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A Wave Based Method for the axisymmetric dynamic analysis of acoustic and poroelastic problems

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Abstract

Element based approaches are most commonly applied to predict the harmonic response of (coupled) acoustic and poroelastic problems. A major disadvantage of these techniques is the practical frequency limitation in that the computational efforts become prohibitively large at higher frequencies. A recently developed Wave Based Method is an efficient alternative deterministic prediction technique which aims at relaxing this frequency limitation by using exact solutions of the governing equations to approximate the field variables. This paper extends the applicability range of this method towards axisymmetric problems. Novel wave function sets, complying with the governing axisymmetric equations, are derived and applied to both coupled and uncoupled acoustic and poroelastic problems. The beneficial efficiency of the method as compared to standard finite element descriptions is illustrated by means of four numerical validation studies.

Keywords: acoustics, poroelastic Biot theory, axisymmetry, Wave Based Method, Trefftz approach

1. Introduction

Given the quest for mass reduction while preserving proper vibration and acoustic comfort levels in industrial machinery and vehicles, lightweight poroelastic materials have gained a lot of importance. However, also in other areas, as for instance for biomedical and geographical applications, efficient and accurate models of the wave propagation in porous materials have become indispensable. A poroelastic material is composed of two constituents, being the elastic solid constituent, also called the frame, and the fluid filling the voids. The most widely used linear theory for wave propagation in poroelastic materials is the Biot theory [1], as adapted and presented in the reference book by Allard and Atalla [2]. This theory predicts the existence of three different types of coupled propagating waves: one shear wave and two longitudinal waves. To account for the effect of these materials in a vibro-acoustic setting, the Finite Element Method (FEM) is the most commonly used prediction technique. Even though several formulations can be found in literature, for instance the $(\mathbf{u}^s, \mathbf{u}^f)$ -formulation [3, 4, 5], the $(\mathbf{u}^s, p)$ -formulation [6, 7, 8] and Dazel's formulation [9], the calculations are generally time consuming due the high number of unknowns per node, the mesh

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requirements to obtain an accurate solution, and due to the complex and frequency dependent material properties that complicates efficient modal reduction schemes.

In the case where there are symmetries present in the problem setting, a large amount of computational effort can be saved by explicitly enforcing the symmetry conditions. This is especially the case when an axisymmetric problem is considered. When the geometry, the boundary and loading conditions, and the material properties are independent of the circumferential angle θ , the solution of the problem itself is also independent of θ . It is then sufficient to express the problem in cylindrical coordinates and to consider just a section in (r,z) -coordinates, instead of using a full 3D model.

For bounded acoustic Helmholtz problems as such, axisymmetric real life applications are not so common, although for instance acoustic transmission systems can be studied [10]. Axisymmetric poroelastic models are for instance used within (bio)-medical applications [11, 12]. Östberg et al. [13] have proposed a weak formulation of Biot's equations with a harmonic expansion in the circumferential direction to efficiently model cylindrical poroelastic models. By only considering zero-order circumferential harmonic functions, axisymmetric solutions can be obtained. When combining axisymmetric acoustic and poroelastic domains, a range of applications comes within reach. Axisymmetric models also allow to predict the sound propagation in circular ducts which are lined with porous materials [14]. A Kundt's tube device can easily be modelled and for instance the influence of boundary conditions on acoustic absorption [15] and on transmission loss [16] can readily be studied. Acoustic measurements within a Kundt tube and numerical models can be combined for the inverse acoustic characterisation of poroelastic parameters [17].

The Wave Based Method (WBM) [18] is a deterministic method which is based on an indirect Trefftz approach [19]. Instead of dividing the problem domain into a large number of small elements, the WBM partitions the problem domain into a small number of large, convex subdomains. In each subdomain, the dynamic response variables are approximated using wave functions which exactly fulfill the governing partial differential equations. In this way, the wave functions may only violate the boundary and interface continuity conditions. Minimising the errors on the boundaries and interfaces by applying a weighted residual formulation yields a small linear system of equations which can be solved for the contribution factor of each wave function. The WB modelling approach has been successfully applied to uncoupled interior and exterior acoustic [20, 21, 22, 23], uncoupled elastic [24, 25] and fully coupled vibro-acoustic problems [26, 27, 28, 29]. A multilevel approach has been developed to take into account scatterers and inclusions in an efficient way [30, 31]. In recent work [32, 33], the WBM has been developed for the solution of the Biot equations. The dynamic field variables are approximated using a superposition of three types of globally defined wave functions, corresponding to the three aforementioned poroelastic wave types. The method has shown to be very efficient, in particular for problems of moderate geometrical complexity.

This paper extends the application range of the WBM towards axisymmetric problem settings and focuses on bounded acoustic and poroelastic problems. Very efficient calculations can be obtained when explicitly exploiting the existing symmetries in the problem setting. Novel wave function sets are derived, fulfilling the axisymmetric form of the governing equations. The errors on the boundary and interface conditions are minimised by the application of a

Galerkin weighted residual formulation. This results in small system matrices and enables an efficient solution. As shown by four numerical validation cases, excellent accuracies and high convergence rates are obtained as compared to FE models. The validation cases cover two uncoupled acoustic, an uncoupled poroelastic problem and finally a coupled acoustic-poroelastic Kundt's tube setting. In literature, Trefftz approaches have been recently applied for axisymmetric incompressible porous media [34]. As compared to this work, the present paper considers compressible poroelastic materials as governed by the Biot equations and different approximation functions and domain decompositions are applied.

This paper is organised as follows. Section 2 discusses the mathematical formulation of a general coupled axisymmetric poroelastic-acoustic problem and explains the boundary conditions as used in the numerical validation cases. Section 3 describes the general WB modelling approach and extends the concepts for axisymmetric problems. The wave function selection is, in a first step, elaborated for an uncoupled acoustic problem domain. In a next step, the wave function sets for poroelastic problem domains are derived. By applying a Galerkin weighted residual approach, a matrix system is obtained which can be solved for the unknown contribution factor of each wave function. Section 4 demonstrates the efficiency of the proposed method as compared to the FEM. Four different cases of increasing complexity are considered. The paper concludes with some remarks and possible topics for future research.

2. Problem definition

This section briefly describes the mathematical problem setting for an axisymmetric coupled acoustic-poroelastic problem, as shown in figure 1. When the geometry, boundary and loading conditions and material properties of the considered problem are independent of the circumferential angle, θ , the solution of the problem is also independent of θ . Consequently, the dynamic fields can be studied in an axisymmetric section with (r, z) coordinates. A time harmonic dependence $e^{j\omega t}$ is assumed, with j is the imaginary unit, ω the circular frequency and t the time. The total problem domain Ω can be divided into two non-overlapping subdomains Ω_a and Ω_{pe} containing an acoustic and a poroelastic medium, respectively. This section briefly reviews the governing equations for both subdomains and gives the boundary conditions applied within this paper as well as the interface coupling conditions between the acoustic and the poroelastic domain.

Figure 1: Axisymmetric problem definition.

2.1. Steady-state acoustic problem

Under the assumption of an inviscid fluid with ambient fluid density ρ_0 and speed of sound c_0 , a linear system behaviour and an adiabatic process and in the absence of sound sources, the steady-state acoustic pressure $p(\mathbf{r})$ inside the problem domain is governed by the homogeneous Helmholtz equation:

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = 0. \quad (1)$$

In this equation, considering axisymmetric coordinates, $\nabla^2 \bullet = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bullet}{\partial r} \right) + \frac{\partial^2 \bullet}{\partial z^2}$ is the Laplacian operator and $k = \omega/c_0$ is the acoustic wave number. Since the Helmholtz equation is a second-order differential equation, one boundary condition needs to be specified at each point of the boundary in order to obtain a well-posed problem. Acoustic boundary conditions and coupling conditions to a poroelastic domain are given in Appendix A.

2.2. Steady-state poroelastic problem

The most commonly applied theory to predict the acoustic behaviour of poroelastic materials is based on the Biot theory [1, 2]. The dynamic equations that describe the coupled motion of both the solid and the fluid phases of a poroelastic material are written as:

$$\nabla \cdot \boldsymbol{\sigma}^s(\mathbf{r}) = -\omega^2 \rho_1 \mathbf{u}^s(\mathbf{r}) - (\omega^2 \rho_a - j\omega b)(\mathbf{u}^s(\mathbf{r}) - \mathbf{u}^f(\mathbf{r})), \quad (2)$$

$$\nabla \cdot \boldsymbol{\sigma}^f(\mathbf{r}) = -\omega^2 \rho_2 \mathbf{u}^f(\mathbf{r}) - (\omega^2 \rho_a - j\omega b)(\mathbf{u}^f(\mathbf{r}) - \mathbf{u}^s(\mathbf{r})). \quad (3)$$

The stress-strain relations are:

$$\boldsymbol{\sigma}^s(\mathbf{r}) = [Ae^s(\mathbf{r}) + Qe^f(\mathbf{r})]\mathbf{I} + 2Ne^s(\mathbf{r}), \quad (4)$$

$$\boldsymbol{\sigma}^f(\mathbf{r}) = [Qe^s(\mathbf{r}) + Re^f(\mathbf{r})]\mathbf{I}. \quad (5)$$

The substitution of the constitutive relations (4)-(5) into the dynamic equations (2)-(3) leads to the well-known Biot equations:

$$N\nabla^2 \mathbf{u}^s(\mathbf{r}) + \nabla[(A + N)e^s(\mathbf{r}) + Qe^f(\mathbf{r})] = -\omega^2(\tilde{\rho}_{11}\mathbf{u}^s(\mathbf{r}) + \tilde{\rho}_{12}\mathbf{u}^f(\mathbf{r})), \quad (6)$$

$$\nabla[Qe^s(\mathbf{r}) + Re^f(\mathbf{r})] = -\omega^2(\tilde{\rho}_{12}\mathbf{u}^s(\mathbf{r}) + \tilde{\rho}_{22}\mathbf{u}^f(\mathbf{r})). \quad (7)$$

In these equations, field variables related with the solid and the fluid phase are indicated by superscripts $\bullet^s$ and $\bullet^f$, respectively. $\mathbf{u}^\bullet(\mathbf{r})$ denotes the displacement vector, $\mathbf{e}^\bullet(\mathbf{r})$ is the strain vector, $e^\bullet(\mathbf{r})$ the longitudinal strain, $\boldsymbol{\sigma}^\bullet$ the stress tensor of phase $\bullet$. The parameter $A = \lambda + Q^2/R$ is the first Lamé coefficient at zero fluid dilatation, λ is the first Lamé coefficient at zero acoustic pore pressure, N is the second Lamé coefficient, Q is a dilatational coupling factor describing the fluid stress dependence of the frame dilatation and conversely frame stress dependence of fluid dilatation and R can be interpreted as the fluid phase bulk stiffness at zero frame dilatation. The bulk densities of the solid and the fluid phase are given by ρ_1 and $\rho_2 = h\rho_0$, respectively, and $\rho_a = h\rho_0(\alpha_\infty - 1)$ is the inertial coupling

term between both phases, with h the porosity of the poroelastic material, defined as the ratio of the volume of fluid phase and the total volume of porous material, ρ_0 the density of the interstitial fluid and α_∞ the geometric tortuosity [35]. The viscous coupling factor b can be related to the static macroscopic flow resistivity σ of the porous material, $b = \sigma h^2 G(\omega)$, taking into account the transition from microscale Stokes flow at very low frequencies to inviscid flow as high frequency asymptote. The corrected densities $\tilde{\rho}_{11}$, $\tilde{\rho}_{12}$, $\tilde{\rho}_{22}$ take into account the total viscous and inertial energy dissipation caused by the relative motion between the solid and the fluid phase. For a complete description and interpretation, the reader is referred to literature [2].

