Controlling excessive waiting times in small service systems with time-varying demand: an extension of the ISA algorithm

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Abstract

In many service systems, the arrival pattern is not constant throughout the day. This raises the question how staffing decisions should be adapted in view of controlling customer's waiting times. Assuming a single-stage queueing system with general abandonment and service times and time-varying demand for service, we suggest a method inspired by the simulation-based Iterative Staffing Algorithm (ISA) proposed by Feldman et al. (2008). The main advantage of our extension is that it enables to control the probability of experiencing an excessive waiting time, in particular in small systems.

Keywords: staffing, personnel planning, time-varying arrival rate, simulation, Iterative Staffing Algorithm

1. Introduction

In many service systems, the demand for service is not constant over time. Fluctuations on a daily, monthly, weekly or yearly basis can be present, which complicates the process of determining appropriate staffing

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levels. Our particular focus lies on emergency departments (ED's): in these systems, capacity is the main lever to control waiting times, as customer service cannot be guaranteed by means of waiting lists or appointment systems. Determining capacity adequately in view of preserving customer service is a complex task, and as a result the capacity strategy in ED departments often mismatches demand ([28]). In an ED, service is mainly related to the length of the customer's waiting time (in particular the longest waits), and hence, controlling excessive waiting times is usually a primary goal. However, most approaches proposed in the literature focus on delay probabilities or expected waiting times.

This paper presents a simulation-based staffing method that enables to stabilize the *probability of excessive waiting* (i.e., the probability that the waiting time exceeds a maximum acceptable value) throughout the day. The suggested method is inspired by the Iterative Staffing Algorithm (ISA), proposed by Feldman et al. [7], which focuses on stabilizing the *delay probability* throughout the day (note that this corresponds to a maximum acceptable wait of zero). The use of discrete-event simulation provides distinct advantages over analytical methods, such as increased flexibility in modeling assumptions and the ability to control the precision of the results. The downside is that evaluation through simulation tends to be more time-consuming. Moreover, choosing appropriate staffing levels is further complicated due to the focus on excess wait probabilities: obtaining the probability of excessive waiting from the simulation's outcome is in general more difficult than calculating the delay probability. This research proposes a computationally efficient way to evaluate the probability of excessive waiting, and adjusts the staffing update function of the original ISA algorithm to account for the

relatively small system size that characterizes an ED. Indeed, as opposed to call centers (which are commonly studied in the literature dealing with time-varying arrival rates), ED arrival rates tend to be rather low: [14] report hourly arrival rates ranging between 1 and 5 (in a New York ED); [27] describe a large urban hospital with hourly arrival rates varying between approximately 2 and 9 customers per hour.

While in reality, the ED resembles a multiserver queueing *network* in which patients move through several process steps to receive treatment, our approach considers the ED as a single-stage multiserver system with customer abandonments. Our experiments indicate that the proposed method (which we call $\text{ISA}(\tau)$) succeeds in finding a staffing vector that meets the performance constraint, irrespective of system size. Large systems (for which the number of servers required is in the order of 100) and extremely small instances (requiring only 1-2 servers) can be solved, although the computation time increases with the problem size. A solution can be obtained for exponential as well as general service and abandonment time distributions, and staffing intervals are taken into consideration.

The remainder of the paper is organized as follows: in Section 2 we zoom in on the research problem. Relevant literature is discussed in Section 3: Section 3.1 discusses the stationary approximations that are most common in our context, while Section 3.2 describes the Iterative Staffing Algorithm proposed in [7] which can be considered as the starting point of the $\text{ISA}(\tau)$ approach we suggest. A detailed description of $\text{ISA}(\tau)$ follows in Section 4. Computational results of $\text{ISA}(\tau)$ are reported in Section 5, and are compared to the staffing results obtained by applying stationary approximation techniques available in the literature. Section 6 summarizes our results, along

with some directions for future research.

2. Problem description

We consider the ED as a single-stage multiserver $M(t)/G/s(t)+G$ queue, with time-varying arrivals and customer impatience (a schematic representation is given in Figure 1). Customers enter the system according to a Poisson pattern with time-varying arrival rate $\lambda(t)$ over the time horizon $[0, T]$ (the commonly used assumption of Poisson arrival patterns is supported by [1]). Only fluctuations on a daily basis are considered, as these tend to be most outspoken ([18]). Customers may leave the system prematurely: the abandonment rate is denoted by θ . Let μ denote the service rate of the service process, i.e. the rate at which one server (or physician) can treat customers (or patients). Once the treatment is finished, the patient leaves the system. Note that in reality, patients that need to be admitted to the hospital often have to wait after treatment until a bed becomes available (see e.g. [2]); as this waiting time results from factors other than the staffing decisions in the ED, we do not take it into account in our model. Contrary to the arrival process, the service and abandonment rates are assumed to be time-invariant and can follow any general distribution. The main goal is to determine an appropriate staffing requirement function $s(\tilde{t})$, which defines the number of servers (or physicians) to be assigned in each staffing interval $\tilde{t}$. The staffing function should guarantee that the excess wait probability (defined as the probability that an arriving customer's waiting time exceeds some maximum acceptable waiting time) is sufficiently low at all time instants. This can be expressed as:

