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DISCUSSION
PAPER

Nonparametric analysis of multi-output production with joint inputs *

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Abstract

We present a novel framework for analyzing cost minimizing production behavior in multi-output settings. Our specific focus is on dealing with joint inputs, i.e. inputs that are simultaneously used for the production of multiple outputs. Here, we distinguish between two possible approaches. The cooperative approach takes a centralized perspective and assumes cost minimization at the aggregate firm level. By contrast, the noncooperative approach adopts a decentralized view and assumes cost minimization at the level of the individual output departments, which implies the possibility of free riding behavior for the joint inputs. Our framework is non-parametric in nature, which means that it allows for analyzing production behavior while avoiding (nonverifiable) prior functional structure for the production technology. We show that it naturally extends the existing nonparametric framework for analyzing single output production. We establish rationalizability conditions for cooperative as well as noncooperative production behavior. In addition, we introduce goodness-of-fit measures for evaluating the degree of violation of these conditions. An empirical application to the English and Welsh drinking water and sewerage sector shows the practical usefulness of

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our framework. Specifically, we compare the empirical validity of the cooperative and noncooperative models for describing the observed production behavior.
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1 Introduction

Whereas introductory economics textbooks usually focus on single output production, most firms in real life simultaneously produce multiple outputs. To rationalize the prevalence of such multi-output firms, economists typically invoke the possibility of joint inputs, i.e. inputs that are simultaneously used for the production of multiple outputs. Joint inputs explain the presence of scope economies, which form a natural economic motivation for multi-output production. Essentially, these joint inputs have a 'public good' character: they satisfy the properties of non-rivalry and non-exclusiveness in a production setting. In the present context, non-rivalry means that using a joint input for one output does not interfere with using the same input for another output, while non-exclusiveness implies that no production process can be excluded from using the joint inputs. In firm practice, examples of joint inputs are general management, brand advertising, research and development, etc.

In what follows, we set up a framework to model multi-output production with joint inputs. In doing so, we maintain two basic assumptions. Firstly, we assume cost minimizing firm behavior. Cost minimization is a standard hypothesis in neoclassical production theory. It prescribes that, for any desired level of outputs, the firm always chooses the inputs that minimize the total cost. From a practical perspective, the assumption of cost minimization has the advantage that it can be used even when output prices are unavailable or of little interest to the workings of the firm (which is the case for e.g. hospitals, non-profit organization, universities or colleges, government agencies, etc). Secondly, we assume that the firm is organized in such a way that each department is responsible for the production of a certain output. In principle, it is possible to relax this assumption and, for example, to consider departments that are individually responsible for more than one output. This would not affect our main arguments, but it would substantially complicate our exposition without really adding new insights.

Cooperative versus noncooperative cost minimization. Cost minimization usually constitutes an uncontested firm objective. However, realizing this objective might be a daunting task, especially for multi-output production processes. Indeed, it is often unrealistic to assume that the central firm management can consistently collect and aggregate all the necessary information concerning the different production processes. In addition, in many practical situations it can be difficult to reach the necessary agreement in order to calculate and implement the optimal cost minimizing input allocation.

Given this, we distinguish between two possible approaches to model multi-output cost minimization. Each approach makes a different assumption regarding a firm's input decision process. The first approach takes a centralized perspective and assumes cost minimization at the aggregate firm level. In this case, the central firm management defines the input allocation that will be used for producing the target outputs. Implicitly, this approach imposes that the different firm departments must cooperate and therefore we will refer to it as the 'cooperative' approach.

The second approach adopts a decentralized view and assumes cost minimization at the level of the different output departments. In this case, the central firm management again imposes the output targets, but now each individual output department is responsible for choosing the (cost minimizing) inputs that achieve its specific output target. Clearly, such a set-up does not automatically imply cooperation between the different departments and, therefore, we call this the 'noncooperative' approach. More formally, we assume that output departments reach a Nash-type equilibrium allocation of the inputs. As we will make explicit in our theoretical discussion, such a noncooperative allocation can be characterized by an inefficient allocation of the joint inputs (in contrast to the cooperative allocation). Essentially, such inefficiencies follow from free-riding behavior that is typically associated with the provision of public goods (i.e. the tragedy of the commons).

Nonparametric production analysis. In the following sections, we will develop a methodology that allows for empirically analyzing firm behavior in terms of the cooperative (centralized) and noncooperative (decentralized) multi-output production models. In practical applications, this enables checking which of the two models best describes the observed firm behavior. A specific feature of our analysis is that it is nonparametric in nature. The term nonparametric here refers to the fact that our methodology abstains from imposing any functional form on the production technology. By contrast, it solely uses information on observed input-output combinations and associated prices in combination with some basic regularity conditions (in casu continuity and quasi-concavity). This is particularly attractive from a practical point of view, as a priori imposed parametric/functional structure is typically non-verifiable from observational data. From this perspective, a nonparametric analysis allows us to draw more robust conclusions regarding the empirical validity of particular behavioral (cooperative or noncooperative) assumptions.

The nonparametric approach to analyzing production behavior was originally developed by Hanoch and Rothschild (1972), Afriat (1972), Diewert and Parkan (1983) and Varian (1984). These authors focused on cost minimization in the case of single-output production. We here complement these earlier studies by introducing a methodology to analyze cost minimization in multi-output settings. The fact that our framework provides a natural extension of the existing nonparametric framework will clearly appear from our following exposition: all our theoretical sections will start by briefly recapturing the single-output case, to subsequently introduce our generalizations that apply under multi-output production.

Related literature. To conclude this Introduction, we indicate two active strands of literature that are related to the work we present here. First, the nonparametric approach to production analysis bears a close relation to the (nonparametric) efficiency measurement methodology that is often referred to as Data Envelopment Analysis (DEA; see, for example, Fried, Lovell, and Schmidt (2008) and Cook and Seiford (2009) for recent reviews).[1] DEA typically focuses on measuring production inefficiencies while imposing minimal consistency conditions on the available production technology. The main aim of our research is to provide a structural approach to modeling cost minimizing behavior in multi-output settings. In addition, we introduce goodness-of-fit measures for evaluating the degree of violation of cost minimization. From a DEA perspective, these goodness-of-fit measures can also be interpreted as efficiency measures.[2]

Next, our following treatment of multi-output production is partly inspired on recent work regarding the modeling of multi-person household consumption. Specifically, our nonparametric methodology for production analysis is formally related to the methodology for consumption analysis that was presented by Cherchye, De Rock, and Vermeulen (2007, 2011b), for the cooperative case, and Cherchye, Demuynck, and De Rock (2011c), for the noncooperative case.[3] Here, it is also worth indicating that parametric methodology has been developed for modeling such multi-person household consumption. See Chiappori (1988), Browning and Chiappori (1998) and Chiappori and Ekeland (2009), for cooperative behavior, and Lechene and Preston (2010) and Browning, Chiappori, and Lechène (2010), for noncooperative behavior. For example, this may provide a useful basis for assessing multi-output cost minimization through parametric efficiency measurement (also referred to as Stochastic Frontier Analysis (SFA); see Kumbhakar and Lovell (2000)). Generally, we believe a further exploration of the link with the literature on multi-person household consumption may open up interesting new avenues for analyzing multi-output production behavior.

Structure. The remainder of this paper is organized as follows. Section 2 states the cost minimization concepts that we will use further on. Section 3 provides nonparametric characterizations of (cooperative and noncooperative) cost minimization in multi-output production. Section 4 presents operational methods for assessing the empirical validity of the different multi-output production models that we study. Specifically, it introduces goodness-of-fit measures that allow for measuring the degree to which observed behavior

[1]See also Banker and Maindiratta (1988) for an early study on the relationship between the nonparametric approach to production analysis and DEA.

[2]Here, it is particularly useful to refer to recent work of Cherchye, De Rock, and Vermeulen (2008) and Cherchye, De Rock, Dierynck, Roodhooft, and Sabbe (2011a). These authors present methodology for DEA-type efficiency measurement that is formally close to the methodology for analyzing cooperative multi-output production that we present in the current paper. From this perspective, our following exposition can also provide a fruitful basis for developing complementary DEA-type methods for efficiency analysis that focus on noncooperative multi-output production.

[3]This formal link is analogous to the one between the nonparametric methodologies for single-output production analysis (discussed above) and single-person consumption analysis (see, for example, Afriat (1967) and Varian (1982)).

is effectively consistent with a particular model specification. Section 5 demonstrates the empirical usefulness of our methodology through an application to the English and Welsh drinking water and sewerage sector. As we will explain, the issue of multi-output production, with jointly used inputs (and possible scope economies), is particularly relevant for this sector. We will compare the empirical validity of the cooperative and noncooperative models for describing the observed multi-output production behavior. Finally, Section 6 summarizes and offers a concluding discussion.

2 Cost minimization: definitions

This section introduces some necessary notation and definitions. We first define the production technology, which characterizes the feasible input-output combinations. Next, we present the different notions of cost minimization that will return in our following exposition. To set the stage, we begin by considering cost minimization in the simplest case, with single-output firms. Subsequently, we consider multi-output cost minimization. Here, we distinguish between cooperative and noncooperative input use.

Production technology. We consider firms that produce J outputs by using N output-specific inputs and M joint inputs. As indicated in the Introduction, joint inputs are simultaneously used for the production of multiple outputs. By contrast, output-specific inputs can only benefit individual outputs; these inputs need to be distributed over the J outputs. Formally, the vector $\mathbf{q} \in \mathbb{R}^N_+$ represents the output-specific inputs, $\mathbf{Q} \in \mathbb{R}^M_+$ denotes the joint inputs and $\mathbf{y} \in \mathbb{R}^J_+$ the outputs. Each vector $\mathbf{q}$ can be split into J separate vectors $\mathbf{q}^1, \ldots, \mathbf{q}^J$ (i.e. $\mathbf{q} = \sum_j \mathbf{q}^j$), with every $\mathbf{q}^j$ capturing the output-specific inputs that are used for the production of output j. Further, we denote by $\mathbf{p}$ the price (row) vector for the output-specific inputs and by $\mathbf{P} \in \mathbb{R}^M_{++}$ the price (row) vector for the joint inputs. Finally, for any vector $\mathbf{x}$, we denote the kth element by $(\mathbf{x})_k$. For example, the price for the mth joint input will be denoted by $(\mathbf{P})_m$. For convenience, however, we also use the notation y^j to represent the level of the jth output, i.e. $y^j \overset{not}{=} (\mathbf{y})_j$.

We assume that a firm's production technology can be represented by J continuous, strictly increasing and quasi-concave production functions f^j ($j \leq J$), where $f^j(\mathbf{q}^j, \mathbf{Q})$ gives the maximal level of output j that can be produced with the input vector $(\mathbf{q}^j, \mathbf{Q}) \in \mathbb{R}^{N+M}_+$. For a given production function f^j and output quantity y^j, we can define the input requirement set

$$V^j(y^j) = \left\{ (\mathbf{q}^j, \mathbf{Q}) \in \mathbb{R}^{N+M}_+ \;\middle|\; f^j(\mathbf{q}^j, \mathbf{Q}) \geq y^j \right\}.$$

This set contains all combinations of inputs that can produce at least the amount y^j of output j. As f^j is continuous and quasi-concave, we have that every set V^j is closed and convex.

In empirical applications, we typically do not observe the production functions f^j (or the sets V^j). Nonparametric production analysis (only) uses technology information that is revealed by a finite set of observed input-output combinations. In our setting, we assume that this data set contains information on the input prices, input quantities and output quantities. Formally, we denote this data set by $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{y}_t\}_{t \in T}$. Here, T is the (finite) set of production observations. For each observation $t \in T$, $(\mathbf{p}_t, \mathbf{P}_t) \in \mathbb{R}^{N+M}_{++}$ gives the input prices, $(\mathbf{q}_t, \mathbf{Q}_t) \in \mathbb{R}^{N+M}_{+}$ the input quantities, and $\mathbf{y}_t \in \mathbb{R}^J_+$ the output quantities. In practice, the production observations pertain to a single firm that is observed over time (under a constant production technology) or to a cross-section of firms facing the same production technology at a given point of time.

In what follows, we will introduce a framework for analyzing cost minimization of individual production observations t (rather than of the full set of observations S). This focus is motivated by the fact that individual cost minimization is usually the most relevant concept in practical applications. For example, this is the case in a cross-section setting where different observations pertain to different firms (as in our own application in Section 5). Clearly, our following cost minimization analysis can be easily extended to apply to the full set S: essentially, for this set S to be consistent with cost minimization it is required that *all* observations t in S are *simultaneously* cost minimizing. For compactness, however, we will not explicitly consider such extensions in the sequel.

Single-output production. We first define cost minimization for the single-output case, i.e. $J = 1$. This is the situation that was originally considered by Hanoch and Rothschild (1972), Afriat (1972), Diewert and Parkan (1983) and Varian (1984). It will provide a useful starting point for our following discussion of the multi-output case. Admittedly, when firms produce only one output the distinction between output-specific and joint inputs becomes artificial. Still, we choose to maintain the distinction here to ease our exposition and to avoid an overload of notation.

