Logarithmic Kodaira dimension and the poles of the Hodge and motivic zeta functions for surfaces

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Abstract

To any polynomial $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ with $f(0) = 0$ one associates the well-known motivic zeta function and its specialization to the level of Hodge polynomials. These zeta functions can be given very explicitly in terms of an embedded resolution of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^n$. In this paper, where we work with polynomials in three variables, i.e., $n = 3$, we find a geometric condition for having a nonzero contribution to the residue at a candidate pole. More precisely, for a given embedded resolution h we fix an exceptional surface E with $h(E) = \{0\}$, which induces in a canonical way a candidate pole q of the motivic zeta function. Then we prove that, when the surface E is non-rational and we are in a generic situation, the maximality of the logarithmic Kodaira dimension of E° implies the non-vanishing of the contribution of E to the residue at q . Here E° denotes the part of E that doesn't belong to any other irreducible component of $h^{-1}(f^{-1}\{0\})$. The same result is already true on the level of Hodge polynomials.

Introduction

(0.1) Let $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ with $f(0) = 0$. Using motivic integration Denef and Loeser associated to such a polynomial an interesting singularity invariant, the well-known (local) *motivic zeta function* [5]. They also derived the following formula for it in terms of an embedded resolution $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^n$. Denote by E_i , $i \in T$, the irreducible components of $h^{-1}(f^{-1}\{0\})$, set $E_I^\circ := (\bigcap_{i \in I} E_i) \setminus (\bigcup_{l \notin I} E_l)$ for $I \subseteq T$ and denote, for each $i \in T$, by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor on X of $f \circ h$ and $h^*(dx_1 \wedge \dots \wedge dx_n)$, respectively. Then the formula is

$$Z_{\text{mot}}(f, s) = \mathbb{L}^{-n} \sum_{I \subseteq T} [E_I^\circ \cap h^{-1}\{0\}] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i + sN_i} - 1}.$$

(Sometimes, as we will do in this paper, the definition is rescaled to eliminate the factor $\mathbb{L}^{-n}$.) Here we denote by $[\cdot]$ the class in the Grothendieck ring of algebraic varieties over $\mathbb{C}$ and by $\mathcal{M}_{\mathbb{C}}$ the localization of this ring with respect to $\mathbb{L} := [\mathbb{A}_{\mathbb{C}}^1]$; see (1.3). Then this zeta function lives in a localization of the polynomial ring $\mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-s}]$.

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Since the Grothendieck ring of algebraic varieties is not an integral domain, see the result of Poonen [14], working with this zeta function, and in particular with its poles, often turns out to be rather complicated. In [19, Section 4] for example we already needed some non-trivial work only to define properly the notion of a pole of the motivic zeta function. Therefore it is often useful to consider its specialization $Z_{\text{Hod}}(f, s)$ to the level of Hodge polynomials. As will be the case for the main theorem of the present paper, most propositions for the motivic zeta function hold already for $Z_{\text{Hod}}(f, s)$; thus, for a lot of purposes the level of Hodge polynomials turns out to be fine enough already. And on this level we at least work over an integral domain. For this *Hodge zeta function* we have the following formula in terms of the embedded resolution h :

$$Z_{\text{Hod}}(f, s) := \sum_{I \subseteq T} H(E_I^\circ \cap h^{-1}\{0\}) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1},$$

where $H(\cdot)$ denotes the Hodge polynomial. For each variety V this Hodge polynomial $H(V)$ is a polynomial over $\mathbb{Z}$ in two variables u and v , see (1.3). The expression above should be considered as a rational function in the variable $T = (uv)^{-s}$.

(0.2) We are especially interested in the poles of these zeta functions. Here we say that a rational number q is a pole of $Z_{\text{Hod}}(f, s)$ if $(uv)^{-q}$ is, considering $Z_{\text{Hod}}(f, s)$ as a rational function in the variable $T = (uv)^{-s}$. For obvious reasons we call the rational numbers $-\frac{\nu_i}{N_i}$, $i \in T$, the *candidate poles*.

Now it is striking that very often a lot of these candidate poles are actually not really a pole. This fact would be elucidated if the Monodromy Conjecture is true, stating that if s_o is a pole of $Z_{\text{Hod}}(f, s)$, then $e^{2\pi i s_o}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}\{0\}$. For $n = 2$ this conjecture is already proved; see [12] or [16]. For $n \geq 3$ it is still open, but we consider the many partial results which are existing by now as a lot of evidence for it; see for example [2], [13], [18] and [21].

The fact that, even when one assumes this conjecture and one removes the candidate poles eliminated by it, still very little was known about the remaining candidate poles, intrigued Veys. He then started the search for “geometric conditions” which would precisely select the poles out of the whole set of candidate poles. More precisely, he studied the question whether or not one can decide “geometrically” when a given candidate pole $-\frac{\nu_i}{N_i}$, $i \in T$, is a pole. In the case of two variables Veys was already able to give the following positive answer.

(0.2.1) Theorem ([22, Theorem 4.3]). *Let $n = 2$ and let h be the minimal embedded resolution of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^2$. Then a rational number s_o is a pole of $Z_{\text{Hod}}(f, s)$ if and only if $s_o = -\frac{1}{N_i}$ for some irreducible component E_i of the strict transform of $f^{-1}\{0\}$ with $0 \in h(E_i)$, or $s_o = -\frac{\nu_i}{N_i}$ for some exceptional curve E_i with $h(E_i) = \{0\}$ and intersecting at least three times other components.*

In fact Veys proved this result for the so-called topological zeta function, see [4] for the definition of this function; but it only needs a trivial verification to see that this immediately implies Theorem 0.2.1. One easily sees that the theorem is also true for an arbitrary embedded resolution, at least when one replaces the condition “intersecting at least three

times other components” by “intersecting at least three times other components E_j with $\nu_j - \frac{\nu_i}{N_i} N_j \neq 1$ ”.

One can also associate very similar zeta functions to a nonconstant regular function on a normal surface germ, see [23]. In joint work with Veys we also obtained in this case a complete geometric determination of the poles [19, Theorem 3.4 and Theorem 5.5].

Recently we generalized Theorem 0.2.1 in the following sense. First note that the formula of $Z_{\text{Hod}}(f, s)$ in terms of an embedded resolution immediately implies that the order of a pole is at most 2. Hence Theorem 0.2.1 actually tells something about the poles of order 1 and 2. For an arbitrary number n of variables we then provided in [17] a complete geometric determination of the poles of order $n - 1$ and n , reducing nicely to the statement of Theorem 0.2.1 in the case $n = 2$.

(0.3) In this paper we put ourselves in the situation of three variables, i.e., $n = 3$, and we continue the search for geometric conditions on a component E_i which would imply that the associated candidate pole $-\frac{\nu_i}{N_i}$ is really a pole. So, fix an exceptional variety E with $h(E) = \{0\}$ and suppose that none of the intersecting varieties E_i , $i \in T$, induces the same candidate pole as E . Consider again the Monodromy Conjecture and recall the formula of A’Campo, see [1, Theorem 3]. From this point of view the framework of Veys in [21], where he wants to prove that the contribution $\mathcal{R}$ of E to the residue at $-\frac{\nu}{N}$ vanishes when $\chi(E^\circ) = 0$, seems to be very natural. Here $\chi(\cdot)$ denotes the topological Euler-Poincaré characteristic and E° stands for the part of E that remains after the subtraction of all other varieties E_i , $i \in T$. Of course, by the contribution of E , being equal to E_j for some $j \in T$, to the residue of $Z_{\text{Hod}}(f, s)$ at $-\frac{\nu}{N}$ we mean the residue at $-\frac{\nu}{N}$ of the function obtained from $Z_{\text{Hod}}(f, s)$ by restricting its defining expression, see (0.1), to the terms which correspond to the subsets I of T containing j . Working with p -adic principal value integrals Denef and Jacobs also arrived at the conjecture that, under certain conditions, the vanishing of $\chi(E^\circ)$ will imply the vanishing of (the p -adic analogue of) $\mathcal{R}$; see [3, Conjecture 1.2.2]. They were already able to prove it in the case of real principal value integrals.

