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MODELING UNCERTAINTY IN THE CONTEXT OF FINITE ELEMENT MODELS WITH DISTANCE-BASED INTERPOLATION

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Abstract. *The engineer is often confronted with a very limited knowledge about the involved uncertainty when dealing with engineering problems. A suitable model for the uncertainty is rarely available. In this paper we present a user-friendly method to model and visualize uncertainty in an environment where information about this uncertainty is limited and comes from measurements at specific points of the model under study or from background knowledge, possibly in the form of linguistic terms. It is shown that distance-based interpolation is an essential tool to achieve this goal, as opposed to traditional interpolation methods based on the Euclidean distance measure. A distance measure is developed that is more consistent than the Euclidean distance in the context of finite element models.*

Keywords. *finite elements, uncertainty, interval, interpolation, visualization*

1 INTRODUCTION

1.1 Purpose of the paper

The main purpose of this paper is to show how uncertainty related to ambiguous information can be modeled in a user-friendly and consistent way in the context of finite element models. It is demonstrated that the technique of distance-based interpolation, a concept described below, plays a crucial role in achieving this goal.
After having modeled the uncertainty, it remains to present this information to the user. Visualization is an important tool to this end. Although not the main topic of this paper, some issues concerning the visualization of uncertainty and its relation to the modeling part are also considered in this paper.
The following sections provide some background to the topics discussed in this article.

1.2 Modeling uncertainty

The finite element method is a numerical analysis technique for obtaining approximate solutions to a wide variety of engineering problems (Huebner et al., 2001). The basic idea is to consider the solution region as built up of many small, interconnected subregions or elements. These elements are interconnected at points called nodes. Thus one assumes that a solution region can be analytically modeled or approximated by replacing it with an assemblage of discrete elements. Since these elements can be put together in a variety of ways, they can be used to represent exceedingly complex shapes.

Uncertainty is an unavoidable aspect of every reasonable model, so it is crucial to represent it in a proper way. Traditionally, probability theory is considered as the key to deal with uncertainties, a theory which dates back to the seventeenth century with de Méré problems about gambling. Only recently a number of non-probabilistic approaches has been invented and are still in full development.
This paper focuses on modeling uncertainty in applications from applied mechanics, such as a cantilever beam subject to loading. More specifically, we consider models in which the uncertainty is due to ambiguity, which is very common for applied mechanics models. Ambiguity is related to the degree to which an event occurs, such as when an engineer wants to construct a cantilever beam that can withstand a 'large' loading. The uncertainty involved in the loading can be described as epistemic uncertainty, since the amount of uncertainty is not reduced by increasing the amount of information about the loading. Notice the difference with uncertainty due to randomness, where the central question is not about the degree to which an event occurs but *whether* it occurs, as for example in the experiment where a player has to draw one ball from an urn containing red and black balls. The uncertainty about the color of the ball the player will draw, completely disappears after the ball has been drawn. This is in contrast to the uncertainty involved in using the term 'large' above, where the uncertainty is due to the fact that the engineer is unable to map this linguistic term to a specific real number. Such a mapping is even undesired, since different customers have their own interpretation of the term 'large' and the construction should be robust enough to satisfy most, if not all, customers. Consequently, one has argued that non-probabilistic methods should be preferred when dealing with epistemic uncertainty (Kosko, 1990).
Two useful techniques to model epistemic uncertainty are the interval method and fuzzy logic.