Consider a firm that produces the (single) output quantity y, and let f and V represent the relevant production function and corresponding input requirement set. The firm is then said to be cost minimizing if, for input prices $(\mathbf{p}, \mathbf{P})$, it chooses the inputs $(\mathbf{q}, \mathbf{Q})$ that solve the optimization problem (**OP-S**)

$$\{\mathbf{q}, \mathbf{Q}\} \in \underset{(\mathbf{x}, \mathbf{X}) \in \mathbb{R}^{N+M}_{+}}{\arg\min}\ \mathbf{px} + \mathbf{PX} \text{ s.t. } (\mathbf{x}, \mathbf{X}) \in V(y).$$

As indicated above, nonparametric production analysis starts from a finite set of production observations. In this case, we have a data set $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, y_t\}_{t \in T}$, with y_t the (one-dimensional) output quantity produced at t. Then, a production observation t is rationalizable if its behavior is consistent with (single-output) cost minimization.

Definition 1 (S-rationalizability). *Consider a data set $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, y_t\}_{t \in T}$. We say that the observation $t \in T$ is single-output (S) rationalizable if there exists a continuous, strictly increasing and quasi-concave production function f such that*

1. *for all $v \in T$, $(\mathbf{q}_v, \mathbf{Q}_v) \in V(y_v)$,*

2. $(\mathbf{q}_t, \mathbf{Q}_t)$ *solves* ***OP-S*** *given the input prices* $\mathbf{p}_t, \mathbf{P}_t$*, the production function* f *and the output level* y_t*.*

In this definition, the first condition requires that the function f (and set V) is such that every observed input-output combination is also technologically feasible. The second condition then imposes cost minimizing behavior at the production observation t.

Cooperative multi-output production. Let us then consider cost minimization under multi-output production, where the distinction between output-specific and joint inputs becomes relevant. We first focus on the situation with outputs produced in a cooperative way (or, production decisions are centralized).

Specifically, we assume a firm that is divided in J departments, where each department j is responsible for the production of the jth output. Cooperative multi-output production then means that the input quantities are chosen such that the firm as a whole is cost minimizing. In other words, the inputs $(\mathbf{q}^1, \ldots, \mathbf{q}^J, \mathbf{Q}) \in \mathbb{R}_+^{J \cdot N+M}$ must solve (**OP-CM**)

$$\{\mathbf{q}^1, \ldots, \mathbf{q}^J, \mathbf{Q}\} \in \underset{(\mathbf{x}^1, \ldots, \mathbf{x}^J, \mathbf{X}) \in \mathbb{R}_+^{J \cdot N+M}}{\arg\min} \sum_j \mathbf{p}\mathbf{x}^j + \mathbf{P}\mathbf{X} \text{ s.t. } (\mathbf{x}^j, \mathbf{X}) \in V^j(y^j) \ (\forall j \leq J).$$

We can now introduce our rationalizability concept for cooperative multi-output production.

Definition 2 (CM-rationalizability)**.** *Consider a data set* $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{y}_t\}_{t \in T}$*. We say that the observation* $t \in T$ *is cooperative multi-output (CM) rationalizable if there exist* J *continuous, strictly increasing and quasi-concave production functions* f^j *such that*

1. *for all* $v \in T$*, there exist output-specific input vectors* $\mathbf{q}_v^j$*, with* $\sum_j \mathbf{q}_v^j = \mathbf{q}_v$*, such that* $(\mathbf{q}_v^j, \mathbf{Q}_v) \in V^j(y_v^j)$ *for all* $j \leq J$*,*
2. $(\mathbf{q}_t^1, \ldots, \mathbf{q}_t^J, \mathbf{Q}_t)$ *solves* ***OP-CM*** *given the input prices* $\mathbf{p}_t, \mathbf{P}_t$*, the production functions* f^j *(*$j \leq J$*) and the output vector* $\mathbf{y}_t$*.*

Just like for the single-output case, the first condition imposes technological feasibility of all observed input-output combinations, while the second condition requires cost minimization (under cooperation) at the observation t.

Noncooperative multi-output production. To conclude this section, we consider the case in which the multiple outputs are produced in a noncooperative way. As discussed in the Introduction, this can be interpreted in terms of a firm that decentralizes the cost minimization decisions, such that each individual department j is responsible for its own expenses on both the output-specific and the joint inputs. In this case, we assume Nash-type equilibrium behavior where each department minimizes the cost of producing its own output given the input decisions of the other departments.

Formally, to distinguish between the joint input purchases of the different departments, we use the vectors $\mathbf{Q}^j \in \mathbb{R}^M_+$ $(j \leq J)$ to represent the joint inputs purchased by every department j. The total amount of joint inputs at the aggregate firm level then equals $\sum_j \mathbf{Q}^j = \mathbf{Q}$. Noncooperative (Nash-type) production behavior requires that, for each department j, the inputs $(\mathbf{q}^j, \mathbf{Q}^j)$ solve (**OP-NM)**

$$\{\mathbf{q}^j, \mathbf{Q}^j\} = \underset{(\mathbf{x}^j, \mathbf{X}^j) \in \mathbb{R}^{N+M}_+}{\arg\min} \; \mathbf{p}\mathbf{x}^j + \mathbf{P}\mathbf{X}^j \text{ s.t. } \left(\mathbf{x}^j, \mathbf{X}^j + \sum_{k \neq j} \mathbf{Q}^k\right) \in V^j(y^j),$$

i.e. each output department j purchases output-specific inputs $\mathbf{q}^j$ and joint inputs $\mathbf{Q}^j$ that imply cost minimization given the joint inputs $\sum_{k\neq j} \mathbf{Q}^k$ purchased by the other departments k.

This leads to the following rationalizability condition for noncooperative multi-output production.

Definition 3 (NM-rationalizability)**.** *Consider a data set* $S = \{\mathbf{p}_t,\, \mathbf{P}_t,\, \mathbf{q}_t,\, \mathbf{Q}_t,\, \mathbf{y}_t\}_{t\in T}$. *We say that the observation* $t \in T$ *is noncooperative multi-output (NM) rationalizable if there exist* J *continuous, strictly increasing and quasi-concave production functions* f^j *such that*

1. *for all* $v \in T$ *there exist output-specific input vectors* $\mathbf{q}^j_v$*, with* $\sum_j \mathbf{q}^j_v = \mathbf{q}_v$*, such that* $(\mathbf{q}^j_v, \mathbf{Q}_v) \in V^j(y^j_v)$ *for all* $j \leq J$*,*
2. *there exist joint input vectors* $\mathbf{Q}^j_t$*, with* $\sum_j \mathbf{Q}^j_t = \mathbf{Q}_t$ *such that each* $(\mathbf{q}^j_t, \mathbf{Q}^j_t)$ $(j \leq J)$ *solves* ***OP-NM*** *given the input prices* $\mathbf{p}_t, \mathbf{P}_t$*, the production functions* f^j*, the output vector* $\mathbf{y}_t$ *and the joint input vectors* $\mathbf{Q}^k_t$ $(k \neq j)$*.*

3 Cost minimization: characterizations

We are now in a position to define the nonparametric conditions for cost minimizing behavior as defined in the previous section. Essentially, these characterizations allow us to check rationalizability while avoiding the specification of the production functions f^j (or the sets V^j). We can test cost minimizing behavior by only using the observed information in the data set S. This is particularly convenient from a practical point of view because, as argued above, the exact production technology (and thus the production functions) are typically not observed in empirical applications. In Section 4, we will show that our following characterizations of cost minimization are easily implemented in practical analysis.

Single-output production. We first concentrate on the single-output conditions in Definition 1. We recall that in this case the empirical analyst can use a data set $S = \{\mathbf{p}_t,$

$\mathbf{P}_t,\ \mathbf{q}_t,\ \mathbf{Q}_t,\ y_t\}_{t\in T}$, where each y_t represents the (one-dimensional) output produced at the observation t.

We will need the following definition.

Definition 4 (SACM). *Consider a data set $S = \{\mathbf{p}_t,\ \mathbf{P}_t,\ \mathbf{q}_t,\ \mathbf{Q}_t,\ y_t\}_{t\in T}$. We say that the observation $t \in T$ satisfies the Strong Axiom of Cost Minimization (SACM) if, for all $v \in T$,*

$$\mathbf{p}_t\mathbf{q}_v + \mathbf{P}_t\mathbf{Q}_v \geq \mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t \text{ whenever } y_v \geq y_t \text{ and} \tag{sacm.1}$$

$$\mathbf{p}_t\mathbf{q}_v + \mathbf{P}_t\mathbf{Q}_v > \mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t \text{ whenever } y_v > y_t. \tag{sacm.2}$$

This SACM condition has two components. The first component (sacm.1) implies consistency with the so-called Weak Axiom of Cost Minimization (WACM; see Varian (1984)). The additional component (sacm.2) is a technical requirement that guarantees continuity of the production function (for S-rationalizability). The SACM condition has a clear interpretation in terms of cost minimizing behavior. For a given observation t, it imposes that if we observe a higher output at observation v (i.e. $y_v \geq (>)\ y_t$), then the cost of producing this higher output must be above the one of producing y_t (i.e. $\mathbf{p}_t\mathbf{q}_v + \mathbf{P}_t\mathbf{Q}_v \geq (>)\ \mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t$). Obviously, if it were cheaper to produce a higher output y_v, then the firm could not be cost minimizing by choosing $(\mathbf{q}_t, \mathbf{Q}_t)$: purchasing the inputs $(\mathbf{q}_v, \mathbf{Q}_v)$ would have produced at least the same output at a lower cost.

The following result states that data consistency with SACM is necessary and sufficient for cost minimization in the single-output case (see Varian (1984) for a proof).

Theorem 1. *Consider a data set $S = \{\mathbf{p}_t,\ \mathbf{P}_t,\ \mathbf{q}_t,\ \mathbf{Q}_t,\ y_t\}_{t\in T}$. The observation t is then S-rationalizable if and only if it satisfies SACM.*

This theorem provides an easy way to nonparametrically verify whether a particular firm observation is cost minimizing: checking the SACM condition in Definition 4 only requires checking linear inequalities that use information captured by the observed set S.

Cooperative multi-output production. Using the SACM concept in Definition 4, we can next characterize cost minimizing behavior in the case of multi-output production (in casu with $S = \{\mathbf{p}_t,\ \mathbf{P}_t,\ \mathbf{q}_t,\ \mathbf{Q}_t,\ \mathbf{y}_t\}_{t\in T}$). Specifically, we will obtain that cost minimization again requires data consistency with SACM, but now we get a separate SACM condition for each of the J outputs. As we will indicate, the specificity of these output-specific SACM conditions is that they require using output-specific prices for evaluating the joint inputs. The essential difference between the cooperative and noncooperative case then pertains to the definition of these output-specific prices.

Let us first consider the nonparametric condition for cost minimization that applies to the cooperative case. (The Appendix contains the proofs of our main theorems.)

Theorem 2. *Consider a data set $S = \{\mathbf{p}_t,\ \mathbf{P}_t,\ \mathbf{q}_t,\ \mathbf{Q}_t,\ \mathbf{y}_t\}_{t\in T}$. Then, the observation t is CM-rationalizable if and only if, for all $v \in T$ and $j \leq J$, there exist input vectors $\mathbf{q}_v^j \in \mathbb{R}_+^N$ and price vectors $\mathbf{P}_t^j \in \mathbb{R}_{++}^M$ such that*

1. *for all* $v \in T$*:* $\sum_j \mathbf{q}_v^j = \mathbf{q}_v$,

2. $\sum_j \mathbf{P}_t^j = \mathbf{P}_t$,

3. *for all* $v \in T$ *and* $j \leq J$*:*

$$\mathbf{p}_t\mathbf{q}_v^j + \mathbf{P}_t^j\mathbf{Q}_v \geq \mathbf{p}_t\mathbf{q}_t^j + \mathbf{P}_t^j\mathbf{Q}_t \text{ whenever } y_v^j \geq y_t^j \text{ and}$$
$$\mathbf{p}_t\mathbf{q}_v^j + \mathbf{P}_t^j\mathbf{Q}_v > \mathbf{p}_t\mathbf{q}_t^j + \mathbf{P}_t^j\mathbf{Q}_t \text{ whenever } y_v^j > y_t^j.$$

The third condition of Theorem 2 shows that CM-rationalizability of a production observation t requires single-output rationalizability (or S-rationalizability) for each individual output separately (i.e. SACM). However, the crucial difference with the characterization in Theorem 1 pertains to the costs that are allocated to the different outputs. First of all, the cost of output-specific inputs is distributed over the output departments according to the vectors $\mathbf{q}_v^j$ defined in the first condition of Theorem 2. Next, for the joint inputs, we should account for output-specific prices. In the cooperative case that we consider here, these output-specific prices $\mathbf{P}_t^j$ must sum to the observed input prices $\mathbf{P}_t$; this is guaranteed by the second condition of Theorem 2. As such, the output-specific prices have a similar interpretation as Lindahl prices in the case of efficient public goods provision (which equally requires that Lindahl prices sum to the market prices of the public goods). This directly complies with the public good interpretation of the joint inputs that we discussed in the Introduction. Like Lindahl prices, the output-specific prices $\mathbf{P}_t^j$ correspond to the marginal production of output j (expressed in monetary terms) that follows from an additional unit of the joint inputs.

Given all this, we can also provide a decentralized representation of cost minimization under cooperative behavior (which parallels the decentralized representation of efficient public goods provision under Lindahl prices). In this representation, the central firm management first sets out the output target for each output department, which defines the quantities y_t^j. In a following step, it then requires every department to produce this output at a minimal cost (i.e. each department separately solves **OP-S**) given the prices $\mathbf{p}_t$ for the output-specific inputs and the prices $\mathbf{P}_t^j$ for the joint inputs (i.e. department j pays its marginal valuation $(\mathbf{P}_t^j)_m$ if it uses an additional unit of the joint input m). The sum contraint $\sum_j \mathbf{P}_t^j = \mathbf{P}_t$ then effectively implies an efficient allocation of the joint inputs: it imposes that the total marginal valuation to the purchase ($= \sum_j \mathbf{P}_t^j$) just equals its expense ($= \mathbf{P}_t$). This sum constraint will imply a main difference with our characterization of cost minimizing production behavior in the noncooperative case. As a consequence, we will obtain that noncooperative behavior can lead to inefficient purchases of the joint inputs.

