Sums of three squareful numbers

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Abstract. We investigate the frequency of positive squareful numbers $x, y, z \leqslant B$ for which $x + y = z$ and present a conjecture concerning its asymptotic behaviour.

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1. Introduction

In this paper we examine the quantitative arithmetic of integral points on certain Campana orbifolds, following the discussions of Abramovich [1], Campana [3] and Poonen [10]. Given rational points $p_i = r_i/s_i \in \mathbf{P}^1(\mathbf{Q})$ with integer multiplicities $m_i \geqslant 2$, for $1 \leqslant i \leqslant n$, we define the divisor $\Delta = \sum_i (1 - \frac{1}{m_i})[p_i]$. The pair $(\mathbf{P}^1, \Delta)$ defines an orbifold curve in the sense of Campana and has associated Euler characteristic

$$\chi = \chi(\mathbf{P}^1) - \deg \Delta = 2 - n + \frac{1}{m_1} + \cdots + \frac{1}{m_n}.$$

A point $r/s \in \mathbf{P}^1(\mathbf{Q})$ is said to be integral if $rs_i - sr_i$ is m_i-powerful for $1 \leqslant i \leqslant n$. Here we recall that an integer k is said to be m-powerful if $p^m \mid k$ whenever p is a prime divisor of k. We will focus our attention here upon the orbifold $(\mathbf{P}^1, \Delta)$ associated to the divisor

$$\Delta = \left(1 - \frac{1}{m}\right)[0] + \left(1 - \frac{1}{m}\right)[1] + \left(1 - \frac{1}{m}\right)[\infty],$$

with Euler characteristic $\chi = -1 + \frac{3}{m}$. The density of integral points on $(\mathbf{P}^1, \Delta)$ with height at most B is captured by the counting function

$$N_{m-1}(B) = \#\left\{(x, y, z) \in \mathbf{N}^3_{\text{prim}} : x + y = z,\ x, y, z \leqslant B,\ x, y, z \ m\text{-powerful}\right\},$$

where $\mathbf{N}$ denotes the set of positive integers and $\mathbf{N}^3_{\text{prim}}$ denotes the set of primitive vectors in $\mathbf{N}^3$.

An old result of Erdős and Szekeres [4] shows that the number of m-powerful integers up to x is $c_m x^{\frac{1}{m}} + O(x^{\frac{1}{m+1}})$, for a certain constant $c_m > 0$. This leads to a basic trichotomy: we expect only finitely many integral points when $\chi < 0$, we expect $N_{m-1}(B)$ to grow at worst logarithmically in B when $\chi = 0$ and we expect $N_{m-1}(B)$ to have order B^{χ} when $\chi > 0$. When $m = 3$ work of Nitaj [7] shows that $N_2(B) \gg \log B$. Our goal in this paper is to provide evidence in support of the expected order $B^{\frac{1}{2}}$ of $N_1(B)$ when $m = 2$.

Conjecture 1. *We have*

$$N_1(B) = cB^{\frac{1}{2}}(1 + o(1)),$$

as $B \to \infty$, with $c = 2.68\ldots$.

The explicit conjectured value of c is too complicated to record here, but may be found in (13) and (14). Our expression for c involves an infinite sum which converges very slowly, thereby making it difficult to evaluate numerically to high accuracy.

We may test Conjecture 1 by naively listing all squareful numbers up to B, and then subsequently sorting them into triples (x, y, z) that are counted by $N_1(B)$. More precisely, the algorithm