The interval method is a very simple way to model uncertainty. The considered variable is assumed to have a value within the interval and it holds that no value is more likely or more true than any other value within the interval. This

concept is very appropriate when the distribution of the uncertainty is completely unknown and only bounds on the values of the considered variable are available. Despite its simplicity, the interval concept has been successfully applied to a wide variety of nontrivial problems (Barker and Rocco S., 2011; Chaturvedi, Prasad and Ranjan, 2006; Muhanna and Mullen, 2001).
Fuzzy logic was introduced in 1965 (Zadeh, 1965) to describe linguistic and, therefore, incomplete information in an intuitive way. Fuzzy logic theory provides the transition from linguistics to mathematics using the concept of fuzzy membership function. Consider for example the temperature of a certain liquid. The current temperature can be described by the linguistic terms 'cold', 'warm' and 'hot', for example. These linguistic terms are then represented by membership functions μ_c, μ_w and μ_h, such that for a given temperature t it holds that $0 \leq \mu_c(t), \mu_w(t), \mu_h(t) \leq 1$. That is, the expression that a given temperature corresponds to, e.g., warm has a certain truth value between 0 and 1. Contrary to the interval concept, the fuzzy concept allows that some values are considered to be more true than other values. The membership functions are chosen by the user, in terms of the considered application. Typically one uses monotonic, triangular, trapezoidal or bell-shaped (e.g. Gaussian, sigmoidal) membership functions. For a membership function μ of this kind an equivalent representation is given by $\{(\alpha, [a_\alpha, b_\alpha]) \,|\, 0 \leq \alpha \leq 1\}$ where $[a_\alpha, b_\alpha] = \{x \in X \,|\, \mu(x) \geq \alpha\}$ with X representing the set of possible values for the considered variable. This provides a relationship between the fuzzy and the interval concept, and this connection can then be exploited to deal with fuzzy uncertainties in terms of interval uncertainties (Moens and Vandepitte, 2005).
Because fuzzy uncertainties can be expressed in terms of interval uncertainties, we only consider the last kind of uncertainties. However, the principles we describe below are equally well applicable to the fuzzy concept.

The interval and the fuzzy concept can be used in the context of finite element models by defining for each element (or for each node) an interval or a fuzzy membership function that defines the uncertainty of the considered variable at that element (or that node). This is only meaningful for variables who vary spatially over the model under study, such as a load on a cantilever beam that varies in intensity. However, if the physical object under study has the form of, say, a disk and the considered variable is its radius, then the considered variable has no local meaning, but only a global meaning, i.e. it is meaningless to model ambiguous information about it by defining the uncertainty for each element or node. The main topic of this paper is to model and visualize the variation of spatial uncertainty. Since this is only meaningful for 'local variables', such as the mentioned varying load on a cantilever beam, we focus on this kind of variables. Global variables, i.e. variables that do not vary spatially, are not considered here.
Although the finite element method provides an intuitive way to model varying uncertainty, by defining the uncertainty for each element or node, the practical application of this idea is more challenging. It would be user-unfriendly to require the user to define, in a non-automatic way, a suitable interval or membership function for each element. Below we propose a user-friendly method that defines the uncertainty for each element in an automated way, taking into account the limited information the user has about the uncertainty for the considered variable.

1.3 Visualizing uncertainty

Although traditionally research has always been directed to the *modeling* of uncertainty, the advent of computers has made the *visualization* of uncertainty to a research topic of equal importance. The purpose of visualizing uncertainty is to make the user aware of the locations and degrees of uncertainties in his data. Possibly due to the fact that visualization research is in its infancy, there is a lot of controversy about how to optimally visualize data sets (Pang, Wittenbrink and Lodha, 1996). This paper does not take a position in this debate. The focus here is on the connection between modeling and visualizing uncertainty, in such a way that even when the user has a very limited knowledge about the uncertainty that is present in the data set, this does not prevent the automated modeling and visualization of it. Connecting modeling and visualization also means that a continuous feedback between modeling and visualization should be possible. That is, if the user analyzes the visualization and concludes that the distribution of uncertainty shows deficiencies, for example the distribution of uncertainty is physically impossible in some region, he should be allowed to reconsider his current knowledge about the uncertainty and quickly obtain another visualization after having adjusted the uncertainty in that region. Thus visualization is an important aid in understanding the data set and the associated uncertainties.