Three boundary conditions have to be specified at each point of the boundary in order to have a well-posed problem. Poroelastic boundary conditions and coupling conditions to an acoustic domain are discussed in Appendix A.

3. The Wave Based Method for axisymmetric problems

The WBM [18] is a deterministic numerical prediction technique for solving boundary value problems in vibro-acoustics and aims at relaxing the frequency limitations encountered with the use of element based approaches, such as the FEM. The WBM is based on the indirect Trefftz principle [19] in that it applies exact solutions of the governing differential equations, called wave functions, to approximate the field variables. The WBM renders solutions of the problem which exactly satisfy the differential equations, but satisfy the boundary and interface coupling conditions only in an approximate way. **A weighted residual approach is used to minimise the errors on the boundary and interfaces.** As a result, the use of very fine domain discretisations as for the FEM is not longer required, leading to smaller model sizes and associated computational cost. **The attainable frequency range (upper frequency limit) depends on the problem complexity. The main drawback of the method is that all subdomains need to be convex. Non-convex problem domains should be partitioned into convex subdomains for the method to converge. As the number of subdomain interfaces increases, so does the integration length and the time to build the system matrices. Contrarily to FEM, the most demanding computational part of the WBM is the construction of the system of equations. As any Trefftz-approach, the WBM yields ill-conditioned matrices and the entries in the matrices should be calculated very accurately. As a consequence, the use of the WBM is restricted to relatively simple geometries, where it shows its best performance. When geometrically complex problems need to be studied, a hybrid FE-WBM can be applied; the WBM is applied for large, convex interior parts of the problem domain whereas the FEM can tackle the geometrical details [28]. A discussion of the WBM's properties and a comparison to those of the FEM, can be found in [21, 32].**

So far, the concept of the WBM has only been applied to 2D Cartesian and polar coordinate systems and to 3D Cartesian and spherical coordinate systems. This section extends the WB concept to axisymmetric acoustic and poroelastic problems. The WB modelling approach generally consists of the following four steps, which will be applied for bounded acoustic and poroelastic axisymmetric problems:

1. partitioning of the considered problem domain into convex subdomains;

2. selection of a suitable set of wave functions for each subdomain;
3. construction of the WB system matrices using a weighted residual formulation of the boundary and interface conditions;
4. solution of the system of equations, yielding the wave function contribution factors and postprocessing of the dynamic variables.

3.1. Partitioning of the considered problem domain into convex subdomains

Desmet [18] has shown that a sufficient condition for convergence of the WBM for bounded problems is the convexity of the considered problem domain. Non-convex problem domains have to be divided into a number of convex subdomains. Coupling conditions on both sides of the interfaces between subdomains ensure the continuity of the dynamic fields in both subdomains.

3.2. Selection of wave functions

The axisymmetric acoustic problem, governed by one Helmholtz equation, is considered first. A suitable set of wave functions, fulfilling both the Helmholtz equation and the axisymmetry condition, is derived. In a second step, the concept is extended to poroelastic axisymmetric problems, governed by two scalar and one vectorial Helmholtz equation.

3.2.1. Wave functions for the axisymmetric acoustic problem

The steady-state acoustic pressure field $p(\mathbf{r})$ in an axisymmetric acoustic domain Ω_a is approximated by a solution expansion $\hat{p}(r, z)$:

$$p(r, z) \simeq \hat{p}(r, z) = \sum_{w_a=1}^{n_{w_a}} u_{w_a} \Phi_{w_a}(r, z) = \mathbf{\Phi}_{w_a}(r, z) \mathbf{u}_{w_a} . \quad (8)$$

The wave function contributions u_{w_a} are the weighting factors for each of the selected wave functions Φ_{w_a} . Together they form the vector of degrees of freedom $\mathbf{u}_{w_a}$. The corresponding *a priori* defined wave functions are collected in the row vector $\mathbf{\Phi}_{w_a}$. Each of the wave functions exactly satisfies the homogeneous Helmholtz equation (1). Since the Helmholtz equation is a second-order differential equation, any pressure field is uniquely defined by its governing Helmholtz equation and the specification of one acoustic quantity at each position of the boundary of the domain. Since the wave functions in the expansion set (8) are chosen to satisfy the Helmholtz equation *a priori*, it is sufficient that the set can represent an arbitrarily imposed normal acoustic velocity on the boundary.

Figure 2 shows a rectangular axisymmetric problem domain with dimension $L_{a,r}$ and $L_{a,z}$. The axisymmetric axis is indicated by a dashed line. Since the response field has to be axisymmetric, the normal velocity on the axis of symmetry has to be zero. The normal velocity distribution is further characterised by $v_1(z)$, $v_2(r)$ and $v_3(r)$, each of them being defined on one of the three remaining side walls of the rectangular boundary. The considered problem can be divided into three subproblems, each of them fulfilling the axisymmetric boundary condition and having two side walls with zero normal fluid velocity distribution and one boundary with a non-zero normal distribution, as indicated

on figure 2. By combining the solution fields for each of the subproblems, a suitable set of wave functions can

Figure 2: General homogeneous pressure field in an axisymmetric rectangular domain.

be defined for a rectangular axisymmetric problem domain. Afterwards, also non-rectangular problem domains are discussed.

Solution of subproblem 1. The first of the three subproblems consists of determining the pressure field $p_1(r, z)$, satisfying the homogeneous axisymmetric Helmholtz equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p_1(r, z)}{\partial r} \right) + \frac{\partial^2 p_1(r, z)}{\partial z^2} + k^2 p_1(r, z) = 0, \quad (9)$$

and the boundary conditions

$$\frac{\partial p_1(r, 0)}{\partial z} = 0, \quad (r \in [0, L_{a,r}]), \quad (10)$$

$$\frac{\partial p_1(r, L_{a,z})}{\partial z} = 0, \quad (r \in [0, L_{a,r}]), \quad (11)$$

$$\frac{j}{\rho_0 \omega} \frac{\partial p_1(L_{a,r}, z)}{\partial r} = v_1(z), \quad (z \in [0, L_{a,z}]). \quad (12)$$

By separation of variables, the pressure field can be written as:

$$p_1(r, z) = R(r)Z(z), \quad (13)$$

where $R(r)$ and $Z(z)$ are given by the following ordinary differential equations:

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - (\lambda^2 - k^2)R = 0, \quad (14)$$

$$\frac{d^2 Z}{dz^2} + \lambda^2 Z = 0, \quad (15)$$

with λ an arbitrary constant. The linear independent solutions of these equations yield a general pressure field of the form [36]:

$$p_1(r, z) = (A \cos(\lambda z) + B \sin(\lambda z)) \left(C J_0(\sqrt{k^2 - \lambda^2} r) + D Y_0(\sqrt{k^2 - \lambda^2} r) \right) + (E + Fz) (G J_0(kr) + H Y_0(kr)). \quad (16)$$

The solutions with unknown constants A, B, C and D are found by assuming $\lambda \neq 0$ and the solutions with unknown constants E, F, G and H are found by assuming $\lambda = 0$. Consequently, in equation (16) λ is a non-zero constant. J_0 and Y_0 the ordinary Bessel functions of zero order and first and second kind, respectively. Since the pressure field at $r = 0$ has to be finite and

$$\lim_{r \rightarrow 0} Y_0(z) = -\infty, \quad (17)$$

the constants $D = H = 0$. From boundary condition (10), it readily follows that $B = F = 0$, yielding a pressure field of the form:

$$p_1(r, z) = I \cos(\lambda z) J_0(\sqrt{k^2 - \lambda^2} r), \quad (18)$$

where I is an unknown constant and λ is a constant which can now be also zero. For boundary condition (11) to be fulfilled, the following condition has to be satisfied at any position $r \in [0, L_{a,r}]$:

$$-I \lambda \sin(\lambda L_{a,z}) J_0(\sqrt{k^2 - \lambda^2} r) = 0. \quad (19)$$

Since $I = 0$ would yield the trivial solution $p_1(r, z) = 0$, λ is confined to

$$\lambda = \frac{a_1 \pi}{L_{a,z}} \quad \text{with } a_1 = 0, 1, 2, \dots, \quad (20)$$

yielding a pressure field of the form:

$$p_1(r, z) = \sum_{a_1=0}^{\infty} p_{a_1} \cos\left(\frac{a_1 \pi z}{L_{a,z}}\right) J_0\left(\sqrt{k^2 - \left(\frac{a_1 \pi}{L_{a,z}}\right)^2} r\right). \quad (21)$$

For the remaining boundary condition (12) to be fulfilled, the constants p_{a_1} have to be defined such that the equation

$$v_1(z) = -\frac{j}{\rho_0 \omega} \sum_{a_1=0}^{\infty} p_{a_1} \sqrt{k^2 - \left(\frac{a_1 \pi}{L_{a,z}}\right)^2} \cos\left(\frac{a_1 \pi z}{L_{a,z}}\right) J_1\left(\sqrt{k^2 - \left(\frac{a_1 \pi}{L_{a,z}}\right)^2} L_{a,r}\right), \quad (22)$$

is satisfied at any position $z \in [0, L_{a,z}]$. Since the functions $\cos(\frac{a_1 \pi}{L_{a,z}} z)$ with $(a_1=0, 1, 2, \dots)$ are the orthogonal eigenfunctions of a Sturm-Liouville problem, it is possible to define the constants p_{a_1} such that the right-hand side of equation (22) converges to $v_1(z)$ at the position $z \in]0, L_{a,z}[$, where $v_1(z)$ is continuous, and to the mean value $0.5(v_1(z^-) + v_1(z^+))$ where $v_1(z)$ is discontinuous [37].