Noncooperative multi-output production. Let us then regard the noncooperative situation. Here, we get a characterization that looks very similar to the one for the cooperative situation. But, as indicated above, an important difference pertains to the output-specific prices for the joint inputs.

We obtain the following nonparametric characterization of cost minimization under noncooperative production.

Theorem 3. *Consider a data set* $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{y}_t\}_{t \in T}$. *Then, the observation* t *is NM-rationalizable if and only if, for all* $v \in T$ *and* $j \leq J$, *there exist input vectors* $\mathbf{q}_v^j \in \mathbb{R}_+^N$ *and price vectors* $\mathfrak{P}_t^j \in \mathbb{R}_{++}^M$ *such that*

1. *for all* $v \in T$: $\sum_j \mathbf{q}_v^j = \mathbf{q}_v$,
2. *for all* $m \leq M$: $\max_j\{(\mathfrak{P}_t^j)_m\} = (\mathbf{P}_t)_m$,
3. *for all* $v \in T$ *and* $j \leq J$:

$$\mathbf{p}_t\mathbf{q}_v^j + \mathfrak{P}_t^j\mathbf{Q}_v \geq \mathbf{p}_t\mathbf{q}_t^j + \mathfrak{P}_t^j\mathbf{Q}_t \text{ whenever } y_v^j \geq y_t^j \text{ and}$$
$$\mathbf{p}_t\mathbf{q}_v^j + \mathfrak{P}_t^j\mathbf{Q}_v > \mathbf{p}_t\mathbf{q}_t^j + \mathfrak{P}_t^j\mathbf{Q}_t \text{ whenever } y_v^j > y_t^j.$$

Thus, the characterization of NM-rationalizability is almost identical to the one of CM-rationalizability. The only difference is that the output-specific prices $\mathbf{P}_t^j$ are now replaced by the vectors $\mathfrak{P}_t^j$, which are subject to the max constraints embedded in the second condition of Theorem 3. In words, such a max constraint imposes, for each joint input m, that the highest output-specific price (defined over all outputs j) must equal the observed price of the input. As a result, it may well be the case that $\sum_j \mathfrak{P}_t^j > \mathbf{P}_t$ (which contrasts with the second condition of Theorem 2).

Similar to before, we can interpret the output-specific prices $\mathfrak{P}_t^j$ as representing the marginal production of output j (in monetary terms) associated with one additional unit of the joint inputs. Then, $\sum_j(\mathfrak{P}_t^j)_m > (\mathbf{P})_m$ implies that the total value added (summed over all outputs j) associated with a one unit increase of the mth joint input exceeds the corresponding input price. In turn, this means that the purchased amount of this joint input is below its efficiency level. The reason for this inefficiency is the free-riders problem that is intrinsic to noncooperative (Nash-type) equilibrium behavior.

In fact, it can be shown that every output department j for which the output-specific input price $(\mathfrak{P}_t^j)_m$ is below the actual price $(\mathbf{P}_t)_m$ will abstain from contributing to this joint input (i.e. $(\mathbf{Q}_t^j)_m = 0$). In other words, this department is effectively free-riding on the other departments k $(\neq j)$ that do contribute to the joint input (because $(\mathfrak{P}_t^k)_m = (\mathbf{P}_t)_m$). Intuitively, in a (decentralized) noncooperative setting, a cost minimizing output department j has every incentive not to contribute to the joint input m (i.e. $(\mathbf{Q}_t^j)_m = 0$) if some other department k already purchased the input amount (i.e. $(\mathbf{Q}_t^k)_m = (\mathbf{Q}_t)_m$ for $k \neq m$) that is necessary for producing the targeted output y_t^j.

Non-nestedness. To conclude this section, we show that CM-rationalizability is non-nested with (or independent from) NM-rationability: a data set S that satisfies the nonparametric conditions for the cooperative model does not necessarily satisfy the ones for the noncooperative model, and vice versa. In particular, Examples 1 and 2 show that there is neither any inclusion nor any exclusion relation between the collection of data sets that satisfy the conditions in Theorem 2 and the collection of data sets that satisfy the conditions in Theorem 3.

This non-nestedness/independence conclusion is particularly interesting from an empirical point of view. It directly follows that we will not have 'by construction' that one model obtains a better empirical fit than the other, simply because it has weaker empirical implications. In our opinion, this effectively makes that we can meaningfully compare the empirical validity of the two model specifications by using our nonparametric conditions. It may actually well be that the appropriate model specification varies depending on the particular firm observation at hand.

Two final observations pertain to the data sets we use in Examples 1 and 2. Firstly, these examples show that we can meaningfully test data consistency with a specific model (and compare the empirical validity of different models) even if only a few observations are available. Secondly, because all inputs are joint in both examples, such an empirical analysis in principle does not require output-specific inputs. In fact, using similar arguments as in Examples 1 and 2, we can show that non-nestedness also applies in the case with (only) a single joint input, provided there is at least one output-specific input.[4] Our application in Section 5 actually considers such a situation with a single joint input.

Example 1. *We first construct a data set S that is NM-rationalizable but not CM-rationalizable. This data set includes 2 observations ($|T| = 2$), 2 outputs ($J = 2$), and 3 joint inputs ($M = 3$):*

$$\mathbf{P}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}', \mathbf{P}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}', \mathbf{Q}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For this set S we have that $\mathbf{P}_2\mathbf{Q}_2 = 14$, which is greater than $\mathbf{P}_2\mathbf{Q}_1 = 12$. If we combine this with the fact that production observation 1 produces more of both outputs than observation 2, we conclude from Theorem 2 that this data set is not CM-rationalizable: any possible specification of the output-specific prices $\mathbf{P}_2^1$ and $\mathbf{P}_2^2$ gives either $\mathbf{P}_2^1\mathbf{Q}_1 < \mathbf{P}_2^1\mathbf{Q}_2$ (while $y_1^1 > y_2^1$) or $\mathbf{P}_2^2\mathbf{Q}_1 < \mathbf{P}_1^2\mathbf{Q}_2$ (while $y_1^2 > y_2^2$).

Next, we can easily verify that the following specification of the vectors $\mathfrak{P}_t^j$ ($j = 1, 2$; $t = 1, 2$) makes the set S satisfy the conditions in Theorem 3:

$$\mathfrak{P}_1^1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}', \mathfrak{P}_1^2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}', \mathfrak{P}_2^1 = \begin{bmatrix} 2 \\ 3 \\ 0.5 \end{bmatrix}', \mathfrak{P}_2^2 = \begin{bmatrix} 2 \\ 0.5 \\ 3 \end{bmatrix}'.$$

Thus, we conclude that the data set is NM-rationalizable.

[4]These example are available from the authors upon request. For compactness, we do not include them here.

Example 2. *We next present a data set S that is CM-rationalizable but not NM-rationalizable. This data set includes 3 observations ($|T| = 3$), 2 outputs ($J = 2$), and 3 joint inputs ($M = 3$):*

$$\mathbf{P}_1 = \begin{bmatrix} 14 \\ 9 \\ 9 \end{bmatrix}', \mathbf{P}_2 = \begin{bmatrix} 9 \\ 14 \\ 9 \end{bmatrix}', \mathbf{P}_3 = \begin{bmatrix} 9 \\ 9 \\ 14 \end{bmatrix}', \mathbf{Q}_1 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{Q}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix},$$
$$\mathbf{y}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

This set S does not satisfy the conditions in Theorem 3. The reason is that, for any possible specification of the output-specific prices associated with observation 2, we have either $(\mathfrak{P}_2^1)_2 = 14$ or $(\mathfrak{P}_2^2)_2 = 14$. Then, for $(\mathfrak{P}_2^1)_2 = 14$ we get $\mathfrak{P}_2^1\mathbf{Q}_1 < \mathfrak{P}_2^1\mathbf{Q}_2$ (while $y_1^1 > y_2^1$) and, similarly, for $(\mathfrak{P}_2^2)_2 = 14$ we get $\mathfrak{P}_2^2\mathbf{Q}_3 < \mathfrak{P}_2^2\mathbf{Q}_2$ (while $y_3^2 > y_2^2$).

Next, we can easily verify that the following specification of the vectors $\mathbf{P}_t^j$ ($j = 1, 2$; $t = 1, 2, 3$) makes the set S satisfy the conditions in Theorem 2:

$$\mathbf{P}_1^1 = \begin{bmatrix} 13 \\ 7 \\ 7 \end{bmatrix}', \mathbf{P}_1^2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}', \mathbf{P}_2^1 = \begin{bmatrix} 8 \\ 7 \\ 1 \end{bmatrix}', \mathbf{P}_2^2 = \begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix}', \mathbf{P}_3^1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}', \mathbf{P}_3^2 = \begin{bmatrix} 7 \\ 7 \\ 13 \end{bmatrix}'.$$

Thus, we conclude that the data set is CM-rationalizable.

4 Goodness-of-fit measures

The rationalizability conditions presented in the previous section are 'sharp' ones: they (only) tell us whether or not observed behavior is exactly consistent with cost minimization. In practice, however, it may well be that a certain firm is close to cost minimization while it is not exactly cost minimizing. As noted by Varian (1990), for most purposes nearly optimizing behavior is just as good as exactly optimizing behavior. This calls for a goodness-of-fit measure that tells us how close observed firm behavior is to cost minimization if it fails the (exact) rationalizability conditions presented above. Such a goodness-of-fit measure then captures the degree of optimization (also referred to as the degree of efficiency) in terms of the behavioral model that is subject to study.

Varian (1990) (based on Afriat (1972)) proposed a nonparametric goodness-of-fit measure for cost minimization in a single-output setting. In what follows, we will extend this idea to our multi-output setting. In this respect, it is also useful to refer to Färe and Grosskopf (1995), who make explicit the relationship between Varian's goodness-of-fit approach and the Data Envelopment Analysis (DEA) literature that we also mentioned in the Introduction. Building on these authors' analysis, our following discussion may provide a useful starting point for exploring new directions of DEA-type efficiency measurement in multi-output settings.

Single-output production. As an introduction to the type of nonparametric goodness-of-fit analysis that we consider here, we briefly recapture Varian (1990)'s original idea and adapt it to our set-up. We start by defining the concept of θ-S-rationalizability.

Definition 5 (θ-S-Rationalizability)**.** *Consider a data set $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, y_t\}_{t\in T}$ and a number $\theta \in [0,1]$. We say that the observation $t \in T$ is single-output θ-rationalizable (θ-S-rationalizable) if there exists a $c \in \mathbb{R}_+$ that solves (for all $v \in T$)*

$$\begin{aligned} \mathbf{p}_t\mathbf{q}_v + \mathbf{P}_t\mathbf{Q}_v &\geq c && \textit{whenever } y_v \geq y_t, \\ \mathbf{p}_t\mathbf{q}_v + \mathbf{P}_t\mathbf{Q}_v &> c && \textit{whenever } y_v > y_t, \\ \theta(\mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t) &\leq c. \end{aligned} \qquad \text{(fp-s)}$$

According to this definition, the observation t is θ-S-rationalizable if there exists a number c that meets a number of linear constraints. The first two constraints imply that c does not exceed the (minimal) cost level associated with any observation v that produces at least the output of observation t (i.e. $y_v \geq (>)\ y_t$). Next, the last constraint imposes a lower bound an c, stating that it cannot lie below θ times the cost level of observation t. Taken together, θ-S-rationalizability requires that the production cost of observation t is not greater than $1/\theta$ (≥ 1) times the minimal cost for producing the output y_t as defined over the set of observations T.

The condition for θ-S-rationalizability in Definition 5 bears a direct relation to the S-rationalizability condition in Theorem 1. For $\theta = 1$ we have that θ-S-rationalizability exactly coincides with S-rationalizability (i.e. the constraints in Definition 5 boil down to requiring that observation t satisfies SACM). More generally, the higher θ, the 'closer' the (θ-S-rationalizable) observation t will be to S-rationalizability.

For any given value of θ, θ-S-rationalizability basically requires feasibility of a set of linear constraints. Using this, we can introduce an easily implementable nonparametric goodness-of-fit measure for cost minimization in the single-output case. Specifically, consider the linear programming problem that maximizes c subject to the constraint (fp-s). For c^* the optimal solution value of this problem, we define the goodness-of-fit measure

$$\theta_t^S = \frac{c^*}{\mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t}.$$

By construction, for any $\theta < \theta_t^S$ it holds that the observation t is θ-S-rationalizable. In addition, the goodness-of-fit measure θ_t^S never exceeds one, and it is equal to unity only if the observation t exactly satisfies the SACM condition. As such, this measure effectively captures how close the firm observation is to cost minimization.

Cooperative multi-output production. We next extend this goodness-of-fit idea to a cooperative multi-output setting. To this end, we focus on the decentralized interpretation of the cooperative production model, which makes use of output-specific quantities $\mathbf{q}_t^j$ and output-specific prices $\mathbf{P}_t^j$ for the joint inputs. Specifically, at the cooperative equilibrium, each output department acts as if it chooses the inputs $\mathbf{q}_t^j$ and $\mathbf{Q}_t$ that solve the cost minimization problem **OP-S** for given output y_t^j and prices $\mathbf{p}_t$, $\mathbf{P}_t^j$. This motivates the following definition.