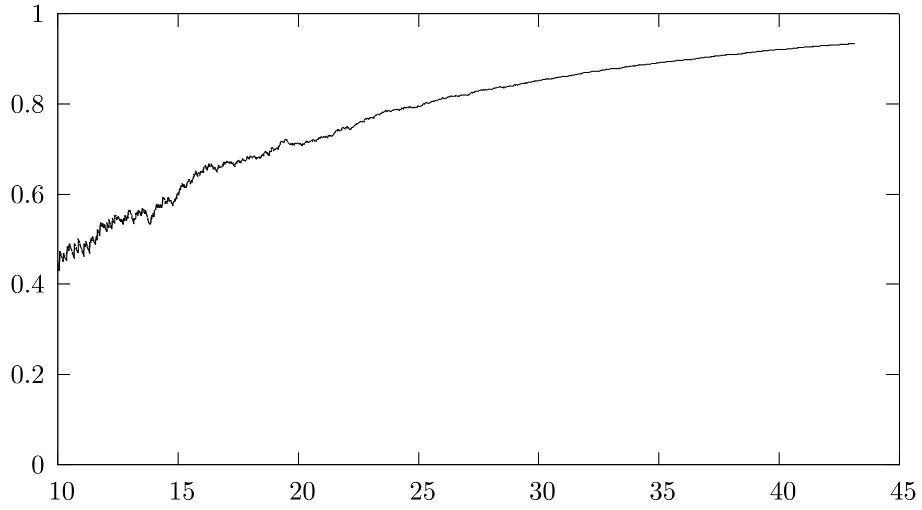

Figure 1. Values of $N_1(B)/(cB^{\frac{1}{2}})$

loops through all squareful numbers z in increasing order, and for each z, it runs over squareful $x \in [z/2, z]$ and uses the list to verify whether or not $y = z - x$ is squareful. If it is, we verify whether $\gcd(x, y) = 1$ and eventually print the two corresponding points (x, y, z) and (y, x, z). The inner code of the two loops is repeated $O(s^2)$ times, where s is the number of squarefuls involved, so that the total complexity is $O(B)$. For $B = 10^{13}$ the compilation of the list took less than 2 minutes on an Intel Core 2 Duo E8400 running at 3GHz, resulting in 6840384 squareful numbers overall. The sorting algorithm required a computing time of 5587.5 minutes. In Figure 1 the values of $N_1(B)/(cB^{\frac{1}{2}})$ are plotted for B up to 10^{13}, where the horizontal axis runs over values of $\log_2 B$. In Table 1 we present some explicit numerical data, including the determination of the quotient $N_1(B)/(cB^{\frac{1}{2}})$ for large values of B.

B	$N_1(B)$	$N_1(B)/(cB^{\frac{1}{2}})$,
10^7	6562	0.774
10^8	21920	0.818
10^9	72124	0.851
10^{10}	235168	0.877
10^{11}	762580	0.900
10^{12}	2465044	0.920
10^{13}	7914884	0.934

Table 1

Any positive squareful integer k can be written uniquely as $k = x^2y^3$, with $x, y \in \mathbf{N}$ and y square-free. Using this description we have

$$N_1(B) = \sum_{\mathbf{y} \in \mathbf{N}^3} \mu^2(y_0y_1y_2) \# \left\{ \mathbf{x} \in \mathbf{N}^3 \cap C_{\mathbf{y}} : \begin{array}{l} \gcd(x_0y_0, x_1y_1, x_2y_2) = 1, \\ x_0^2y_0^3, x_1^2y_1^3, x_2^2y_2^3 \leqslant B \end{array} \right\}, \tag{1}$$

where μ is the Möbius function and $C_{\mathbf{y}}$ denotes the conic $x_0^2y_0^3 + x_1^2y_1^3 = x_2^2y_2^3$. One is naturally led to analyse $N_1(B)$ by counting points on each conic and then summing the contribution over the $\mathbf{y}$. This is the point of view adopted by the second author [11], where the structure of the orbifold $(\mathbf{P}^1, \Delta)$ is generalised to a higher-dimensional analogue $(\mathbf{P}^{n-1}, \Delta)$, corresponding to a hyperplane of squareful numbers. An asymptotic formula of the expected order of magnitude is then obtained when there are $n + 1 \geqslant 5$ terms present in the hyperplane. In addition to this [11] contains an interpretation of the leading constant in terms of local densities for the underlying quadric. We will revisit this discussion in §2 in order to justify the numerical value of the constant in Conjecture 1.

Ignoring all but the term with $\mathbf{y} = (1,1,1)$ in (1), one readily arrives at the lower bound $N_1(B) \gg B^{\frac{1}{2}}$, via the familiar parametrisation for Pythagorean triples. Building on this observation suitably, we will sketch a proof of the following result in §3.

Theorem 1. *We have $N_1(B) \geqslant cB^{\frac{1}{2}}(1+o(1))$, where c is the constant in Conjecture 1.*

The problem of producing an upper bound of the expected order of magnitude is much more challenging. In §4 we shall establish the following estimate.

Theorem 2. *We have $N_1(B) = O(B^{\frac{3}{5}} \log^{12} B)$.*

With more work it ought to be possible to remove the factor involving $\log B$ from Theorem 2. The proof of Theorem 2 involves two estimates. The first is based on fixing the $\mathbf{y}$ and counting points on the conic $C_{\mathbf{y}}$, uniformly in the coefficients. The second involves switching the rôles of $\mathbf{y}$ and $\mathbf{x}$, viewing the equation as a family of plane cubics instead. For both of these the determinant method of Heath-Brown [6] is a key tool. The same argument has been observed by a number of mathematicians, including Valentin Blomer in private communication with the first author. In order to improve the exponent of B in Theorem 2 one requires a new means of treating the contribution from $\mathbf{x}, \mathbf{y}$ for which each x_i and y_i has order of magnitude $B^{\frac{1}{5}}$. It would be desirable, for example, to have better control over the $\mathbf{y}$ which produce conics $C_{\mathbf{y}}$ containing at least one rational point of small height.