Solution of subproblems 2 & 3. In a similar manner, the solution of the two remaining subproblems can be defined. By separation of variables, the pressure field p_2 can be written as:

$$p_2(r, z) = R(r)Z(z), \quad (23)$$

where $R(r)$ and $Z(z)$ are given by the following ordinary differential equations:

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \lambda^2 R = 0, \quad (24)$$

$$\frac{d^2 Z}{dz^2} + (\lambda^2 - k^2)Z = 0. \quad (25)$$

In this equations, λ is again an arbitrary constant. The imposed boundary conditions for the second subproblem are

$$\frac{-j}{\rho_0 \omega} \frac{\partial p_2(r, 0)}{\partial z} = v_2(r), \quad (r \in [0, L_{a,r}]), \quad (26)$$

$$\frac{\partial p_2(r, L_{a,z})}{\partial z} = 0, \quad (r \in [0, L_{a,r}]), \quad (27)$$

$$\frac{\partial p_2(L_{a,r}, z)}{\partial r} = 0, \quad (z \in [0, L_{a,z}]). \quad (28)$$

The linear independent solutions of equations (24) and (25) yield a general pressure field of the form [36]:

$$p_2(r, z) = \left(A e^{j\sqrt{k^2 - \lambda^2}z} + B e^{-j\sqrt{k^2 - \lambda^2}z} \right) (C J_0(\lambda r) + D Y_0(\lambda r)) + \left(E e^{jkz} + F e^{-jkz} \right) (G + H \ln(r)), \quad (29)$$

The solutions with unknown constants A , B , C and D are found by assuming $\lambda \neq 0$ and the solutions with unknown constants E , F , G and H are found by assuming $\lambda = 0$. Consequently, in equation (29), λ a non-zero constant. D and H have to be equal to zero due to the finiteness of the pressure field at $r = 0$. Consequently, the resulting pressure field can be written as:

$$p_2(r, z) = \left(I e^{j\sqrt{k^2 - \lambda^2}z} + J e^{-j\sqrt{k^2 - \lambda^2}z} \right) J_0(\lambda r), \quad (30)$$

in which I and J are constants and λ is a constant which can now also be zero. By applying boundary condition (27), the following relation between I and J is obtained:

$$I = J e^{-2j\sqrt{k^2 - \lambda^2}L_{a,z}}. \quad (31)$$

For boundary condition (28) to be fulfilled, the constants J and λ need to be defined such that the equation

$$-J\lambda \left(e^{-2j\sqrt{k^2 - \lambda^2}L_{a,z}} e^{j\sqrt{k^2 - \lambda^2}z} + e^{-j\sqrt{k^2 - \lambda^2}z} \right) J_1(\lambda L_{a,r}) = 0, \quad (32)$$

is satisfied at any position $z \in [0, L_{a,z}]$. Since $J = 0$ would yield the trivial solution, λ is confined to solutions of

$$J_1(\lambda L_{a,r}) = 0. \quad (33)$$

The a_2 th zero of this equation is denoted by λ_{a_2} . This yields a pressure field p_2 of the following form

$$p_2(r, z) = \sum_{a_2=1}^{\infty} p_{a_2} \left(e^{-2j\sqrt{k^2 - \lambda_{a_2}^2}L_{a,z}} e^{j\sqrt{k^2 - \lambda_{a_2}^2}z} + e^{-j\sqrt{k^2 - \lambda_{a_2}^2}z} \right) J_0(\lambda_{a_2} r). \quad (34)$$

For the remaining boundary condition (26) to be fulfilled, the constants p_{a_2} have to be determined such that the equation

$$v_2(r) = \frac{1}{\rho_0 \omega} \sum_{a_2=1}^{\infty} p_{a_2} \sqrt{k^2 - \lambda_{a_2}^2} \left(e^{-2j\sqrt{k^2 - \lambda_{a_2}^2}L_{a,z}} - 1 \right) J_0(\lambda_{a_2} r) \quad (35)$$

is satisfied at any position $r \in [0, L_{a,r}]$. Applying the same reasoning as for the first subproblem, it follows that equation (35) can be fulfilled by a proper selection of the components p_{a_2} .

In a similar way, it can be shown that a homogeneous axisymmetric pressure field $p_3(r, z)$, having a normal fluid velocity field $v_3(r)$ imposed along the side wall at $z = L_{a,z}$ and a zero normal velocity imposed along the two other edges, can be expressed as:

$$p_3(r, z) = \sum_{a_3=1}^{\infty} p_{a_3} \left(e^{j\sqrt{k^2 - \lambda_{a_3}^2} z} + e^{-j\sqrt{k^2 - \lambda_{a_3}^2} z} \right) J_0(\lambda_{a_3} r), \quad (36)$$

with λ_{a_3} also solutions of $J_1(\lambda L_{a,r}) = 0$, similar as for subproblem 2.

Definition of the wave function sets for a rectangular problem domain. The requested homogeneous axisymmetric pressure field $p(r, z) = (p_1(r, z) + p_2(r, z) + p_3(r, z))$ in a rectangular domain with dimensions $L_{a,r}$ and $L_{a,z}$ and with an arbitrary normal fluid velocity distribution on the boundaries, and in extension arbitrarily imposed boundary conditions, can be approximated as indicated by equation (8) by a weighted sum of wave functions. The wave functions are subdivided into two sets, the s-set and the t-set:

$$\sum_{w_a=1}^{n_a} p_{w_a} \Phi_{w_a}(r, z) = \sum_{w_{a,s}=1}^{n_{a,s}} p_{w_{a,s}} \Phi_{w_{a,s}}(r, z) + \sum_{w_{a,t}=1}^{n_{a,t}} p_{w_{a,t}} \Phi_{w_{a,t}}(r, z), \quad (37)$$

with $n_a = n_{a,s} + n_{a,t}$. The wave functions are defined as:

$$\begin{cases} \Phi_{w_{a,s}}(r, z) = J_0(k_{rw_{a,s}} r) \cos(k_{zw_{a,s}} z) \\ \Phi_{w_{a,t}}(r, z) = J_0(k_{rw_{a,t}} r) e^{-jk_{zw_{a,t}} z} \end{cases}, \quad (38)$$

with

$$\begin{cases} (k_{rw_{a,s}}, k_{zw_{a,s}}) = \left(\sqrt{k^2 - \left(k_{zw_{a,s}}\right)^2}, \frac{b_{a,1}\pi}{L_z} \right) \\ (k_{rw_{a,t}}, k_{zw_{a,t}}) = \left(\lambda_{b_{a,2}}, \pm \sqrt{k^2 - \left(k_{rw_{a,t}}\right)^2} \right) \end{cases}, \quad (39)$$

with coefficients $b_{a,1}=0,1,2,\dots$ and the indices $b_{a,2}=0,1,2,\dots$ associated with positive roots, $\lambda_{b_{a,2}}$, of $J_1(L_{a,r}r)$. The wave functions in the s-set are defined based on the solutions of subproblem 1 and consequently on the dimension $L_{a,z}$. The wave functions in the t-set are defined based on the solutions of subproblem 2 and 3, which have been recombined, and consequently the dimension $L_{a,r}$. For the full set of wave functions (an infinite number of wave functions), the exact solution will be obtained.

Non-rectangular acoustic axisymmetric domain. For non-rectangular, however convex, problem domains, the same wave function sets can be applied, based on the dimensions of the smallest rectangular bounding box circumscribing the considered domain. The convergence is preserved, provided that the domain is convex. Desmet [18] explains for two-dimensional cartesian acoustic problems, how the convergence requirements can be extended to non-rectangular convex domains. For axisymmetric problems, the derivation is identical and the reader is referred to [18] for more details.

Truncation of the wave function set. The infinite set of wave functions (37) forms a complete set and the WBM will converge towards the exact solution of the problem domain given that the subdomains are convex. In order to apply the WBM in a numerical scheme, the wave function sets are truncated to a finite number of wave functions, n_{w_a} . The truncation rule followed here is analogous to the truncation rule applied for cartesian coordinates. The adopted truncation rule is frequency dependent and selects the maximum values for the integers $b_{a,1,max}$ and $b_{a,2,max}$ such that the corresponding wave number of the highest oscillating wave function of the corresponding set is approximately equal to a user-defined truncation factor T times the largest wave number k_{max} occurring in the considered problem:

$$\begin{aligned} b_{a,1,max} &:= \max \left\{ b_{a,1} \in \mathbb{N} \mid \frac{b_{a,1}\pi}{L_{a,z}} < T|k_{max}| \right\}, \\ b_{a,2,max} &:= \max \left\{ b_{a,2} \in \mathbb{N} \mid \frac{b_{a,2}\pi}{L_{a,r}} < T|k_{max}| \right\}. \end{aligned} \quad (40)$$

In case of a purely acoustic problem, $k_{max} = k$. In case also a poroelastic domain is present, k_{max} is the maximum of the absolute value of the largest wave number occurring in either the acoustic or the poroelastic domain. In this way, a similar spatial resolution is obtained for the oscillating wave numbers between adjacent domains, which was shown to lead to the most stable convergence, for instance see [30]. The number of wave functions in the s and the t-wave function set, $n_{a,s}$ and $n_{a,t}$ (50), are consequently defined as:

$$\begin{aligned} n_{a,s} &:= (b_{a,1,max} + 1), \\ n_{a,t} &:= 2(b_{a,2,max} + 1). \end{aligned} \quad (41)$$

The number of wave functions results from the value of T and increases approximately linearly with frequency. The presented truncation rule selects a number of wave functions for the s-set such that the spatial resolution is approximately similar as for the r-set. This is the case since the distance between two subsequent zeros of an ordinary Bessel function converges to π in a monotonic way [38].