$$Pr(W(t) > \tau) \leq \alpha \quad 0 \leq t < T \quad (1)$$

Figure 1: Schematic representation a single-stage queueing system with time-varying demand

where $W(t)$ represents the virtual waiting time¹ of a customer arriving at time t , τ indicates the maximum acceptable waiting time, and α denotes the targeted (i.e. the maximum allowed) excess wait probability. We assume that τ is significantly smaller than the expected time-to-abandon (otherwise τ would probably be misspecified, especially in the context of an ED). Note that, by controlling excess wait probabilities, the number of abandoning customers is influenced as well. The number of servers is only allowed to change at fixed points in time; the planning period over which the number of servers is assumed to remain unchanged is referred to as the staffing interval (hence the distinction between time t and staffing interval $\tilde{t}$).

Note that Expression 1 reduces to a constraint on the delay probability for $\tau = 0$. Consequently, our performance requirement is more general than the (commonly used) delay probability. Measuring performance based on excess wait probabilities gives more leeway to the staffing level function,

¹The virtual waiting time $W(t)$ is defined as the time that a customer would have to spend in queue if he were to arrive at time t (cf. [16], pp. 13-14).

while at the same time preserving patient service, as the decision maker can decide on the value of τ .

3. Related literature

In this section, we briefly describe the literature that is most relevant to our research: we provide a description of the so-called stationary approximations (Section 3.1) and discuss the Iterative Staffing Algorithm (ISA) introduced in [7] (Section 3.2). For a more comprehensive overview of the literature on staffing systems with time-varying demand, the interested reader is referred to [15], [32], [4], and references therein.

3.1. Stationary approximations

The most common way to deal with the time-varying nature of the demand for service, is to approximate the non-stationary, time-varying system by one (or more) related stationary model(s). In Section 5, the performance of $\text{ISA}(\tau)$ is tested against a number of these approaches, so we prefer to briefly discuss them here.

In the *Pointwise Stationary Approximation* (PSA), the arrival rate at each time instant is plugged into a separate stationary model to obtain steady state performance measures for each moment in time ([10], [22], [29]). PSA is most appropriate in large systems with limited nonstationarity, characterized by high service rates, high targeted quality of service, and low to moderate loads.

To improve performance in case of low service rates, a *lagged* variant of PSA (denoted LagPSA) has been proposed ([11], [12]). This approach is identical to PSA, except for the use of a lagged arrival rate $\lambda(t - E[S])$,

where $E[S]$ (i.e. the expected service time) represents a *time lag* (further theoretical background supporting this choice, can be found in [6]).

In the *Modified Offered Load* (MOL, [21],[25], [26]) approach, a stationary model is solved at each point in time, using a modified arrival rate that equals the product of the service rate and the infinite server offered load (i.e. the number of servers that would be used if infinitely many servers were available). As the number of servers declines, the MOL approximation becomes less accurate because of a lower resemblance to the infinite server system ([26]).

The PSA, lagPSA and MOL approaches vary staffing levels continuously and do not account for the presence of staffing intervals. To this purpose, two refinements to the PSA approach have been proposed: Segmented PSA and *Stationary Independent Period-by-Period* (SIPP). In the *Segmented PSA* approach (see e.g. [15]), the staffing levels are set equal to the maximum of the PSA staffing requirements over the staffing interval ([15]). The SIPP approach ([12], [14]) uses a stationary model in each staffing interval, with the arrival rate averaged over that interval. As shown in [12], SIPP does not perform well when staffing intervals are long or when the arrival rate changes substantially over the staffing interval.

Further refinements to the SIPP approach have been proposed in [12] and [13]: Lag SIPP (which uses a lagged arrival rate), SIPPmax (which uses the maximum arrival rate over the staffing interval instead of the average, and hence coincides with Segmented PSA), and Lag SIPPmax (which is a combination of both). As shown in [13], Lag SIPP and Lag SIPPmax tend to have better performance than SIPP and SIPPmax.

Once the nonstationary system has been transformed into one or more stationary models, approximations are often needed to obtain the steady state performance measures, in particular when service times and abandonment times are generally distributed (as in our setting). For further details on approximations for the $M/G/s + G$ model, we refer to [20] and [31].

Often however, explicit performance calculations are avoided by using a rule of thumb, known as the *square-root staffing rule* (SRS; see [8], [30] and [23]). The main benefit of SRS lies in its simplicity and robustness: at each time instant, staffing is determined as the offered load (denoted m) augmented with an amount of safety capacity (cf. Expression 2). The required safety capacity is proportional to the offered load, and depends on the desired quality of service (which is reflected in the quality of service parameter β):

$$s = m + \beta\sqrt{m} \quad (2)$$

The appropriate β can be obtained through the inverse of the *Halfin-Whitt delay function* (for $M/M/s$ models, cf. [17]) or the *Garnett delay function* (for $M/M/s + M$ models, cf. [9]). An extension towards $M/M/s + G$ models can be found in [33]. As a rule of thumb, the SRS rule is easy to apply. However, it provides no firm guarantee that the desired performance constraint is actually met.