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2. The constant

Recall the expression for $N_1(B)$ in (1), in which $C_{\mathbf{y}}$ denotes the conic

$$x_0^2 y_0^3 + x_1^2 y_1^3 = x_2^2 y_2^3,$$

for given $\mathbf{y} = (y_0, y_1, y_2) \in \mathbf{N}^3$. Let $H_{\mathbf{y}} : C_{\mathbf{y}}(\mathbf{Q}) \to \mathbf{R}_{\geqslant 0}$ denote the height function

$$[x_0, x_1, x_2] \mapsto \max\{|x_0^2 y_0^3|, |x_1^2 y_1^3|, |x_2^2 y_2^3|\}^{\frac{1}{2}},$$

if $x_0, x_1, x_2 \in \mathbf{Z}$ satisfy $\gcd(x_0, x_1, x_2) = 1$. On noting that $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in $\mathbf{P}^2$ we easily infer that $N_1(B)$ is approximated by the sum

$$\frac{1}{4} \sum_{\mathbf{y} \in \mathbf{N}^3} \mu^2(y_0 y_1 y_2) \# \left\{ x \in C_{\mathbf{y}}(\mathbf{Q}) : H_{\mathbf{y}}(x) \leqslant B^{\frac{1}{2}},\ \gcd(x_0 y_0, x_1 y_1, x_2 y_2) = 1 \right\}. \tag{2}$$

Following the framework developed by the second author [11, §5], we are therefore led to take the value

$$c = \frac{1}{4} \sum_{\mathbf{y} \in \mathbf{N}^3} \mu^2(y_0 y_1 y_2) c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+), \tag{3}$$

in Conjecture 1. Here, if $C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+$ denotes the open subset of the adelic space $C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})$ carved out by the condition $\min_{0 \leqslant i \leqslant 2}\{v_p(x_{i,p} y_i)\} = 0$ for each prime p, then $c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+)$ is a special case of the constant conjecturally introduced by Peyre [8, Définition 2.5] in the broader context of Fano varieties. In particular it follows that

$$c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+) = \alpha(C_{\mathbf{y}}) \boldsymbol{\omega}_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+), \tag{4}$$

where $\boldsymbol{\omega}_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+)$ denotes the Tamagawa measure of $C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+$ associated to the height $H_{\mathbf{y}}$ and $\alpha(C_{\mathbf{y}})$ is the volume of a certain polytope contained in the cone of effective divisors.

Let $\mathbf{y} \in \mathbf{N}^3$ with $\mu^2(y_0y_1y_2) = 1$. In the present setting we have $\mathrm{Pic}(C_{\mathbf{y}}) \cong \mathbf{Z}$ and one finds, using [8, Définition 2.4], that

$$\alpha(C_{\mathbf{y}}) = \frac{1}{2}. \tag{5}$$

In [11], wherein non-singular quadrics in $\mathbf{P}^n$ feature for $n \geqslant 4$, it is worth highlighting that the corresponding value of the constant is found to be $\frac{1}{n-1}$ using the Lefschetz hyperplane theorem. This is no longer true when considering conics in $\mathbf{P}^2$, since the anticanonical divisor is not a generator for the Picard group.

Turning to the Tamagawa constant we let $S = \{\infty, 2\} \cup \{p \mid y_0y_1y_2\}$, a finite set of places. The Tamagawa measure on $C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})$ associated to the height function $H_{\mathbf{y}}$ is given by

$$\boldsymbol{\omega}_{H_{\mathbf{y}}} = \lim_{s\to 1}(s-1)L_S(s, \mathrm{Pic}(\overline{C_{\mathbf{y}}})) \prod_{v\in \mathrm{Val}(\mathbf{Q})} \lambda_v^{-1}\boldsymbol{\omega}_{H_{\mathbf{y}},v}, \tag{6}$$

where

$$\lambda_v = \begin{cases} (1-\frac{1}{p})^{-1}, & \text{if } v \in \mathrm{Val}(\mathbf{Q}) - S,\\ 1, & \text{otherwise} \end{cases} \tag{7}$$

and

$$L_S(s, \mathrm{Pic}(\overline{C_{\mathbf{y}}})) = \prod_{v\in \mathrm{Val}(\mathbf{Q})-S} \left(1-\frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p\mid 2y_0y_1y_2} \left(1-\frac{1}{p^s}\right).$$

Hence

$$\lim_{s\to 1}(s-1)L_S(s, \mathrm{Pic}(\overline{C_{\mathbf{y}}})) = \prod_{p\mid 2y_0y_1y_2}\left(1-\frac{1}{p}\right). \tag{8}$$

In the next few sections, we will calculate the v-adic densities at the different places.