3.2.2. Wave functions for the axisymmetric poroelastic problem

As indicated by Biot [1], poroelastic materials can sustain three wave types simultaneously: two longitudinal waves and one shear wave. Within the WB approach, these three wave types are explicitly taken into account. In earlier work, the WBM has been applied to 2D cartesian poroelastic problem domains [32]. Two possible decompositions are described for the Biot equations in the case of material isotropy: in terms of strains and in terms of potentials. Also an FE approach has been proposed, applying a Helmholtz decomposition, and using scalar and vector potentials as primary variables [39]. A similar approach is followed here, based on the potential formulation. The transformation is based on the Helmholtz decomposition of a vector field [40] which states that any vector field can be decomposed into an irrotational and a solenoidal part, provided that the vector field is piecewise differentiable. The derivation, also in axisymmetric coordinates, is given in [41] and leads to a decomposition into two scalar potentials, $\varphi_1^s(r, z)$ and $\varphi_2^s(r, z)$, and a vector potential, $\psi^s(r, z)$:

$$\begin{Bmatrix} u_r^s(r, z) \\ u_z^s(r, z) \end{Bmatrix} = \nabla \left(\varphi_1^s(r, z) + \varphi_2^s(r, z) \right) + \nabla \times \psi^s(r, z). \quad (42)$$

Due to the axial symmetry, the displacement associated with the vector potential, has only components in the r - and z -direction. The vector potential can thus be written as:

$$\boldsymbol{\psi}^s(r, z) = \psi^s(r, z)\mathbf{e}_\theta. \quad (43)$$

By substituting (42) in the Biot equations (6)-(7), one obtains the following set of decoupled Helmholtz equations:

$$\left(\nabla^2 \varphi_1^s(r, z) + k_{l_1}^2 \varphi_1^s(r, z)\right) \left(\nabla^2 \varphi_2^s(r, z) + k_{l_2}^2 \varphi_2^s(r, z)\right) = 0, \quad (44)$$

$$\nabla^2 \psi^s(r, z)\mathbf{e}_\theta + k_t^2 \psi^s(r, z)\mathbf{e}_\theta = \mathbf{0}, \quad (45)$$

in which $\varphi^s(r, z) = \varphi_1^s(r, z) + \varphi_2^s(r, z)$ is the scalar potential. Equation (44) indicates the existence of two dilatational wave types with distinct wave numbers k_{l_1} and k_{l_2} :

$$\begin{cases} k_{l_1} = \sqrt{\frac{A_1}{2} + \sqrt{\frac{A_1^2}{4} - A_2}}, \\ k_{l_2} = \sqrt{\frac{A_1}{2} - \sqrt{\frac{A_1^2}{4} - A_2}}, \end{cases} \quad \text{with} \quad \begin{cases} A_1 = \omega^2 \frac{\tilde{\rho}_{11}R - 2\tilde{\rho}_{12}Q + \tilde{\rho}_{22}P}{RP - Q^2}, \\ A_2 = \omega^4 \frac{\tilde{\rho}}{RP - Q^2}. \end{cases}, \quad (46)$$

with $P = A + 2N$ and $\tilde{\rho} = \tilde{\rho}_{11}\tilde{\rho}_{22} - \tilde{\rho}_{11}^2$. Equation (45) indicates the existence of one rotational wave type with wave number k_t :

$$k_t = \omega \sqrt{\frac{\tilde{\rho}}{N\tilde{\rho}_{22}}}. \quad (47)$$

By working out the ∇^2 -operator acting on a vector field and applying the axisymmetric assumption, equation (45) can be rewritten as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) \psi^s(r, z) + k^2 \psi^s(r, z) = 0. \quad (48)$$

Each of the three fields $\varphi_1^s(\mathbf{r})$, $\varphi_2^s(\mathbf{r})$ and $\psi^s(\mathbf{r})$ in a poroelastic subdomain Ω_{pe} can now be approximated according to the WB principle. Similar as for cartesian coordinates [32], each of these three variables is approximated by a solution expansion in terms of n_* wave functions Φ_{w_*} ($w_* = 1, \dots, n_*$):

$$\begin{cases} \varphi_1^s(r, z) \simeq \hat{\varphi}_1^s(r, z) = \sum_{w_{\varphi_1}=1}^{n_{\varphi_1}} u_{w_{\varphi_1}} \Phi_{w_{\varphi_1}}(r, z) = \mathbf{\Phi}_{\mathbf{w}_{\varphi_1}}(r, z) \mathbf{u}_{\mathbf{w}_{\varphi_1}}, \\ \varphi_2^s(r, z) \simeq \hat{\varphi}_2^s(r, z) = \sum_{w_{\varphi_2}=1}^{n_{\varphi_2}} u_{w_{\varphi_2}} \Phi_{w_{\varphi_2}}(r, z) = \mathbf{\Phi}_{\mathbf{w}_{\varphi_2}}(r, z) \mathbf{u}_{\mathbf{w}_{\varphi_2}}, \\ \psi^s(r, z) \simeq \hat{\psi}^s(r, z) = \sum_{w_\psi=1}^{n_\psi} u_{w_\psi} \Phi_{w_\psi}(r, z) = \mathbf{\Phi}_{\mathbf{w}_\psi}(r, z) \mathbf{u}_{\mathbf{w}_\psi}. \end{cases} \quad (49)$$

Each of the wave functions $\Phi_{w_*}(\mathbf{r})$ has to satisfy *a priori* one of the three associated homogeneous equations as defined in (44) or (48). Each of the wave functions $\Phi_{w_{\varphi_\star}}(\mathbf{r})$, with $\star = 1, 2$ can be defined in a similar way as for the acoustic wave functions, since they have to be solutions of a homogeneous Helmholtz equation as well. For the wave functions Φ_{w_ψ} , associated with the vector potential, the wave functions have to be exact solutions of equation (48) and can be obtained in a similar way as for the scalar Helmholtz equation. Due to the additional $-\frac{1}{r^2}$ -term as compared with the

scalar Helmholtz equation (1), these wave functions contain Bessel functions of first kind and first order. For the three field variables of poroelastic wave functions, s-, t- and u-sets can be defined:

$$\sum_{w_s=1}^{n_s} u_{w_s} \Phi_{w_s}(r, z) = \sum_{w_{s,s}=1}^{n_{s,s}} u_{w_{s,s}} \Phi_{w_{s,s}}(r, z) + \sum_{w_{s,t}=1}^{n_{s,t}} u_{w_{s,t}} \Phi_{w_{s,t}}(r, z) + \sum_{w_{s,u}=1}^{n_{s,u}} u_{w_{s,u}} \Phi_{w_{s,u}}(r, z) , \quad \text{with } * = \varphi_1, \varphi_2, \psi, \quad (50)$$

with $n_* = n_{*,s} + n_{*,t} + n_{*,u}$. The wave functions are defined as:

$$\begin{cases} \Phi_{w_{\varphi_*},s}(r, z) = J_0(k_{rw_{\varphi_*},s} r) \cos(k_{zw_{\varphi_*},s} z) \\ \Phi_{w_{\varphi_*},t}(r, z) = J_0(k_{rw_{\varphi_*},t} r) \sin(k_{zw_{\varphi_*},t} z) \\ \Phi_{w_{\varphi_*},u}(r, z) = J_0(k_{rw_{\varphi_*},u} r) e^{-jk_{zw_{\varphi_*},u} z} \end{cases} , \quad \text{with } \star = 1, 2 \quad (51)$$

and

$$\begin{cases} \Phi_{w_{\psi},s}(r, z) = J_1(k_{rw_{\psi},s} r) \cos(k_{zw_{\psi},s} z) \\ \Phi_{w_{\psi},t}(r, z) = J_1(k_{rw_{\psi},t} r) \sin(k_{zw_{\psi},t} z) \\ \Phi_{w_{\psi},u}(r, z) = J_1(k_{rw_{\psi},u} r) e^{-jk_{zw_{\psi},u} z} \end{cases} , \quad (52)$$

As compared, to the acoustic wave function sets, also the sine functions are included. Although the two sets are not strictly necessary for theoretical convergence, it was shown by Vanmaele et al. [24] that inclusion of both cosine and sine functions is beneficial for the convergence rate for 2D solid problems. The wave number components are defined in the same manner as for an acoustic subdomain:

$$\begin{cases} (k_{rw_{\varphi_*},s}, k_{zw_{\varphi_*},s}) = (k_{rw_{\varphi_*},t}, k_{zw_{\varphi_*},t}) = \left(\sqrt{k_{l_*}^2 - (k_{zw_{\varphi_*},s})^2}, \frac{b_{\varphi_*},1\pi}{L_{pe,z}} \right) \\ (k_{rw_{\varphi_*},s}, k_{zw_{\varphi_*},s}) = \left(\lambda_{b_{\varphi_*},2}, \pm \sqrt{k_{l_*}^2 - (k_{rw_{\varphi_*},s})^2} \right) \end{cases} , \quad \text{with } \star = 1, 2 \quad (53)$$

and

$$\begin{cases} (k_{rw_{\psi},s}, k_{zw_{\psi},s}) = (k_{rw_{\psi},t}, k_{zw_{\psi},t}) = \left(\sqrt{k_t^2 - (k_{zw_{\psi},s})^2}, \frac{b_{\psi},1\pi}{L_{pe,z}} \right) \\ (k_{rw_{\psi},s}, k_{zw_{\psi},s}) = \left(\lambda_{b_{\psi},2}, \pm \sqrt{k_t^2 - (k_{rw_{\psi},s})^2} \right) \end{cases} , \quad (54)$$

with $b_{*,1} = 0, 1, 2, \dots$ and the coefficients $b_{*,2} = 0, 1, 2, \dots$ are associated with positive roots, $\lambda_{b_{*,2}}$, of $J_1(L_{pe,r}r)$, with $* = \psi, \varphi_1, \varphi_2$. The wave functions that are strictly zero, for instance the sine functions with wave number component zero, are excluded from the resulting wave function sets. $L_{pe,r}$ and $L_{pe,z}$ are the dimensions of the smallest rectangular bounding box circumscribing the poroelastic problem domain Ω_{pe} . The wave functions in each of the sets also need to be truncated to a finite number in order to be used in a numerical scheme. In case of a poroelastic problem, where three different wave types coexist, the applied truncation rule for each of the sets is similar to the one presented for the acoustic case (40), but the bounding box dimensions of the poroelastic domain are used and k_{max} is in this case the largest of the three physical wave numbers k_{l_1} , k_{l_2} and k_t occurring in the poroelastic domain. Practically, the shear wave type or the fast wave are setting the limit. Consequently, for the three types of wave functions in a poroelastic

domain, a similar number of wave functions is used in each of the wave function sets:

$$\begin{aligned} b_{*,1,max} &:= \max\{b_{*,1} \in \mathbb{N} \mid \frac{b_{*,1}\pi}{L_{pe,z}} < T|k_{max}|\}, \\ b_{*,2,max} &:= \max\{b_{*,2} \in \mathbb{N} \mid \frac{b_{*,2}\pi}{L_{pe,r}} < T|k_{max}|\}. \end{aligned}, \quad \text{with } * = \varphi_1, \varphi_2, \psi. \quad (55)$$