As stated in [12], the use of stationary models implicitly assumes that: (1) delays between separate intervals are statistically independent; (2) steady state is reached in each interval, (3) the arrival rate remains constant over the staffing interval, and (4) no overloading is present within any given interval, as this would cause instability in the stationary model. These assumptions

are unlikely to hold in an ED context, as an ED represents a system that is often temporarily overloaded (violating assumption 4) and where substantial changes in the arrival rate during the staffing interval are common (violating assumption 3). Moreover, due to the combination of typically large service times and rather short staffing intervals, assumptions 1 and 2 are likely to be violated as well.

3.2. The Iterative Staffing Algorithm (ISA)

In [7], a promising simulation-based technique for determining staffing requirements in time-varying queues is proposed. As the name suggests, the *Iterative Staffing Algorithm* (ISA) repeatedly evaluates and alters the staffing function, until the desired performance is attained. For each staffing function, system performance is evaluated by means of simulation and the staffing level is updated based on the observed performance. This sequence of evaluating performance and updating staffing levels is called an *iteration*.

Performance is expressed in terms of a constraint on the delay probability, that is, the delay probability must lie below a target value α at all time instants:

$$Pr(W(t) > 0) \leq \alpha \quad 0 \leq t \leq T \quad (3)$$

Equivalently, the delay probability equals the probability that the number of customers in the system $N(t)$ surpasses the available capacity $s(\tilde{t})$, leading to the following constraint:

$$Pr(N(t) \geq s(\tilde{t}|t \in \tilde{t})) \leq \alpha \quad 0 \leq t \leq T \quad (4)$$

The ISA assumes staffing changes can be made almost continuously. The planning horizon T is divided into very small intervals: staffing changes are only allowed at the start of each interval and the number in system is evaluated once every staffing interval. ISA then proceeds as follows. Initially, all staffing levels are set equal to an arbitrarily large number. Subsequently, system performance is simulated by performing a fixed number of independent replications, which results in a distribution of the number of customers in system at each moment in time. Then, staffing levels are improved (simultaneously for all staffing intervals) such that in each staffing interval $\tilde{t}$, the staffing level corresponds to the smallest value of $s(\tilde{t})$ satisfying the performance requirement in Expression 4. Formally, the evaluation of the distribution of the number of patients in system at the start of staffing interval $\tilde{t}$ in iteration i (denoted $N_i(\tilde{t})$) determines the staffing level in interval $\tilde{t}$ in iteration $i + 1$ (denoted $s_{i+1}(\tilde{t})$):

$$s_{i+1}(\tilde{t}) = \arg \min \{k \in \mathbb{N} : Pr(N_i(\tilde{t}) \geq k) \leq \alpha\} \quad \forall \tilde{t} \quad (5)$$

The algorithm stops when the staffing changes in subsequent iterations become sufficiently small for all staffing intervals (i.e., staffing levels differ by at most 1, for all $\tilde{t}$).

The major advantage of the ISA lies in the use of simulation to evaluate system performance. As a result, the appropriateness of the staffing function generated by ISA is validated automatically (that is, under the assumption that the simulation model is adequate). Moreover, the method has potential to be applied to general settings, for which analytical results are no longer available. However, some aspects make the traditional ISA less appropriate

in an ED context:

- Firstly, an ED is commonly characterized by a rather *small system size*; arrival rates tend to be much lower than in the examples provided in [7]. As discussed in the original publication, the delay probabilities obtained by the original ISA tend to be less stable in periods with low demand, as even a small change in capacity has a substantial impact on performance. Moreover, the conventional stopping rule of the ISA might pose problems: the algorithm stops when the change in staffing requirements is at most 1 unit in all staffing intervals and thus staffing changes of ± 1 server are disregarded. In small systems however, the addition or removal of one server can result in substantial differences in performance.
- Secondly, the ISA does not explicitly deal with the *length of the staffing intervals*, i.e. the time period over which capacity remains constant (all examples used in [7] assume small staffing intervals with a length of $0.1/\mu$). As the number of customers in system is measured only once every interval, it can be expected that an increase in the staffing interval length will lead to a decrease in accuracy, which will negatively impact the algorithm's performance. In the method we present in Section 4, this problem is addressed by making a distinction between staffing intervals and (smaller) calculation intervals.
- Finally, the results in [7] indicated that a staffing function that stabilizes delay probability does not automatically stabilize other performance measures (such as abandonment probabilities, average queue lengths and average waiting times). As discussed in Section 2, the delay probability is not the most appropriate performance measure in an ED environment, as most patients are able to tolerate a (small)

waiting time. Focusing on the probability of excessive waiting is more relevant and allows more flexibility, as the decision maker can decide both on the maximum acceptable waiting time that should be met, and on the service level α (see Expression 1). Focusing on excess wait probabilities, however, implies that staffing levels can no longer be set using Expression 5.

These observations justify the search for a method which (1) is suitable for very small system sizes such as an ED, (2) is capable of dealing with staffing intervals over which the capacity remains constant and (3) emphasizes low probabilities of excessive wait rather than delay probabilities. In the next sections, a more detailed description of the $\text{ISA}(\tau)$ method is given.