Our point of interest here is mainly the other implication, i.e.,

$$\begin{aligned} (\chi(E^\circ) \neq 0) &\Rightarrow (\mathcal{R} \neq 0), \text{ or perhaps} \\ (\bar{\kappa}(E^\circ) \text{ maximal}) &\Rightarrow (\mathcal{R} \neq 0), \end{aligned}$$

where $\bar{\kappa}(\cdot)$ denotes the logarithmic Kodaira dimension, see Definition 1.6. Since the logarithmic Kodaira dimension $\bar{\kappa}(V)$ of a nonsingular variety V can only take on the values $-\infty, 0, 1, \dots, \dim(V)$, the maximality of the logarithmic Kodaira dimension of E° is equivalent to the condition that $\bar{\kappa}(E^\circ) = \dim(E^\circ)$. Till now nothing was known (for $n \geq 3$) about these two possible implications.

(0.4) Let us give some motivation and plausibility for the latter implication. First, when we extend a given resolution by a superfluous blowing-up, with exceptional variety E , the logarithmic Kodaira dimension $\bar{\kappa}(E^\circ)$ will not be maximal. Secondly, consider the case of curves, i.e., $n = 2$. Let h be the minimal embedded resolution and E an exceptional curve with $h(E) = \{0\}$ and numerical data (ν, N) . Suppose that E intersects k times other

components E_i and that $\alpha_i = \nu_i - \frac{\nu}{N}N_i \neq 0$. (Note that $\alpha_i \neq 1$ in this situation.) Then we know by (the proof of) Theorem 0.2.1 that the contribution of the rational curve E to the residue at $-\frac{\nu}{N}$ is different from zero if and only if $k \geq 3$. But below in Figure 1 we see that this is equivalent to $\bar{\kappa}(E^\circ) = 1 = \dim(E^\circ)$. The same situation occurs in the case of (functions on) normal surfaces. Indeed, here we cannot suppose that $\alpha_i \neq 1$, even not for the minimal embedded resolution, see [19, Example 2.13.1], so we assume that the exceptional curve E intersects k times other components E_i with $\alpha_i \neq 1$. Then by (the proof of) [19, Theorem 3.4] we know that the contribution of E is not zero if and only if E is non-rational, or E is rational and $k \geq 3$. When we are in the case that none of the numbers α_i equals 1, then we see again in Figure 1 that the non-vanishing of the contribution is equivalent to $\bar{\kappa}(E^\circ) = 1$. Note that it is quite remarkable that the trouble causing curves E_i , i.e., those with $\alpha_i = 1$, are precisely the ones which do not contribute to the canonical divisor on E in the expression

$$K_E = \sum (\alpha_i - 1)E'_i,$$

where E'_i is the intersection of E with E_i , see [20, Proposition 2.1].

$\bar{\kappa}(C)$	C
$-\infty$	$\mathbb{P}^1, \mathbb{A}^1$
0	(projective) elliptic curve, $\mathbb{A}^1 \setminus \{\text{point}\}$
1	the others

Figure 1

Finally recall that in [13, Subsection 6.6] also Loeser included the maximality of the logarithmic Kodaira dimension in his conditions.

(0.4.1) Remark. Consider again the situation of three variables. In [21] Veys already proved in a lot of cases that the condition $\chi(E^\circ) \leq 0$ implies that $\mathcal{R} = 0$, where as usual $\mathcal{R}$ stands for the contribution of E to the residue at $-\frac{\nu}{N}$. In the present paper we will work on the possible implication

$$(\bar{\kappa}(E^\circ) = 2) \Rightarrow (\mathcal{R} \neq 0).$$

Recall that $\bar{\kappa}(E^\circ) = 2$ implies that $\chi(E^\circ) > 0$ when the divisor on E consisting of the intersections with the other surfaces E_i , $i \in T$, has a connected support, see for example [7, Proposition 2]; so at least in these cases our intentions are compatible with the results of Veys.

(0.5) In the first section we fix notations and recall some important facts. In Section 2 we find some strong conditions on the genera of the irreducible components of the curves $E \cap E_i$, for $i \in T$, which should be fulfilled when the contribution of E to the residue at $-\frac{\nu}{N}$ is zero. These conditions will enable us in Section 3 to prove the following theorem.

(0.5.1) Theorem. *Let $f \in \mathbb{C}[x_1, x_2, x_3] \setminus \mathbb{C}$ and fix an embedded resolution $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^3$ of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^3$. Let E be a non-rational exceptional surface with $h(E) = \{0\}$ and numerical*

data (ν, N) . Let F_i , $i = 1, \dots, r$, be the irreducible components of the intersections $E \cap E_k$, $k \in T$, where E_k is another exceptional surface or an irreducible component of the strict transform of $f^{-1}\{0\}$. When F_i is a component of $E \cap E_{k(i)}$, then we put $\alpha_i := \nu_{k(i)} - \frac{\nu}{N} N_{k(i)}$. Suppose that $\alpha_i \neq 0, 1$ for each $i \in \{1, \dots, r\}$. If $\bar{\kappa}(E^\circ) = 2$, then the contribution $\mathcal{R}$ of E to the residue of $Z_{\text{Hod}}(f, s)$ at $-\frac{\nu}{N}$ is nonzero.

Theorem 0.5.1 also holds for the motivic zeta function $Z_{\text{mot}}(f, s)$.

1 Preliminaries

(1.1) Let f be a nonconstant polynomial over $\mathbb{C}$ in n variables, i.e., $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$. An *embedded resolution* of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^n$ consists of a nonsingular variety X and a proper birational morphism $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ such that the restriction $h : X \setminus h^{-1}(f^{-1}\{0\}) \rightarrow \mathbb{A}_{\mathbb{C}}^n \setminus f^{-1}\{0\}$ is an isomorphism and $h^{-1}(f^{-1}\{0\})$ has only normal crossings in X . An embedded resolution always exists in characteristic zero by Hironaka [9].

A reduced closed subscheme E of everywhere codimension one of a nonsingular variety X is said to have *only normal crossings with* a closed subscheme D of X , if for all $x \in D$ there exists a regular system of parameters $t_1, \dots, t_n$ in the local ring $O_{X,x}$ of X at x such that the ideal in $O_{X,x}$ of each irreducible component of E containing x is generated by one of the t_i , and the ideal of D in $O_{X,x}$ is generated by some of the t_i . When $D = X$ we say that E has *only normal crossings*.

(1.2) Fix an embedded resolution $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^n$ for a given polynomial $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$. Let E_i , $i \in T = T_e \cup T_s$, be the (reduced) irreducible components of $h^{-1}(f^{-1}\{0\})$, where E_i is a (reduced) irreducible component of the exceptional divisor for $i \in T_e$ and of the strict transform of $f^{-1}\{0\}$ in X for $i \in T_s$. For each $i \in T$, let N_i and $\nu_i - 1$ be the multiplicities of E_i in the divisor on X of $f \circ h$ and $h^*(dx_1 \wedge \dots \wedge dx_n)$, respectively. The (ν_i, N_i) , for $i \in T$, are called the *numerical data* of the resolution (X, h) . Although we denote by definition the irreducible components of $h^{-1}(f^{-1}\{0\})$ by E_i , we will often denote them freely by E , $E_i^{[m]}$, et cetera. For the numerical data we then use the corresponding notation (ν, N) , $(\nu_i^{[m]}, N_i^{[m]})$, et cetera. For $i \in T$ and $I \subseteq T$ we denote

$$E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j, \quad E_I := \bigcap_{i \in I} E_i \quad \text{and} \quad E_I^\circ := E_I \setminus \bigcup_{j \in T \setminus I} E_j.$$

In particular when $I = \emptyset$, we have $E_\emptyset = X$. Remark that X is the disjoint union of the E_I° .

Let E_j be an irreducible component of $h^{-1}(f^{-1}\{0\})$ and let F_i , $i = 1, \dots, r$, be the irreducible components of the intersections $E_j \cap E_k$ for $k \in T \setminus \{j\}$. For $i \in \{1, \dots, r\}$ we will also denote

$$F_i^\circ := F_i \setminus \bigcup_{j \neq i} F_j.$$

(1.3) We give a brief description of the Grothendieck ring of complex varieties. In this context a variety is not necessarily irreducible, it is a reduced separated scheme of finite

type over $\mathbb{C}$. We denote by $K_0(\text{Var}_{\mathbb{C}})$ the *Grothendieck group* (or *ring*) of complex varieties. It is the abelian group generated by the symbols $[X]$, for X a complex variety, with the relations $[X] = [Y]$ if X and Y are isomorphic as complex varieties, and $[X] = [Y] + [X \setminus Y]$ if Y is Zariski closed in X . There is a natural ring structure on $K_0(\text{Var}_{\mathbb{C}})$, the product of $[X]$ and $[Y]$ being equal to $[X \times_{\mathbb{C}} Y]$. We set $\mathbb{L} := [\mathbb{A}_{\mathbb{C}}^1]$ and denote by $\mathcal{M}_{\mathbb{C}}$ the ring obtained from $K_0(\text{Var}_{\mathbb{C}})$ by inverting $\mathbb{L}$.