Definition 6 (θ-CM-rationalizability)**.** *Consider a data set $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{y}_t\}_{t\in T}$ and a number $\theta \in [0,1]$. We say that the observation $t \in T$ is cooperative multi-output θ-rationalizable (θ-CM-rationalizable) if there exist $c^j \in \mathbb{R}_+$, $\mathbf{P}_t^j \in \mathbb{R}^M_{++}$ and $\mathbf{q}_v^j \in \mathbb{R}^N_+$ that solve (for all $v \in T$ and $j \leq J$)*

$$\sum\nolimits_j \mathbf{q}_v^j = \mathbf{q}_v, \tag{cm-1}$$

$$\sum\nolimits_j \mathbf{P}_t^j = \mathbf{P}_t, \tag{cm-2}$$

$$\begin{aligned} \mathbf{p}_t\mathbf{q}_v^j + \mathbf{P}_t^j\mathbf{Q}_v &\geq c^j && \textit{whenever } y_v^j \geq y_t^j, \qquad \text{(cm-3)}\\ \mathbf{p}_t\mathbf{q}_v^j + \mathbf{P}_t^j\mathbf{Q}_v &> c^j && \textit{whenever } y_v^j > y_t^j,\\ \theta(\mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t) &\leq \sum\nolimits_j c^j. \end{aligned}$$

The interpretation is analogous to the one of Definition 5. The specificity of Definition 6 reflects the decentralized representation of cooperative multi-output production. In particular, for θ-CM-rationalizability we need for each output j that there exists a number c^j satisfying a number of constraints. The first two constraints in Definition 6 put restrictions on the quantities ($\mathbf{q}_v^j$) and the output-specific prices ($\mathbf{P}_t^j$), which are specific to the cooperative model. The next two constraints require that no c^j exceeds the cost level (for output j) for any observation v that produces at least the same amount of output j as observation t (i.e. $y_v^j \geq (>)\ y_t^j$). Finally, the last constraint imposes that the total production cost of observation t must not exceed $1/\theta$ times the minimal cost of producing the (multi-dimensional) output associated with observation t, where the reference (minimal) cost $\sum_j c^j$ is defined over the set of observations T. Like before, we get that the condition for θ-CM-rationalizability reduces to the one for CM-rationalizability (in Theorem 2) if $\theta = 1$; and, more generally, lower values for θ imply less stringent rationalizability requirements.

Similar to the single-output case, we can check θ-CM-rationalizability by verifying feasibility of a set of linear constraints, which actually suggests an easy-to-use goodness-of-fit measure. Specifically, we solve the linear programming problem that maximizes $\sum_j c^j$ subject to (cm-1), (cm-2) and (cm-3). For $\sum_j c^{j*}$ the optimal value of the problem, we define the goodness-of-fit measure

$$\theta_t^{CM} = \frac{\sum_j c^{j*}}{\mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t}.$$

Once more, this measure is situated between zero and one. And we have that the observation t will be θ-CM-rationalizable whenever $\theta < \theta_t^{CM}$. In effect, θ_t^{CM} measures the degree to which the firm under study is cost minimizing at the observation t under the assumption of cooperative multi-output production.

Noncooperative multi-output production. For the noncooperative multi-output production setting, we construct a similar goodness-of-fit measure as for the cooperative case. In the noncooperative equilibrium, each output department chooses the inputs $\mathbf{q}_t^j$ and $\mathbf{Q}_t$ that solve the cost minimization problem **OP-S** for given output y_t^j and prices $\mathbf{p}_t$, $\mathfrak{P}_t^j$. We recall that an important difference with the cooperative scenario is that the output-specific prices for the joint inputs ($\mathfrak{P}_t^j$) need not sum to the observed market prices. Specifically, we now get the following definition.

Definition 7 (θ-NM-rationalizability). *Consider a data set $S = \{\mathbf{p}_t, \mathbf{P}_t, \mathbf{q}_t, \mathbf{Q}_t, \mathbf{y}_t\}_{t\in T}$ and a number $\theta \in [0,1]$. We say that the observation $t \in T$ is noncooperative multi-output θ-rationalizable (θ-NM-rationalizable) if there exist $\mathfrak{c}^j \in \mathbb{R}_+$, $\mathfrak{P}_t^j \in \mathbb{R}_{++}^M$ and $\mathbf{q}_v^j \in \mathbb{R}_+^N$ that solve (for all $v \in T$ and $j \leq J$)*

$$\begin{aligned} \sum\nolimits_j \mathbf{q}_v^j &= \mathbf{q}_v, && && \text{(nm-1)}\\ \max\nolimits_j\{(\mathfrak{P}_t^j)_m\} &= (\mathbf{P}_t)_m && \textit{for all } m \leq M, && \text{(nm-2)}\\ \mathbf{p}_t\mathbf{q}_v^j + \mathfrak{P}_t^j\mathbf{Q}_v &\geq \mathfrak{c}^j && \textit{whenever } y_v^j \geq y_t^j, && \text{(nm-3)}\\ \mathbf{p}_t\mathbf{q}_v^j + \mathfrak{P}_t^j\mathbf{Q}_v &> \mathfrak{c}^j && \textit{whenever } y_v^j > y_t^j, &&\\ \theta\left(\mathbf{p}_t\mathbf{q}_t + \sum\nolimits_j \mathfrak{P}_t^j\mathbf{Q}_t\right) &\leq \sum\nolimits_j \mathfrak{c}_t^j. && && \end{aligned}$$

This definition has exactly the same interpretation as Definition 6, except for one subtle (but important) difference. As indicated above, in the noncooperative case the output-specific prices $\mathfrak{P}_t^j$ are no longer subject to a sum constraint (i.e. cm-2 in Definition 6). Instead, we now get the max constraint (nm-2).

Given Definition 7, we can define a goodness-of-fit measure in an analogous way as for the cooperative case. Specifically, we let $\sum_j \mathfrak{c}^{j*}$ represent the outcome of maximizing $\sum_j \mathfrak{c}^j$ subject to the constraints (nm-1), (nm-2) and (nm-3), and let $\mathfrak{P}_t^{*j}$ be the optimal value of $\mathfrak{P}_t^j$ for this optimization problem. Then, define

$$\theta_t^{NM} = \frac{\sum_j \mathfrak{c}^{j*}}{\mathbf{p}_t\mathbf{q}_t + \sum_j \mathfrak{P}_t^{*j}\mathbf{Q}_t}.$$

Once more, we have $\theta_t^{NM} \in [0,1]$ by construction. Generally, the value of this goodness-of-fit measure reveals the degree to which the observed production behavior is θ-NM-rationalizable.[5]

As a final remark, we note that the constraint (nm-2) is nonlinear, which means that feasibility of the constraints in Definition 7 cannot be verified through linear programming methods. However, we can check feasibility by using standard mixed integer programming methods. Specifically, the max constraint (nm-2) is equivalent to the requirement that there exist binary variables $z_m^j \in \{0,1\}$ such that, for all $m \leq M$ and

[5]We have to note though that there is a subtle difference between the denominators of θ_t^{CM} and θ_t^{NM}. By construction the optimal values for the $\mathbf{P}_t^j$ add up to the observed price $\mathbf{P}_t$, which makes that the denominator of θ_t^{CM} is equal to the observed cost $\mathbf{p}_t\mathbf{q}_t + \mathbf{P}_t\mathbf{Q}_t$. Because of constraint (nm-2), this does not need to hold for the denominator of θ_t^{NM}.

$j \leq J$,

$$(\mathfrak{P}_t^j)_m - (\mathbf{P}_t)_m \leq 0,$$
$$(\mathbf{P}_t)_m - (\mathfrak{P}_t^j)_m \leq (1 - z_m^j)(\mathbf{P}_t)_m,$$
$$\sum_j z_m^j \geq 1.$$

It is easily verified that, for every joint input m, these constraints guarantee $(\mathfrak{P}_t^j)_m \leq (\mathbf{P}_t)_m$ for all j, while $(\mathbf{P}_t)_m = (\mathfrak{P}_t^j)_m$ for at least one j (with $z_m^j = 1$). Thus, replacing constraint (nm-2) by these mixed integer linear constraints effectively obtains a mixed integer programming problem. In turn, this provides an easy way to compute the goodness-of-fit measure θ_t^{NM}.

5 Empirical Application

We apply our newly proposed methodology to the English and Welsh (E&W) drinking water and sewerage sector. We will start by briefly introducing the sector. Here, it is indicated that multi-output production and joint input use (with, correspondingly, possible scope economies) form important issues in modeling the production behavior. Next, we present the data (including the input and output selection) that are used in the application. After that, we discuss our empirical results. First, we assess whether the cooperative model or the noncooperative model does the better job in explaining the observed behavior in our sample. Subsequently, we conduct an explanatory analysis that correlates the goodness-of-fit (or efficiency) measures that we obtain for the two models with alternative contextual variables that have been studied in the relevant literature.

The E&W drinking and sewerage sector. Multi-output production has recently become an important issue in the privatized E&W drinking water and sewerage sector (see Ofwat (2008)). In 1974, the majority of municipal drinking water companies were merged and nationalized into 10 'Regional Water Authorities' (RWAs), which were responsible for water quality, drinking water production, distribution and sanitation. These water and sewerage companies (WSCs) account for about 80% of the water provision.[6] The Water Act, issued under the Tatcher government, privatized the RWAs in 1989.

To avoid monopoly profits in a privatized environment, a strong economic regulator has been established: the Office of Water Services (Ofwat). Ofwat applies a price-cap regulation which limits the annual growth rate of the water price for every company by a factor K (RPI-X). The variable K is calculated as the growth rate of the Retail Prices Index (RPI) minus a productivity factor X, which is determined by comparing the performances of the water utilities. The firm-specific maximum price is determined

[6] Besides the RWAs, about 30 'Water only Companies' (WoCs) produced and distributed (only) drinking water. For simplicity we do not focus on WoCs in our analysis below.

once in each regulatory cycle, which consists of five years (although initially intended to last for 10 years).

Besides setting tariffs, Ofwat determines the industry structure. Recently, it considered the benefits of increased competition by separated companies (see Ofwat (2008)) as a response to the so-called 'Cave report' (see Cave (2009)). Both vertical separation of elements in the supply chain (e.g. separating abstraction, treatment and distribution) and horizontal unbundling of WSCs could create a more competitive environment. As a drawback, existing scope economies would be lost. Despite the fact that joint water and sewerage companies service about 80% of the E&W population, the literature is inconclusive on the existence of scope economies. Some studies find diseconomies of scope (Hunt and Lynk (1995) and Saal and Parker (2000) for E&W; Marques and De Witte (2010) for the Portuguese water sector) while other studies conclude the opposite (Lynk (1993) and Stone and Webster (2004) in E&W but only if water quality is accounted for).

Importantly, our methodology does not require us to take a prior stance to whether or not scope economies are effectively present. Instead, it allows us to focus upon the cooperative versus noncooperative nature of joint input use. Our specific interest is in identifying which model best describes the observed production behavior. Clearly, a better understanding of the behavioral model that underlies the observed multi-output production can only benefit the regulatory policy (in casu by Ofwat).

Data. Our selection of the output-specific inputs falls in line with the existing literature (see, for example, Stone and Webster (2004)). First, for water production we include the number of employees assigned to drinking water production, distribution and purification, as well as the total length of the network. Similarly, for sewerage the output-specific inputs consist of the number of employees in sewerage collection and purification, as well as the length of the sewerage network, which is considered as a proxy for capital use.

Next, we use the number of employees that produce both water and sewerage as a joint input. Although this input is not directly observed from the annual accounts, we can deduce it from the financial statements. In particular, the Ofwat June returns report the 'Water direct costs to employment', the 'Sewerage direct costs to employment' as well as the 'Total manpower costs'. This allows us to retrieve the employment shares that are specific to water and sewerage production as well as the share of employment that is simultaneously used for both outputs.

As outputs, we consider the measured volume of water and sewerage. Here, we can reasonably argue that companies effectively aim at producing these (exogenously defined) outputs at a minimal cost. In other words, our behavioral assumption of cost minimization is a plausible one for the setting at hand.

The data we use arise from the Ofwat June returns, which is a public database with detailed information on the water and sewerage companies. Our data cover the period from 1991 (two years after privatization) to 2009. This obtains data on 4 regulatory cycles: we have 40 production observations in 1991-1994 and 50 observations in the subsequent regulatory cycles 1995-1999, 2000-2004, and 2005-2009. Our sample comprises 190 WSC production observations in total. Table 1 provides summary statistics for our selection of inputs and outputs.