4. $\text{ISA}(\tau)$ algorithm

4.1. Notation

Let $s_i(\tilde{t})$ represent the staffing level in staffing interval $\tilde{t}$ at the start of i^{th} iteration of the algorithm, so that $\mathbf{s}_i$ denotes the staffing vector. The performance vector corresponding to $\mathbf{s}_i$ is represented as $\mathbf{P}_i$; its elements $P_i(t)$ denote the excess wait probability for a customer arriving at the start of each *calculation* interval t . The squared coefficient of variation of $\mathbf{P}_i$ will be represented by SCV_i . For a given staffing interval $\tilde{t}$ starting at t_s and ending at t_f , $P_{\max,i}(\tilde{t})$ denotes the maximum value of the excess wait probabilities $P_i(t)$ that directly result from the capacity $s_i(\tilde{t})$ (as detailed below in Section 4.2, this implies $t \in [t_s - \tau, t_f - \tau]$). Each vector $\mathbf{P}_i$ (containing excess wait probabilities per calculation interval t) can thus be translated into vector $\mathbf{P}_{\max,i}$ (containing maximum excess wait probabilities per staffing interval $\tilde{t}$). For a given staffing vector $\mathbf{s}_i$, e_i denotes the number of staffing intervals

for which the performance target is not met (i.e. $P_{max,i}(\tilde{t}) > \alpha$). Additionally, we associate a cost c_i (expressed in man-hours) with each staffing vector $\mathbf{s}_i$, and define c^* to be the lowest cost found so far, for a solution that meets the performance constraint at all times. The cost (expressed in man-hours) associated to using one unit of capacity during one staffing interval is represented as u .

4.2. Procedure

The ISA(τ) algorithm proceeds as follows: initially, the capacity in each staffing interval is set equal to the offered load that would prevail if the system were stationary, namely the ratio of the average arrival rate over the time horizon to the service rate (rounded up, since capacity is integer). A first phase in the algorithm (referred to as PHASE I, and summarized in Algorithm 1) aims at quickly finding a staffing vector for which performance is *close* to the target (but not necessarily below the target value at all times). To this purpose, the current staffing level function $s_i(\tilde{t})$ is altered iteratively, based on the simulation output.

During each iteration i of the algorithm, $s_i(\tilde{t})$ is updated as follows:

$$s_{i+1}(\tilde{t}) = \begin{cases} \lceil s_i(\tilde{t})A_i(\tilde{t}) \rceil & \text{if } A_i(\tilde{t}) \geq 1, \quad \forall \tilde{t} \\ \lfloor s_i(\tilde{t})A_i(\tilde{t}) \rfloor & \text{if } A_i(\tilde{t}) < 1, \quad \forall \tilde{t} \end{cases} \quad (6)$$

where $A_i(\tilde{t})$ refers to an amplification factor, which is determined based on the deviation between $P_{max,i}(\tilde{t})$ and the target α (in percent):

$$A_i(\tilde{t}) = 1 + \frac{P_{max,i}(\tilde{t}) - \alpha}{\alpha i} \quad \forall \tilde{t} \quad (7)$$

$P_{max,i}(\tilde{t})$ is derived from the simulation results, using the evaluation method

described in [3]. For each staffing interval $\tilde{t}$ (starting at t_s and ending at t_f), $P_{max,i}(\tilde{t}) = \max \{P_i(t) : t \in [t_s - \tau, t_f - \tau]\}$. Indeed, if a capacity shock occurs at t_s , the excess wait probabilities are affected for all arrivals after $(t_s - \tau)$, as τ represents the time window within which these patients should start service in order to have a waiting time below the maximum wait (as illustrated in Figure 2). Thus, the capacity in a staffing interval $\tilde{t}$ (starting at t_s and ending at t_f) has a direct² effect on the performance of patients arriving during interval $[t_s - \tau, t_f - \tau]$.

Figure 2: Interval over which capacity impacts performance

Excess wait probabilities below (above) target will result in an $A_i(\tilde{t})$ below (above) 1 and thus a decrease (increase) in capacity in the corresponding interval (note that due to the rounding in Expression 6, capacity is always increased or decreased with at least one unit).

The use of the scaling factor i in the denominator of Expression 7 ensures that $A_i(\tilde{t})$ approaches 1 ($\forall \tilde{t}$) as the number of iterations increases. This forces the algorithm to decrease the size of the staffing changes as it

²It is clear that an indirect effect is present as well; i.e. the capacity level in any staffing interval also has an impact on the performance at all later time instants, through the number of customers in system.