Let $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ with $f(0) = 0$ and let $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ be an embedded resolution of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^n$. The *motivic zeta function* associated to f can be defined in terms of h as follows:

$$Z_{\text{mot}}(f, s) := \sum_{I \subseteq T} [E_I^{\circ} \cap h^{-1}\{0\}] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i + sN_i} - 1},$$

where $[\cdot]$ denotes the class in $\mathcal{M}_{\mathbb{C}}$ and where the expression $\mathbb{L}^{-s}$ should be considered as a variable.

Another interesting singularity invariant of $f^{-1}\{0\}$ is the *Hodge zeta function*. It is defined as

$$Z_{\text{Hod}}(f, s) := \sum_{I \subseteq T} H(E_I^{\circ} \cap h^{-1}\{0\}) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1} \in \mathbb{Q}(u, v)(T),$$

where we consider $(uv)^{-s}$ as a variable T and where $H(\cdot)$ denotes the Hodge polynomial. For a variety V this Hodge polynomial is given by $H(V) := \sum_{p,q} e^{p,q}(V) u^p v^q \in \mathbb{Z}[u, v]$; here $e^{p,q}(V) = \sum_{i \geq 0} (-1)^i h^{p,q}(H_c^i(V, \mathbb{C}))$ with $h^{p,q}(H_c^i(V, \mathbb{C}))$ the rank of the (p, q) -Hodge component of the i -th cohomology group with compact support of V .

The zeta function $Z_{\text{mot}}(f, s)$ is a finer invariant than $Z_{\text{Hod}}(f, s)$; indeed, we can specialize $Z_{\text{mot}}(f, s)$ into $Z_{\text{Hod}}(f, s)$, heuristically by taking Hodge polynomials.

We are especially interested in the poles of these zeta functions. For the zeta function on the level of Hodge polynomials we say that a rational number q is pole if $(uv)^{-q}$ is, considering $Z_{\text{Hod}}(f, s)$ as a rational function in the variable $T = (uv)^{-s}$. Being the only possible poles, we will call the elements of the set $\{-\frac{\nu_i}{N_i} \mid i \in T\}$ the *candidate poles*. So, to each irreducible component E_i of $h^{-1}(f^{-1}\{0\})$ we associate one candidate pole, namely the rational number $-\frac{\nu_i}{N_i}$. One should remark that a candidate pole s_0 can only be a pole when there exists a component E_i with $s_0 = -\frac{\nu_i}{N_i}$ and satisfying $E_i \cap h^{-1}\{0\} \neq \emptyset$.

For the motivic zeta function we use the definition of a pole introduced in [19]. Very briefly, we just have to represent $Z_{\text{mot}}(f, s)$ as a quotient of two polynomials in the variable $\mathbb{L}^{-s}$ over a “well-chosen” ring A , say for example $Z_{\text{mot}}(f, s) = N(\mathbb{L}^{-s})/D(\mathbb{L}^{-s})$. The main point about A is that in this ring the elements $1 - \mathbb{L}^q$ and $b\mathbb{L}^a$, for $q \in \mathbb{Q} \setminus \{0\}$, $b \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{Q}$, are units; see [19, Subsection 4.4] for the rigorous definition of A . Now we simply say that a rational number q is a pole of $Z_{\text{mot}}(f, s)$ if there exists a number $m \in \mathbb{N} \setminus \{0\}$ such that $(\mathbb{L}^{-s} - \mathbb{L}^{-q})^m \mid D(\mathbb{L}^{-s})$ in $A[\mathbb{L}^{-s}]$, but $(\mathbb{L}^{-s} - \mathbb{L}^{-q})^m \nmid N(\mathbb{L}^{-s})$ in $A[\mathbb{L}^{-s}]$.

(1.4) Let $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$. We fix an embedded resolution $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^n$ in the sense of Hironaka’s Main Theorem II [9, p. 142] by means of blowing-ups. We explain what we mean by this.

Set $X_0 = \mathbb{A}_{\mathbb{C}}^n$ and $Y^{[0]} = f^{-1}\{0\}$. For $i = 0, \dots, m-1$ we have a finite succession of blowing-ups $g_i : X_{i+1} \rightarrow X_i$ with irreducible nonsingular center $D_i \subset X_i$ and exceptional

variety $E_{i+1}^{[i+1]} \subset X_{i+1}$ subject to the following conditions. Let $E_j^{[i+1]}$ and $Y^{[i+1]}$ denote the strict transform of $E_j^{[i]}$ and $Y^{[i]}$, respectively, in X_{i+1} by g_i for $j = 1, \dots, i$. Then

- (1) $X = X_m$ and $h = g_0 \circ \dots \circ g_{m-1}$;
- (2) for $i = 0, \dots, m-1$ we have $D_i \subset Y^{[i]}$, $\text{codim}(D_i, X_i) \geq 2$, and the multiplicity on $Y^{[i]}$ of all $x \in D_i$ equals the maximal multiplicity on $Y^{[i]}$ (i.e., $\mu_x(Y^{[i]}) = \max_{y \in Y^{[i]}} \mu_y(Y^{[i]})$ for all $x \in D_i$);
- (3) $\cup_{1 \leq j \leq i} E_j^{[i]}$ has only normal crossings and only normal crossings with D_i (in X_i) for $i = 1, \dots, m-1$; and
- (4) $\left(\cup_{1 \leq j \leq m} E_j^{[m]}\right) \cup Y^{[m]} = h^{-1}(f^{-1}\{0\})$ has only normal crossings in X_m .

Let V be a closed subscheme of everywhere codimension one of a nonsingular variety X . For all $x \in X$ we define the *multiplicity* $\mu_x(V)$ of x on V as the maximal integer m , such that the m -th power of the maximal ideal of the local ring $O_{X,x}$ of X at x contains the ideal of V in $O_{X,x}$.

Obviously we can apply all the notations of (1.2) and (1.3) to this kind of embedded resolutions. For the exceptional variety $E_{i+1}^{[i+1]}$, $i = 0, \dots, m-1$, of the blowing-up $g_i : X_{i+1} \rightarrow X_i$ we also denote

$$(E_{i+1}^{[i+1]})^\circ := E_{i+1}^{[i+1]} \setminus \left(\left(\cup_{j=1}^i E_j^{[i+1]} \right) \cup Y^{[i+1]} \right).$$

(1.4.1) Important notes. From now on we will mean by “embedded resolution” an embedded resolution in the sense of Hironaka. When we want to use a more arbitrary embedded resolution, as in (1.1), we will simply say “arbitrary embedded resolution”.

In what follows we will often suppose, without explicitly mentioning it anymore, that a polynomial $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ and an embedded resolution $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ (in the sense of Hironaka) are given. Recall that we will also always assume that $f(0) = 0$.

When we talk about an exceptional variety without any further specification we always mean an exceptional variety of the morphism h , i.e., we consider the final stage of the resolution process. We should also note that sometimes, by abuse of notation, we will denote a variety and its strict transform by a morphism in exactly the same way.

(1.5) Fix $i \in \{0, \dots, m-1\}$ and set $E = E_{i+1}^{[i+1]}$, $D = D_i$, $\pi = g_{i|E} : E \rightarrow D$ and $k = \text{codim}(D, X_i)$. Let $\mathcal{I}$ be the ideal sheaf of D in X_i . Then we know by [8, Theorem II.8.24] that E , together with the induced projection map π , is isomorphic to the *projective space bundle* $\mathbf{P}(\mathcal{E})$ over D , associated to the locally free sheaf $\mathcal{E} = j^*(\mathcal{I}/\mathcal{I}^2)$ of rank k on D , where $j : D \rightarrow X_i$ denotes the inclusion of D in X_i .

