Electric and Magnetic Fields of two Infinitely Long Parallel Cylindrical Conductors Carrying a DC Current

Kristof Engelen, Pieter Jacqmaer and prof. dr. Johan Driesen

University of Leuven, Department of Electrical Engineering (ESAT), Heverlee, Belgium
E-mail: kristof.engelen1@esat.kuleuven.be

***Résumé* — This paper calculates the electric and magnetic fields and the Poynting vector around two infinitely long parallel cylindrical conductors, carrying a DC current. Also the charges on the surface of the wire are calculated, and their distribution is visualized. The wire is assumed to be perfectly electrically conducting. Furthermore, the Hall effect is ignored. In literature [1], the problem of determining the electric field is usually tackled using an equivalent model consisting of two line charge densities, producing the same electric field. In this work, the Laplace equation is rigorously solved. The authors found no work explaining the solution of the Laplace equation with boundary conditions for this problem and hence thought it was useful to dedicate a paper to this topic. The method of separation of variables is employed and a bipolar coordinate system is used. After solving the appropriate Sturm-Liouville problems, the scalar potential is obtained. Taking the gradient yields the electric field.**

I. PROBLEM DESCRIPTION

Two perfectly conducting cylindrical wires, each conducting a DC current I are parallel to the z-axis (Fig. 1). The first wire is located at $x = -d$ and conducts the current in the positive z-direction. The other wire is located at $x = d$ and conducts the current in the negative z-direction. The left wire has a potential V_1 and the right wire $-V_1$. We wish to determine the scalar potential, the electric and the magnetic field in the region outside the conductors. The cross-section of the right wire is bounded by a circle, called C_1.

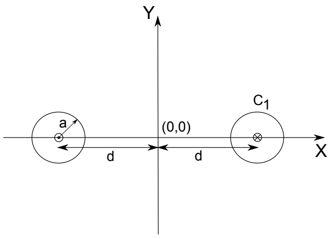

Fig. 1. Geometry of the problem

The electric potential ϕ is given by the Laplace equation:

$$\nabla^2 \phi = 0 \tag{1}$$

We'll solve it using an appropriate coordinate system: bipolar coordinates.

II. BIPOLAR COORDINATES AND BOUNDARY CONDITIONS

The bipolar coordinate system [3] is defined as ($\tau \in [-\infty, \infty]$ and $\sigma \in [0, 2\pi]$):

$$\begin{cases} x = \alpha \dfrac{\sinh \tau}{\cosh \tau - \cos \sigma} & (2) \\ y = \alpha \dfrac{\sin \sigma}{\cosh \tau - \cos \sigma} & (3) \end{cases}$$

Thanks are due to prof. André Koch Torres Assis, professor of physics at the University of Campinas - UNICAMP, in Brazil, for the illuminating discussions.

For the problem of determining the electric scalar potential in the region outside the two wires, also the boundary conditions are rewritten in terms of bipolar coordinates.

A. *First boundary condition: $\phi = 0$ on the y-axis*

The y-axis has the equation $x = 0$. Thus, according to (2), this corresponds to points in the (τ, σ)-plane with $\tau = 0$.

B. *Second boundary condition: $\phi = V_1$ on the circumference of the circle C_1*

The circle C_1, with origin point $(0, d)$ and radius a, coincides with a circle of constant τ (cfr. [3]), if

$$\begin{cases} d = \alpha \coth \tau \\ a = \frac{\alpha}{\sinh \tau} \end{cases} \tag{4}$$

We thus have:

$$d = \alpha \coth \tau = \underbrace{\frac{\alpha}{\sinh \tau}}_{=a} \cosh \tau = a \cosh \tau$$

Let's call the τ for which this circle C_1 is defined, τ_c. We have $\cosh \tau_c = d/a$, and C_1 is thus defined by the single τ-coordinate:

$$\tau_c = \cosh^{-1} \frac{d}{a} \tag{5}$$

We can also calculate α from the second equation of (4):

$$a^2 = \frac{\alpha^2}{\sinh^2 \tau_c} = \frac{\alpha^2}{\cosh^2 \tau_c - 1} = \frac{\alpha^2}{\left(\frac{d^2 - a^2}{a^2}\right)}$$

Thus:

$$\alpha = \sqrt{d^2 - a^2} \tag{6}$$

III. FORMULATION OF THE PROBLEM IN BIPOLAR COORDINATES

The scalar potential ϕ has to obey the Laplace partial differential equation in bipolar coordinates [2]:

$$\nabla^2 \phi = \frac{1}{\alpha^2} (\cosh \tau - \cos \sigma)^2 \left(\frac{\partial^2 \phi}{\partial \sigma^2} + \frac{\partial^2 \phi}{\partial \tau^2} \right) = 0 \tag{7}$$

with boundary conditions:

$$\begin{aligned} \phi(\sigma, \tau = 0) &= 0, \qquad \sigma \in [0, 2\pi] \quad (8) \\ \phi(\sigma, \tau = \tau_c) &= V_1, \qquad \sigma \in [0, 2\pi] \quad (9) \end{aligned}$$

where ϕ is a periodic function in σ with a period equal to 2π, and τ_c as defined in (5).