In the case of a coupled problem where also an acoustic domain is present, k_{max} is the maximum of the absolute value of the largest wave number occurring in either the acoustic or the poroelastic domain, which is practically always a poroelastic wave number. The truncation rule ensures that a similar resolution of the oscillating wave number components is obtained within a physical domain but also over subdomain interfaces. It has been validated for different kind of problems that this ensures the most stable convergence. The numbers $n_{*,r}$, $n_{*,s}$ and $n_{*,t}$, defining the s-, t- and u-wave function sets (50), are consequently found as:

$$\begin{aligned} n_{*,s} &:= b_{*,1,max} + 1, \\ n_{*,u} &:= b_{*,1,max}, \\ n_{\varphi_*,t} &:= 2(b_{\varphi_*,2,max} + 1), \\ n_{\psi,t} &:= 2(b_{\psi,2,max}). \end{aligned} \quad \text{with } * = \varphi_1, \varphi_2, \psi \quad \text{and } \star = 1, 2. \quad (56)$$

3.3. Construction of the WB system matrices

With the use of the proposed solution expansions (8) and (49) and the accompanying definitions of the wave functions, the governing equations, in the acoustic as well as in the poroelastic domain, are always exactly satisfied, irrespective of the values of the unknown wave function contribution factors $u_{w,*}$. The wave functions may, however, violate the imposed boundary and interface conditions. The residuals on the boundaries and interfaces, are given by equations (A.1)-(A.3) and (A.5)-(A.8). Appendix B explains how each of the dynamic field variables in the acoustic as well as in the poroelastic subdomain can be determined by applying differential operators on the approximation sets of the field variables. The mutual coupling between the three wave field components in the poroelastic domain is entirely contained within the conditions specified along the boundaries and interfaces. The wave functions in the acoustic and poroelastic domain couple along the common interface Γ_C . For the acoustic Ω_a and poroelastic subdomain Ω_{pe} , the error residual functions are weighted in an integral sense with respect to some arbitrary weighting functions, indicated by $\tilde{\bullet}$:

$$\begin{aligned} &2\pi \left(\int_{\Gamma_{ki} \cup \Gamma_{mi}} r \tilde{\sigma}_n^s(\mathbf{r}) R_{u_n^s}(\mathbf{r}) d\Gamma + \int_{\Gamma_{ki}} r \tilde{\sigma}_s^s(\mathbf{r}) R_{u_s^s}(\mathbf{r}) d\Gamma + \int_{\Gamma_{ki} \cup \Gamma_{mi}} r \tilde{\sigma}^f(\mathbf{r}) R_{u_n^f}(\mathbf{r}) d\Gamma - \int_{\Gamma_{me}} r \tilde{u}_n^s(\mathbf{r}) R_{\sigma_n^s}(\mathbf{r}) d\Gamma \right. \\ &- \int_{\Gamma_{me} \cup \Gamma_{mi}} r \tilde{u}_s^s(\mathbf{r}) R_{\sigma_s^s}(\mathbf{r}) d\Gamma - \int_{\Gamma_{me}} r \tilde{u}_n^f(\mathbf{r}) R_{\sigma_n^f}(\mathbf{r}) d\Gamma - \int_{\Gamma_C} r \tilde{u}_n^s(\mathbf{r}) R_{\sigma_{n,C}^s}(\mathbf{r}) d\Gamma - \int_{\Gamma_C} r \tilde{u}_s^s(\mathbf{r}) R_{\sigma_{s,C}^s}(\mathbf{r}) d\Gamma \\ &- \int_{\Gamma_C} r \tilde{u}^f(\mathbf{r}) R_{\sigma_C^f}(\mathbf{r}) d\Gamma + \int_{\Gamma_C} r \tilde{p}(\mathbf{r}) R_{u,C}(\mathbf{r}) d\Gamma + \int_{\Gamma_v} r \tilde{p}(\mathbf{r}) R_v(\mathbf{r}) d\Gamma + \int_{\Gamma_z} r \tilde{p}(\mathbf{r}) R_z(\mathbf{r}) d\Gamma \\ &\left. - \int_{\Gamma_p} r \mathcal{L}_v(\tilde{p}(\mathbf{r})) R_p(\mathbf{r}) d\Gamma \right) = 0. \end{aligned} \quad (57)$$

This weighted residual formulation only considers the boundary conditions that are applied in the numerical examples in this paper and takes into account one acoustic and one poroelastic subdomain. A similar procedure can be applied for other kinds of boundary conditions and a larger number of subdomains [32, 23]. The factor $2\pi r$ in each integral takes into account the transformation to the actual 3D geometry. The weighting functions $\tilde{a}$ (with $a=u_n^s, u_s^s, u_n^f, \sigma_n^s, \sigma_s^s, \sigma^f, p$) are expressed in terms of the same set of wave functions as used for the field variable expansions, following a Galerkin-like weighting procedure.

Substituting the field variable expansions and the weighting function expansions into the weighted residual formulation for both the acoustic and the poroelastic subdomain yields an algebraic equation which links together the wave function contribution factors of both subdomains. By enforcing that this equation should hold for any weighting function results in a fully populated and complex system of equations:

$$\begin{bmatrix} \mathbf{A}_{pe} & \mathbf{C}_{pe,a} \\ \mathbf{C}_{a,pe} & \mathbf{A}_a \end{bmatrix} \cdot \begin{bmatrix} [\mathbf{u}_{w\varphi_1} \mathbf{u}_{w\varphi_2} \mathbf{u}_{w\psi}]^T \\ \mathbf{u}_{wa} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{pe} \\ \mathbf{f}_a \end{bmatrix} \quad (58)$$

The matrices $\mathbf{A}_\bullet$ are the system matrices, the matrices $\mathbf{C}_{\bullet,\bullet}$ are the coupling matrices, mutually linking the wave functions of the poroelastic and the acoustic subdomains and $\mathbf{f}_\bullet$ are the loading vectors resulting from non-homogeneous boundary conditions.

3.4. Solution of the system of equations and postprocessing

The fourth step in the WB modelling procedure is the solution of matrix equation (58). This yields the unknown contribution factors belonging to each of the wave functions. The back substitution of these values into the field variable expansions (8) and (49), yields an approximation of the acoustic pressure field and the three poroelastic potential fields, respectively. Dynamic field variables, like the acoustic velocities and poroelastic displacements and stresses can be post-processed by analytically applying the differential operators as defined in Appendix B.

4. Numerical examples

This section shows the potential of the novel WBM for axisymmetric problems for four problem cases of increasing complexity. The first two examples contain a rectangular and a non-rectangular acoustic problem domain, respectively, evaluating the WB concepts for a single axisymmetric Helmholtz equation. The third validation case studies an uncoupled poroelastic problem and verifies that also more complex problems, involving both scalar and vector Helmholtz equations can be tackled in an efficient manner. The last example applies the axisymmetric WBM to a model of a Kundt tube measuring device containing a poroelastic sample, illustrating how the method can be used for a problem of practical relevance. One of the possible future applications of the novel WBM, is to employ the method for inverse material characterisation by matching measurements with numerical predictions.

A comparison is made to predictions obtained with the FEM in terms of calculation time as well as accuracy. In order to obtain independent references, highly refined FE models are used. The FE predictions are obtained

using Comsol 3.5, using an axisymmetric formulation. For the uncoupled acoustic problems, quadratic triangular Lagrangian elements are applied. For the poroelastic and the coupled acoustic-poroelastic case, cubic triangular Lagrangian elements are used. Between two successive FE models, an adaptive refinement based on the L^2 -norm of the error in the domain is performed. In this way, the mesh is refined in the most efficient way to speed up convergence. The FE models are solved using an UMFPAK solver. All WB routines are implemented in Matlab R2010a and the models are solved using Matlab's backslash, i.e. by Gaussian elimination. All calculations are performed on a Linux-based 2.66GHz Intel Xeon system with 32 GB RAM.

4.1. Case 1: Cylindrical acoustic problem

The first validation case considers a simple acoustic cylinder. The axisymmetric problem description is given in figure 3. The bounding box dimensions are $L_{a,r} \times L_{a,z} = 0.5\text{m} \times 0.4\text{m}$. The fluid properties are $\rho_0 = 1.225\text{kg/m}^3$ and $c_0 = 340\text{m/s}$. The axis of symmetry is indicated with a dashed line. On the three remaining boundaries, Neumann boundary conditions are applied. The imposed normal velocities are given by v_1 , v_2 and v_3 respectively. These imposed velocity fields are arbitrarily chosen, but are not constant along the boundaries since in that case the exact solution could be reconstructed with a very low number of wave functions. The location of four post-processing points are indicated by x-marks on the figure.

Figure 3: Problem description validation case 1.

Figure 4(a) shows the absolute value of the predicted pressure field by using the WBM at 2750 Hz. In this case a truncation factor $T=2$ is used, leading to 47 wave functions in the model. Figure 4(b) shows the relative error ϵ on this pressure field, calculated as:

$$\epsilon(\hat{p}(\mathbf{r})) = \frac{|\hat{p}(\mathbf{r}) - p_{ref}(\mathbf{r})|}{|p_{ref}(\mathbf{r})|}. \quad (59)$$

As a reference, a highly refined FE model has been used consisting of 1,941,599 degrees of freedom (DOFs). This model is obtained starting from a coarse model containing 991 DOFs and then performing 11 successive adaptive refinements based on the L_2 norm of the error inside the domain. As can be seen on the figure, a very good accuracy

is obtained, although only a small number of wave functions is used in the WB model. The error is only larger at the pressure nodal lines, where the relative error is most sensitive to small variations due to a division by nearly zero.

