda}}^{\mathring{\sigma}}_k + \tilde{\mathring{\lambda}}^{\mathring{\sigma}}_{n+k}}{2} = \frac{\mathring{\lambda}^{\mathring{\sigma}}_{n+k} + \tilde{\mathring{\lambda}}^{\mathring{\sigma}}_{n+k}}{2} = \hat{\mathring{\lambda}}^{\mathring{\sigma}}_{n+k}, \quad k = 1, \ldots, n,$$

where the second and fourth equality follows from the fact that the measure $\mathring{\sigma}$ is symmetric. This way, we deduce from Theorem 2 that

$$J^{\sigma}_{n(0)}(f) = \frac{1}{2}\hat{I}^{\mathring{\sigma}}_{N(0)}(\mathring{f}), \quad \sum_{k=1}^{n(0)} |\lambda^{\sigma}_k| = \frac{1}{2}\sum_{k=1}^{N(0)} \left|\hat{\mathring{\lambda}}^{\mathring{\sigma}}_k\right|, \quad \text{and} \quad E^{\sigma}_{n(s)}(f) = \frac{1}{2}\hat{\mathring{E}}^{\mathring{\sigma}}_{N(s)}(\mathring{f}).$$

Taking into account that the auxiliary quadrature rule $\hat{I}^{\mathring{\sigma}}_{N(0)}(\mathring{f})$ has domain of validity $\mathring{\mathcal{S}}_{2n-2}$, the error bound (18) can again be obtained with the aid of (20) by setting $m = p = q$, $\rho = r = 1/R$, and $\tilde{\beta}_j = J^{inv}(\tilde{\alpha}_j)$.

To conclude this section, we will give an estimate for the rate of convergence of the sequence of RIQs considered in the previous section. Let us first consider the case of the unit circle. For this, we need to know something about the distribution of the complex numbers $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$. Let $\mathring{\nu}^{\beta}_n$ be the normalized counting measure which assigns a point mass to β_j, taking the multiplicity of β_j into account. We then have the following generalization of [7, Thm. 5.2].

Theorem 15. *Suppose $\mathring{\mu}$ is a positive measure on $\mathbb{T}$ that satisfies the Szegő condition $\int_{-\pi}^{\pi} \log \mathring{\mu}'(\theta) d\theta > -\infty$, where $\mathring{\mu}'$ is the Radon-Nikodym derivative of the measure $\mathring{\mu}$ with respect to the Lebesgue measure on $\mathbb{T}$, and let $\mathring{\sigma}$ be a complex measure on $\mathbb{T}$, such that $\mathring{\sigma} \ll \mathring{\mu}$ and*

$$\int_{-\pi}^{\pi} |\mathring{g}(\theta)|^2 \, d\mathring{\mu}(\theta) < \infty, \quad \mathring{g}(\theta) = \frac{d\mathring{\sigma}(\theta)}{d\mathring{\mu}(\theta)}.$$

Let the sequence $\mathcal{B}$ be contained in a compact subset of $\mathbb{D}$, and assume that $\mathring{\nu}^{\beta}_n$ converges to some measure $\mathring{\nu}^{\beta}$ in weak star topology. Consider the RIQs $I^{\mathring{\sigma}}_n(\mathring{f})$, $n = 1, 2, \ldots$, based on the sets of n $\mathring{\mu}$-nodes, such that $I^{\mathring{\sigma}}_n(\mathring{f}) = I_{\mathring{\sigma}}(\mathring{f})$ for all $\mathring{f} \in \mathring{\mathcal{L}}^c_{p(n-1)} \cdot \mathring{\mathcal{L}}_{q(n-1)*}$, with $p(n-1) + q(n-1) = n-1$, and $\lim_{n\to\infty} q(n)/n = r \in (0, 1)$. Suppose that $\mathring{f}$ is analytic in a closed and connected region G for which $\mathbb{T} \subset G$, $G \cap (\mathcal{B} \cup \mathcal{B}^c_* \cup \{0, \infty\}) = \emptyset$, where $\mathcal{B}^c_* = \{1/\beta_1, 1/\beta_2, \ldots\}$, and such that the boundary ∂G is a finite union of Jordan curves. Then it holds *that*

$$\limsup_{n\to\infty} \left|\mathring{E}_n^{\mathring{\sigma}}(\mathring{f})\right|^{1/n} \leqslant \gamma < 1,$$

where

$$\gamma = \max\left\{ \max_{z\in\partial G\cap\mathbb{D}} \{\exp[r\lambda(z)]\}, \max_{z\in\partial G\cap\mathbb{E}} \{\exp[(1-r)\lambda(1/z)]\} \right\},$$

with $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ *and*

$$\lambda(z) = \int \log|\zeta_z(u)|\, d\mathring{\nu}^\beta(u), \quad \zeta_z(u) = \frac{u-z}{1-\overline{z}u}.$$