Table 1: Summary statistics

		Mean	Std.	Min	Max
Joint input	Employees	2058.11	1086.31	596.98	5048.93
Private input	W. Employees	578.61	447.84	65.81	2758.78
	Sew. Employees	545.94	336.52	180.54	2104.31
	W. Mains (km)	26290.67	11240.20	7658.44	46573.69
	Sew. Mains (km)	30454.65	19627.19	7498.03	83791.43
Output water	W. volume (Ml/d)	1253.40	748.96	345.68	2874.31
	Sew. volume (Ml/d)	935.92	662.79	229.23	2909.09
Control	Service area	12918.93	5824.22	3850.00	22090.00
	Leakage	328.56	256.07	72.12	1108.69
	River water	0.65	3.43	0.00	45.00
	Ground water	0.36	0.74	0.03	9.40
	Impound water	0.54	3.49	0.00	45.70
	Bulk supply imports	49.00	110.42	0.00	404.49
	Bulk supply exports	55.71	98.20	0.00	373.52
	Connected water properties	1849.09	1051.34	448.10	3736.40
	Connected sewerage properties	2228.86	1358.57	585.00	5737.00

Goodness-of-fit results. Before presenting the results of our goodness-of-fit analysis, we briefly recall the non-nestedness result that we demonstrated at the end of Section 3. In particular, we showed that CM-rationalizability does not necessarily imply NM-rationalizability, and vice versa. As such, there is no a priori reason why one model should have weaker empirical implications (or less discriminatory power) than the other. In our opinion, this provides a strong motivation for our following exercise, where we investigate which model effectively does provide the better empirical fit of the production behavior in the sector under study.

Figure 1 displays the empirical decumulative distribution for our goodness-of-fit (or efficiency) measures introduced in Section 4: it gives the percentage of production observations t (vertical axis) of which the value of the goodness-of-fit measures θ_t^{CM} (cooperative model) and θ_t^{NM} (noncooperative model) equals at least the value on the horizontal axis. To account for technological shifts, over different regulatory cycles, we evaluate a particular company by (only) comparing it to companies in the same regulatory cycle. For a given goodness-of-fit value, a better performing model corresponds to a higher percentage of firms that can be rationalized. The overall picture that emerges from Figure 1 is that the noncooperative model outperforms the cooperative model. The difference is actually rather pronounced: the distribution for the noncooperative model stochastically dominates the one for the cooperative model. Interestingly, the difference also appears to be statistically significant: if we conduct a two-sided Kolmogorov-Smirnov test, then we reject the null hypothesis that the two distributions coincide at a significance level of 10% (Kolmogorov-Smirnov test statistic amounts to 0.1316, and the associated p-value equals 0.075).

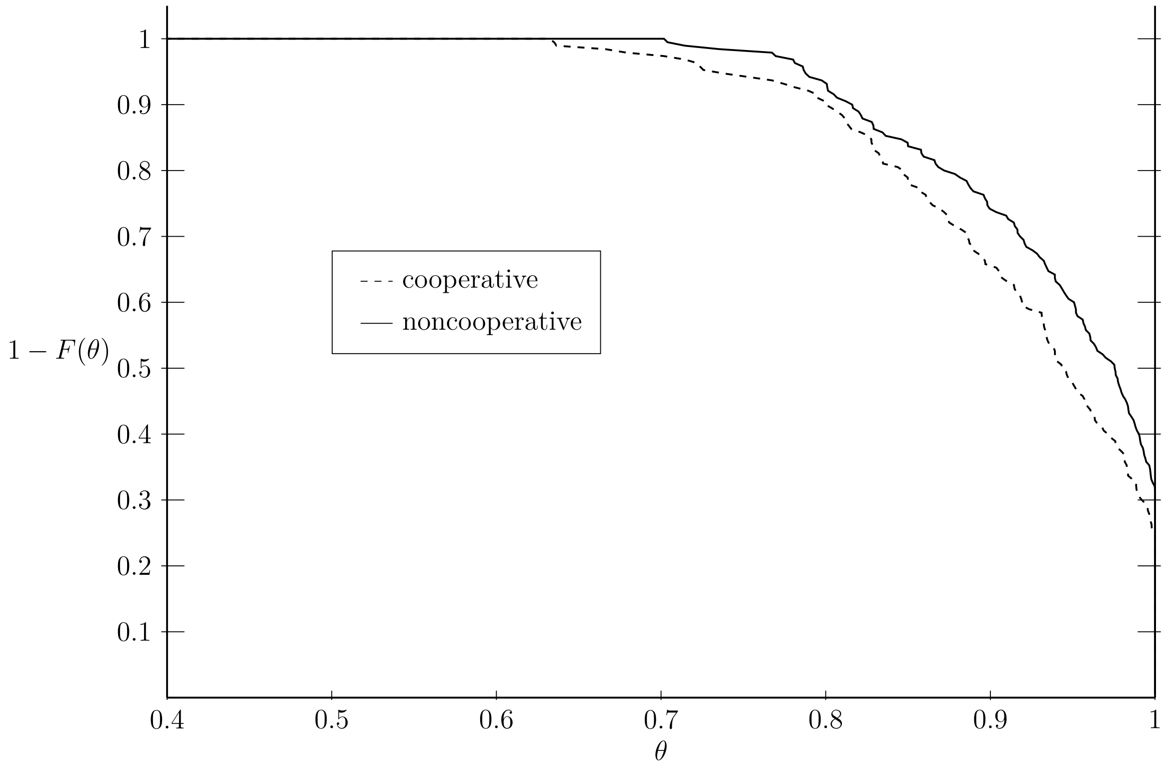
Table 2 summarizes the same information in tabulated form; but here we distinguish between the 4 regulatory cycles captured by our data set. The results in this table allow for a more detailed analysis. We obtain a median goodness-of-fit value above 93% for each model specification in every regulatory cycle. This shows that, on average, both models provide a reasonably good fit of the observed production behavior in every different time period. But, again, the noncooperative model systematically dominates the cooperative model. Even though the difference is not very substantial in many cases, it turns out to be quite pronounced in some instances (see, for example, the difference between the minimum and first quartile values for the two model specifications). Overall, the results in Table 2 support the same conclusions as the results in Figure 1.

Generally, our results suggest a better empirical support for the noncooperative model than for the cooperative model. But the difference between the goodness-of-fit of the two models seems to depend on the company (environment) at hand. This directly brings us to our next exercise, where we relate the (cooperative and noncooperative) goodness-of-fit values to particular variables describing the production context of every company that we studied.

Table 2: Goodness-of-fit estimations for each regulatory cycle

period	number of observations	min	1st quart	median	3rd quart	max
1991-1994						
cooperative	40	0.632	0.864	0.949	0.998	1.000
noncooperative		0.704	0.934	0.987	1.000	1.000
1995-1999						
cooperative	50	0.722	0.886	0.971	1.000	1.000
noncooperative		0.736	0.919	0.979	1.000	1.000
2000-2004						
cooperative	50	0.726	0.869	0.937	0.986	1.000
noncooperative		0.767	0.873	0.957	0.991	1.000
2005-2009						
cooperative	50	0.666	0.862	0.932	0.999	1.000
noncooperative		0.702	0.878	0.958	1.000	1.000

Figure 1: Empirical decumulative distribution function of goodness of fit



Explanatory analysis. To examine the influence of background factors, we make use of a two-step approach in which our goodness-of-fit measures are regressed on a number of observable contextual variables by using both ordinary least squares (OLS) and tobit (because of the truncated nature of our goodness-of-fit measures). The appropriateness of this two-step approach for the type of questions we want to address here has been advocated in particular by Banker and Natarajan (2008), McDonald (2009) and Banker (2011) (in a DEA context). Essentially, our following analysis will evaluate which environmental factors explain the validity of a particular behavioral (cooperative/noncooperative) model for describing the observed production behavior.

As a preliminary note, we emphasize that our following analysis is mainly meant to be illustrative and should be interpreted as explorative rather than conclusive. This also explains why we opt for a most simple methodological set-up. In this respect, two remarks are in order. First, Simar and Wilson (2007) suggested an alternative two-stage approach for assessing the impact of contextual variables on goodness-of-fit (or efficiency) measures, claiming that this other approach more adequately deals with a number of statistical issues associated with explanatory analysis such as the one we consider here. See, for example, Banker and Natarajan (2008), Banker (2011) and Simar and Wilson (2011) for a comparison between this approach and the one we follow here. At this point, we restrict to indicating that our methodology for assessing the goodness-of-fit of cooperative and noncooperative production models can also easily be combined with the two-step approach proposed by Simar and Wilson. Second, our following regression results must be interpreted with sufficient care as we do not explicitly correct for possible endogeneity bias. From this perspective, while we will not always explicitly indicate this in what follows, our results below actually reveal correlations rather than causal relationships.

We draw on the existing literature to select our contextual variables (see, for example, Stone and Webster (2004)). We consider three specifications of the regression model. Our first specification (model 1) is our base model and includes four explanatory variables. First, service area figures as a proxy for the scale of operations. Second, the percentage of leakage captures the geographical relief (i.e. more hilly landscape requires more pressure on the network of pipes, which could cause leakages more easily) and the extent of maintenance (i.e. more leakages correspond to less expenses with maintenance). Third, we assessed water quality information as a potentially important contextual factor. It is defined in terms of the source of water production: ground water has a higher quality and therefore lower purification costs than river and impound water.[7]

Our second specification (model 2) adds water imports and exports to the control variables. The underlying idea is that companies which import or export water might be structurally different from other companies. A high export of water might indicate the presence of relatively cheap water or, alternatively, cost disadvantages (especially as the transportation of water is very expensive). Our third specification (model 3) adds the number of connections as an explanatory variable; this variable can be conceived as an alternative output measure and figures as a proxy for the size of the operations. Note that we allow for cycle fixed effects in our different model specifications, as maximum prices vary over regulatory cycles. Our main qualitative conclusions are, however, robust to this fixed effect assumption.[8]

Table 3 presents the results of our OLS second stage regressions for the three model specifications under study. In line with the findings of Banker and Natarajan (2008) and McDonald (2009), the estimates for the truncated tobit model are very similar and therefore omitted (but available upon request). We observe that the sign of the regression coefficients are generally the same for the cooperative and noncooperative model specifications, although the significance levels differ for some variables.

If we look at Table 3 in more detail, we find for all three model specifications that service area is significantly negatively correlated with goodness-of-fit: the larger the supply area, the less rationalizable the production behavior of water utilities (on average). This negative impact is about the same for the noncooperative model and the cooperative model. Furthermore, we observe for the three models that volume of leakage is positively but (almost always) insignificantly correlated to cost minimization. The proportion of river water also seems to have a positive correlation with the (cooperative

[7]In the regression analysis, we consider impound water as the reference category.

[8]We note that year fixed effects also deliver robust outcomes.

and noncooperative) goodness-of-fit. One possible interpretation is that, because it is more expensive to purify this type of water, a higher dependence on river water urges utilities to effectively behave efficiently (i.e. in a cost minimizing way). An alternative explanation is that companies save on employee costs if the share of water purification in the price gets higher. Analogous interpretations can explain the negative effect for the share of ground water. We observe that the source of water explains better goodness-of-fit for the cooperative model than for the noncooperative model (where it has lower significance, if significant at all). This may indicate that companies with a larger proportion of river water act more cooperatively than noncooperatively. This actually seems intuitive given the higher purification costs and, consequently, the higher incentives for public benefits; in such a case, the costs for these benefits are more easily shared by both water and sewerage customers.

Let us then consider the specific variables that have been included in regression models 2 and 3. First, looking at our results for model 2, which captures water export and import, we find that import does not seem to have a significant influence on goodness-of-fit (for any model specification), whereas export apparently does have a significantly positive effect for the noncooperative model: water utilities with a large volume of water export produce the volume of water and sewerage with proportionally fewer employees and capital (mains). Finally, we consider our results for model 3. Here we find that, in contrast to the number of sewerage connections (which has a positive but insignificant correlation with cost minimization), the number of connected properties for water provision does appear to have a significantly negative impact on cost minimization, which suggests that smaller scale companies suffer less from inefficiencies (i.e. behavior inconsistent with cost minimization). This last finding falls in line with the existing literature, which indeed suggests diseconomies of scale for water utilities that have about the same size as the E&W companies that we study here (see Berg and Marques (2010) for a literature review of the water sector).

From these results, we can draw the overall conclusion that some of the variables we selected seem particularly related with the cooperative or noncooperative model. For example, the proportion of river water correlates significantly with goodness-of-fit only for the cooperative model, and the proportion of ground water shows a stronger correlation for the cooperative model than for the noncooperative model. Our final regression exercise allows us to investigate these patterns a little bit further. Specifically, we now take the difference between the cooperative and noncooperative goodness-of-fit measures as the dependent variable, while using the same contextual factors (and related model specifications) as before.

The results are presented in Table 4. For models 1 and 2, we find that in particular the proportion of river and ground water abstraction explains the difference in goodness-of-fit between the cooperative and noncooperative models. The larger the proportion of the more costly river water abstraction, the better the observed production behavior is explained by the cooperative (centralized) model (relative to the noncooperative model). Intuitively, this can be explained by cross-subsidies between the departments that are necessary to compensate the more expensive water source, or by an exchange of know-how on water purification. The larger the proportion of the cheaper (in terms of purification) ground water, the more noncooperatively (i.e. decentralized) companies seem to act. Intuitively, in this case the water source is already relatively pure, and so there is less need for cooperation. In Model 2, the service area explains the cooperative model: the larger the service area, the more companies cooperate. The volume of bulk supply exports rationalizes the non-cooperative model better than the cooperative model. The more water exports, the better the observed production behavior is explained by the noncooperative model. The other specific variables of model 3 seem to have roughly the same effect on the goodness-of-fit for the two specifications under study. The assumption of fixed effects associated with the different regulatory cycles does not change this conclusion.

Summarizing, we believe this application clearly illustrates the kind of questions that can be addressed by using the newly proposed methodology. First, our nonparametric toolkit allows for checking whether the noncooperative or cooperative model best describes the observed multi-output production behavior. A second stage regression analysis may then investigate which environmental factors drive the appropriateness of a specific behavioral (cooperative/noncooperative) model. In our application, we did identify a number of such contextual factors that seem to significantly impact on the goodness-of-fit of both the cooperative and noncooperative models. Moreover, we were able to distinguish factors that specifically drive the better fit of one particular model.