Algorithm 1 ISA(τ): PHASE I : EXPLORATION

Initial staffing vector: $s_0(\tilde{t}) = \lceil \frac{\bar{\lambda}}{\mu} \rceil \quad \forall \tilde{t}$
 Initialize stop criterion: $stop \leftarrow \text{FALSE}$
 Initialize iteration counter: $i \leftarrow 0$
while $stop = \text{FALSE}$ **do**
 $i \leftarrow i + 1$
 Simulate staffing vector $\mathbf{s}_i$ (result: performance vector $\mathbf{P}_{max,i}$)
 Update capacity in all $\tilde{t}$:
 $A_i(\tilde{t}) \leftarrow 1 + \frac{P_{max,i}(\tilde{t}) - \alpha}{\alpha i}$
 $s_{i+1}(\tilde{t}) \leftarrow \begin{cases} \lceil s_i(\tilde{t}) A_i(\tilde{t}) \rceil & \text{if } A_i(\tilde{t}) \geq 1, \quad \forall \tilde{t} \\ \lfloor s_i(\tilde{t}) A_i(\tilde{t}) \rfloor & \text{if } A_i(\tilde{t}) < 1, \quad \forall \tilde{t} \end{cases}$
 Determine $\mathbb{I}(SCV_i) \leftarrow \begin{cases} 1 & \text{if } SCV_i \leq SCV_{i-1} \\ 0 & \text{otherwise} \end{cases}$
 if $\exists j < (i + 1) | \forall t : s_j(\tilde{t}) = s_{i+1}(\tilde{t})$ **then**
 $stop \leftarrow \text{TRUE}$; repetition in staffing levels, so proceed to PHASE II
 else if Alternating behavior in $\mathbb{I}(SCV_i)$ and $SCV_i \leq 1$ **then**
 $stop \leftarrow \text{TRUE}$; SCV performance is stabilizing, so proceed to PHASE II
 end if
end while

progresses, eventually switching to unit-size changes (and, in the limit, converging to a final staffing vector despite the fact that possible deviations from the target may still remain). In fact, the choice of the scaling factor in the denominator of Expression 7 is rather arbitrary; other factors may be considered, such as $i/2$ or i^2 . Especially in large systems, high values for the factor should be used with caution: they may lead the algorithm to switch to unit-size capacity changes too soon, causing the number of iterations in the exploration phase to increase substantially. This issue is illustrated numerically in Section 5.1.

We allow the algorithm to stop exploring when either (1) cycling occurs (meaning that the staffing vector put forward in the current iteration has already been assessed during a previous iteration), or (2) when the excess

wait probability is stabilizing³. To check condition (2), we keep track of the squared coefficient of variation of the excess wait probability over the time horizon in each iteration i (denoted as SCV_i), and record whether it increases or decreases with respect to the previous iteration (denoted as $\mathbb{I}(SCV_i) = 1$ if $SCV_i \leq SCV_{i-1}$, 0 otherwise). The algorithm stops when SCV_i is sufficiently low (we obtained good results with the criterion $SCV_i \leq 1$ in our experiments) and a cycle occurs in $\mathbb{I}(SCV_i)$, signaling that SCV_i starts alternating between increase(s) and decrease(s) for subsequent iterations without any significant improvement. Though condition (2) is more ad hoc, it is particularly useful in large systems: we observed that although many iterations are needed before cycling in the staffing levels occurs (i.e. before condition (1) is met), SCV_i usually stabilizes far more quickly. An illustration of the typical evolution of SCV_i throughout the algorithm in a large problem setting is given in Figure 3. As such, criterion (2) can substantially lower the computational time in PHASE I for large systems.

Figure 3: Evolution SCV_i

³Recall that the original ISA uses a different terminating condition: it stops when the staffing level changes at most with one unit in each interval, compared to the previous iteration. We opt not to use this stop criterion due to the focus on small system sizes, where one unit capacity changes may occur more frequently (causing the algorithm to stop prematurely).

Note that the exploration phase does not necessarily result in a feasible staffing vector. Consequently, an additional fine-tuning procedure (PHASE II or “exploitation phase”, detailed in Algorithm 2) is needed. The aim of PHASE II is to deduce feasible solutions from the infeasible staffing levels obtained in PHASE I, in hope of finding a solution which outperforms the best feasible solution found so far (if any) in terms of labor cost. To that end, all infeasible solutions encountered during PHASE I are first sorted based on increasing maximum excess wait probability over the time horizon (i.e. increasing values of $\max_{\tilde{t}}\{P_{max,k}(\tilde{t})\}$, where index k denotes an infeasible solution). Indeed, a lower value indicates smaller deviations from the target, which makes the corresponding solution more promising to fine-tune. In case of a tie, the projected staffing cost (expressed in man-hours) is considered, i.e. the staffing cost that results when adding 1 unit of capacity to each staffing interval that causes the excess wait probability to surpass the target. Indeed, if this target is exceeded in just a limited number of intervals, a small number of capacity increases may be sufficient to obtain a feasible staffing level vector, which is appealing from a labor cost perspective.

Next, all infeasible solutions are improved one by one (in sorted order). For a given infeasible solution j , an improved staffing vector $\mathbf{s}'_j$ (with corresponding cost c'_j) is constructed by adding one unit of capacity in all staffing intervals where performance is unsatisfactory. The new staffing vector’s performance is evaluated through simulation, provided that its cost is strictly lower than c^* (further exploiting the current infeasible solution is futile if $c'_j \geq c^*$). Based on the simulation results, two cases can be distinguished:

- The performance constraint is not yet met, in which case the exploitation continues. Vector $\mathbf{s}'_j$ is improved further (unit size capacity in-

creases are made) and its performance is simulated (if the cost is lower than c^*).