Proof. The proof is exactly the same as the one of [7, Thm. 5.2] when replacing '$\pi_{q(n)}$' and '$\omega_{q(n)}$' (i.e.; those with index '$q(n)$') in the proof of [7, Thm. 5.1] with '$\pi^c_{q(n)}$' and '$\omega^c_{q(n)}$' respectively. This way it holds for the para-orthogonal rational function '$\tilde{\chi}_n(z)$' in [7, Eq. (5.8)] that (see also [7, Thm. 2.4])

$$\limsup_{n\to\infty} |\tilde{\chi}_n(z)|^{1/n} = \exp\{r\lambda(z) + (1-r)\lambda(\overline{z})\}, \quad \forall z \in \mathbb{E},$$

where it is easily verified that $\lambda(\overline{z}) = -\lambda(1/z)$. ∎

Finally, we can prove the following estimate for the rate of convergence for the case of the interval.

Theorem 16. *Suppose* μ *is a positive measure on* I *that satisfies the Szegő condition* $\int_{-1}^{1} \frac{\log\mu'(x)}{\sqrt{1-x^2}}dx > -\infty$, *where* μ' *is the Radon-Nikodym derivative of the measure* μ *with respect to the Lebesgue measure on* I, *and let* $\mathring{\sigma}$ *be a complex measure on* I, *such that* $\sigma \ll \mu$ *and* $\|g\|_{\mu,2} < \infty$, *where* g *is defined as before in* (13). *Let the sequence* $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\}$ *be bounded away from* I, *and assume that the corresponding normalized counting measure converges in weak star topology. Consider the RIQs* $J^\sigma_{n(s)}(f)$, $n = 1, 2, \ldots$, *based on the sets of* $n(s)$ (μ, s)*-nodes, such that* $J^\sigma_{n(s)}(f) = J_\sigma(f)$ *for all* $f \in \mathcal{L}_{n(s)-1}$. *Suppose that* f *is analytic in a closed connected region* H *for which* $I \subset H$, $H \cap (\mathcal{A} \cup \{\infty\}) = \emptyset$, *and such that the boundary* ∂H *is a finite union of Jordan curves. Then it holds that*

$$\limsup_{n(s)\to\infty} \left|E^\sigma_{n(s)}(f)\right|^{1/n(s)} \leqslant \kappa < 1,$$

where

$$\kappa = \max_{x\in\partial H} \left\{\exp[\lambda(J^{inv}(x))]\right\},$$

and $\lambda(z)$, *with* $x = J(z)$ *and* $\beta_k = J^{inv}(\alpha_k)$ *for every* $k > 0$, *is defined as above in Theorem 15.*

Proof. From (21) it follows that

$$\limsup_{n(s)\to\infty}\left|E^{\sigma}_{n(s)}(f)\right|^{1/n(s)} = \limsup_{n(s)\to\infty}\left(\frac{1}{2}\right)^{1/n(s)}\left|\mathring{E}^{\mathring{\sigma}}_{N(s)}(\mathring{f})\right|^{1/n(s)}$$
$$= \limsup_{n(s)\to\infty}\left\{\left|\mathring{E}^{\mathring{\sigma}}_{N(s)}(\mathring{f})\right|^{1/N(s)}\right\}^{N(s)/n(s)} \leqslant \gamma^2 < 1.$$

So, set $\mathring{H} := \{z \in \mathbb{C} : J(z) \in H\}$ and let $\partial\mathring{H}$ denote the boundary of $\mathring{H}$. Then it follows from Theorem 15, with $r = \frac{1}{2}$ and $\partial G = \partial\mathring{H}$, that

$$\gamma^2 = \max\left\{\max_{z\in\partial\mathring{H}\cap\mathbb{D}}\{\exp[\lambda(z)]\},\ \max_{z\in\partial\mathring{H}\cap\mathbb{E}}\{\exp[\lambda(1/z)]\}\right\} = \max_{z\in\partial\mathring{H}\cap\mathbb{D}}\{\exp[\lambda(z)]\},$$

where the last equality is because $z \in \partial\mathring{H}\cap\mathbb{E}$ iff $1/z \in \partial\mathring{H}\cap\mathbb{D}$. This concludes the proof. ■