When $n = 3$, the center D can only be a point or a nonsingular curve. If D is a point, then $E \cong \mathbb{P}^2$; if D is a nonsingular curve, then E together with the morphism $\pi : E \rightarrow D$ is a ruled surface over D , see [8, Proposition V.2.2].

We proceed by stating one of the relations of Veys between the numerical data.

(1.5.1) Theorem ([20, Theorem 6.1 and Theorem 6.2]). Let E'_j , $j = 1, \dots, s$, be the reduced irreducible components of the intersections of E with $Y^{[i+1]}$ and $E_l^{[i+1]}$ for $l \in \{1, \dots, i\}$. Then the strict transform of E'_j in $E_{i+1}^{[m]}$ by the consecutive $g_{l|E_{i+1}^{[l+1]}} : E_{i+1}^{[l+1]} \rightarrow E_{i+1}^{[l]}$, $i+1 \leq l < m$, is equal to some irreducible component of the intersection of $E_{i+1}^{[m]}$ with another irreducible component of $h^{-1}(f^{-1}\{0\})$, say with $E_{k(j)}$. Set $\alpha_j := \nu_{k(j)} - \frac{\nu_{i+1}^{[m]}}{N_{i+1}^{[m]}} N_{k(j)}$ for $j \in \{1, \dots, s\}$. Then

$$\sum_{j=1}^s d_j(\alpha_j - 1) + k = 0,$$

where d_j , $j = 1, \dots, s$, is the degree of the intersection cycle $E'_j \cdot F$ on F for any fiber $F \cong \mathbb{P}^{k-1}$ of $\pi : E \rightarrow D$ over a point of D .

Let us consider again the case that $n = 3$. When D is a point, the number d_j is just the degree of the curve E'_j in $E \cong \mathbb{P}^2$. When D is a nonsingular curve, d_j is nothing but the number of intersections of the curve E'_j with a general fiber F of π , see also [20, Example 6.6(b)].

(1.6) Let V be a nonsingular variety and $\overline{V}$ a nonsingular completion of V with normal crossings boundary D ; by this we mean that $\overline{V}$ is a nonsingular complete variety containing V as an open dense subset and that $D = \overline{V} \setminus V$ is a divisor on $\overline{V}$ with only normal crossings.

Definition ([10, Chapter 11]). Let $K_{\overline{V}}$ be the canonical divisor of $\overline{V}$. Denote by $\mathbb{N}(K_{\overline{V}} + D)$ the set $\{m \in \mathbb{N} \setminus \{0\} \mid \dim H^0(V, \mathcal{L}(m(K_{\overline{V}} + D))) > 0\}$. Here $\mathcal{L}(m(K_{\overline{V}} + D))$ is the invertible sheaf corresponding to $m(K_{\overline{V}} + D)$. Then the *logarithmic Kodaira dimension* $\overline{\kappa}(V)$ of V is defined to be

$$\overline{\kappa}(V) = \begin{cases} \max_{m \in \mathbb{N}(K_{\overline{V}} + D)} \dim \phi_{m(K_{\overline{V}} + D)}(\overline{V}) & \text{if } \mathbb{N}(K_{\overline{V}} + D) \neq \emptyset, \\ -\infty & \text{otherwise,} \end{cases}$$

where $\phi_{m(K_{\overline{V}} + D)} : \overline{V} \rightarrow \mathbb{P}^N$, with $N = \dim H^0(V, \mathcal{L}(m(K_{\overline{V}} + D))) - 1$, is the rational map associated with the complete linear system $|m(K_{\overline{V}} + D)|$.

This definition is independent of the choice of the completion $\overline{V}$. Clearly $\overline{\kappa}(V)$ can only take on the values $-\infty, 0, 1, \dots, \dim V$; furthermore, we have $\overline{\kappa}(W) \geq \overline{\kappa}(V)$ whenever W is an open dense subset of the nonsingular variety V .

Replacing the divisor $K_{\overline{V}} + D$ by $K_{\overline{V}}$ in the definition above precisely gives us the ordinary Kodaira dimension $\kappa(V)$ of V . Now, for $n = 3$, let E be an exceptional surface with $h(E) = \{0\}$. Then we know that E is birationally equivalent to a ruled surface. So, since for varieties V and W we have $\kappa(V \times W) = \kappa(V) + \kappa(W)$, we easily see that $\kappa(E^\circ) = \kappa(E) = -\infty$. Thus obviously this notion is useless in our context. We need an invariant that takes into account the curves $E \cap E_k$, $k \in T$, where E_k is another exceptional surface or an irreducible component of the strict transform of $f^{-1}\{0\}$. This is provided by the logarithmic Kodaira dimension.

For further use in the next section we state the following two theorems. But for the time being we can also use them to give us some more intuition. We need the notion of a fibered

variety. For varieties V and W we call a morphism $f : V \rightarrow W$ a *fibred variety* if f is dominating and the extension of fields $\text{Rat}(V)/\text{Rat}(W)$ is algebraically closed. Here $\text{Rat}(\cdot)$ denotes the function field.

(1.6.1) Easy Addition Theorem ([10, Theorem 11.9]). *Let $f : V \rightarrow W$ be a fibred variety. Then for a general point x of W ,*

$$\bar{\kappa}(V) \leq \bar{\kappa}(f^{-1}\{x\}) + \dim W.$$

(1.6.2) Theorem ([11], [10, Theorem 11.15]). *Let $f : V \rightarrow W$ be a fibred variety such that a general fiber $f^{-1}\{x\}$ is a curve. If x is a general point, then*

$$\bar{\kappa}(V) \geq \bar{\kappa}(f^{-1}\{x\}) + \bar{\kappa}(W).$$

2 Necessary conditions for residue zero

(2.1) From now on we consider the case $n = 3$, i.e., $f \in \mathbb{C}[x_1, x_2, x_3]$. In this case of surfaces only a very limited kind of exceptional surfaces can occur. They arise either by blowing up at a point or by blowing up at an (irreducible) nonsingular curve. After their creation, they only “change” when they have a nonempty intersection with the center D of a blowing-up appearing later on in the resolution process. This intersection consists of a finite number of points (or the whole curve D , but we may neglect this second possibility since blowing up a nonsingular surface at a nonsingular curve results in an isomorphic surface). So one obtains the strict transform, i.e., the “changed” exceptional surface, by performing on the surface a finite number of point-centered blowing-ups. Remember that this important fact is true since at any stage in the resolution process the already created exceptional locus has only normal crossings with D , see [6, page 605] or [20, Proposition 3.1].

Notation. Let E be an exceptional surface. Then we denote by m_E the number of point-centered blowing-ups that we performed on E after its creation.

(2.2) Lemma. *Let E be an exceptional surface.*

- (i) *If E is created by a blowing-up at a projective nonsingular curve D of geometric genus g , then $H(E) = (uv)^2 - gu^2v - guv^2 + (2 + m_E)uv - gu - gv + 1$.*
- (ii) *If E is created by a blowing-up at a point P , then $H(E) = (uv)^2 + (1 + m_E)uv + 1$.*

Proof. (i) Remember that at its creation E actually was a ruled surface $\pi : E_0 \rightarrow D$ over D . Since $H(\cdot)$ is an additive invariant, we immediately see that the Hodge polynomial of a ruled surface over D equals

$$H(\mathbb{P}^1 \times D) = H(\mathbb{P}^1)H(D) = (uv + 1)(uv - gu - gv + 1).$$

Again by the additivity property of $H(\cdot)$ we obtain that for each point-centered blowing-up we have to change the Hodge polynomial of a point by that of a projective line, i.e., we have to change 1 by $1 + uv$, or we just have to add uv . This implies that

$$\begin{aligned} H(E) &= H(\mathbb{P}^1)H(D) + m_E uv \\ &= (uv)^2 - gu^2v - guv^2 + (2 + m_E)uv - gu - gv + 1. \end{aligned}$$

(ii) In this case E was isomorphic with $\mathbb{P}^2$ at its creation. So by the discussion in (i) we only have to prove that $H(\mathbb{P}^2) = (uv)^2 + uv + 1$. But, again by the additivity of $H(\cdot)$, this is obvious, since one can write $\mathbb{P}^2$ as the disjoint union $\mathbb{A}^2 \sqcup \mathbb{A}^1 \sqcup \{\text{point}\}$. ■

(2.3) Fix an exceptional surface E which is created by a blowing-up at a nonsingular curve D of genus g and which satisfies $h(E) = \{0\}$. (Note that E itself is not supposed to be a ruled surface over the curve D ; we only suppose that we obtain E as a ruled surface E_0 over D “changed” by the remaining sequence of blowing-ups in the resolution process after the creation of E_0 .) Since h is proper, the assumption $h(E) = \{0\}$ immediately implies that D is a complete, and hence projective, nonsingular curve. Also the surface E itself is a complete variety.