IV. SOLUTION OF THE LAPLACE EQUATION WITH BOUNDARY CONDITIONS

The method of separation of variables is applied. This means that the solution for the potential ϕ is written as the product of single-variable functions:

$$\phi(\sigma,\tau) = S(\sigma)T(\tau) \tag{10}$$

When we substitute this equation in the Laplace equation (7), we must solve a system of two differential equations:

$$\begin{cases} S''(\sigma) - kS(\sigma) = 0 & (11) \\ T''(\tau) + kT(\tau) = 0 & (12) \end{cases}$$

This has different solutions according to the sign of k. These are in detail discussed in the full paper. After using the two boundary conditions, we obtain, for the scalar potential:

$$\phi(\tau) = \frac{V_1}{\tau_c}\tau \tag{13}$$

V. EXPRESSING THE SCALAR POTENTIAL IN CARTESIAN COORDINATES

In the full paper, the complete solution for the scalar potential in Cartesian coordinates is also derived:

$$\phi(x,y) = \frac{V_1}{\ln\left(\frac{d}{a} + \sqrt{\frac{d^2}{a^2}-1}\right)} \ln\sqrt{\frac{(x+\sqrt{d^2-a^2})^2+y^2}{(x-\sqrt{d^2-a^2})^2+y^2}} \tag{14}$$

VI. ELECTRIC FIELD

The electric field outside the two conducting wires, is thus:

$$\vec{E}(x,y) = -\vec{\nabla}\phi \tag{15}$$

and can be expressed in Cartesian coordinates. This expression will be given in the full paper.

VII. MAGNETIC FIELD

A DC current I flows in the positive z-direction in the wire at $x = -d$, and a DC current of I flows in the negative z-direction in the wire at $x = d$. In the full paper, the expression for the magnetic field will be derived in detail. There must be made distinction between three regions of space:

A. *Magnetic field in the region between the two wires: $-d \le x \le d$*

$$\begin{aligned}\vec{H}(x,y) &= \frac{Iy\vec{e}_x}{2\pi}\left(\frac{-1}{(x+d)^2+y^2} + \frac{1}{(x-d)^2+y^2}\right) + \\ &\quad \frac{I\vec{e}_y}{2\pi}\left(\frac{(x+d)}{(x+d)^2+y^2} + \frac{-(x-d)}{(x-d)^2+y^2}\right)\end{aligned} \tag{16}$$

We can see that inside this region, the dot product of the electric and the magnetic field is not zero, except on the x- and y-axes.

B. *Magnetic field in the regions for which $x \le -d$ or $x \ge d$*

We can again express the magnetic field in Cartesian coordinates. However, the expression is so long that for numerically evaluating it, in the full paper we recommend a different approach, consisting of a method using polar coordinates.

VIII. SURFACE CHARGES ON THE WIRES

The surface density of the free charges on the surface of the right wire, centered around $x = d$, is called σ_R, and is equal to ϵ_0 times the normal component of the electric field, perpendicular to the circle C_1. Here, ϵ_0 is the permittivity of vacuum. The surface density of the charges on the surface of the left wire is then, by symmetry, the mirrored image and the negative of σ_R. Let's determine σ_R. We choose a polar coordinate system (r,β) where r is the distance from $(x,y) = (d,0)$ to the observed point, and β is the angle between the x-axis, pointing from $(x,y) = (d,0)$ to infinity and the line from $(x,y) = (d,0)$ to the observed point.
We find a very simple form for the radial component of the electric field; hence, the surface charge density $[C/m^2]$ can be determined and is:

$$\sigma_R = \epsilon_0 E_r(r=a,\beta) = \frac{\epsilon_0 V_1\sqrt{d^2-a^2}}{a(d+a\cos\beta)\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)} \tag{17}$$

This corresponds with the expression derived with the equivalent line charge model, in [4].

IX. VISUALISATION OF SOLUTIONS

With $I = 1$ A, $V_1 = 0.5$ V, $d = 0.01$ m and $a = 0.5$ mm, the full paper will show figures depicting, with some contours, the Poynting field, the electric field, the magnetic field and the scalar potential outside the wires. Also, some fieldlines of the electric field, starting at the right wire will be shown, and the surface charge density in the surface of the right wire.

As a check, the Poynting vector was numerically integrated over the xy-plane. We obtained 1.004 W, corresponding well with the theoretical value of 1 V $\times$ 1 A = 1 W.

REFERENCES

[1] S. J. Orfanidis, *Electromagnetic waves and antennas*, online available: www.ece.rutgers.edu/ orfanidi/ewa (2008), chapter 10.

[2] E. W. Weisstein, *Laplace's Equation–Bipolar Coordinates*, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LaplacesEquationBipolarCoordinates.html

[3] J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York and London (1941), pp. 55– 56.

[4] A. K. T. Assis and J. A. Hernandes, *The Electric Force of a Current: Weber and the Surface Charges of Resistive Conductors Carrying Steady Currents*, Apeiron, Montreal, (2007), 239 pages