5 Numerical examples

In this section we will illustrate the effectiveness of the quadrature formulas introduced in Section 3. For this, we consider the Chebyshev weight function of the first kind

$$d\mu(x) = \frac{dx}{\sqrt{1-x^2}}, \quad x \in I,$$

which satisfies the Szegő condition $\int_{-1}^{1}\frac{\log\mu'(x)}{\sqrt{1-x^2}}dx > -\infty$, and for which the corresponding measure on the unit circle is the Lebesgue measure $d\mathring{\mu}(\theta) = d\theta$. Explicit expressions for the orthonormal rational functions φ_k with respect to this measure and inner product (2) are given in [14, Thm. 3.2], while expressions to compute the (μ, s)-nodes and (μ, s)-weights in the corresponding Gauss-type quadrature formulas can be found in [11, Sect. 4]. Further, we take

$$d\sigma(x) = \frac{dx}{(1-x^2)^{\mathbf{i}/4}}, \quad x \in I, \tag{22}$$

for which $\int_{-1}^{1}|d\sigma(x)| = 2$ and $\left\|\frac{d\sigma}{d\mu}\right\|_{\mu,2} = \sqrt{\frac{\pi}{2}}$. The corresponding measure on the unit circle is then given by

$$d\mathring{\sigma}(\theta) = |\sin\theta|^{1-\mathbf{i}/2}\,d\theta = \frac{\left|z^2-1\right|^{1-\mathbf{i}/2}}{2^{1-\mathbf{i}/2}}d\theta, \quad z = e^{\mathbf{i}\theta}.$$

Given a function $f_i(x)$ on I, we approximate the integral $J_\sigma(f_i)$ by means of an $n(s)$-point RIQ $J^{\sigma}_{n(s)}(f_i)$, based on the set of (μ, s)-nodes $\mathbf{x}^{[s]}_{n(s)} \subset I$, where we replaced $\alpha_n \notin \overline{\mathbb{R}}_I$ with $\tilde{\alpha}_n = \infty$ for $s \in \{0, 2\}$, and with the weights $\{\lambda^{\sigma}_k\}_{k=1}^{n(s)}$ as defined in Lemma 8. In the examples that follow, the computations were done with the aid of MAPLE®10, with 30 digits. Since the calculation of the projection coefficients $J_\sigma(\varphi_k)$ could take a lot of time (especially for higher degrees and/or different poles[4]), we considered the auxiliary functions (see also [19, Sect. 3])

$$f^{(\alpha_k)}(x) = \left(\frac{1-\alpha_k x}{x-\alpha_k}\right)^{m_k}, \quad m_k = \#\alpha_k \text{ in } \{\alpha_1,\ldots,\alpha_k\},$$

to speed-up the computations. The integrals $J_\sigma(\varphi_k)$ were then computed by solving the lower-triangular system of equations

$$\begin{aligned} J_\sigma(f^{(\alpha_k)}) &= \sum_{j=0}^{k} J_\mu(f^{(\alpha_k)}\varphi_j^c)\cdot J_\sigma(\varphi_j) \\ \Leftrightarrow J_\sigma(f^{(\alpha_k)}) - \varphi_0^2 J^\mu_{n(s)}(f^{(\alpha_k)})\cdot J_\sigma(1) &= \sum_{j=1}^{k} J^\mu_{n(s)}(f^{(\alpha_k)}\varphi_j^c)\cdot J_\sigma(\varphi_j), \\ & \qquad k=1,\ldots,n-1, \end{aligned}$$

where $J^\mu_{n(s)}(\cdot)$ is the $n(s)$-point rational Gauss-type quadrature formula. In the case in which $s=2$, we also needed to compute the constant $a_{n,2}$. For this, we have that

$$\begin{aligned} J_\mu(f^{(\tilde\alpha_n)}\varphi_n^c)\cdot J_\sigma(\varphi_n) &= J_\sigma(f^{(\tilde\alpha_n)}) - \sum_{j=0}^{n-1} J_\mu(f^{(\tilde\alpha_n)}\varphi_j^c)\cdot J_\sigma(\varphi_j) \\ &= J_\sigma(f^{(\tilde\alpha_n)}) - \sum_{j=0}^{n-1} J^\mu_{n(2)}(f^{(\tilde\alpha_n)}\varphi_j^c)\cdot J_\sigma(\varphi_j), \end{aligned}$$