Let F_i , $i = 1, \dots, r$, be the irreducible components of the intersections $E \cap E_k$, $k \in T$, where E_k is another exceptional surface or an irreducible component of the strict transform of $f^{-1}\{0\}$. When F_i is a component of $E \cap E_{k(i)}$, then we put $\nu_i := \nu_{k(i)}$, $N_i := N_{k(i)}$ and $\alpha_i := \nu_i - \frac{\nu}{N}N_i$, where (ν, N) are the numerical data of E . Since E is complete and $h^{-1}(f^{-1}\{0\})$ is closed in X , we clearly see that also the curves F_i are complete.

We are going to study the contribution of E to the residue (of order 1) of $Z_{\text{Hod}}(f, s)$ at $-\frac{\nu}{N}$; so, we put ourselves in the generic situation that

$$\alpha_i \neq 0 \quad \text{for } i \in \{1, \dots, r\},$$

meaning that there is no other component E_k which intersects E and satisfies $\frac{\nu}{N} = \frac{\nu_k}{N_k}$. Note that there eventually can be other contributions to the residue at $-\frac{\nu}{N}$, but that those will not interfere with the contribution of E because of this genericity condition. Finally we also suppose that

$$\alpha_i \neq 1 \quad \text{for } i \in \{1, \dots, r\}.$$

Note that when F_i is a curve with $\alpha_i = 1$, simply “forgetting” F_i will not change the contribution of E to the residue at $-\frac{\nu}{N}$.

(2.4) Clearly the contribution of E to the residue of $Z_{\text{Hod}}(f, s)$ at $-\frac{\nu}{N}$ is

$$\mathcal{R} := \frac{(1 - uv)}{N(uv)^{-\nu/N}} \left(H(E^\circ) + \sum_{i=1}^r H(F_i^\circ) \frac{uv - 1}{(uv)^{\alpha_i} - 1} + \sum_{\substack{i,j=1 \\ i < j}}^r m_{ij} \frac{(uv - 1)^2}{((uv)^{\alpha_i} - 1)((uv)^{\alpha_j} - 1)} \right),$$

where $m_{ij} := F_i \cdot F_j$ for $i, j \in \{1, \dots, r\}$. In this section we will search for relations between the genus $g(D)$ of D and the genera $g(F_i)$ of the curves F_i , assuming that $\mathcal{R} = 0$.

Write $r = r^{\text{pos}} + r^{\text{neg}}$, where r^{pos} and r^{neg} stand for the number of curves F_i with $\alpha_i > 0$ and $\alpha_i < 0$, respectively; and suppose for convenience that $\alpha_i > 0$ precisely for $i \in \{1, \dots, r^{\text{pos}}\}$. Also we write $\beta_i = -\alpha_i$ if $\alpha_i < 0$. Now we can state the following proposition.

(2.5) Proposition. *We use the notations and conditions of (2.3) and (2.4). If $\mathcal{R} = 0$, then*

- (i) $g = \sum_{i=r^{\text{pos}}+1}^r g_i$,
- (ii) $g = \sum_{i=1}^{r^{\text{pos}}} g_i + \sum_{\{i|\beta_i=1\}} g_i \quad \text{and}$
- (iii) $\sum_{\{i|\alpha_i=x\}} g_i = \sum_{\{i|\beta_i=x\}} g_i \quad \text{for any } x \in \mathbb{Q}^+ \setminus \{1\}.$

Here we denote by g the genus of D and by g_i the genus of F_i , for $i = 1, \dots, r$.

Proof. Using the notation of (2.1) we have by Lemma 2.2(i) that

$$\begin{aligned}
R &:= \frac{N(uv)^{-\nu/N} \mathcal{R}}{1 - uv} \\
&= (uv)^2 - gu^2v - guv^2 + (2 + m_E)uv - gu - gv + 1 - \sum_{i=r^{\text{pos}}+1}^r H(F_i^\circ) - \sum_{\substack{i,j=1 \\ i < j}}^r m_{ij} \\
&\quad + \left(\sum_{i=1}^{r^{\text{pos}}} H(F_i^\circ) \frac{uv - (uv)^{\alpha_i}}{(uv)^{\alpha_i} - 1} + \sum_{i=r^{\text{pos}}+1}^r H(F_i^\circ) \frac{(uv - 1)(uv)^{\beta_i}}{1 - (uv)^{\beta_i}} \right) \\
&\quad + \left(\sum_{\substack{\{i,j\} \subseteq \{1, \dots, r\} \\ \alpha_i, \alpha_j > 0}} m_{ij} \frac{(uv - 1)^2}{((uv)^{\alpha_i} - 1)((uv)^{\alpha_j} - 1)} \right. \\
&\quad + \sum_{\substack{\{i,j\} \subseteq \{1, \dots, r\} \\ \alpha_i > 0, \alpha_j < 0}} m_{ij} \frac{(uv - 1)^2 (uv)^{\beta_j}}{((uv)^{\alpha_i} - 1)(1 - (uv)^{\beta_j})} \\
&\quad \left. + \sum_{\substack{\{i,j\} \subseteq \{1, \dots, r\} \\ \alpha_i, \alpha_j < 0}} m_{ij} \frac{(uv - 1)^2 (uv)^{\beta_i + \beta_j}}{(1 - (uv)^{\beta_i})(1 - (uv)^{\beta_j})} \right),
\end{aligned}$$

where $H(F_i^\circ) = uv - g_i u - g_i v + 1 - \sum_{\substack{k=1 \\ k \neq i}}^r m_{ik}$.

Now we consider $R(0, v)$ and we look at the coefficient of the linear term of this polynomial in v . This coefficient is

$$-g + \sum_{i=r^{\text{pos}}+1}^r g_i,$$

which is equal to zero, since we assume that $\mathcal{R} = 0$. So the first statement is already proven.

When we also take into account that $R(0,0) = 0$, we can rewrite the expression for R as follows :

$$\begin{aligned}
R &= (uv)^2 - gu^2v - guv^2 + (2 + m_E - r^{\text{neg}})uv \\
&+ \sum_{i=1}^{r^{\text{pos}}} H(F_i^\circ) \frac{uv - (uv)^{\alpha_i}}{(uv)^{\alpha_i} - 1} + \sum_{i=r^{\text{pos}}+1}^r H(F_i^\circ) \frac{(uv-1)(uv)^{\beta_i}}{1 - (uv)^{\beta_i}} \\
&+ \sum_{\substack{\{i,j\} \subseteq \{1,\dots,r\} \\ \alpha_i, \alpha_j > 0}} m_{ij} \frac{(uv)^2 - 2uv - (uv)^{\alpha_i+\alpha_j} + (uv)^{\alpha_i} + (uv)^{\alpha_j}}{((uv)^{\alpha_i} - 1)((uv)^{\alpha_j} - 1)} \\
&+ \sum_{\substack{\{i,j\} \subseteq \{1,\dots,r\} \\ \alpha_i > 0, \alpha_j < 0}} m_{ij} \frac{(uv-1)^2 (uv)^{\beta_j}}{((uv)^{\alpha_i} - 1)(1 - (uv)^{\beta_j})} + \sum_{\substack{\{i,j\} \subseteq \{1,\dots,r\} \\ \alpha_i, \alpha_j < 0}} m_{ij} \frac{(uv-1)^2 (uv)^{\beta_i+\beta_j}}{(1 - (uv)^{\beta_i})(1 - (uv)^{\beta_j})}.
\end{aligned}$$

By putting all the terms of R on the same denominator and by looking at the terms of the numerator whose power of u is exactly one higher than the power of v , we deduce from the vanishing of R that

$$\begin{aligned}
0 &= -gu^2v \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l}) \\
&- \sum_{i=1}^{r^{\text{pos}}} g_i u (uv - (uv)^{\alpha_i}) \prod_{\substack{k=1 \\ k \neq i}}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l}) \\
&- \sum_{i=r^{\text{pos}}+1}^r g_i u (uv - 1) (uv)^{\beta_i} \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{\substack{l=r^{\text{pos}}+1 \\ l \neq i}}^r (1 - (uv)^{\beta_l}). \quad (*)
\end{aligned}$$