Table 3: Regressing goodness-of-fit on contextual variables for cooperative and noncooperative models

	Model 1		Model 2		Model 3	
Dependent variable	coop	noncoop	coop	noncoop	coop	noncoop
Constant	9.671E-01***	1.009E+00***	9.666E-01***	1.014E+00***	9.613E-01***	1.003E+00***
	2.756E-02	2.359E-02	2.834E-02	2.372E-02	2.771E-02	2.351E-02
Service area	-3.452E-06**	-3.893E-06***	-4.047E-06**	-5.131E-06***	-2.128E-06	-2.450E-06*
	1.131E-06	9.683E-07	1.348E-06	1.128E-06	1.283E-06	1.089E-06
Leakage	2.569E-05	3.649E-05	2.966E-05	3.813E-05	1.068E-04	1.227E-04*
	2.667E-05	2.283E-05	2.686E-05	2.248E-05	6.493E-05	5.508E-05
Proportion river water	1.502E-02**	7.087E-03	1.230E-02*	3.799E-03	1.684E-02*	9.207E-03
	5.308E-03	4.544E-03	5.375E-03	4.497E-03	6.471E-03	5.489E-03
Proportion ground water	-9.230E-02***	-5.333E-02*	-7.866E-02**	-3.663E-02	-1.018E-01**	-6.437E-02*
	2.506E-02	2.146E-02	2.545E-02	2.129E-02	3.090E-02	2.621E-02
Bulk supply imports			-6.401E-05	-4.176E-05		
			6.185E-05	5.176E-05		
Bulk supply exports			1.418E-04	2.098E-04**		
			7.536E-05	6.306E-05		
Connected water properties					-3.996E-05*	-4.370E-05*
					2.021E-05	1.714E-05
Connected sewerage properties					1.194E-05	1.353E-05
					1.587E-05	1.347E-05
Regulatory cycle fixed effects	Yes	Yes	Yes	Yes	Yes	Yes

Note: Standard errors below. ***, **, and * denote significance at, respectively, 1, 5 and 10%-level.

Table 4: Regressing the difference in goodness-of-fit (cooperative - noncooperative) on contextual variables

Dependent variable	Model 1 coop-noncoop	Model 2 coop-noncoop	Model 3 coop-noncoop
Constant	-4.170E-02***	-4.699E-02***	-4.136E-02***
	1.082E-02	1.114E-02	1.104E-02
Service area	4.412E-07	1.084E-06*	3.222E-07
	4.443E-07	5.299E-07	5.113E-07
Leakage	-1.080E-05	-8.475E-06	-1.593E-05
	1.048E-05	1.056E-05	2.587E-05
Proportion river water	7.932E-03***	8.501E-03***	7.636E-03**
	2.085E-03	2.112E-03	2.578E-03
Proportion ground water	-3.897E-02***	-4.203E-02***	-3.748E-02**
	9.845E-03	1.000E-02	1.231E-02
Bulk supply imports		-2.226E-05	
		2.431E-05	
Bulk supply exports		-6.801E-05*	
		2.962E-05	
Connected water properties			3.742E-06
			8.052E-06
Connected sewerage properties			-1.589E-06
			6.324E-06
Regulatory cycle fixed effects	Yes	Yes	Yes

Note: Standard errors below. ***, **, and * denote significance at, respectively, 1, 5 and 10%-level.

6 Conclusion

We have presented a novel framework for analyzing multi-output production behavior. Such behavior typically involves jointly used inputs, which raises the issue of whether these joint inputs are allocated in a cooperative (centralized) or noncooperative (decentralized) way. We introduced a methodology to empirically analyze multi-output production behavior in terms of the cooperative model and the noncooperative model. A distinguishing feature of our methodology is that it is nonparametric in nature, which means that it avoids imposing prior (nonverifiable) functional structure on the production technology.

An empirical application to the English and Welsh drinking water and sewerage sector demonstrated the practical usefulness of our framework. Here, a specific focus was on assessing (and comparing) the goodness-of-fit of the two model specifications for this particular sector. We found that the noncooperative model systematically provided a better description of the production behavior in our sample. Subsequently, an explanatory analysis allowed us to identify a number of company-specific contextual factors that significantly correlate with our goodness-of-fit measures for both models. Moreover, our data did enable us to distinguish particular factors that specifically seem to drive the better fit of one model (but not the other). In particular, we found that the behavior of companies with a higher proportion of river water abstraction, which is a more expensive resource to purify, is better explained by the cooperative model. On the contrary, companies with a higher proportion of ground water act more in line with the noncooperative model. Moreover, we observed that a larger service area explains cooperative behavior, while an increasing volume of bulk supply exports explains the noncooperative model.

We see different avenues for follow-up research. First, to focus our analysis we have only considered characterizing multi-output production under (cooperative and noncooperative) cost minimization, and empirically assessing the goodness-of-fit of alternative model specifications. If observed production behavior is found consistent with a particular model (i.e. can be rationalized), then interesting next questions pertain to recovering/identifying the decision model (including the production technology) that underlies the (rationalizable) production behavior, and to forecasting/simulating production behavior in new situations (e.g. characterized by new input prices and/or output levels). Nonparametric recovery and forecasting issues have been addressed in the case of single-output production (see, for example, the original contributions of Hanoch and Rothschild (1972), Afriat (1972), Diewert and Parkan (1983) and Varian (1984)). As indicated above, our newly proposed methodology naturally extends existing tools for assessing single-output production. Given this, it provides a useful basis for developing the multi-output generalizations of recovery and forecasting tools that apply to the single output case.

Second, referring to our discussion in the Introduction, we believe it is interesting to further exploit the formal link with models for multi-person household consumption, to develop novel tools for analyzing multi-output production. Specifically, existing contributions on parametric analysis of multi-person consumption can provide a fruitful basis for developing the parametric counterpart of the nonparametric framework we set out here. In turn, this may imply useful multi-output extensions of the parametric efficiency measurement literature commonly referred to as Stochastic Frontier Analysis (SFA).

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Appendix

Proof of Theorem 2

(necessity) In order to demonstrate necessity we begin by introducing some notation. We denote the input vector, which solves **OP-CM**, as an element of $\mathbb{R}_+^{J \cdot N+M} = \Omega_+$, by stacking all the output-specific inputs $\mathbf{q}_t^j$ on top of each other and ending with the joint inputs $\mathbf{Q}_t$. We denote this vector by $\mathcal{Q}_t$:

$$\mathcal{Q}_t = [\mathbf{q}_t^{1\prime} \ \ \dots \ \ \mathbf{q}_t^{j\prime} \ \ \dots \ \ \mathbf{q}_t^{J\prime} \ \ \mathbf{Q}_t^{\prime}]^{\prime}.$$

Similarly, any other vector $\mathcal{X} \in \Omega_+$ is decomposed as

$$\mathcal{X} = [\mathbf{x}^{1\prime} \ \ \dots \ \ \mathbf{x}^{j\prime} \ \ \dots \ \ \mathbf{x}^{J\prime} \ \ \mathbf{X}^{\prime}]^{\prime}.$$

Likewise, we denote price vectors by elements in the set $\mathbb{R}_{++}^{J \cdot N+M} = \Omega_{++}$, which are obtained by replicating the price vectors $\mathbf{p}_t$ (J times) and ending with the vector $\mathbf{P}_t$. Let $\mathcal{P}_t$ represent this vector

$$\mathcal{P}_t = [\mathbf{p}_t \ \ \dots \ \ \mathbf{p}_t \ \ \mathbf{P}_t].$$

Consider a convex set S and an element $\mathbf{a} \in S$. The normal cone of S at the point $\mathbf{a}$ is denoted by $C(\mathbf{a}|S)$ and is defined as

$$C(\mathbf{a}|S) = \{\mathbf{w} | \forall \mathbf{x} \in S, \mathbf{w}(\mathbf{x} - \mathbf{a}) \leq 0\}.$$

Now, consider an output j and the input requirement set $V^j(y^j)$. We define the set $\tilde{V}^j(y^j)$ by

$$\tilde{V}^j(y_t^j) = \{\mathcal{X} \in \Omega_+ | (\mathbf{x}^j, \mathbf{X}) \in V^j(y_t^j)\}.$$

Fact 1. *The set $\tilde{V}^j(y_t^j)$ is convex.*

Proof. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are in $\tilde{V}^j(y_t^j)$ and let $\alpha \in [0,1]$. Let $\mathcal{Z} = \alpha\mathcal{X} + (1-\alpha)\mathcal{Y}$. Then $(\mathbf{x}^j, \mathbf{X}) \in V^j(y_t^j)$, $(\mathbf{y}^j, \mathbf{Y}) \in V^j(y_t^j)$, $\mathbf{z}^j = \alpha\mathbf{x}^j + (1-\alpha)\mathbf{y}^j$ and $\mathbf{Z} = \alpha\mathbf{X} + (1-\alpha)\mathbf{Y}$. By convexity of the set $V^j(y_t^j)$, we obtain that $(\mathbf{z}^j, \mathbf{Z}) \in V^j(y_t^j)$ and we can conclude that $\mathcal{Z} \in \tilde{V}^j(y_t^j)$. □

Fact 2. *Let $\mathcal{U}_j \in C(\mathcal{Q}_t | \tilde{V}^j(y_t^j))$ (i.e. $\mathcal{U}_j$ is in the normal cone of $\tilde{V}^j(y_t^j)$ at $\mathcal{Q}_t$), where*

$$\mathcal{U}_j = [\mathbf{u}_j^1 \ \dots \ \mathbf{u}_j^j \ \dots \ \mathbf{u}_j^J \ \mathbf{U}_j].$$

Then, it must be that

- *for all $k \neq j$, $\mathbf{u}_j^k = \mathbf{0}$,*
- *$\mathbf{u}_j^j \leq \mathbf{0}$,*
- *$\mathbf{U}_j \leq \mathbf{0}$.*

Proof. Let $\mathcal{X} \in \tilde{V}^j(y_t^j)$ be equal to $\mathcal{Q}_t$ except for $(\mathbf{x}^k)_m$ ($k \neq j, m \leq N$), where

$$(\mathbf{x}^k)_m = (\mathbf{q}_t^j)_m + \delta,$$

We consider values of $\delta \in]-\varepsilon, \varepsilon[$ for a small number $\varepsilon > 0$. Clearly, $\mathcal{X} \in \tilde{V}^j(y_t^j)$ for all possible values of δ. Then, if $\mathcal{U}_j$ is normal to $\tilde{V}^j(y_t^j)$ at $\mathcal{Q}_t$, it must be that

$$\begin{aligned} \mathcal{U}_j\mathcal{X} &\leq \mathcal{U}_j\mathcal{Q}_t \\ \Leftrightarrow (\mathbf{u}_j^k)_m(\mathbf{x}^k)_m &\leq (\mathbf{u}_j^k)_m(\mathbf{q}_t^k)_m \\ &= (\mathbf{u}_j^k)_m\left((\mathbf{x}^k)_m - \delta\right). \end{aligned}$$

Setting $\delta > 0$ shows that $(\mathbf{u}_j^k)_m \leq 0$. On the other hand, if $\delta < 0$, then $(\mathbf{u}_j^k)_m \geq 0$. As such, if the condition must hold for all δ in the interval, it must be that $(\mathbf{u}_j^k)_m = 0$. Given that m and k were arbitrarily chosen (except for the fact that $k \neq j$), it follows that for all $k \neq j$, $\mathbf{u}_j^k = \mathbf{0}$.