where it holds for the left hand side that

$$\begin{aligned} J_\mu(f^{(\tilde\alpha_n)}\varphi_n^c)\cdot J_\sigma(\varphi_n) &= \frac{J_\mu(f^{(\tilde\alpha_n)}\varphi_n^c)}{J^\mu_{n(2)}(f^{(\tilde\alpha_n)}\varphi_n^c)}\cdot J_\sigma(\varphi_n)\cdot J^\mu_{n(2)}(f^{(\tilde\alpha_n)}\varphi_n^c) \\ &= a_{n,2}\cdot J^\mu_{n(2)}(f^{(\tilde\alpha_n)}\varphi_n^c). \end{aligned}$$

Example 17. The first function $f_{n(s),1}(x)$ to be considered is given by

$$f_{n(s),1}(x) = \frac{x^{n(s)-1}}{(x-\omega)^{n-1}}, \quad \omega\in\mathbb{C}_I, \quad n>1, \tag{23}$$

which has poles of order $n-1$ in ω, and one pole at infinity for the case in which $s=2$. So let $\alpha_k=\omega$, $k=1,2,\ldots$, with $\omega=\frac{3+\mathbf{i}}{4}$. Table 1 then gives the relative error

$$r_{n(s),1} := \left|\frac{J_\sigma(f_{n(s),1}) - J^\sigma_{n(s)}(f_{n(s),1})}{J_\sigma(f_{n(s),1})}\right| \tag{24}$$

[4] For certain degrees or choices of poles, MAPLE®10 even completely failed to compute the integral $J_\sigma(\varphi_k)$.

Table 1
The relative error, given by (24), in the RIQs for the estimation of $J_\sigma(f_{n(s),1})$, where σ is given by (22) and $f_{n(s),1}$ is given by (23).

n	$r_{n(0),1}$	$r_{n(1),1}(+1)$	$r_{n(1),1}(-1)$	$r_{n(2),1}$
2	$9.3780e-30$	$7.2139e-30$	$7.3567e-30$	$8.1616e-30$
3	$1.5784e-29$	$1.4216e-29$	$1.6002e-29$	$4.1922e-30$
4	$1.9734e-29$	$9.3931e-30$	$1.5795e-29$	$2.0998e-29$
5	$2.6967e-29$	$1.9363e-29$	$3.4059e-29$	$2.6656e-29$
6	$2.1534e-29$	$3.5910e-29$	$5.3360e-29$	$2.1231e-29$
7	$5.7863e-29$	$1.5959e-29$	$1.5255e-29$	$3.8053e-29$

for several values of n. The relative errors in Table 1 clearly show that the integrals are approximated exactly by the RIQs.

Example 18. The second function $f_2(x)$ to be considered is given by

$$f_2(x) = \sin\left(\frac{1}{x^2+\omega^2}\right), \quad \omega \in \mathbb{R}_0. \tag{25}$$

This function has an essential singularity in $x = \mathbf{i}\omega$ and $x = -\mathbf{i}\omega$. For $\omega > 0$ but very close to 0, this function is extremely oscillatory near these singularities. Since an essential singularity can be viewed as a pole of infinite multiplicity, this suggests taking $\alpha_k = (-1)^{k+1}\mathbf{i}\omega$, $k = 1, 2, \ldots$. So, let $\omega = \frac{3}{4}$. Table 2 then gives the relative error

$$r_{n(s),2} := \left|\frac{J_\sigma(f_2) - J^\sigma_{n(s)}(f_2)}{J_\sigma(f_2)}\right| \tag{26}$$

for several values of n. With $\beta = J^{inv}(3\mathbf{i}/4) = -\mathbf{i}/2$ and $\mathring{\nu}^\beta = \frac{1}{2}(\delta_\beta + \delta_{\overline{\beta}})$ (where δ_z is the unit measure whose support is the point z) we obtain from Theorem 16 the following estimation for the rate of convergence:

$$\kappa = \max_{z\in\partial\mathring{H}\cap\mathbb{D}} \left\{\sqrt{\left|\frac{z^2-\beta^2}{1-\overline{z}^2\beta^2}\right|}\right\} > \exp[\lambda(0)] = 0.5\ .$$

Figure 1 graphically shows the actual rate of convergence

$$\kappa_{n(s)} := \left|J_\sigma(f_2) - J^\sigma_{n(s)}(f_2)\right|^{1/n(s)} \tag{27}$$

as a function of the number of interpolation points $n(s)$ for the case in which $s = 0$. The graph suggests that $\lim_{n(0)\to\infty} \kappa_{n(0)} \approx 0.1$, which is indeed less than or equal to 0.5. However, this also suggests that for a desired accuracy that is sufficiently small, the accuracy will be reached approximately three times faster than indicated by the estimated rate of convergence.