Figure 4: Result contours at 2750Hz.

To compare the convergence rate of the axisymmetric WBM and the FEM, a quadratic prediction error estimator $\langle \varepsilon \rangle$ of the obtained pressure field is calculated as a function of CPU time. Whereas the overall accuracy in the domain has been illustrated by figure 4(b), a point wise error estimator has been chosen here. The absolute quadratic difference in a certain response point $p(\mathbf{r}_j)$ is calculated as the absolute value of the square of the error between the calculated value of the pressure in the response point and the reference value in that point. This error is then summed over n response points and divided by the absolute value of the sum of the squares of the reference values in the n response points:

$$\langle \varepsilon \rangle = \sqrt{\frac{\sum_{j=1}^n |\hat{p}(\mathbf{r}_j) - p^{ref}(\mathbf{r}_j)|^2}{\sum_{j=1}^n |p^{ref}(\mathbf{r}_j)|^2}} \quad (60)$$

In this case, the 4 response points indicated in figure 3 are used. Convergence curves are calculated with the axisymmetric WBM and FEM for three different frequencies: 500Hz, 1000Hz and 1500Hz respectively. Quadratic triangular Lagrangian elements are applied. For the three frequencies, the coarsest mesh contains 991 DOFs. Between two subsequent FE models, one adaptive refinement is performed, based on the L_2 -norm of the error in the domain. The finest FE models, applying 11 subsequent adaptive refinements, serve as a reference for the FE and WB convergence curves. The FE reference models contain 2,615,721 DOFs, 2,647,391 DOFs and 1,788,837 DOFs at 500Hz, 1000Hz and 1500Hz respectively. Counterintuitively, the reference mesh is the coarsest at the highest frequency. This is possible due to the adaptive mesh refinement procedure, which does not always refine the mesh maximally at each step. However, as can be seen from the FE curves, for a certain calculation time, which is associated with a number of FE DOFs, the obtained accuracy is in general lower for higher frequency, according to expectations. The coarsest WB model applies a truncation factor $T = 1$ and the finest WB model applies a truncation factor of $T=50$. The shown WB

convergence curve is obtained by applying the following values of T : [1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5 8.5 10 12 15 20 25 30 35 40 45 50]. This results in respectively 6, 9 and 14 DOFs for the coarsest and 207, 414 and 619 DOFs for the finest WB models for the three different frequencies. Figure 5 shows the convergence curves obtained with both the WBM and the FEM. As can be seen, the WB models have a high convergence rate as compared to the FE models. The WB convergence curves stagnate at the accuracies of the FE reference models, which are different for the three considered frequencies. As frequency increases and wavelengths shorten, more elements are required to retain a similar accuracy, due to interpolation and pollution errors [42, 43]. This explains the lower accuracy at higher frequencies of the FE reference models and consequently the plateaus in the WB convergence curves. For a given accuracy, using the WBM leads to a strong reduction in calculation time. In analogy with other applications, the WB methods proves to be superior in efficiency for problems with a geometrically simple domain.

Figure 5: Convergence diagram, WBM vs. FEM for three different frequencies.

4.2. Case 2: Acoustic truncated cone

The second validation case considers a truncated cone, resulting in a non-rectangular axisymmetric problem domain, as presented in figure 6. The same fluid properties as for the previous example are applied, $\rho_0=1.225\text{kg/m}^3$ and $c_0=340\text{m/s}$. The bounding box dimensions are $L_{a,r} \times L_{a,z}=0.5\text{m} \times 0.4\text{m}$. The axis of symmetry is indicated by a dashed line. The cavity is excited by an imposed normal velocity of 1m/s at the top boundary. The remaining boundaries are assumed to be rigid. The four post-processing points are indicated by x-marks.

Figure 7 shows the frequency response function of the pressure calculated in response point p_1 . The WBM curve is obtained by using a truncation factor $T = 4$. The reference FE curve is obtained starting from a mesh consisting of 870 DOFs and applying 10 adaptive refinements for each frequency line. Also the relative error between both curves is shown, indicating that a very good accuracy is obtained over the whole frequency range.

Convergence curves are again calculated for the WBM as well as for the FEM at three different frequencies: 500, 1000 and 1500Hz. The average relative prediction error is calculated using equation (60). For the FE convergence

Figure 6: Problem description for case 2.

Figure 7: FRF and error FRF, with the FEM as a reference, obtained in response point p_1 .

curves, quadratic triangular elements are used and an adaptive refinement is performed between two subsequent models. The coarsest mesh consists of 870 DOFs. The finest FE models, applying 12 subsequent adaptive refinements, serve as a reference for the FE and WB convergence curves. The FE reference models contain 2,840,398 DOFs, 4,638,167 DOFs and 4,856,223 DOFs at 500Hz, 1000Hz and 1500Hz, respectively. The WB convergence curve is obtained by applying the following truncation factors T : [1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5 8.5 9 9.5 10 10.5 11 11.5 12 12.5 13 13.5 14 14.5 15 20 25 30 35 40 45 50]. Since the dimensions of the bounding box are the same as for the previous example, exactly the same number of DOFs are used in the expansion set. The convergence curves

of both methods at the three selected frequencies are given in figure 8. Also for a non-rectangular problem domain, a high convergence rate and a significant gain in calculation time are obtained as compared to FE calculations.

Figure 8: Convergence curves, WBM vs. FEM for three different frequencies, applying the finest FE model as a reference.

4.3. Poroelastic problem domain

The third problem case considers a cylindrical poroelastic domain as presented in figure 9. The bounding box dimensions are $L_{pe,r} \times L_{pe,z} = 0.5\text{m} \times 0.2\text{m}$. The axis of symmetry is indicated with a dashed line. On the top boundary, an arbitrary but axisymmetric acoustic pressure $\bar{p} = -4r^3 - 3r^2 + 1$ is imposed. Consequently, mechanical boundary conditions (A.6) are imposed: $\sigma_n^s = -(1-h)\bar{p}$, $\sigma^f = -h\bar{p}$ and $\sigma_s^s = 0$. On the two remaining boundaries, sliding edge boundary conditions are applied, as given by equation (A.7) with zero imposed values. Similar as for the previous examples, the post-processing points are indicated by x-marks. The considered material is a polyurethane foam, as given in [5]. The material properties of air and material 1 in table 1 are used. Figure 10 shows the three wave numbers occurring in this type of material in function of frequency. Clearly, the shear wave number is setting the limit for the truncation rule.

Figure 9: Problem description for the poroelastic problem domain.

Air properties	
Thermal conductivity	$k_c = 2.57 \cdot 10^{-2} \text{ W(mK)}$
Specific heat	$c_p = 1.005 \cdot 10^3 \text{ J/(kgK)}$
Gas constant	$R=286.7 \text{ m}^2/(\text{s}^2 \text{ K})$
Temperature	$T=293.15 \text{ K}$
Ratio of specific heats	$\gamma=1.4$
Fluid kinematic viscosity	$\nu_f=15.11 \cdot 10^{-6} \text{ m}^2/\text{s}$
Fluid density	$\rho_0=1.205 \text{ kg/m}^3$
Material 1: Polyurethane foam material [5]	
Young's modulus of the frame material	$E_s=70 \cdot 10^3 \text{ Pa}$
Loss factor	$\eta_l = 0.265$
Poisson ratio	$\nu = 0.39$
Bulk density of the solid phase	$\rho_1=22.1 \text{ kg/m}^3$
Porosity	$h = 0.98$
Viscous characteristic length	$\Lambda=1.1 \cdot 10^{-4} \text{ m}$
Thermal characteristic length	$\Lambda'=7.42 \cdot 10^{-4} \text{ m}$
Static flow resistivity	$\sigma=3.75 \cdot 10^3 \text{ kg/(m}^3 \text{ s)}$
Tortuosity	$\alpha_{\infty} = 1.17$
Material 2: polyurethane foam Fireflex, fabricated by Recticel, Wetteren, Belgium [15]	
Young's modulus of the frame material	$E = 4.3 \cdot 10^5 + j \cdot 1 \cdot 10^5 \text{ Pa}$
Shear modulus of the frame material	$N=1.6 \cdot 10^5 + j \cdot 3 \cdot 10^4 \text{ Pa}$
Bulk density of the solid phase	$\rho_1=30 \text{ kg/m}^3$
Porosity	$h = 0.93$
Viscous characteristic length	$\Lambda=10 \cdot 10^{-6} \text{ m}$
Thermal characteristic length	$\Lambda'=100 \cdot 10^{-6} \text{ m}$
Static flow resistivity	$\sigma=80 \cdot 10^3 \text{ kg/(m}^3 \text{ s)}$
Tortuosity	$\alpha_{\infty} = 2.5$

Table 1: Material properties of air and two poroelastic media.

Figure 10: Poroelastic wave numbers in function of frequency

Figure 11 shows the contour plots of two poroelastic field variables, $u_r^f(\mathbf{r})$ and $\sigma_{rs}^s(\mathbf{r})$ at 1000Hz, calculated with the WBM. A truncation factor $T=4$ is applied. The imposed boundary conditions are accurately represented: the shear stresses are almost zero along the boundaries and the shown displacement field is almost zero along the axis of symmetry and on the boundary at the right hand side. For the FE results, the axisymmetric weak formulation of the Biot equations in $(\mathbf{u}^s, \mathbf{u}^f)$ -degrees of freedom, applying four degrees of freedom per node, as given by Kang et al. [16] has been implemented in Comsol 3.5. Cubic triangular Langrangian elements are used.

Figure 11: Result contours at 1000Hz

To assess the potential of the WBM for axisymmetric poroelastic materials, one FRF is shown in figure 12. The figure shows the predicted stress component σ_{rr}^s at post-processing point 3, obtained with both the WBM and the FEM for a wide frequency range. For the WBM, again a truncation factor $T = 4$ is applied. For the FE simulation, 9 adaptive refinements are performed at every frequency line, starting from a mesh consisting of 316 DOFs. The two curves coincide, indicating the WBM's capability to capture the same results as the FEM, also for poroelastic problems.