2.1. Density at the good places

Let p be a prime such that $p \nmid 2y_0y_1y_2$. Recall that $C_{\mathbf{y}}(\mathbf{Q}_p)^+$ is defined as the subset of points $[x_{0,p}, x_{1,p}, x_{2,p}] \in C_{\mathbf{y}}(\mathbf{Q}_p)$, with $x_{i,p} \in \mathbf{Z}_p$ and $\min_{0\leqslant i\leqslant 2}\{v_p(x_{i,p})\} = 0$, for which

$$\min_{0\leqslant i\leqslant 2}\{v_p(x_{i,p}y_i)\} = 0. \tag{9}$$

Since $p \nmid y_0y_1y_2$ this latter condition is automatically satisfied, whence $C_{\mathbf{y}}(\mathbf{Q}_p)^+ = C_{\mathbf{y}}(\mathbf{Q}_p)$. By Lemmas 3.2 and 3.4 in [9] and [8, Lemme 5.4.6], we have

$$\boldsymbol{\omega}_{H_{\mathbf{y}},p}(C_{\mathbf{y}}(\mathbf{Q}_p)) = \frac{\#C_{\mathbf{y}}(\mathbf{F}_p)}{p}.$$

Since $C_{\mathbf{y}}(\mathbf{F}_p) \neq \emptyset$ by Chevalley–Warning, we deduce that $\#C_{\mathbf{y}}(\mathbf{F}_p) = \#\mathbf{P}^1(\mathbf{F}_p) = p + 1$. This implies that for the good places we have

$$\begin{aligned}\prod_{v\in \mathrm{Val}(\mathbf{Q})-S} \lambda_v^{-1}\boldsymbol{\omega}_{H_{\mathbf{y}},v}(C_{\mathbf{y}}(\mathbf{Q}_v)^+) &= \prod_{p\nmid 2y_0y_1y_2}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)\\ &= \frac{8}{\pi^2}\cdot \prod_{\substack{p\mid y_0y_1y_2\\ p>2}}\left(1-\frac{1}{p^2}\right)^{-1},\end{aligned} \tag{10}$$

since $\prod_{p>2}\left(1-\frac{1}{p^2}\right) = \frac{4}{3}\cdot\frac{6}{\pi^2} = \frac{8}{\pi^2}$.

2.2. Density at the bad places

We now suppose that p is a prime divisor of $2y_0y_1y_2$. In this case, when considering $C_{\mathbf{y}}(\mathbf{Q}_p)^+$, the condition (9) will no longer be satisfied trivially. Let

$$N_{\mathbf{y}}^*(p^r) = \#\left\{\mathbf{x} \in (\mathbf{Z}/p^r\mathbf{Z})^3 - (p\mathbf{Z}/p^r\mathbf{Z})^3 : \begin{array}{l} y_0^3x_0^2 + y_1^3x_1^2 \equiv y_2^3x_2^2 \pmod{p^r}, \\ \min_{0\leqslant i\leqslant 2}\{v_p(x_iy_i)\} = 0 \end{array}\right\}.$$

Using Lemmas 3.2 and 3.4 in [9] and [8, Lemme 5.4.6], we deduce the existence of $r_0 \in \mathbf{N}$ such that

$$\boldsymbol{\omega}_{H_{\mathbf{y}},p}(C_{\mathbf{y}}(\mathbf{Q}_p)^+) = \left(1 - \frac{1}{p}\right)^{-1} \cdot \frac{N_{\mathbf{y}}^*(p^r)}{p^{2r}}, \tag{11}$$

for each $r \geqslant r_0$. The following pair of results are concerned with the calculation of $N_{\mathbf{y}}^*(p^r)$ for primes $p \mid 2y_0y_1y_2$.

Lemma 1. *If $p \mid y_0y_1y_2$ and $p > 2$, we have*

$$\frac{N_{\mathbf{y}}^*(p^r)}{p^{2r}} = \left(1 - \frac{1}{p}\right) \times \begin{cases} \left(1 + \left(\frac{y_1y_2}{p}\right)\right), & \text{if } p \mid y_0, \\ \left(1 + \left(\frac{y_0y_2}{p}\right)\right), & \text{if } p \mid y_1, \\ \left(1 + \left(\frac{-y_0y_1}{p}\right)\right), & \text{if } p \mid y_2. \end{cases}$$

Proof. Suppose, for example, that p divides y_0. In this case $p \nmid y_1y_2$. Modulo p we obtain the equation $y_1^3x_1^2 \equiv y_2^3x_2^2 \pmod{p}$. If $y_1^{-3}y_2^3$ is a square modulo p, then we can choose x_2 arbitrarily in $\mathbf{F}_p^\times$ and for each choice of x_2 there are two solutions for x_1. It follows that there are $2p(p-1)$ solutions modulo p in this case. If $y_1^{-3}y_2^3$ is not a square modulo p, then there are no solutions. We conclude that $N_{\mathbf{y}}^*(p) = (1 + (\frac{y_1y_2}{p}))p(1-p)$. Using Hensel's lemma we deduce that $N_{\mathbf{y}}^*(p^r)$ is equal to $p^{2(r-1)}(1 + (\frac{y_1y_2}{p}))p(1-p)$ for each $r \geqslant 1$, which thereby completes the proof. ☐