Table 2
The relative error, given by (26), in the RIQs for the estimation of $J_\sigma(f_2)$, where σ is given by (22) and f_2 is given by (25).

n	$r_{n(0),2}$	$r_{n(1),2}(+1)$	$r_{n(1),2}(-1)$	$r_{n(2),2}$
3	$4.1088e-2$	$1.1244e-2$	$1.1244e-2$	$1.9547e-2$
5	$8.5077e-4$	$2.3263e-4$	$2.3263e-4$	$4.1060e-4$
9	$9.1766e-7$	$3.3781e-7$	$3.3781e-7$	$2.5981e-7$
17	$5.0893e-13$	$2.2097e-13$	$2.2097e-13$	$6.9810e-14$
33	$5.7247e-27$	$2.6760e-27$	$2.6780e-27$	$3.8649e-28$

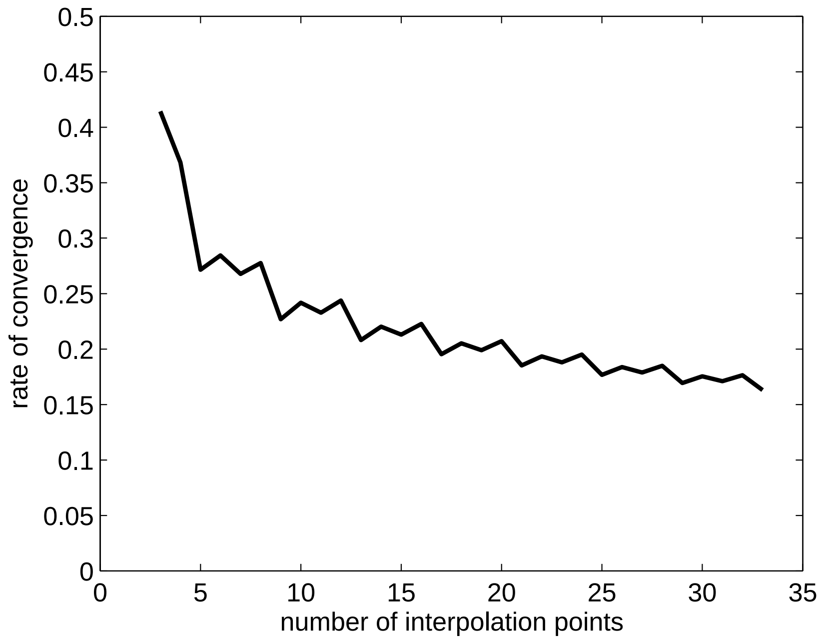

Fig. 1. Rate of convergence $\kappa_{n(0)}$, given by (27), in the RIQ $J^\sigma_{n(0)}(f_2)$ for the estimation of $J_\sigma(f_2)$, where σ is given by (22) and f_2 is given by (25).

Example 19. The last function $f_3(x)$ to be considered is given by

$$f_3(x) = \frac{\pi x/\omega}{\sinh(\pi x/\omega)}, \quad \omega \in \mathbb{R}_0, \tag{28}$$

which has simple poles at the integer multiples of $\mathbf{i}\omega$; thus, let

$$\alpha_k = (-1)^k \lceil k/2 \rceil \mathbf{i}\omega, \quad k = 1, 2, \ldots . \tag{29}$$

Note that (see e.g. [1, p. 85])

$$\left| \frac{\sinh(\pi x/\omega)}{\pi x/\omega} \right| = \prod_{j=1}^{\infty} \left| 1 + \frac{x^2}{(j\omega)^2} \right|,$$

Table 3
The absolute error $a_{n(s),3}$, $s \in \{0,2\}$, and upper bound $a_{n(s),3}^{[u]}$, given by (30), in the RIQs for the estimation of $J_\sigma(f_3)$, where σ is given by (22) and f_3 is given by (28).