Figure 12: Frequency response function of a stress component, obtained in response point p_3 with the WBM and the FEM.

Convergence curves are calculated for both methods, in order to assess their efficiency. Three different frequencies are considered: 200, 400 and 800Hz. The average relative prediction error is calculated according to equation (60), using the four post-processing points as indicated in figure 9. For the FE convergence curves, one adaptive refinement

is performed between two successive models. The coarsest mesh consists of 316 DOFs. The most refined FE models, applying 14 successive adaptive refinements, serve as a reference for the FE and WB convergence curves. The FE reference models contain 1,258,108 DOFs, 1,236,676 DOFs and 1,230,424 DOFs at 500Hz, 1000Hz and 1500Hz, respectively. The coarsest WB model applies a truncation factor $T=1$ and the finest WB model applies a truncation factor of $T=8.5$. In between, T is linearly increased with steps of 0.5. This results in respectively 18, 30 and 54 DOFs for the coarsest and 108, 210 and 408 DOFs for the finest WB models for the three different frequencies. The convergence curves of both methods at the three selected frequencies are given in figure 13 for two different poroelastic variables: u_z^f and σ_{rr}^s . As can be seen from the figures, the higher the frequency, the less accurate the reference model becomes, which is expected due to interpolation and pollution errors. The WB convergence curves stagnate at the accuracy of the FE reference model. Comparison of both subfigures shows that the accuracy of the FE models for secondary variables, such as stresses, is a number of orders lower as compared to the accuracy for primary variables. Consequently, also the convergence rate is lower. For the WBM on the other hand, there is no additional loss in prediction accuracy for secondary field variables, since the derivatives of the wave functions are wave functions as well with the same spatial resolution.

Figure 13: Convergence curves at 200, 400 and 800Hz of the WBM and the FEM for two different field variables, using the finest FE model as a reference.

4.4. Kundt's tube simulation

Figure 14: Problem description Kundt tube set-up.

The final example consists of a commonly used measurement setup: the Kundt's tube, as shown in figure 14. The set-up, dimensions and material properties are inspired by the paper of Vigran et al. [15]. The studied poroelastic

material is a polyurethane foam Fireflex, of which the properties are given in table 1. The poroelastic sample is a slab of 5cm thickness, indicated by a grey rectangle on figure 14. The Kundt tube has a diameter of 4cm and the sample is glued onto a rigid backing, applying fixed boundary conditions (A.6). The white rectangle in the figure indicates the acoustic domain. On the top edge of the acoustic domain, a normal velocity of zero is imposed, taking into account that the tube is rigid. The acoustic domain is excited with a normal velocity of 1 m/s, imposed on the right hand side of the domain. The x-marks show the positions of two microphones, which are separated by 5cm. In this way, the two-microphone method can be reproduced, measuring the acoustic pressure at the two nodes and applying the formulas developed for this measurement procedure in order to determine the absorption coefficient. The damping in the acoustic domain is small, but needs to be taken into account when precise measurements, and consequently simulations, are required. The first order high frequency approximation of the acoustic wave number is then given by [41]:

$$k = \frac{\omega}{c_0} \left[1 + \frac{1}{2}(1-j)\frac{\delta}{\Lambda} \left(1 + \frac{\gamma-1}{\sqrt{N_{pr}}} \right) \right], \quad (61)$$

with N_{pr} Prandtl's number, c_0 the speed of sound in free air and δ the viscous skin depth [41]:

$$\delta = \sqrt{\frac{2\eta}{\omega\rho_0}}, \quad (62)$$

with η the fluid dynamic viscosity. This wave number k is taken into account in the acoustic domain of the problem. Figure 15 shows the three wave numbers occurring in the poroelastic material and the wave number in the acoustic domain in function of frequency. Depending on the frequency, the shear wave number or the first longitudinal wave number is setting limit for the truncation rule.

Figure 15: Poroelastic and acoustic wave numbers in function of frequency

In this numerical example, the influence of two different ways of mounting the sample in the tube is compared. In a first set-up, the sample is free to slide along the wall of the tube, applying sliding edge boundary conditions (A.7).

In a second set-up, the sample is glued to the side wall, applying fixed boundary conditions (A.6).

Figures 16 and 17 show contour plots of the displacement field $u_z^s(\mathbf{r})$ and the acoustic stress field $\sigma^f(\mathbf{r})$ within the poroelastic domain obtained with the WBM. A truncation factor $T = 4$ has been applied. The left figures show the results for a poroelastic sample which is allowed to slide along the side wall of the tube, applying mixed boundary conditions, and the right figures show the same dynamic variables obtained for a fixed sample. The figures show that for the first set-up a perfect 1D behaviour is obtained. This was to be expected, since the shear wave is not excited. For this simple set-up analytical formulas could be employed to predict the dynamic fields. For the second setup, a more complex behaviour is obtained.

Figure 16: Contour of the amplitude of $u_z^s(\mathbf{r})$ [m] at 2500Hz with two different types of boundary conditions.

Figure 17: Contour of the amplitude of $\sigma^f(\mathbf{r})$ [Pa] at 2500Hz with two different types of boundary conditions.

By calculating the absorption coefficient α , the influence of different boundary conditions can be verified. The absorption coefficient is given by:

$$\alpha = 1 - |R|^2, \quad (63)$$

with R the reflection coefficient. The reflection coefficient is calculated using the obtained pressure field $p(p_1)$ and $p(p_2)$ at the microphone locations [41]:

$$R = \frac{e^{jkd_1} - e^{jkd_2} p(p_1)/p(p_2)}{e^{-jkd_2} p(p_1)/p(p_2) - e^{-jkd_1}}, \quad (64)$$

with d_1 and d_2 the distances from the poroelastic-acoustic interface to the microphone positions p_1 and p_2 , respectively. Figure 18 shows the absorption coefficient obtained with the WBM and the FEM for both configurations of

boundary conditions and for a frequency range between 250Hz and 5000Hz in steps of 25Hz. In case sliding edge boundary conditions are applied, the absorption coefficient can be exactly calculated [2]. The exact absorption curve is also shown in Figure 18(a). For the WBM a truncation factor $T = 4$ is applied, resulting in 86 wave functions at 250Hz and 419 wave functions at 5000Hz. For the FEM, 6 adaptive mesh refinements are performed, starting from a mesh consisting of 791 acoustic and poroelastic DOFs, resulting in approximately 30,000 DOFs at each frequency line. The results coincide, indicating again that the same results can be represented with the WBM, applying a much lower number of DOFs.

The effect of the presence of the shear wave is clear when comparing figure 18(a) and figure 18(b), indicating the importance of the mounting conditions. The shear wave is not present when sliding edge boundary conditions are imposed. The effect of imposing fixed edge boundary conditions is that the sample exhibits a stiffer behaviour, shifting the peak in the absorption curve related to a resonance in the poroelastic material towards higher frequencies. The obtained results are in good agreement with the results obtained by Vigran et al. [15]. This example indicates future possibilities for the presented method: the prediction and the validation of measurements for axisymmetric geometries containing poroelastic samples. Possibly, the method can furthermore be used for inverse material characterisation.

Figure 18: Absorption coefficient of the porous material obtained with a Kundt tube set-up for two types of boundary conditions, as calculated with the WBM and the FEM.

5. Conclusions

This paper presents a Wave Based prediction technique to solve coupled and uncoupled axisymmetric acoustic and poroelastic problems. The basic principles of the WBM are discussed and are applied to the axisymmetric acoustic equation. Novel wave function sets are derived, exactly fulfilling the Helmholtz equation and the axisymmetry conditions. In a next step, the poroelastic axisymmetric problem is considered. The Biot equations can be decoupled into two scalar and one vector Helmholtz equation. For each of these three equations, suitable wave function sets are defined, analogously to the acoustic problem. In this way, the physics of the three propagating wave types in a porous material is explicitly taken into account. The definition of axisymmetric wave functions allows to solve the

problem in a 2.5D setting, alleviating the computational load of a full 3D problem. Four verification cases show the potential of the proposed method and compare the convergence of the method to quadratic and cubic adaptive Finite Element models. The first two validation cases consider a rectangular and a non-rectangular acoustic axisymmetric problem domain. A very high convergence rate is obtained. A third validation case verifies the method for poroelastic problems, illustrating again the potential of the method. By embedding *a priori* knowledge of the solution field in the expansion functions, a low number of degrees of freedom is sufficient to represent the response field accurately. The last example shows one of the possible application areas of the developed method. The axisymmetric WBM can be used to validate and predict, for instance, Kundt tube measurements. The effect of mounting conditions, ending conditions, etc. can be easily predicted. The inverse characterisation of poroelastic parameters by using the axisymmetric WBM forms an interesting topic of further research.

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Appendix A. Boundary and interface conditions

Appendix A.1. Boundary conditions for an acoustic subdomain Ω_a

The boundary of the considered acoustic subdomain Ω_a is denoted as $\Gamma_a = \partial\Omega_a$ and consists of two parts: the exterior boundary Γ_{ae} and the coupling interface Γ_C with the poroelastic domain: ($\Gamma_a = \Gamma_{ae} \cup \Gamma_C$).

The exterior boundary $\Gamma_{ae} = \Gamma_v \cup \Gamma_p \cup \Gamma_Z$ represents the part of the boundary Γ_a on which Neumann, Dirichlet or Robin boundary conditions can be imposed, respectively. The following boundary condition residuals apply:

$$\mathbf{r} \in \Gamma_v : R_v = \mathcal{L}_v(p(\mathbf{r})) - \bar{v}_n(\mathbf{r}) = 0, \quad (\text{A.1})$$

$$\mathbf{r} \in \Gamma_p : R_p = p(\mathbf{r}) - \bar{p}(\mathbf{r}) = 0, \quad (\text{A.2})$$

$$\mathbf{r} \in \Gamma_Z : R_Z = \mathcal{L}_v(p(\mathbf{r})) - \frac{p(\mathbf{r})}{\bar{Z}_n(\mathbf{r})} = 0, \quad (\text{A.3})$$

where the quantities $\bar{v}_n$, $\bar{p}$ and $\bar{Z}_n$ are, respectively, the imposed normal velocity, pressure and normal impedance. The velocity operator $\mathcal{L}_v$ is defined as:

$$\mathcal{L}_v = \frac{j}{\rho_0 \omega} \frac{\partial}{\partial \gamma_n}, \quad (\text{A.4})$$

with γ_n the normal direction on the boundary. The coupling conditions between an acoustic and a poroelastic subdomain are explained in section Appendix A.3.