Lemma 2. *If $r \geqslant 3$, we have*

$$\frac{N_{\mathbf{y}}^*(2^r)}{2^{2r}} = \begin{cases} 1, & \text{if } 2 \nmid y_0y_1y_2 \text{ and } \neg\{y_0 \equiv y_1 \equiv -y_2 \pmod{4}\}, \\ 2, & \text{if } 2 \mid y_0 \text{ and } y_1 \equiv y_2 \pmod{8}, \\ 2, & \text{if } 2 \mid y_1 \text{ and } y_0 \equiv y_2 \pmod{8}, \\ 2, & \text{if } 2 \mid y_2 \text{ and } y_0 \equiv -y_1 \pmod{8}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows from direct calculation for the case $r = 3$. The formula for $r > 3$ follows from Hensel's lemma. ☐

2.3. Density at the infinite place

It remains to consider the infinite place $v = \infty$. Let

$$D_1 = \left\{(y_0^3x_0^2, y_1^3x_1^2, y_2^3x_2^2) \in (\mathbf{R} \cap [-1,1])^3 : y_0^3x_0^2 + y_1^3x_1^2 = y_2^3x_2^2\right\}.$$

Using [8, Lemme 5.4.7], we obtain

$$\boldsymbol{\omega}_{H_{\mathbf{y}},\infty}(C_{\mathbf{y}}(\mathbf{R})^+) = \frac{1}{2} \cdot \int_{D_1} \boldsymbol{\omega}_{L,\infty},$$

where

$$\boldsymbol{\omega}_{L,\infty} = \frac{\mathrm{d}x_0\mathrm{d}x_1}{2y_2^{\frac{3}{2}}\sqrt{y_0^3x_0^2 + y_1^3x_1^2}}$$

is the Leray form. Let $D_2 = \{(x_0, x_1) \in (\mathbf{R} \cap [-1,1])^2 : x_0^2 + x_1^2 \leqslant 1\}$. Then it follows that

$$\begin{aligned} \boldsymbol{\omega}_{H_{\mathbf{y}},\infty}(C_{\mathbf{y}}(\mathbf{R})^+) &= \frac{1}{2} \cdot \frac{1}{(y_0y_1y_2)^{\frac{3}{2}}} \int_{D_2} \frac{1}{\sqrt{x_0^2 + x_1^2}} \mathrm{d}x_0\mathrm{d}x_1 \\ &= \frac{\pi}{(y_0y_1y_2)^{\frac{3}{2}}}. \end{aligned} \tag{12}$$

2.4. Conclusion

Recall the definition (6) of the Tamagawa measure, in which the convergence factors are given by (7). Combining (8), (10), (11) with Lemma 1 and (12) we deduce that $\boldsymbol{\omega}_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+)$ is equal to

$$\frac{1}{(y_0y_1y_2)^{\frac{3}{2}}}\cdot\frac{8}{\pi}\cdot\sigma_{2,\mathbf{y}}\cdot\prod_{\substack{p\mid y_0\\ p>2}}\frac{\left(1+\left(\frac{y_1y_2}{p}\right)\right)}{\left(1+\frac{1}{p}\right)}\cdot\prod_{\substack{p\mid y_1\\ p>2}}\frac{\left(1+\left(\frac{y_0y_2}{p}\right)\right)}{\left(1+\frac{1}{p}\right)}\cdot\prod_{\substack{p\mid y_2\\ p>2}}\frac{\left(1+\left(\frac{-y_0y_1}{p}\right)\right)}{\left(1+\frac{1}{p}\right)},$$

where $\sigma_{2,\mathbf{y}}=\lim_{r\to\infty}2^{-2r}N^*_{\mathbf{y}}(2^r)$ is given by Lemma 2. Substituting this into the definition of the conjectural constant (4), and combining it with (5), we deduce from (3) that

$$\begin{aligned}c=\frac{1}{\pi}\cdot\sum_{\mathbf{y}\in\mathbf{N}^3}\frac{\mu^2(y_0y_1y_2)}{(y_0y_1y_2)^{\frac{3}{2}}}\cdot\sigma_{2,\mathbf{y}}\\ \times\prod_{\substack{p\mid y_0\\ p>2}}\frac{\left(1+\left(\frac{y_1y_2}{p}\right)\right)}{\left(1+\frac{1}{p}\right)}\cdot\prod_{\substack{p\mid y_1\\ p>2}}\frac{\left(1+\left(\frac{y_0y_2}{p}\right)\right)}{\left(1+\frac{1}{p}\right)}\cdot\prod_{\substack{p\mid y_2\\ p>2}}\frac{\left(1+\left(\frac{-y_0y_1}{p}\right)\right)}{\left(1+\frac{1}{p}\right)}.\end{aligned}\tag{13}$$