Denote by q_1 the minimum of the set $\{|\alpha_i| \mid 1 \leq i \leq r\} \cup \{1\}$. If $q_1 < 1$, then the coefficient of $u^{1+q_1}v^{q_1}$ is

$$(-1)^{r^{\text{pos}}-1} \left(\sum_{\{i \mid \alpha_i = q_1\}} g_i \right) + (-1)^{r^{\text{pos}}} \left(\sum_{\{i \mid \beta_i = q_1\}} g_i \right).$$

Since the expression $(*)$ is zero, we obtain that

$$\sum_{\{i \mid \alpha_i = q_1\}} g_i = \sum_{\{i \mid \beta_i = q_1\}} g_i.$$

If we now add the terms of $(*)$ which contribute to the coefficient above, we get

$$\begin{aligned}
&- \sum_{\{i \mid \alpha_i = q_1\}} g_i u (uv - (uv)^{q_1}) \prod_{\substack{k=1 \\ k \neq i}}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l}) \\
&- \sum_{\{i \mid \beta_i = q_1\}} g_i u (uv - 1) (uv)^{q_1} \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{\substack{l=r^{\text{pos}}+1 \\ l \neq i}}^r (1 - (uv)^{\beta_l}) \\
&= - \left(\sum_{\{i \mid \alpha_i = q_1\}} g_i \right) u \frac{(uv - (uv)^{q_1})}{(uv)^{q_1} - 1} \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l})
\end{aligned}$$

$$\begin{aligned}
& - \left(\sum_{\{i|\beta_i=q_1\}} g_i \right) u \frac{(uv-1)(uv)^{q_1}}{1-(uv)^{q_1}} \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1-(uv)^{\beta_l}) \\
& = \left(\sum_{\{i|\alpha_i=q_1\}} g_i \right) u^2 v \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1-(uv)^{\beta_l}).
\end{aligned}$$

So, in the expression (*) we can simply omit the terms contributing to the coefficient above and replace the coefficient

$$-g \quad \text{by} \quad -g + \sum_{\{i|\alpha_i=q_1\}} g_i.$$

Now we can consider the minimum $q_2 := \min(\{|\alpha_i| \mid 1 \leq i \leq r, |\alpha_i| > q_1\} \cup \{1\})$ and proceed in the same way when $q_2 < 1$, yielding the equality

$$\sum_{\{i|\alpha_i=q_2\}} g_i = \sum_{\{i|\beta_i=q_2\}} g_i,$$

omitting the terms associated to $\alpha_i = q_2$ and $\beta_i = q_2$, and replacing

$$-g + \sum_{\{i|\alpha_i=q_1\}} g_i \quad \text{by} \quad -g + \sum_{\{i|\alpha_i=q_1\}} g_i + \sum_{\{i|\alpha_i=q_2\}} g_i.$$

We can keep on doing this until the minimum q_n equals 1. (Remark that this process necessarily has to stop after a finite number of steps.) Then the coefficient of $u^2 v$ is

$$\begin{aligned}
0 &= (-1)^{r^{\text{pos}}} \left(-g + \sum_{\{i|0 < \alpha_i < 1\}} g_i \right) - (-1)^{r^{\text{pos}}-1} \left(\sum_{\{i|\alpha_i > 1\}} g_i \right) + (-1)^{r^{\text{pos}}} \left(\sum_{\{i|\beta_i=1\}} g_i \right) \\
&= (-1)^{r^{\text{pos}}} \left(-g + \sum_{i=1}^{r^{\text{pos}}} g_i + \sum_{\{i|\beta_i=1\}} g_i \right).
\end{aligned}$$

This proves (ii); but in the discussion above we also showed already the third statement for $x < 1$.

In order to prove (iii) for $x > 1$ we first calculate the sum of all the terms contributing to the coefficient of $u^2 v$:

$$\begin{aligned}
& \left(-g + \sum_{\{i|0 < \alpha_i < 1\}} g_i \right) u^2 v \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1-(uv)^{\beta_l}) \\
& - \sum_{\{i|\alpha_i > 1\}} g_i u (uv - (uv)^{\alpha_i}) \prod_{\substack{k=1 \\ k \neq i}}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1-(uv)^{\beta_l}) \\
& - \sum_{\{i|\beta_i=1\}} g_i u (uv - 1) uv \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{\substack{l=r^{\text{pos}}+1 \\ l \neq i}}^r (1-(uv)^{\beta_l}) \\
& = \left(-g + \sum_{\{i|0 < \alpha_i < 1\}} g_i + \sum_{\{i|\beta_i=1\}} g_i \right) u^2 v \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1-(uv)^{\beta_l})
\end{aligned}$$

$$\begin{aligned}
& - \sum_{\{i|\alpha_i > 1\}} g_i u(uv - (uv)^{\alpha_i}) \prod_{\substack{k=1 \\ k \neq i}}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l}) \\
& = - \sum_{\{i|\alpha_i > 1\}} g_i u(uv - 1)(uv)^{\alpha_i} \prod_{\substack{k=1 \\ k \neq i}}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l}).
\end{aligned}$$

Thus we can rewrite (*) in the following way :

$$\begin{aligned}
0 & = - \sum_{\{i|\alpha_i > 1\}} g_i u(uv - 1)(uv)^{\alpha_i} \prod_{\substack{k=1 \\ k \neq i}}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{l=r^{\text{pos}}+1}^r (1 - (uv)^{\beta_l}) \\
& \quad - \sum_{\{i|\beta_i > 1\}} g_i u(uv - 1)(uv)^{\beta_i} \prod_{k=1}^{r^{\text{pos}}} ((uv)^{\alpha_k} - 1) \prod_{\substack{l=r^{\text{pos}}+1 \\ l \neq i}}^r (1 - (uv)^{\beta_l}).
\end{aligned}$$

If we denote by t_1 the minimum of the set $\{|\alpha_i| \mid 1 \leq i \leq r, |\alpha_i| > 1\}$, then the coefficient of $u^{1+t_1}v^{t_1}$ is

$$0 = (-1)^{r^{\text{pos}}-1} \left(\sum_{\{i|\alpha_i=t_1\}} g_i - \sum_{\{i|\beta_i=t_1\}} g_i \right),$$

which is precisely statement (iii) for $x = t_1$. Note that the contributing terms cancel each other; so the summation sets $\{i|\alpha_i > 1\}$ and $\{i|\beta_i > 1\}$ can simply be replaced by $\{i|\alpha_i > t_1\}$ and $\{i|\beta_i > t_1\}$, respectively. Repeat the same procedure. Because there are only finitely many numbers α_i , this proves the proposition. $\blacksquare$

(2.6) Remarks.

- (i) Proposition 2.5 clearly imposes some new conditions on the possible configurations in the case that $\mathcal{R} = 0$. As we will see in the proof of Theorem 3.4, when $g > 0$ there will be at least one and at most two curves F_i with $g_i > 0$. When $g = 0$, we immediately find that all curves F_i need to be rational.
- (ii) It is easily seen that Proposition 2.5 still holds when some of the curves F_i have $\alpha_i = 1$; we only have to replace the second statement by

$$g = \sum_{\substack{i=1 \\ \alpha_i \neq 1}}^{r^{\text{pos}}} g_i + \sum_{\{i|\beta_i=1\}} g_i.$$

3 Logarithmic Kodaira dimension and candidate poles of order 1

(3.1) In this section we fix a non-rational exceptional surface E with $h(E) = \{0\}$. We already know that at its creation E was a ruled surface $\pi : E_0 \rightarrow D$ over a nonsingular

projective curve D of genus $g \geq 1$, see (2.1) and (2.3). To pass from E_0 to E we only have to perform the remaining sequence of blowing-ups in the resolution process after the creation of E_0 . We denote the restriction to E of the composition of these blowing-ups by $\varphi : E \rightarrow E_0$. Forgetting some isomorphisms, φ consists only of a finite number of point-centered blowing-ups.