- The performance constraint is satisfied at all times; in this case a new feasible solution is found, which is stored if it is less costly than the current best feasible solution in terms of labor cost. The exploitation of solution j is terminated and the algorithm then proceeds to the next infeasible solution in the sorted list.

Note that the procedures described in PHASE I and PHASE II are suitable for small system sizes, largely avoid cyclic behavior and moreover guarantee that the algorithm yields a staffing vector meeting the performance constraint.

Algorithm 2 ISA(τ): PHASE II: EXPLOITATION

Define c_i the cost of a staffing vector $\mathbf{s}_i$
Define c^* the cost of the cheapest feasible solution found so far
Define e_i the number of staffing intervals with $P_{max,i}(\tilde{t}) > \alpha$, for a given staffing vector $\mathbf{s}_i$
Define u the cost associated to using one unit of capacity during one staffing interval (expressed in man-hours)
Initialize $c^* = \text{cost of best feasible solution found during PHASE I (if any), } \infty \text{ otherwise}$

Sort all infeasible solutions k considered in PHASE I, based on

- 1) increasing $\max_{\tilde{t}} \{P_{max,k}(\tilde{t})\}$
- 2) $c_k + ue_k$

for all infeasible solutions j (in sorted order) **do**

$$s'_j(\tilde{t}) \leftarrow \begin{cases} s_j(\tilde{t}) + 1 & \text{if } P_{max,j}(\tilde{t}) > \alpha \quad \forall \tilde{t} \\ s_j(\tilde{t}) & \text{if } P_{max,j}(\tilde{t}) \leq \alpha \quad \forall \tilde{t} \end{cases}$$

Calculate cost of $\mathbf{s}'_j$: $c'_j \leftarrow c_j + ue_j$

while $c'_j < c^*$ **do**

Simulate $s'_j(\tilde{t})$

if $\max_{\tilde{t}} \{P_{max,j}(\tilde{t})\} > \alpha$ **then**

$$s'_j(\tilde{t}) \leftarrow \begin{cases} s'_j(\tilde{t}) + 1 & \text{if } P_{max,j}(\tilde{t}) > \alpha \quad \forall \tilde{t} \\ s'_j(\tilde{t}) & \text{if } P_{max,j}(\tilde{t}) \leq \alpha \quad \forall \tilde{t} \end{cases}$$

Update cost of $\mathbf{s}'_j$: $c'_j \leftarrow c'_j + ue_j$

else

if $c'_j < c^*$ **then**

Better feasible solution found: store $s'_j(\tilde{t})$ and update $c^* \leftarrow c'_j$

end if

end if

end while

end for

5. Computational results

The method we proposed in Section 4 was tested for two settings: a large-scale example taken from [7] (we added the assumption of 15 minute staffing intervals), and a small-scale setting based on real-life arrival data at the ED of a Belgian hospital. Unfortunately, no detailed process data were available from the hospital, hence we assume service times that are exponentially distributed with mean 30 minutes (similar to Green et al. (2006), where service times were chosen based on physician workload estimations that are available in Graff et al. (1993)). Table 1 summarizes the main characteristics for both examples; in Figure 4 the corresponding arrival rates are plotted (note that the example derived from [7] represents a large scale system, with arrival rates varying between 80 and 120 customers per hour). For both examples, a 24-hour time horizon is considered and performance was calculated quasi-continuously: the calculation interval length is set equal to 1 minute. The length of the staffing interval equals 15 minutes and the maximum acceptable waiting time τ is set to 10 minutes. Per iteration of the algorithm, 2500 replications are performed. The algorithm's performance for the 2 examples is first evaluated in a $M(t)/M/s(t)+M$ context in Section 5.1. However, the algorithm remains applicable in a $M(t)/G/s(t)+G$ context, as the experiments in Section 5.2 indicate. In Section 5.3, we compare the proposed $ISA(\tau)$ method with some heuristics available in the literature.

5.1. $ISA(\tau)$: Exponential service and abandonment times

A comparison of the results for both small and large settings in an $M(t)/M/s(t)+M$ setting (see Table 2), leads to the conclusion that our algorithm results in staffing levels that indeed meet the desired performance targets in relatively few iterations. That is, a staffing vector is found for

Figure 4: Arrival rate (a) Belgian hospital (b) example based on [7]

	Belgian hospital	Example based on [7]
Service rate (customers/hour)	2	1
Abandonment rate (customers/hour)	0.25	1
Time horizon	24 hours	
Calculation interval	1 minute	
Staffing interval	15 minutes	
Maximum acceptable wait τ	10 minutes	
Target excess wait probability α	0.1	
Performance constraint	$Pr(W(t) > \tau) \leq \alpha \quad \forall t$	
Number of replications (per iteration)	2500	

Table 1: System parameters

which the probability of excessive wait lies below the chosen acceptable waiting time threshold and does not depend on the customer's arrival time. However, the number of iterations needed increases with system size: small systems (which we focus on) require less iterations than large systems. Moreover, as the capacity shocks are more frequent and larger in size in the large scale system (see Figure 6), so are the performance shocks (for both excess wait probability and delay probability). Finally, the performance graphs in Figures 5 and 6 clearly indicate that the delay probability is indeed a more restrictive performance measure; therefore, overstaffing might occur if the delay probability is used as a performance constraint (assuming that

customers tolerate a waiting time of $\tau > 0$).