In the remainder of this section we shall attempt to simplify this expression, in order to facilitate its numerical evaluation. Writing S for the set of $\mathbf{y}\in\mathbf{N}^3$ for which $\mu^2(y_0y_1y_2)=1$, we can partition S into subsets

$$S_{-1}=\{\mathbf{y}\in S: 2\nmid y_0y_1y_2\},\quad S_i=\{\mathbf{y}\in S: 2\mid y_i\},$$

for $0\leqslant i\leqslant 2$. We then split (13) into sums c_i over S_i, for each $-1\leqslant i\leqslant 2$. To streamline the notation, we define

$$\gamma(n)=\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1}$$

and, for $a,b\in\mathbf{N}$ with a,b squarefree and $b>1$ odd, we set $(\frac{a}{b})_*=1$ if and only if $(\frac{a}{p})=1$ for each $p\mid b$, with the convention that $(\frac{a}{1})_*=1$.

We begin by examining c_{-1}, in which case y_0, y_1 and y_2 are all odd. We get

$$\begin{aligned}c_{-1}=\frac{1}{\pi}\cdot\sum_{\substack{\mathbf{y}\in S_{-1}\\ \neg\{y_0\equiv y_1\equiv -y_2 \pmod 4\}}}\frac{\gamma(y_0y_1y_2)}{(y_0y_1y_2)^{\frac{3}{2}}}\\ \times\prod_{p\mid y_0}\left(1+\left(\frac{y_1y_2}{p}\right)\right)\cdot\prod_{p\mid y_1}\left(1+\left(\frac{y_0y_2}{p}\right)\right)\cdot\prod_{p\mid y_2}\left(1+\left(\frac{-y_0y_1}{p}\right)\right).\end{aligned}$$

Substituting $d=y_0y_1y_2$, we obtain

$$c_{-1}=\frac{1}{\pi}\cdot\sum_{\substack{d=1\\ 2\nmid d}}^{\infty}\frac{\mu^2(d)\cdot\gamma(d)\cdot 2^{\omega(d)}}{d^{\frac{3}{2}}}\cdot\Delta_{-1}(d),$$

where $\omega(d)$ denotes the number of distinct prime divisors of d and

$$\Delta_{-1}(d)=\#\left\{y_0y_1y_2=d:\ \begin{array}{c}\neg\{y_0\equiv y_1\equiv -y_2 \pmod 4\},\\ \left(\frac{y_1y_2}{y_0}\right)_*=\left(\frac{y_0y_2}{y_1}\right)_*=\left(\frac{-y_0y_1}{y_2}\right)_*=1\end{array}\right\}.$$

We next consider c_0, noting that $c_0=c_1=c_2$, by symmetry. If y_0 is even, we set $y_0=2y_0'$, where y_0' is odd. It then holds that

$$\begin{aligned}c_0=\frac{1}{\pi}\cdot\sum_{\substack{(y_0',y_1,y_2)\in S_{-1}\\ y_1\equiv y_2 \pmod 8}}\frac{2\gamma(y_0'y_1y_2)}{(2y_0'y_1y_2)^{\frac{3}{2}}}\\ \times\prod_{p\mid y_0'}\left(1+\left(\frac{y_1y_2}{p}\right)\right)\cdot\prod_{p\mid y_1}\left(1+\left(\frac{2y_0'y_2}{p}\right)\right)\cdot\prod_{p\mid y_2}\left(1+\left(\frac{-2y_0'y_1}{p}\right)\right).\end{aligned}$$

Putting $d = y_0' y_1 y_2$ we deduce as above that

$$c_0 = \frac{1}{\pi} \cdot \sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \frac{\mu^2(d) \cdot \gamma(d) \cdot 2^{\omega(d)}}{d^{\frac{3}{2}}} \cdot \frac{\Delta_0(d)}{\sqrt{2}},$$

where now

$$\Delta_0(d) = \# \left\{ y_0' y_1 y_2 = d : \begin{array}{c} y_1 \equiv y_2 \pmod 8, \\ \left(\frac{y_1 y_2}{y_0'}\right)_* = \left(\frac{2y_0' y_2}{y_1}\right)_* = \left(\frac{-2y_0' y_1}{y_2}\right)_* = 1 \end{array} \right\}.$$

Bringing these expressions together in (13), we conclude that

$$c = \frac{1}{\pi} \cdot \sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \frac{\mu^2(d) \cdot \gamma(d) \cdot 2^{\omega(d)}}{d^{\frac{3}{2}}} \cdot \left(\Delta_{-1}(d) + \frac{3}{\sqrt{2}} \Delta_0(d) \right). \tag{14}$$

One finds by numerical computation that $c = 2.68\ldots$, as in Conjecture 1.