Figure 2

In the figure above the sketched curves on E are the (reduced) irreducible components F_i , $i = 1, \dots, r$, of the intersections of E with the other components E_k of $h^{-1}(f^{-1}\{0\})$; the same note applies to the curves G_i , $i = 1, \dots, s$, on E_0 where one has to consider the resolution process at the stage of the creation of E_0 . We use the same notation α_i , for $i = 1, \dots, r$, as in (2.3), meaning that when F_i is a component of $E \cap E_{k(i)}$ we put $\alpha_i = \nu_{k(i)} - \frac{\nu}{N} N_{k(i)}$; and we suppose that $\alpha_i \neq 0, 1$. Then we will prove in Theorem 3.4 that if $\bar{\kappa}(E^\circ) = 2$, then the contribution $\mathcal{R}$ of E to the residue of $Z_{\text{Hod}}(f, s)$ at $-\frac{\nu}{N}$ is different from zero. (Recall that “forgetting” a curve F_i with $\alpha_i = 1$ will not change $\mathcal{R}$, but note that it might affect $\bar{\kappa}(E^\circ)$. This is the reason why we make the assumption that $\alpha_i \neq 1$ for $i = 1, \dots, r$.) We will need the following facts. The first one is an immediate corollary of Hurwitz’s Theorem [8, Corollary IV.2.4].

(3.2) Lemma. *Let G_i be an intersection curve on E_0 which is not a fiber of π . Let $\tilde{G}_i$ be the strict transform of G_i by φ and $f_i : \tilde{G}_i \rightarrow D$ the restriction of $\pi \circ \varphi$ to $\tilde{G}_i$. Then we have*

$$g_i = g + (n_i - 1)(g - 1) + \frac{\deg R_i}{2},$$

where g_i is the geometric genus of G_i , n_i the degree of f_i and R_i the ramification divisor of f_i . In particular this implies that $g_i \geq g$.

(3.2.1) Remarks.

- (i) We denote by d_i the number of intersections of the curve G_i with a general fiber F of π , recall the remark at the end of Subsection 1.5. In particular, the fibers are precisely the curves G_i with $d_i = 0$. Then we have

$$n_i = \deg f_i = d_i.$$

- (ii) Let G_i be a curve with $d_i = 1$. Then we know by the first remark that $f_i : \tilde{G}_i \rightarrow D$ is an isomorphism. But in this case we even know that G_i is already nonsingular and that $\pi|_{G_i} : G_i \rightarrow D$ is an isomorphism. Indeed, put $\pi_i := \pi|_{G_i} : G_i \rightarrow D$.

Figure 3

Since any two fibers of a ruled surface are algebraically equivalent, we see that

$$1 = d_i = G_i \cdot (\text{any fiber of } \pi),$$

implying already that G_i is nonsingular. Because $\deg f_i = d_i = 1$, we also have $\deg \pi_i = 1$, yielding that π_i is an isomorphism. Remark also that the composition

$$D \xrightarrow{\pi_i^{-1}} G_i \hookrightarrow E_0$$

is a section of $\pi : E_0 \rightarrow D$.

(3.3) Lemma. *The ruled surface $\pi : E_0 \rightarrow D$ is a fibered variety.*

Proof. Because π is obviously dominating, we only have to prove that the field extension $\text{Rat}(E_0)/\text{Rat}(D)$, given by π^* , is algebraically closed. Let U be an affine open subset of D such that $\pi^{-1}(U) \cong \mathbb{P}^1 \times U$ and such that under this isomorphism π coincides with the second projection $p_2 : \mathbb{P}^1 \times U \rightarrow U$. Then we equivalently can study the field extension $\text{Rat}(\mathbb{A}^1 \times U)/\text{Rat}(U)$, given by p_2^* . Now suppose that $U = \text{Spec } A$ with A a finitely generated $\mathbb{C}$ -algebra. Then

$$\mathbb{A}^1 \times U = \text{Spec}(\mathbb{C}[x] \otimes_{\mathbb{C}} A) \cong \text{Spec}(A[x]),$$

where x is a variable. If the field of fractions of A is denoted by K , we get

$$\text{Rat}(U) = K,$$

$$\text{Rat}(\mathbb{A}^1 \times U) = K(x);$$

so the lemma follows from the easy fact that the field extension $K(x)/K$, given by the canonical embedding, is algebraically closed. $\blacksquare$

Now we are ready to prove the main result of this paper.

(3.4) Theorem. *Let $f \in \mathbb{C}[x_1, x_2, x_3] \setminus \mathbb{C}$. Fix an embedded resolution $h : X \rightarrow \mathbb{A}_{\mathbb{C}}^3$ of $f^{-1}\{0\}$ in $\mathbb{A}_{\mathbb{C}}^3$ and use the notation of Section 1. Let E be a non-rational exceptional surface with*

$h(E) = \{0\}$ and numerical data (ν, N) . Let F_i , $i = 1, \dots, r$, be the irreducible components of the intersections $E \cap E_k$, $k \in T$, where E_k is another exceptional surface or an irreducible component of the strict transform of $f^{-1}\{0\}$. When F_i is a component of $E \cap E_{k(i)}$, then we put $\alpha_i := \nu_{k(i)} - \frac{\nu}{N}N_{k(i)}$. Suppose that $\alpha_i \neq 0, 1$ for each $i \in \{1, \dots, r\}$. If $\bar{\kappa}(E^\circ) = 2$, then the contribution $\mathcal{R}$ of E to the residue of $Z_{\text{Hod}}(f, s)$ at $-\frac{\nu}{N}$ is nonzero.

Proof. We use the same notation as in (3.1). Suppose that all the conditions of the theorem are fulfilled, but $\mathcal{R} = 0$. Remark that $E^\circ \cong E_0^\circ$, since the centers of the blowing-ups always belong to the strict transform of $f^{-1}\{0\}$ (by the resolution process up to the right stage). Thus $\bar{\kappa}(E_0^\circ) = \bar{\kappa}(E^\circ) = 2$.

There can be at most two curves G_i with $d_i \geq 1$. Indeed, if G_i is such a curve, then we know by Lemma 3.2 that $g_i \geq g \geq 1$, where g_i is the geometric genus of G_i . So also the strict transform $\tilde{G}_i$ of G_i by φ , which equals one of the curves F_j , has geometric genus $g_i \geq g$. By Proposition 2.5(i) and (ii) we see that there can be at most two such curves F_j .

On the other hand, since D did belong to an exceptional surface (of the resolution process at the stage just before the blowing-up with center D), there is at least one G_i with $d_i = 1$, say G_1 .

By Lemma 3.3 we can easily deduce that also $\pi|_{E_0^\circ} : E_0^\circ \rightarrow D$ is a fibered variety. So Theorem 1.6.1 implies that

$$\bar{\kappa}(E_0^\circ) \leq \bar{\kappa}\left(\mathbb{P}^1 \setminus \left\{\sum_{i=1}^s d_i \text{ points}\right\}\right) + 1,$$

which yields by Figure 1 in (0.4) that

$$\sum_{i=1}^s d_i \geq 3.$$

Thus there are exactly two curves G_i with $d_i \neq 0$. Let G_2 be the second one. Then $d_2 \geq 2$. When $g \geq 2$, we are done now, since by Lemma 3.2

$$g_2 = g + (d_2 - 1)(g - 1) + \frac{\deg R_i}{2} > g,$$

contradicting Proposition 2.5.

When $g = 1$, we are not ready yet. Let $\tilde{G}_1$ and $\tilde{G}_2$ be equal to $F_{j(1)}$ and $F_{j(2)}$, respectively. Since the only curves F_j with nonzero genus are among the $\tilde{G}_i$, we can conclude by Proposition 2.5(iii) that

$$\alpha_{j(1)} + \alpha_{j(2)} = 0.$$

But by Theorem 1.5.1 we also know that

$$d_1(\alpha_{j(1)} - 1) + d_2(\alpha_{j(2)} - 1) = -2.$$

Combining the two relations yields that

$$(d_2 - 1)(\alpha_{j(2)} - 1) = 0,$$

which is again a contradiction since $d_2 \geq 2$ and $\alpha_{j(2)} \neq 1$. This proves the theorem. ■

(3.4.1) Remark. As explained in [19, Proposition 5.4.1], the appropriate definition of the contribution of E to the residue at $-\frac{\nu}{N}$ of the motivic zeta function is the expression

$$\mathcal{R}_{\text{mot}} := \frac{(\mathbb{L} - 1)}{-N\mathbb{L}^{-\nu/N}} \left([E^\circ] + \sum_{i=1}^r [F_i^\circ] \frac{(\mathbb{L} - 1)\mathbb{L}^{-\alpha_i}}{1 - \mathbb{L}^{-\alpha_i}} + \sum_{\substack{i,j=1 \\ i < j}}^r m_{ij} \frac{(\mathbb{L} - 1)^2 \mathbb{L}^{-(\alpha_i + \alpha_j)}}{(1 - \mathbb{L}^{-\alpha_i})(1 - \mathbb{L}^{-\alpha_j})} \right)$$

in the ring A of Subsection 1.3. Then Theorem 3.4 also holds for the motivic zeta function $Z_{\text{mot}}(f, s)$. This can easily be proved by using the technique of specialization. For a very similar argument, see for example [19, Proof of Theorem 5.5], [15, Proof of Theorem 5.2] and [17, Remark 3.8.2(ii)].