n	$a_{n(0),3}$	$a_{n(0),3}^{[u]}$	$a_{n(2),3}$	$a_{n(2),3}^{[u]}$
2	$2.6931e-1$	$1.0237e+1$	$3.2052e-2$	$5.8884e+0$
3	$1.0475e-2$	$2.8423e-1$	$4.0725e-3$	$2.8559e-1$
4	$3.5376e-3$	$4.5227e-1$	$8.1381e-5$	$1.4787e-1$
5	$7.7270e-5$	$5.0176e-3$	$3.8165e-5$	$5.0353e-3$
6	$2.0835e-5$	$8.4449e-3$	$3.9569e-7$	$1.8290e-3$
7	$2.5942e-7$	$4.5704e-5$	$9.0437e-8$	$4.5791e-5$

so that

$$\left\|F_{n(s)-1}\right\|_{\mathcal{E}_{\rho_{n(s)}}} = \|F_{n-1}\|_{\mathcal{E}_{\rho_{n(s)}}} = \left|F_{n-1}\left(J((-1)^{n-1}\mathbf{i}\rho_{n(s)})\right)\right|, \quad \rho_{n(s)} \in (|\beta_n|, 1),$$

where $|\beta_n| = \sqrt{(\lceil n/2\rceil\omega)^2+1} - \lceil n/2\rceil\omega$. So, let $\omega = 1$. Tables 3–4 then give the absolute error $a_{n(s),3} := \left|J_\sigma(f_3) - J_{n(s)}^\sigma(f_3)\right|$ for several values of n, together with the upper bound

$$a_{n(s),3}^{[u]} := \min_{\rho_{n(s)}\in(|\beta_n|,1)} \left\{ \left(\frac{2\rho_{n(s)}^{m_{n,s}}}{1-\rho_{n(s)}^2}\right) |\Upsilon_{n-1}| \cdot \left|F_{n-1}\left(J((-1)^{n-1}\mathbf{i}\rho_{n(s)})\right)\right| \times \left(2 + \sum_{k=1}^{n(s)} |\lambda_k^\sigma|\right) G_{n-1}(\rho_{n(s)}) \right\}, \quad (30)$$

with

$$m_{n,s} = \begin{cases} 2\lceil n/2\rceil, & s = 0 \\ n, & s = 1 \\ n+1, & s = 2, \end{cases}$$

where the expression for the case in which $s = 0$ follows from the fact that $\varphi_n(-x) \equiv (-1)^n\varphi_n(x)$ for the given sequence of poles (29) and for $\tilde{\alpha}_n = \infty$, so that $J_\sigma(\varphi_n) = 0$ whenever n is odd (see also Remark 5). Since $|J_\sigma(f_3)| \approx 1.3272$, the relative errors are of the same order.

6 Conclusion

We presented a connection between rational interpolatory quadrature formulas (RIQs) for complex bounded measures σ on the interval and certain RIQs for

Table 4
The absolute error $a_{n(1),3}$ and upper bound $a^{[u]}_{n(1),3}$, given by (30), in the RIQs for the estimation of $J_\sigma(f_3)$, where σ is given by (22) and f_3 is given by (28).

n	$a_{n(1),3}(+1)$	$a^{[u]}_{n(1),3}(+1)$	$a_{n(1),3}(-1)$	$a^{[u]}_{n(1),3}(-1)$
2	$2.2220e-1$	$1.0210e+1$	$2.2071e-1$	$1.0245e+1$
3	$3.5521e-3$	$8.4133e-1$	$3.5521e-3$	$8.4133e-1$
4	$3.5242e-3$	$4.5203e-1$	$3.5207e-3$	$4.5314e-1$
5	$2.1264e-5$	$2.2622e-2$	$2.1264e-5$	$2.2622e-2$
6	$2.0992e-5$	$8.4495e-3$	$2.0986e-5$	$8.4561e-3$
7	$8.7345e-8$	$2.7964e-4$	$8.7345e-8$	$2.7964e-4$

complex bounded measures $\mathring{\sigma}$ on the unit circle. Conditions have been given to ensure the convergence of these RIQs for the case of the interval (conditions for the case of the unit circle are easily obtained in a similar way), and an upper bound for the error on the nth approximation and an estimate for the rate of convergence have been provided for the case of the interval as well as for the case of the unit circle. We concluded with some numerical experiments.

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