Now, consider a vector $\mathcal{X}$ which equals $\mathcal{Q}_t$ except for the element $(\mathbf{x}^j)_m$, where

$$(\mathbf{x}^j)_m = (\mathbf{q}_t^j)_m + \delta.$$

Here, we have to assume $\delta > 0$, since otherwise we can no longer guarantee that $\mathcal{X} \in \tilde{V}^j(y_t^j)$. By monotonicity of the set $V^j(y_t^j)$, we see that $(\mathbf{x}^j, \mathbf{X}) \in V^j(y_t^j)$. As such, $\mathcal{X} \in \tilde{V}^j(y_t^j)$. Now, if $\mathcal{U}_j$ is normal to $\tilde{V}^j(y_t^j)$ at $\mathcal{Q}_t$, it must be that

$$\begin{aligned} (\mathbf{u}_j^j)_m(\mathbf{x}^j)_m &\leq (\mathbf{u}_j^j)_m(\mathbf{q}_t^j)_m \\ &= (\mathbf{u}_j^j)_m\left((\mathbf{x}^j)_m - \delta\right). \end{aligned}$$

This shows that $(\mathbf{u}_j^j)_m \leq 0$. As m was arbitrarily chosen, we see that $\mathbf{u}_j^j \leq \mathbf{0}$. Straightforwardly, we can conduct a similar reasoning with respect to the vector $\mathbf{Q}_t$ in order to show that that the vector $\mathbf{U}_j \leq \mathbf{0}$. □

Given the definition of the sets $\tilde{V}^j(y_t^j)$, we see that the cost minimization program **OP-CM** can be rewritten as:

$$\min_{\mathcal{X}\in\Omega_+} \mathcal{P}_t\mathcal{X} \text{ s.t. } \mathcal{X} \in \tilde{V}^j(y_t^j) \qquad (\forall j \leq J).$$

The sets $\tilde{V}^j(y_t^j)$ are convex. Hence, a necessary and sufficient condition for $\mathcal{Q}_t$ to be a solution to this problem is that there exist vectors $\mathcal{U}_j$ in $C(\mathcal{Q}_t|\tilde{V}^j(y_t^j))$ such that[9]

$$\mathbf{0} = \mathcal{P}_t + \sum_j \mathcal{U}_j.$$

By Fact 2, we have that $\mathcal{U}_j$ is of the form

$$\mathcal{U}_j = [\mathbf{0} \;\; \ldots \;\; \mathbf{u}_j^j \;\; \ldots \;\; \mathbf{0} \;\; \mathbf{U}_j],$$

where $\mathbf{u}_j^j \leq \mathbf{0}$ and $\mathbf{U}_j \leq \mathbf{0}$. Then, we can rewrite the equilibrium conditions as

$$\begin{aligned} \mathbf{p}_t &= -\mathbf{u}_j^j, \\ \mathbf{P}_t &= -\sum_j \mathbf{U}_j. \end{aligned}$$

Further, given that $\mathcal{U}_j$ is a normal vector for the set $\tilde{V}^j(y_t^j)$, we must have that for all $\mathcal{X} \in \tilde{V}^j(y_t^j)$:

$$\mathcal{U}_j(\mathcal{X} - \mathcal{Q}_t) \leq 0.$$

Let us define $\mathbf{P}_t^j = -\mathbf{U}_j$, which gives a solution for condition 2 of Theorem 2. Given the above, we obtain that, for all $(\mathbf{x}^j, \mathbf{X}) \in V^j(y_t^j)$,

$$-\mathbf{p}_t(\mathbf{x}^j - \mathbf{q}_t^j) - \mathbf{P}_t^j(\mathbf{X} - \mathbf{Q}) \leq 0.$$

Now, consider an observation v such that $y_v^j \geq y_t^j$. As t is rationalizable, we have that $(\mathbf{q}_v^j, \mathbf{Q}_v) \in V^j(y_t^j)$. As such, we obtain

$$\mathbf{p}_t\mathbf{q}_v^j + \mathbf{P}_t^j\mathbf{Q}_v \geq \mathbf{p}_t\mathbf{q}_t^j + \mathbf{P}_t^j\mathbf{Q}_t \text{ whenever } y_v^j \geq y_t^j.$$

This shows the first part of condition 3 of Theorem 2 (or equivalently the first condition of SACM).

The second condition part of condition 3 of Theorem 2 can be established by using continuity of f^j (the proof is similar to the one of Theorem 2 in Varian (1984)). In particular, let $y_v^j > y_t^j$, which implies that $(\mathbf{q}_v^j, \mathbf{Q}_v) \in V^j(y_t^j)$. By continuity and strict monotonicity of f^j, there exists a $\theta < 1$, such that $(\theta\mathbf{q}_v^j, \theta\mathbf{Q}_v) \in V^j(y_t^j)$, and therefore

$$\mathbf{p}_t\mathbf{q}_t^j + \mathbf{P}_t\mathbf{Q}_t \leq \mathbf{p}_t\theta\mathbf{q}_v^j + \mathbf{P}_t\theta\mathbf{Q}_v < \mathbf{p}_t\mathbf{q}_v^j + \mathbf{P}_t\mathbf{Q}_v.$$

(sufficiency) Let us fix the observation t. We proceed by constructing for every output j, a production function f^j which will rationalize the data.

Towards this end, consider the output j. For every observation $v \in T - \{t\}$, let C_v^j be the convex hull of all vectors $(\mathbf{q}_s^j, \mathbf{Q}_s)$ with $y_s^j \geq y_v^j$. We denote by R^j the collection of all observations $v \in T - \{t\}$ for which $(\mathbf{q}_v^j, \mathbf{Q}_v)$ is not in the interior of C_v^j.

[9] See, for example Rockafellar (1970, p. 283)

Fact 3. *For each of the elements $v \in R^j$, there exist vectors $\mathbf{w}^j \in \mathbb{R}^N_{++}$ and $\mathbf{W}^j \in \mathbb{R}^M_{++}$ such that*

$$\begin{aligned} \mathbf{w}^j_v \mathbf{q}^j_v + \mathbf{W}^j_v \mathbf{Q}_v &\leq \mathbf{w}^j_v \mathbf{q}^j_z + \mathbf{W}^j_v \mathbf{Q}_z \qquad (\forall z \in T \textit{ with } y^j_z \geq y^j_v), \\ \mathbf{w}^j_v \mathbf{q}^j_v + \mathbf{W}^j_v \mathbf{Q}_v &< \mathbf{w}^j_v \mathbf{q}^j_z + \mathbf{W}^j_v \mathbf{Q}_z \qquad (\forall z \in T \textit{ with } y^j_z > y^j_v). \end{aligned}$$

Proof. This follows from the separating hyperplane theorem and the fact that the production functions are strictly increasing and continuous. □

Now, we construct an artificial data set K^j such that

- the observation $\{\mathbf{p}_t, \mathbf{P}^j_t, \mathbf{q}^j_t, \mathbf{Q}_t\}$ is in K^j,
- for all $v \in R^j$, the observation $\{\mathbf{w}^j_v, \mathbf{W}^j_v, \mathbf{q}^j_v, \mathbf{Q}_v\}$ is in K^j.

Fact 4. *The data set K^j satisfies the generalized axiom of revealed preference (GARP).*[10]

Proof. This follows from the fact that K^j satisfies SACM for all observations (if t is rationalizable) and the fact that this is a stronger condition than GARP. □

By Afriat's Theorem (Afriat, 1967), we have that there exist nonnegative numbers U^j_t, U^j_v ($v \in R^j$) and strict positive numbers λ^j_t, λ^j_v ($v \in R^j$) such that, for all $v, s \in R^j$,

$$\begin{aligned} U^j_v - U^j_s &\leq \lambda^j_s \left[\mathbf{w}^j_s(\mathbf{q}^j_v - \mathbf{q}^j_s) + \mathbf{W}^j_s(\mathbf{Q}_v - \mathbf{Q}_s)\right], \\ U^j_t - U^j_v &\leq \lambda^j_v \left[\mathbf{w}^j_v(\mathbf{q}^j_t - \mathbf{q}^j_v) + \mathbf{W}^j_v(\mathbf{Q}_t - \mathbf{Q}_v)\right], \\ U^j_v - U^j_t &\leq \lambda^t_t \left[\mathbf{p}_t(\mathbf{q}^j_v - \mathbf{q}^j_t) + \mathbf{P}^j_t(\mathbf{Q}_v - \mathbf{Q}_t)\right]. \end{aligned}$$

Furthermore, it is possible to impose that $y^j_v < y^j_s$ if and only if $U^j_v < U^j_s$ (for all $v, s \in R^j \cup \{t\}$). As such, we can plot the values of U^j_s against the corresponding values of y^j_s ($s \in R^j \cup \{t\}$) in a graph, and connect the dots. We call this function g^j (i.e. for all $s \in R_t \cup \{t\}$, $y^j_s = g^j(U^j_s)$). The function g^j is strictly increasing and, therefore, one to one and invertible. Let h^j be the inverse of g^j.

Now, we construct the function $U^j(\mathbf{x}^j, \mathbf{X})$ defined by

$$U^j(\mathbf{x}^j, \mathbf{X}) = \min \left\{ \begin{array}{c} \min_{v \in R^j} \{U^j_v + \lambda^j_v \left[\mathbf{w}_v(\mathbf{x}^j - \mathbf{q}^j_v) + \mathbf{W}^j_v(\mathbf{X} - \mathbf{Q}_v)\right]\}; \\ U^j_t + \lambda^j_t \left[\mathbf{p}_t(\mathbf{x}^j - \mathbf{q}^j_t) + \mathbf{P}^j_t(\mathbf{X} - \mathbf{Q}_t)\right] \end{array} \right\}.$$

This function is concave (as it is the minimum of a finite set of linear functions), strictly increasing, and satisfies $U^j(\mathbf{q}^j_s, \mathbf{Q}_s) = U^j_s$ (for all $s \in R^j \cup \{t\}$). Given the function $U^j(.)$, we define the production function $f^j(.)$ by

$$f^j(\mathbf{x}^j, \mathbf{X}) = g^j(U^j(\mathbf{x}^j, \mathbf{X})).$$

Observe that for all $s \in R^j \cup \{t\}$, $f^j(\mathbf{q}^j_s, \mathbf{Q}_s) = y^j_s$, and that the function f^j is continuous, strictly monotonic, and quasi-concave (as it is a strictly monotonic transformation of a strictly monotonic and concave function).

[10] See, for example, Varian (1982) for a discussion of GARP.

We can repeat the above procedure for every output $j = 1, \ldots, J$, creating the functions $f^1, \ldots, f^J$. Let us show that observation t solves **OP-CM** for these production functions. We do this ad absurdum. Specifically, we assume that there exist inputs $\mathbf{x}^j$ and $\mathbf{X}$ such that $\sum_j \mathbf{p}_t \mathbf{x}^j + \mathbf{P}_t \mathbf{X} < \sum_j \mathbf{p}_t \mathbf{q}_t^j + \mathbf{P}_t \mathbf{Q}_t$ and, for all $j \leq J$, $f^j(\mathbf{x}^j, \mathbf{X}) \geq y_t^j$. Then,

$$
\begin{aligned}
\sum_j \frac{1}{\lambda_t^j} h^j\left(f^j(\mathbf{x}^j, \mathbf{X})\right) &= \sum_j \frac{1}{\lambda_t^j} U^j(\mathbf{x}^j, \mathbf{X}) \\
&\leq \sum_j \frac{1}{\lambda_t^j} U_t^j + \sum_j \left[\mathbf{p}_t(\mathbf{x}^j - \mathbf{q}_t^j) + \mathbf{P}_t^j(\mathbf{X} - \mathbf{Q}_t)\right] \\
&< \sum_j \frac{1}{\lambda_t^j} U_t^j.
\end{aligned}
$$

By the pigeonhole principle, we see that for at least one $j \leq J$, it must be the case that $U^j(\mathbf{x}^j, \mathbf{X}) < U_t^j$. This implies $f^j(\mathbf{x}^j, \mathbf{X}) < y_t^j$, which gives the desired contradiction.

The only thing we still need to establish is that, for all $v \in T$, $f^j(\mathbf{q}_v^j, \mathbf{Q}_v) \geq y_v^j$. If $v \in R^j \cup \{t\}$, we have that $f^j(\mathbf{q}_v^j, \mathbf{Q}_v) = y_v^j$, by construction. On the other hand, if $v \notin R^j \cup \{t\}$, then $(\mathbf{q}_v^j, \mathbf{Q}_v)$ can be written as the convex combination of vectors $(\mathbf{q}_s^j, \mathbf{Q}_s)$ for which $y_s^j \geq y_v^j$. In fact, we can restrict ourselves to observations s in R^j. As such, we have that for all these observations s, $f^j(\mathbf{q}_s^j, \mathbf{Q}_s) = y_s^j \geq y_v^j$. The result $f^j(\mathbf{q}_v^j, \mathbf{Q}_v) \geq y_v^j$ follows from quasi-concavity of the function f^j.

Proof of Theorem 3

(necessity) As a preliminary step, we note that the problem **OP-NM** can be rewritten as

$$\min_{\mathbf{x}^j, \mathbf{X}^j} \mathbf{p}_t \mathbf{x}^j + \mathbf{P}^j \mathbf{X} \qquad \text{s.t. } (\mathbf{x}_j, \mathbf{X}) \in V(y_t^j), \text{ and } \mathbf{X} \geq \sum_{k \neq j} \mathbf{Q}_t^k.$$

Analogous to the proof of Theorem 2, consider the space $\Omega_+ = \mathbb{R}^{N+M_+}$, with typical element $\mathcal{X}$ given as

$$\mathcal{X} = [\mathbf{x}^{j\prime} \ \ \mathbf{X}']'.$$

We denote by $\mathcal{Q}_t^j$, which contains the solutions of **OP-NM**, the vector

$$\mathcal{Q}_t^j = [\mathbf{q}_t^{j\prime} \ \ \mathbf{Q}_t']'.$$

and by $\mathcal{P}$ the vector

$$\mathcal{P} = [\mathbf{p} \ \ \mathbf{P}].$$

As before, let $C(\mathbf{a}|S)$ be the normal cone of a convex set S at the point $\mathbf{a} \in S$. Consider, then, the following set:

$$\tilde{W}^j = \left\{ \mathcal{X} \in \Omega_+ \,\middle|\, \mathbf{X} \geq \sum_{k \neq j} \mathbf{Q}_t^k \right\}.$$

Fact 5. *The set $\tilde{W}^j$ is convex.*

Now, consider the normal cone $C(\mathcal{Q}_t|\tilde{W}^j)$. We obtain the following fact about its elements:

Fact 6. *Let $\mathcal{R}_j \in C(\mathcal{Q}_t|\tilde{W}^j)$, with*

$$\mathcal{R}_j = [\mathbf{r}_j^j \;\; \mathbf{R}_j].$$

Then

- $\mathbf{r}_j^j = \mathbf{0}$,
- *if* $(\mathbf{Q}_t)_m > \sum_{k\neq j}(\mathbf{Q}_t^k)_m$, *then* $(\mathbf{R}_j)_m = 0$,
- *if* $(\mathbf{Q}_t)_m = \sum_{k\neq j}(\mathbf{Q}_t^k)_m$, *then* $(\mathbf{R}_j)_m \leq 0$.