(3.5) We keep on using for a moment the notation of the proof of Theorem 3.4. An important ingredient in this proof is that the condition of maximal logarithmic Kodaira dimension implies that $\sum_{i=1}^s d_i \geq 3$. Actually we then proved that under the latter condition the residue $\mathcal{R}$ is not zero. In the following proposition we show that in “most of these cases” we do have maximal logarithmic Kodaira dimension.

(3.5.1) Proposition. *Use the same notation as in (the proof of) Theorem 3.4; in particular $\pi : E_0 \rightarrow D$ denotes a ruled surface over the curve D with $g(D) = g$. Suppose that $\sum_{i=1}^s d_i \geq 3$. Then*

- (i) *if $g \geq 2$, we have $\overline{\kappa}(E_0^\circ) = 2$,*
- (ii) *if $g = 1$ and at least one of the curves G_i is a fiber, we have $\overline{\kappa}(E_0^\circ) = 2$,*
- (iii) *if $g = 0$ and at least three of the curves G_i are fibers, we have $\overline{\kappa}(E_0^\circ) = 2$.*

Remark that in (iii) we freely use the notation of this section for a ruled surface over a rational curve D , although we assume in (3.1) that E_0 is non-rational.

Proof. Consider the morphism $\pi|_{E_0^\circ} : E_0^\circ \rightarrow U$, where U is the open subset of D which is obtained from D by removing the points $\pi(G_i)$ with G_i a fiber of π . We apply Theorem 1.6.2 to this morphism, which is a fibered variety by Lemma 3.3. This yields that

$$\overline{\kappa}(E_0^\circ) \geq \overline{\kappa} \left(\mathbb{P}^1 \setminus \left\{ \sum_{i=1}^s d_i \text{ points} \right\} \right) + \overline{\kappa}(U).$$

Since $\sum_{i=1}^s d_i \geq 3$, we know by Figure 1 in (0.4) that the right-hand side of the inequality above is equal to 2. This proves the proposition. $\blacksquare$

(3.5.2) Remark. We already mentioned above that we actually showed in the proof of Theorem 3.4 that under the condition $\sum_{i=1}^s d_i \geq 3$ the residue $\mathcal{R}$ is not zero. Possibly this includes cases with $\overline{\kappa}(E_0^\circ) < 2$. But Proposition 3.5.1 learns us that these special cases can only occur when $g = 1$ and none of the curves G_i is a fiber. Consider for example the following situation where G_1, G_2 and G_3 are three nonintersecting sections of π .

Figure 4

Note that in this case we even have $\chi(E_0^\circ) = 0$. It is interesting to recall that this is exactly one of the exceptions in the framework of Veys where he wants to prove that $\mathcal{R} = 0$ when $\chi(E_0^\circ) = \chi(E^\circ) \leq 0$, see [21, Theorem 5.9]. It is an intriguing question whether or not this situation can occur in an embedded resolution. The impossibility of occurrence is already proved by Veys when all the singular points of $f^{-1}\{0\}$ are absolutely isolated, see [21, Definition 5.10 and Proposition 5.12].

(3.6) Remark. When some of the curves F_i have $\alpha_i = 1$ the conclusion of Theorem 3.4 remains true if we consider E° to be

$$E \setminus \bigcup_{\{i|\alpha_i \neq 1\}} F_i,$$

at least when $g \geq 2$. Indeed, let U be the open subset of E° , obtained from E° by also removing the curves F_i which have $\alpha_i = 1$ and which are not equal to one of the curves $\tilde{G}_j$. Then we have

$$\bar{\kappa}(U) = 2.$$

It is not so hard to verify that

$$U \cong E_0^\circ \setminus \{P_1, \dots, P_m\}$$

for certain points P_j on E_0 , where also E_0° has to be interpreted as

$$E_0 \setminus \bigcup_{\{i|\alpha_{j(i)} \neq 1\}} G_i$$

with $j(i)$ such that $\tilde{G}_i$ equals $F_{j(i)}$. Actually, each point P_j only belongs to some curves G_i with $\alpha_{j(i)} = 1$ and somewhere in our process towards E we performed a blowing-up at it. Because Theorem 1.6.1 is only interested in a general fiber, we can conclude that

$$\sum_{\substack{i=1 \\ \alpha_{j(i)} \neq 1}}^s d_i \geq 3.$$

On the other hand, following the proof of Theorem 3.4, we know by Remark 2.6(ii) that there can be at most two curves G_i with $d_i \geq 1$ and $\alpha_{j(i)} \neq 1$. This again completes the proof for

$g \geq 2$ by Lemma 3.2. Remark that for $g = 1$ we needed in the last and technical part of the proof of Theorem 3.4 the existence of a curve G_i with $d_i = 1$. In our new situation it may happen that this curve G_i , which of course still exists, has $\alpha_{j(i)} = 1$, making it impossible (for us) to end the argumentation.

(3.7) Remarks.

- (i) For a rational exceptional surface E with $h(E) = \{0\}$ and all $\alpha_i \neq 0$ we have that $\mathcal{R} \neq 0$ as soon as one of the curves F_i on it, with $\alpha_i \neq 1$, is non-rational. Indeed, when E is created by a blowing-up at a rational curve, this easily follows from Proposition 2.5 and Remark 2.6(ii). When E is created by a point-centered blowing-up, we argue as follows. Recall the expression for $\mathcal{R}$:

$$\begin{aligned} \mathcal{R} = & \frac{1-uv}{N(uv)^{-\nu/N}} \left(H(E^\circ) + \sum_{i=1}^r H(F_i^\circ) \frac{uv-1}{(uv)^{\alpha_i}-1} \right. \\ & \left. + \sum_{\{i,j\} \subseteq \{1,\dots,r\}} m_{ij} \frac{(uv-1)^2}{((uv)^{\alpha_i}-1)((uv)^{\alpha_j}-1)} \right), \end{aligned}$$

where $m_{ij} := F_i \cdot F_j$. Suppose that $\mathcal{R} = 0$. Remark that the expression for $\mathcal{R}$ can be obtained from the one in the proof of Proposition 2.5 by replacing in $H(E^\circ)$ the coefficient $(2 + m_E)$ by $(1 + m_E)$ and g by 0, see Lemma 2.2(ii). So, after performing those replacements in the expression for $\mathcal{R}$ in the proof of Proposition 2.5, we can go again through the details of that proof. This time the conclusion will be that $g_i = 0$ for all curves F_i with $\alpha_i \neq 1$.

- (ii) When E is a rational exceptional surface such that $h(E) = \{0\}$, all $\alpha_i \neq 0$ and all curves F_i , with $\alpha_i \neq 1$, are rational, we have some partial results, all obtained by tedious and very laborious calculations.

- (a) When $E = \mathbb{P}^2$ and the curves F_i , with $\alpha_i \neq 1$, form a configuration of lines having only normal crossings, we can show that

$$\mathcal{R} \neq 0 \quad \text{if and only if} \quad \bar{\kappa}(E \setminus \bigcup_{\{i|\alpha_i \neq 1\}} F_i) = 2.$$

- (b) When E is created as a projective plane E_0 and there is only one intersection curve G on E_0 and the singularities of G are all cusps or ordinary r -fold points, i.e., analytically equivalent to a configuration of r lines through one point, then we have

$$(\bar{\kappa}(E_0 \setminus G) = 2) \Rightarrow (\mathcal{R} \neq 0).$$

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