	Belgian hospital	Example based on [7]
Number iterations PHASE I	6 (no feasible)	17 (no feasible)
Number iterations PHASE II	5 (3 feasible)	6 (2 feasible)
$\max_t \{P(t)\}$	0.090	0.100
Staffing cost	74	2257.25

Table 2: Results ISA(τ): exponential service and abandonment times

Figure 5: Small system: staffing vector and resulting performance (exponential service and abandonment times)

As mentioned in Section 4.2, the speed of convergence in the exploration phase can be influenced by changing the scaling factor i in Expression 7. Appendix A compares the results for both systems, using alternative scaling factors. As evident from these results, the number of iterations in the exploration phase increases drastically when the algorithm is forced to move

Figure 6: Large system: staffing vector and resulting performance (exponential service and abandonment times)

to unit-size capacity changes too soon (e.g. when a factor i^2 is used). Moreover, there is no evidence that this decrease in efficiency in the exploration phase pays off in the exploitation phase, nor does it seem to impact the quality of the final solution (both in terms of excess wait probability and staffing cost).

5.2. Lognormal service and abandonment times

One of the major advantages of a simulation-based method is its general applicability. As such, experiments were repeated assuming an $M(t)/G/s(t)+G$ setting, assuming service and abandonment times are lognormally distributed with squared coefficient of variation (SCV) equal to 1.5 and 0.5. As the results in Table 3 indicate, the algorithm's effectiveness does not change when departing from the exponential service time and abandonment

time distributions. Also, the number of iterations needed by the algorithm does not change substantially.

	Small system ($\mu = 2$)		Large system ($\mu = 1$)	
	SCV = 0.5	SCV = 1.5	SCV = 0.5	SCV = 1.5
Number iterations PHASE I	6	5	13	22
Number iterations PHASE II	5	4	16	6
$\max_t \{P(t)\}$	0.090	0.083	0.099	0.099
Staffing cost	73.25	74.5	2492.25	2363.4

Table 3: Results ISA(τ): Lognormal service and abandonment times

5.3. Comparison to other staffing heuristics

In this section, the staffing vector obtained by ISA(τ) is compared to some readily implementable staffing heuristics based on the stationary approximations as discussed in Section 3.1. The key features of the selected heuristics are summarized in Table 4.

The SRS_SIPPmax, SRS_lagSIPPmax and SRS_MOL heuristics are all based on the *square root staffing rule* (SRS). The value for β is determined using the Garnett delay function ([9]). The CF_MOL heuristic is based on the available exact results for stationary M/M/s models (i.e. *without* abandonments): at each point in time, the closed form formula for the excess wait probability (cf. [16] pp. 66-72) is used to assess the performance corresponding to a staffing vector. Again, a MOL approximation is used to capture the time-varying nature of the system. As the MOL approximation (used for SRS_MOL and CF_MOL) is calculated at each time t , the staffing level in a given staffing interval $\tilde{t}$ is set equal to the maximum staffing requirement over the interval.

Abbreviation	Staffing by means of	Applied to	Used arrival rate or offered load
CF_MOL	Closed form formula $M/M/s$	each time instant t	MOL arrival rate
SRS_SIPPmax	SRS formula $M/M/s + M$ (using Garnett delay function[9])	each staffing interval $\tilde{t}$	SIPPmax offered load
SRS_lagSIPPmax	SRS formula $M/M/s + M$ (using Garnett delay function[9])	each staffing interval $\tilde{t}$	lagged SIPPmax offered load
SRS_MOL	SRS formula $M/M/s + M$ (using Garnett delay function[9])	each time instant t	MOL offered load

Table 4: Other heuristics available in the literature

Each of the four heuristics was applied to both the small and large system (cf. Table 1). Figures 7 and 8 show the corresponding staffing vectors and the resulting excess wait probabilities for the large system; Figures 9 and 10 show the results for the small system (assuming exponential service and abandonment times). In each case, the results are compared to those of $\text{ISA}(\tau)$.

Figure 7: Comparison $\text{ISA}(\tau)$ staffing vs. other heuristics: Staffing (large system, exponential service and abandonment times)

The obtained staffing levels for the large system (represented in Figure 7) clearly illustrate the importance of the time lag between the arrival rate and the offered load. For SRS_SIPPmax, which does not account for this time lag, the peak staffing levels clearly occur earlier than the peaks in offered

Figure 8: Comparison $\text{ISA}(\tau)$ staffing vs. other heuristics: Performance (large system, exponential service and abandonment times)

load, leading to extremely poor performance. The time-varying nature of the offered load is captured better if a time lag is added (cf. SRS_lagSIPPmax) or if the MOL-approximation is used (cf. CF_MOL and SRS_MOL).