Proof. Let $\mathcal{X}$ be the vector in Ω_+ which equals $\mathcal{Q}_t^j$ except for the element $(\mathbf{x}^j)_m$ with

$$(\mathbf{x}^j)_m = (\mathbf{q}_t^j)_m + \delta.$$

Here we take $\delta \in]-\varepsilon, \varepsilon[$ for some small $\varepsilon > 0$. We see that $\mathcal{X} \in \tilde{W}^j$. Then, if $\mathcal{R}_j$ is in the normal cone of $\tilde{W}^j$ at $\mathcal{Q}_t^j$, it follows that

$$\begin{aligned} \mathcal{R}_j\mathcal{X} &\leq \mathcal{R}_j\mathcal{Q}_t^j \\ \Leftrightarrow (\mathbf{r}_j^j)(\mathbf{x}^j)_m &\leq (\mathbf{r}_j^j)_m(\mathbf{q}_t^j)_m \\ &= (\mathbf{r}_j^j)_m\left((\mathbf{x}^j)_m - \delta\right). \end{aligned}$$

This must hold for all δ in the interval, and hence, $(\mathbf{u}_j^j)_m = 0$. As m was arbitrarily chosen, we must have that $\mathbf{r}_j^j = \mathbf{0}$. If $(\mathbf{Q}_t)_m > \sum_{k\neq j}(\mathbf{Q}_t^k)_m$, we can use a similar reasoning to show that $(\mathbf{R}_j)_m = 0$.

Then, consider the case where $(\mathbf{Q}_t)_m = \sum_{k\neq j}(\mathbf{Q}_t^k)_m$ and assume that the vector $\mathcal{X}$ equals $\mathcal{Q}_t^j$ except for the element $(\mathbf{X})_m$, which is given by

$$(\mathbf{X})_m = (\mathbf{Q}_t)_m + \delta.$$

Here, we have to take $\delta > 0$ to guarantee that the vector $\mathcal{X}$ is in $\tilde{W}^j$. Then, if $\mathcal{R}_j$ is in the normal cone of $\tilde{W}^j$ at $\mathcal{Q}_t^j$, it follows that

$$(\mathbf{R}_j)_m(\mathbf{X})_m \leq (\mathbf{R}_j)_m(\mathbf{Q}_t)_m = (\mathbf{R}_j)_m\left((\mathbf{X})_m - \delta\right).$$

This can only be the case when $(\mathbf{R}_j)_m \leq 0$. □

Fact 7. *Let $\mathcal{U}_j \in C(\mathcal{Q}_t|V^j(y_t^j)$, with*

$$\mathcal{U}_j = [\mathbf{u}_j^j, \mathbf{U}_j].$$

Then

- $\mathbf{u}_j^j \leq \mathbf{0}$,
- $\mathbf{U}_j \leq \mathbf{0}$.

Proof. The proof of this is very similar to the proof of Fact 6. $\square$

The optimization problem can be written as

$$\min_{\mathcal{X} \in \Omega_+} \mathcal{P}\mathcal{X} \text{ s.t. } \mathcal{X} \in V^j(y_t^j) \text{ and } \mathcal{X} \in \tilde{W}^j.$$

Again using Rockafellar (1970), we have that a necessary and sufficient condition for a solution of this problem is that there exist vectors $\mathcal{U}_j \in C(\mathcal{Q}_t^j | V^j(y_t^j))$ and $\mathcal{R}_j \in C(\mathcal{Q}_t^j | \tilde{W}^j)$ such that

$$\mathbf{0} = \mathcal{P} + \mathcal{U}_j + \mathcal{R}_j.$$

Thus, we get

$$\begin{aligned} -\mathbf{u}_j^j &= \mathbf{p}_t, \\ -\mathbf{U}_j &= \mathbf{P}_t + \mathbf{R}_j. \end{aligned}$$

Let us define $\mathfrak{P}_t^j = -\mathbf{U}_j \geq \mathbf{0}$, which gives a solution for condition 2 of Theorem 3. As $\mathcal{U}_j$ is in the normal cone of $V^j(y_t^j)$ at $\mathcal{Q}_t^j$, it must be that, for all $(\mathbf{x^j}, \mathbf{X}) \in V(y_t^j)$,

$$\mathbf{p}_t(\mathbf{x}^j - \mathbf{q}_t^j) + \mathfrak{P}_t^j(\mathbf{X} - \mathbf{Q}_t) \geq 0.$$

Now, if $y_v^j \geq y_t^j$, it must be that $(\mathbf{q}_v^j, \mathbf{Q}_v) \in V^j(y_t^j)$ and, therefore,

$$\mathbf{p}_t(\mathbf{q}_v^j - \mathbf{q}_t^j) + \mathfrak{P}_t^j(\mathbf{Q}_v - \mathbf{Q}_t) \geq 0.$$

This shows the first part of condition 3 of Theorem 3 (or equivalently the first condition of SACM). The second part of condition 3 of Theorem 3 can be established by using continuity of f^j (just like for Theorem 2).

Also, because $(\mathbf{Q}_t)_m > 0$ for all m, it must be that there is at least one j such that $(\mathbf{Q}_t)_m > \sum_{j \neq k} (\mathbf{Q}_t^k)_m$, i.e, there must be at least one j such that $(\mathbf{Q}_t^j)_m > 0$. For this j it follows that $(\mathbf{R}_j)_m = 0$ and therefore $(\mathfrak{P}_t^j)_m = (\mathbf{P}_t)_m$. Else, if $\sum_{j \neq k} (\mathbf{Q}_t^k)_m = (\mathbf{Q}_t)_m$, we have that $(\mathbf{R}_j)_m \leq 0$ and therefore, $(\mathfrak{P}_t^j)_m \leq (\mathbf{P}_t)_m$. From this, it follows that

$$\max_j (\mathfrak{P}_t^j)_m = (\mathbf{P}_t)_m.$$

(sufficiency) Fix an observation t. As in the proof of Theorem 3, we construct the set R^j of observations such that, for all $v \in R^j$, there exist vectors $\mathbf{w}^j \in \mathbb{R}_{++}^N$ and $\mathbf{W}^j \in \mathbb{R}_{++}^M$ that yield

$$\begin{aligned} \mathbf{w}_v^j \mathbf{q}_v^j + \mathbf{W}_v^j \mathbf{Q}_v &\leq \mathbf{w}_v^j \mathbf{q}_z^j + \mathbf{W}_v^j \mathbf{Q}_z \qquad (\forall z \in T | y_z^j \geq y_v^j), \\ \mathbf{w}_v^j \mathbf{q}_v^j + \mathbf{W}_v^j \mathbf{Q}_v &< \mathbf{w}_v^j \mathbf{q}_z^j + \mathbf{W}_v^j \mathbf{Q}_z \qquad (\forall z \in T | y_z^j > y_v^j). \end{aligned}$$

Next, we construct the artificial data set K^j such that

- $\{\mathbf{p}_t, \mathfrak{P}_t, \mathbf{q}_t^j, Q_t\}$ is in K^j,
- for all $v \in R^j$, $\{\mathbf{w}_v, \mathbf{W}_v, \mathbf{q}_v, \mathbf{Q}_v\}$ is in K^j.

Fact 8. *The data set K^j satisfies GARP.*

Next, we can apply Afriat's Theorem to obtain that there exist nonnegative numbers U_t^j, U_v^j $(v \in R^j)$ and strict positive numbers λ_t^j, λ_v^j $(v \in R^j)$ such that, for all $v, s \in R^j$,

$$\begin{aligned}
U_v^j - U_s^j &\leq \lambda_s^j \left[\mathbf{w}_s(\mathbf{q}_v^j - \mathbf{q}_s^j) + \mathbf{W}_s^j(\mathbf{Q}_v - \mathbf{Q}_s)\right], \\
U_t^j - U_v^j &\leq \lambda_v^j \left[\mathbf{w}_v(\mathbf{q}_t^j - \mathbf{q}_v^j) + \mathbf{W}_v^j(\mathbf{Q}_t - \mathbf{Q}_v)\right], \\
U_v^j - U_t^j &\leq \lambda_t^t \left[\mathbf{p}_t(\mathbf{q}_v^j - \mathbf{q}_t^j) + \mathfrak{P}_t^j(\mathbf{Q}_v - \mathbf{Q}_t)\right].
\end{aligned}$$

We can plot the corresponding values of U_s^j against y_s^j $(s \in R^j \cup \{t\})$ in a graph and connect the dots, calling this function g^j (i.e. for all $s \in R^j \cup \{t\}$, $y_s^j = g^j(U_s^j)$). This function is strictly increasing and, therefore, one to one and invertible. Let h^j be the inverse of g^j.

Now, for each $j \leq J$ consider the function $U^j(\mathbf{q}^j, \mathbf{Q})$ defined by

$$U^j(\mathbf{q}^j, \mathbf{Q}) = \min \left\{ \begin{array}{c} \min_{v \in R^j} \{U_v^j + \lambda_v^j [\mathbf{w}_v(\mathbf{q}^j - \mathbf{q}_v^j) + \mathbf{W}_v^j(\mathbf{Q} - \mathbf{Q}_v)]\}; \\ U_t^j + \lambda_t^j \left[\mathbf{p}_t(\mathbf{q}^j - \mathbf{q}_t^j) + \mathfrak{P}_t^j(\mathbf{Q} - \mathbf{Q}_t)\right] \end{array} \right\}.$$

This functions is concave, strictly increasing, and satisfies $U^j(\mathbf{q}_s^j, \mathbf{Q}_s) = U_s^j$ for all $s \in T^j \cup \{t\}$. Then, define $f^j(\mathbf{q}^j, \mathbf{Q}) = g^j(U^j(\mathbf{q}^j, \mathbf{Q})))$. For all $s \in T^j \cup \{t\}$ we have $f^j(\mathbf{q}_s^j, \mathbf{Q}_s) = y_s^j$, and the function f^j is continuous, strictly monotonic, and quasi-concave. We can repeat this procedure for all outputs j, so obtaining the functions $f^1, \ldots, f^J$.

Next, let us show that observation t solves **OP-NM**. For all $j \leq J$, if $(\mathfrak{P}_t^j)_m < (\mathbf{P}_t)_m$, we set $(\mathbf{Q}_t^j)_m = 0$ and if $(\mathfrak{P}_t^j)_m = (\mathbf{P}_t)_m$ we set $(\mathbf{Q}_t^j)_m$ arbitrarily under the restriction that $\sum_j (\mathbf{Q}_t^j)_m = (\mathbf{Q}_t)_m$.

We prove the wanted result ad absurdum. Specifically, we assume that there exist inputs $\mathbf{x}^j$ and $\mathbf{X}^j$ such that $\mathbf{p}_t\mathbf{x}^j + \mathbf{P}_t\mathbf{X}^j < \mathbf{p}_t\mathbf{q}_t^j + \mathbf{P}_t\mathbf{Q}_t^j$ and $f^j(\mathbf{x}^j, \mathbf{X}^j + \sum_{k \neq j} \mathbf{Q}_t^k) \geq y_t^j$.

Observe that our construction is such that $\mathfrak{P}_t^j(\mathbf{X}^j - \mathbf{Q}_t^j) \leq \mathbf{P}_t(\mathbf{X}^j - \mathbf{Q}_t^j)$. We have that:

$$\begin{aligned}
\frac{1}{\lambda_t^j} h^j \left(f^j \left(\mathbf{x}^j, \mathbf{X}^j + \sum_{k \neq j} \mathbf{Q}_t^k \right) \right) &= \frac{1}{\lambda_t^j} U^j \left(\mathbf{x}^j, \mathbf{X}^j + \sum_{k \neq j} \mathbf{Q}_t^k \right) \\
&\leq \frac{1}{\lambda_t^j} U_t^j + \left[\mathbf{p}_t(\mathbf{x}^j - \mathbf{q}_t^j) + \mathfrak{P}_t^j \left(\mathbf{X}^j + \sum_{k \neq j} \mathbf{Q}_t^k - \mathbf{Q}_t \right) \right] \\
&= \frac{1}{\lambda_t^j} U_t^j + \mathbf{p}_t(\mathbf{x}^j - \mathbf{q}_t^j) + \mathfrak{P}_t(\mathbf{X}^j - \mathbf{Q}_t^j) \\
&\leq \frac{1}{\lambda_t^j} U_t^j + \mathbf{p}_t(\mathbf{x}^j - \mathbf{q}_t^j) + \mathbf{P}_t(\mathbf{X}^j - \mathbf{Q}_t^j) \\
&< \frac{1}{\lambda_t^j} U_t^j.
\end{aligned}$$

This implies $f^j(\mathbf{x}^j, \mathbf{X}^j + \sum_{k\neq j} \mathbf{Q}_t^k) < y_t^j$, a contradiction.

The only thing we still need to establish is that, for all $v \in T$, $f^j(\mathbf{q}_v^j, \mathbf{Q}_v) \geq y_v^j$. If $v \in R^j \cup \{t\}$, we have $f^j(\mathbf{q}_v^j, \mathbf{Q}_v) = y_v^j$ by construction. On the other hand, if $v \notin R^j \cup \{t\}$, then $(\mathbf{q}_v^j, \mathbf{Q}_v)$ can be written as the convex combination of vectors $(\mathbf{q}_s^j, \mathbf{Q}_s)$ in R^j (for which $f^j(\mathbf{q}_s^j, \mathbf{Q}_s) = y_s^j \geq y_v^j$). The result $f^j(\mathbf{q}_v^j, \mathbf{Q}_v) \geq y_v^j$ follows from quasi-concavity of the function f^j.