Appendix A.2. Boundary conditions for an acoustic subdomain Ω_{pe}

The boundary of the subdomain $\Gamma_{pe} = \partial\Omega_{pe}$ consists of an exterior boundary Γ_{pee} and the interface with the acoustic domain Γ_C . The exterior boundary $\Gamma_{pee} = \Gamma_{ki} \cup \Gamma_{me} \cup \Gamma_{mi}$ represents the part of the boundary Γ_{pe} along which kinematic, mechanical or mixed boundary condition are defined respectively. On each part of the boundary, three residuals are defined:

- *kinematic boundary conditions along Γ_{ki}* ; the displacement components are prescribed:

$$\mathbf{r} \in \Gamma_{ki} : \begin{cases} R_{u_n^s}(\mathbf{r}) = u_n^s(\mathbf{r}) - \bar{u}_n^s(\mathbf{r}) = 0, \\ R_{u_s^s}(\mathbf{r}) = u_s^s(\mathbf{r}) - \bar{u}_s^s(\mathbf{r}) = 0, \\ R_{u_n^f}(\mathbf{r}) = u_n^f(\mathbf{r}) - \bar{u}_n^f(\mathbf{r}) = 0, \end{cases} \quad (\text{A.5})$$

with $\bar{u}_n^s(\mathbf{r})$, $\bar{u}_s^s(\mathbf{r})$ and $\bar{u}_n^f(\mathbf{r})$ the prescribed values of the displacement components of the solid phase in the normal and tangential direction to the boundary and the prescribed value of the displacement of the fluid phase in the normal direction to the boundary, respectively. For a fixed edge, the imposed values are zero.

- *mechanical boundary conditions*; the stress resultants are prescribed:

$$\mathbf{r} \in \Gamma_{me} : \begin{cases} R_{\sigma_n^s}(\mathbf{r}) = \sigma_n^s(\mathbf{r}) - \bar{\sigma}_n^s(\mathbf{r}) = 0, \\ R_{\sigma_s^s}(\mathbf{r}) = \sigma_s^s(\mathbf{r}) - \bar{\sigma}_s^s(\mathbf{r}) = 0, \\ R_{\sigma^f}(\mathbf{r}) = \sigma^f(\mathbf{r}) - \bar{\sigma}^f(\mathbf{r}) = 0, \end{cases} \quad (\text{A.6})$$

with $\bar{\sigma}_n^s(\mathbf{r})$, $\bar{\sigma}_s^s(\mathbf{r})$ and $\bar{\sigma}^f(\mathbf{r})$ the prescribed values of the normal and tangential stress resultant components of the solid phase in the normal and tangential direction to the boundary and the prescribed hydrostatic stress of the fluid phase, respectively.

- *mixed boundary conditions*; displacements as well as a stress resultant are prescribed:

$$\mathbf{r} \in \Gamma_{mi} : \begin{cases} R_{u_n^s}(\mathbf{r}) = u_n^s(\mathbf{r}) - \bar{u}_n^s(\mathbf{r}) = 0, \\ R_{u_n^f}(\mathbf{r}) = u_n^f(\mathbf{r}) - \bar{u}_n^f(\mathbf{r}) = 0, \\ R_{\sigma_s^s}(\mathbf{r}) = \sigma_s^s(\mathbf{r}) - \bar{\sigma}_s^s(\mathbf{r}) = 0. \end{cases} \quad (\text{A.7})$$

For a so called sliding edge, the prescribed values of $\bar{u}_n^s(\mathbf{r})$, $\bar{u}_n^f(\mathbf{r})$ and $\bar{\sigma}_s^s(\mathbf{r})$ are zero.

Appendix A.3. Coupling conditions on the interface between an acoustic and a poroelastic subdomain

On the interface Γ_C between the acoustic and the poroelastic subdomain, the following four coupling conditions are imposed:

$$\mathbf{r} \in \Gamma_C : \begin{cases} R_{\sigma_{n,C}^s}(\mathbf{r}) = \sigma_{n,C}^s(\mathbf{r}) + (1-h)p(\mathbf{r}) = 0, \\ R_{\sigma_{n,C}^f}(\mathbf{r}) = \sigma_{n,C}^f(\mathbf{r}) + hp(\mathbf{r}) = 0, \\ R_{\sigma_{s,C}^s}(\mathbf{r}) = \sigma_{s,C}^s(\mathbf{r}) = 0, \\ R_{u,C}(\mathbf{r}) = \mathcal{L}_v(p(\mathbf{r})) - j\omega((1-h)u_n^s(\mathbf{r}) + hu_n^f(\mathbf{r})) = 0, \end{cases} \quad (\text{A.8})$$

The first condition requires that the normal stress acting on the solid phase of the porous material is equal to $-(1 - h)$ times the pressure in the acoustic field. The second equation gives a similar relation for the stress acting on the fluid phase. Since the air in the acoustic cavity is considered to be inviscid, the shear stress in the solid phase of the poroelastic material has to be zero. This is only valid on the interface with the acoustic domain. The final condition represents the continuity of the normal volume velocity.

Appendix B. Field variable expansions

All field variables, in an acoustic subdomain as well as in a poroelastic subdomain, can be expanded in terms of the set(s) of wave functions in the considered subdomain. In the acoustic subdomain, the acoustic pressure $p(r, z)$ is directly given by the wave function expansion (8). The acoustic velocity components are determined by:

$$v_{\bullet}(r, z) = \mathcal{L}_{v_{\bullet}}(p(r, z)), \quad (\text{B.1})$$

with $\mathcal{L}_{v_r}$ and $\mathcal{L}_{v_z}$ differential operators, defined as

$$\mathcal{L}_{v_r} = \frac{j}{\rho_0 \omega} \frac{\partial}{\partial r}, \quad (\text{B.2})$$

$$\mathcal{L}_{v_z} = \frac{j}{\rho_0 \omega} \frac{\partial}{\partial z}. \quad (\text{B.3})$$

Concerning the poroelastic subdomain, each field variable $a(r, z)$ can be written in terms of the potential field by applying a corresponding differential operator acting on the three potential fields:

$$a(\mathbf{r}) = \mathcal{L}_a \begin{bmatrix} \varphi_1^s(r, z) \\ \varphi_2^s(r, z) \\ \psi^s(r, z) \end{bmatrix}. \quad (\text{B.4})$$

The differential operators for the displacements $u_{\bullet}^*$ and the stresses $\sigma_{\bullet\bullet}^*$ in both the solid ($=^s$) and the fluid ($=^f$) phase are defined as:

$$\mathcal{L}_{u_r^s} = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial r} & -\frac{\partial}{\partial z} \end{bmatrix}, \quad (\text{B.5})$$

$$\mathcal{L}_{u_z^s} = \begin{bmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \end{bmatrix}, \quad (\text{B.6})$$

$$\mathcal{L}_{u_r^f} = \begin{bmatrix} \mu_{l_1} \frac{\partial}{\partial r} & \mu_{l_2} \frac{\partial}{\partial r} & -\mu_t \frac{\partial}{\partial z} \end{bmatrix}, \quad (\text{B.7})$$

$$\mathcal{L}_{u_z^f} = \begin{bmatrix} \mu_{l_1} \frac{\partial}{\partial z} & \mu_{l_2} \frac{\partial}{\partial z} & \mu_t \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \end{bmatrix}, \quad (\text{B.8})$$

$$\mathcal{L}_{\sigma_{rr}^s} = \begin{bmatrix} -k_{l_1}^2 (A + \mu_{l_1} Q) + 2N \frac{\partial^2}{\partial r^2} & -k_{l_2}^2 (A + \mu_{l_2} Q) + 2N \frac{\partial^2}{\partial r^2} & 2N \left(\frac{\partial^2}{\partial r \partial z}\right) \end{bmatrix}, \quad (\text{B.9})$$

$$\mathcal{L}_{\sigma_{rz}^s} = \begin{bmatrix} 2N \frac{\partial^2}{\partial r \partial z} & 2N \frac{\partial^2}{\partial r \partial z} & N \left(-\frac{1}{r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial z^2}\right) \end{bmatrix}, \quad (\text{B.10})$$

$$\mathcal{L}_{\sigma_{zz}^s} = \begin{bmatrix} -k_{l_1}^2 (A + \mu_{l_1} Q) + 2N \frac{\partial^2}{\partial z^2} & -k_{l_2}^2 (A + \mu_{l_2} Q) + 2N \frac{\partial^2}{\partial z^2} & 2N \left(\frac{1}{r} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial r \partial z}\right) \end{bmatrix}, \quad (\text{B.11})$$

$$\mathcal{L}_{\sigma^f} = \begin{bmatrix} -k_{l_1}^2 (Q + \mu_{l_1} R) & -k_{l_2}^2 (Q + \mu_{l_2} R) & 0 \end{bmatrix}. \quad (\text{B.12})$$

μ_{l_1} , μ_{l_2} and μ_t are the ratios of the fluid and the solid displacements of the different wave types [2] and are given by:

$$\mu_{l_1} = \frac{Pk_{l_1}^2 - \omega^2 \tilde{\rho}_{11}}{\omega^2 \tilde{\rho}_{12} - Qk_{l_1}^2}, \quad (\text{B.13})$$

$$\mu_{l_2} = \frac{Pk_{l_2}^2 - \omega^2 \tilde{\rho}_{11}}{\omega^2 \tilde{\rho}_{12} - Qk_{l_2}^2}, \quad (\text{B.14})$$

$$\mu_t = -\frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}}. \quad (\text{B.15})$$

Furthermore, for the practical implementation of the differential operators, the following relations between Bessel functions have been used [36]:

$$\frac{dJ_0(qx)}{dx} = -qJ_1(qx), \quad (\text{B.16})$$

$$\frac{dJ_1(qx)}{dx} = qJ_0(qx) - \frac{1}{x}J_1(qx). \quad (\text{B.17})$$

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