3. The lower bound

Let $C \subset \mathbf{P}^2$ be a conic defined over $\mathbf{Q}$ and let $H : C(\mathbf{Q}) \to \mathbf{R}_{\geqslant 0}$ be an exponential height function. Suppose that C is defined by a non-singular quadratic form defined over $\mathbf{Z}$ with relatively prime coefficients all bounded in modulus by M. A number of results in the literature are directed at estimating the counting function $N_{C,H}(P) = \#\{x \in C(\mathbf{Q}) : H(x) \leqslant P\}$, as $P \to \infty$, with the outcome that there exist absolute constants $\delta, \psi > 0$ such that

$$N_{C,H}(P) = c_H(C(\mathbb{A}_{\mathbf{Q}}))P + O(M^{\psi} P^{1-\delta}), \tag{15}$$

where $c_H(C(\mathbb{A}_{\mathbf{Q}}))$ is the constant predicted by Peyre [8]. This is a special case of the work of Franke, Manin and Tschinkel [5] on flag varieties $P \setminus G$, with G taken to be the orthogonal group in three variables. Typically the uniformity in M is not actually recorded, but it transpires that the dependence on M is at worst polynomial.

We are now ready to establish Theorem 1. For any choice of $\mathbf{y}$ there are clearly $O(1)$ rational points on $C_{\mathbf{y}}$ which correspond to a solution with $x_0 x_1 x_2 = 0$. Beginning with (2) we deduce that

$$N_1(B) \geqslant \frac{1}{4} \sum_{\substack{\mathbf{y} \in \mathbf{N}^3 \\ y_0, y_1, y_2 \leqslant B^{\theta}}} \mu^2(y_0 y_1 y_2) N^{+}_{C_{\mathbf{y}}, H_{\mathbf{y}}}(B^{\frac{1}{2}}) + O(B^{3\theta}),$$

for any $\theta \leqslant \frac{1}{3}$, where $N^{+}_{C_{\mathbf{y}}, H_{\mathbf{y}}}$ is defined as for $N_{C_{\mathbf{y}}, H_{\mathbf{y}}}$, but with the additional constraint that $\gcd(x_0 y_0, x_1 y_1, x_2 y_2) = 1$. Once taken in conjunction with the fact that $y_0 y_1 y_2$ is square-free and $\gcd(x_0, x_1, x_2) = 1$, we see that the coprimality condition $\gcd(x_0 y_0, x_1 y_1, x_2 y_2) = 1$ on $C_{\mathbf{y}}$ is equivalent to demanding that $\gcd(x_i, x_j, y_k) = 1$ for each permutation $\{i, j, k\} = \{0, 1, 2\}$. Using the Möbius function to remove these coprimality conditions gives

$$N^{+}_{C_{\mathbf{y}}, H_{\mathbf{y}}}(B^{\frac{1}{2}}) = \sum_{k_0 | y_0} \sum_{k_1 | y_1} \sum_{k_2 | y_2} \mu(k_0 k_1 k_2) N_{C_{\mathbf{k},\mathbf{y}'}, H_{\mathbf{k},\mathbf{y}'}} \left(\frac{B^{\frac{1}{2}}}{k_0 k_1 k_2} \right),$$

where $y_i = k_i y_i'$ for $0 \leqslant i \leqslant 2$, $C_{\mathbf{k},\mathbf{y}'}$ is the conic $k_0 y_0'^3 x_0^2 + k_1 y_1'^3 x_1^2 = k_2 y_2'^3 x_2^2$ and $H_{\mathbf{k},\mathbf{y}'}$ is defined as for $H_{\mathbf{y}}$ but with y_i^3 replaced by $k_i y_i'^3$, for $0 \leqslant i \leqslant 2$. The conic $C_{\mathbf{k},\mathbf{y}'}$ has an underlying quadratic form with coefficients of size at most $B^{3\theta}$. Applying (15) we conclude that

$$N^{+}_{C_{\mathbf{y}}, H_{\mathbf{y}}}(B^{\frac{1}{2}}) = B^{\frac{1}{2}} \sum_{k_0 | y_0} \sum_{k_1 | y_1} \sum_{k_2 | y_2} \frac{\mu(k_0 k_1 k_2)}{k_0 k_1 k_2} \cdot c_{H_{\mathbf{k},\mathbf{y}'}}(C_{\mathbf{k},\mathbf{y}'}(\mathbb{A}_{\mathbf{Q}})) + O_{\varepsilon}(B^{\frac{1-\delta}{2} + 3\theta\psi + \varepsilon}),$$

for any $\varepsilon > 0$. One finds that the main term here is precisely equal to $c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+)B^{\frac{1}{2}}$, in the notation of §2. Noting that

$$\sum_{y \leqslant B^\theta} \frac{f(y)}{y^{\frac{3}{2}}} = \sum_{y=1}^{\infty} \frac{f(y)}{y^{\frac{3}{2}}} + O(B^{-\frac{\theta}{2}+\varepsilon}),$$

for any arithmetic function f satisfying $f(n) = O_\varepsilon(n^\varepsilon)$, we deduce that

$$N_1(B) \geqslant cB^{\frac{1}{2}} + O(B^{3\theta}) + O_\varepsilon(B^{\frac{1-\delta}{2}+3\theta(1+\psi)+\varepsilon}) + O_\varepsilon(B^{\frac{1-\theta}{2}+\varepsilon}),$$

for any $\varepsilon > 0$. We therefore conclude the proof of Theorem 1 by taking θ to satisfy the inequalities $0 < \theta < \frac{\delta}{6(1+\psi)}$.