The observed performance for CF_MOL in the large system (represented in Figure 8) indicates that it leads to overstaffing, a result that can be explained by the fact that the underlying closed form $M/M/s$ formula ignores the presence of abandonments. For the large system, the SRS_MOL heuristic performs pretty well. This is not surprising, as the offered load used in the

SRS_MOL heuristic is exact⁴ and in addition the conditions for the Garnett delay function are met (i.e., exponential service and abandonment times and a sufficiently large number of servers). For the large system, the SRS_MOL staffing vector yields the best results among all stationary approximations. The excess wait probability occasionally exceeds the target though, which might be explained in part by the presence of staffing intervals, and by the fact that the SRS rule is a rule of thumb and therefore provides no guarantee for the performance constraint being met. The excess wait probabilities obtained through $\text{ISA}(\tau)$ closely resemble those of SRS_MOL, yet for $\text{ISA}(\tau)$ the performance constraint is met at all times.

Figure 9: Comparison $\text{ISA}(\tau)$ staffing vs. other heuristics: Staffing (small system, exponential service and abandonment times)

The results for the small example setting (given in Figures 9 and 10) show that none of the SRS-based heuristics result in adequate staffing. This might be addressed to the SRS rule performing best for moderate to large offered loads [9], whereas we applied it to a very small system. The closed-

⁴The distribution of number in system in any $M(t)/G/s(t) + G$ system is identical to that of the infinite server model, if the specific condition holds that the abandonment rate is equal to the service rate [32], as is the case in this setting.

Figure 10: Comparison $\text{ISA}(\tau)$ staffing vs. other heuristics: Performance (small system, exponential service and abandonment times)

form formula results in slightly better staffing vectors, although again, it has the tendency to overstaff as the presence of abandonments is ignored (for this example, the overstaffing remains limited due to the low abandonment rate).

Figures 11 and 12 reveal a key advantage of $\text{ISA}(\tau)$. Whereas in a Markovian setting, the SRS_MOL heuristic yielded performance comparable to $\text{ISA}(\tau)$ for the large system, this no longer holds when general service and abandonment times prevail: the excess wait probability is considerably above target.

For the remaining heuristics, the insights from the Markovian setting largely remain valid: i.e., performance is poor. Consequently, we may conclude that $\text{ISA}(\tau)$ is the only heuristic that yields consistent and satisfactory performance, both for small and large systems, and in particular in settings where the Markovian assumptions do not hold.

Figure 11: Comparison $\text{ISA}(\tau)$ staffing vs. other heuristics: Performance (large system, lognormal service and abandonment times, $\text{SCV} = 0.5$)

Figure 12: Comparison $\text{ISA}(\tau)$ staffing vs. other heuristics: Performance (large system, lognormal service and abandonment times, $\text{SCV} = 1.5$)

6. Conclusions and future research

This paper suggests an extension of the simulation-based Iterative Staffing Algorithm (ISA) proposed by Feldman et al. [7] as a method to set staffing levels in small-scale systems with time-varying demand. This extension (called $\text{ISA}(\tau)$) enables to measure performance based on the probability of

an excessive wait, instead of the common focus on delay probability as a performance metric. Moreover, it takes into account the sensitivity of small scale systems to changes in the staffing levels, and the presence of staffing intervals. Meanwhile, the advantages of the traditional ISA (namely general applicability, automatic validation) remain valid.

Experiments illustrate that $\text{ISA}(\tau)$ is both effective and efficient in determining staffing requirements for small-scale and large-scale systems, and consistently outperforms heuristics based on stationary approximations, in particular for settings in which the service and abandonment processes are not Markovian. In general, the efficiency of the algorithm tends to depend on its parameters (the amplification factor in Expression 7 and the SCV-based stop criterion can be tuned), and the size of the system (larger systems require more computation time).

Given that $\text{ISA}(\tau)$ is both effective and efficient in detecting required capacities, and requires no specific tools other than simulation, we are confident that the method offers opportunities to support decisions in practice. Though this paper has focused on the context of staffing in an emergency department, the methodology is in fact applicable to any setting in which demand is time-varying, and in which the decision maker relies primarily on capacity to ensure adequate customer service. In future research, we plan to apply the algorithm to a real-life ED setting including patients of different priority classes (due to the lack of detailed process data, the current ED test setting only mimicked real-life arrival data). This requires the inclusion of a triage (or “sorting”) step, which may result in a reduction of work at the treatment step. As such, it may impact the service rate of the treatment step, and opens up the opportunity to use different waiting time targets for different priority classes ([5]). In addition, we plan to introduce shift

constraints in the model, to further increase its practical relevance.

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Appendix A: Impact of scaling factor on algorithm convergence

	Small system				Large system			
	$i/2$	i	$2i$	i^2	$i/2$	i	$2i$	i^2
Number iterations PHASE I	2	6	5	7	17	17	18	79
Number iterations PHASE II	3	5	5	5	10	6	7	10
$\max_t \{P(t)\}$	0.090	0.090	0.090	0.086	0.100	0.100	0.099	0.100
Staffing cost	74	74	74	74	2299.75	2257.25	2257.75	2258

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