4. The upper bound

The aim of this section is to prove Theorem 2, for which our starting point is (1). In order to estimate $N_1(B)$ we will view the equation in two basic ways: either as a family of conics or as a family of plane cubic curves. The work of Heath-Brown [6] allows one to estimate rational points of bounded height on plane curves, uniformly in the coefficients of the underlying equation. We will invoke this theory through the prism of the first author's work [2, Lemma 4.10], which yields the following bound for any integer $d \geqslant 2$.

Lemma 3. *Let $\mathbf{c} \in \mathbf{Z}^3$ with $c_1c_2c_3 \neq 0$ and pairwise coprime coordinates. Then we have*

$$\#\left\{\mathbf{z} \in \mathbf{Z}^3 : \begin{array}{l} \gcd(z_1,z_2,z_3)=1,\ |z_i| \leqslant Z_i, \\ c_1z_1^d + c_2z_2^d + c_3z_3^d = 0 \end{array}\right\} \ll_d \left(1 + \frac{Z_1Z_2Z_3}{|c_1c_2c_3|^{\frac{2}{d}}}\right)^{\frac{1}{3}} d^{\omega(c_1c_2c_3)}.$$

We will also make use of the familiar bound $\sum_{n \leqslant x} k^{\omega(n)} \ll x \log^{k-1} x$, which is valid for any $k \in \mathbf{N}$. We consider the contribution $N(\mathbf{X},\mathbf{Y})$, say, to $N_1(B)$ from $\mathbf{x},\mathbf{y}$ such that

$$X_i \leqslant x_i < 2X_i, \quad Y_i \leqslant y_i < 2Y_i,$$

for $0 \leqslant i \leqslant 2$. Clearly $N(\mathbf{X},\mathbf{Y}) = 0$ unless $X_i^2Y_i^3 \leqslant B$ and $X_i, Y_i > 1/2$, for $0 \leqslant i \leqslant 2$. It will be convenient to set $X = X_0X_1X_2$ and $Y = Y_0Y_1Y_2$. In particular we may henceforth assume that $X^2Y^3 \leqslant B^3$. On summing over dyadic intervals we see that

$$N_1(B) \ll \log^6 B \max_{\mathbf{X},\mathbf{Y}} N(\mathbf{X},\mathbf{Y}), \tag{16}$$

where the maximum is over $\mathbf{X},\mathbf{Y}$ satisfying the above inequalities.

Viewing the underlying equation as a family of conics first, we take $d = 2$ in Lemma 3 and deduce that

$$\begin{aligned} N(\mathbf{X},\mathbf{Y}) &\ll \sum_{\mathbf{y}} 2^{\omega(y_0y_1y_2)} \left(1 + \frac{X}{Y^3}\right)^{\frac{1}{3}} \\ &\ll \left(Y + X^{\frac{1}{3}}\right) \log^3 B. \end{aligned}$$

Alternatively, regarding the equation as a family of cubics, we take $d = 3$ in Lemma 3 and obtain

$$\begin{aligned} N(\mathbf{X},\mathbf{Y}) &\ll \sum_{\mathbf{x}} 3^{\omega(x_0x_1x_2)} \left(1 + \frac{Y}{X^{\frac{4}{3}}}\right)^{\frac{1}{3}} \\ &\ll \left(X + Y^{\frac{1}{3}}X^{\frac{5}{9}}\right) \log^6 B. \end{aligned}$$

Bringing these two estimates together we conclude that

$$N(\mathbf{X},\mathbf{Y}) \ll \left(\min\{X,Y\} + \min\{Y, Y^{\frac{1}{3}}X^{\frac{5}{9}}\} + X^{\frac{1}{3}}\right) \log^6 B.$$

Now it is clear that $\min\{X,Y\} \leqslant X^{\frac{2}{5}}Y^{\frac{3}{5}} \leqslant B^{\frac{3}{5}}$ and

$$\min\{Y, Y^{\frac{1}{3}}X^{\frac{5}{9}}\} \leqslant Y^{\frac{9}{25}} \cdot (Y^{\frac{1}{3}}X^{\frac{5}{9}})^{\frac{18}{25}} = X^{\frac{2}{5}}Y^{\frac{3}{5}} \leqslant B^{\frac{3}{5}},$$

since $X^2Y^3 \leqslant B^3$. Finally we note that $X^{\frac{1}{3}} \leqslant B^{\frac{1}{2}}$. Inserting our estimate for $N(\mathbf{X}, \mathbf{Y})$ into (16), we therefore arrive at the statement of Theorem 2.

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