2 OUTLINE OF THE PAPER

In section 3 the use of the interval method is demonstrated by means of a practical problem in car industry. In section 4 the notion of distance is considered in the context of finite element models. It is demonstrated that the widely used Euclidean distance measure fails to adequately describe the distance between two nodes or elements in a finite element model. An alternative distance measure that is more consistent in this respect is proposed. In section 5 we discuss the topic of interpolation. Section 5.1 argues the importance of interpolation when modeling uncertainty with the interval method. Section 5.2 shows the relationship between distance and interpolation, thereby providing the link between sections 4 and 5. It is emphasized that traditional interpolation techniques, such as polynomial interpolation, assumes that distance is measured according to the Euclidean distance measure, implying that these techniques are not reliable to model uncertainty with the interval method in finite element models. The alternative is to use distance-based interpolation, described in section 5.3.
Sections 6 and 7 provide two case studies. The first case study demonstrates the use of the interval method and distance-based interpolation, using the distance measure defined in section 4. The second case study compares two existing distance-based interpolation methods in approximating some chosen functions. The merits of both methods are compared.

3 THE INTERVAL METHOD

As described higher (see section 1.2), the interval method has been used in a variety of nontrivial problems. Here we present an interesting application that emphasizes the use of the interval method.
In Fig. 1 we show a section of a trimmed car FE model. The model consists of 371800 nodes and 324694 elements. In car industry it is preferred that design is validated *before* the start of the production, i.e. already in the design phase, since the involved costs are then much lower than the costs related to testing in the actual production stage. Testing in this early phase of the process is done in a virtual design environment, by making use of computer simulations. The available data come from background knowledge about previous developments, since many new vehicles brought to market are actually enhancements or incremental developments of legacy vehicles. On the other hand, uncertainty is unavoidably present in this very early phase of the development process.
Now it is interesting to do an analysis of the structural vibration as a vertical force is applied at the front right suspension mount and a vertical displacement output in the middle of the driver's seat. The analysis is based on a modal analysis taking into account 100 modes and is carried out using the MSC.Nastran solver. The frequency response function is calculated for a range from 10 to 50 Hz.
The uncertainty considered here affects the roof panel. One possibility to apply the interval method is to use the Free-Form Deformation (FFD) method (Sederberg and Parry, 1986). This method allows to define a control box in the form of a parallelepiped to delimit the uncertainty zone, as shown in Fig. 2(a). On this control box a grid of control points is specified. Any control point can be selected and displaced using the mouse in a computer environment to deform the control box. As the control box is deformed, the complete part of the model lying inside this box is automatically deformed too, according to the deformation function as defined in (Sederberg and Parry, 1986). For example, the control point in the middle of the shown control box can be selected and displaced to represent the uncertainty about the height of the roof at that control point. That is, by displacing such a control point, an interval for the height of the roof at that point is defined, with minimum value equal to the original, nominal height, and with maximum value equal to the height after displacement of the point. For all other points within the control box an interval is automatically defined according to the deformation function. In Fig. 2(b) we show the FE model after displacing the control point in the middle of the control box by a small amount. The difference with the original model is small, but visible at close inspection.
An analysis of the input uncertainty can then be done by, for example, constructing a Kriging response surface, and subsequently applying optimization and anti-optimization on this response surface model to yield the bounds on the considered output quantity. The details of this analysis are not of interest here, since the focus of the paper is on *defining* uncertainty. We limit ourselves by presenting the results and referring the interested reader to (De Munck, 2009) and (Verhaeghe et al., 2011) for details about the performed analysis method. The bounds on the frequency response function are shown in Fig. 3. The influence of the uncertain geometry in a limited number of frequency zones is clear.

It is clear that the FFD method is a very practical method to define interval uncertainty. However, the freedom given to the user to define the uncertainty is limited, since one has to define a control box and it is only within this control box that the variable is considered to be uncertain. Of course, the complete model can be placed inside the control box, but this is often not meaningful, since, e.g., in the considered case the uncertainty on a window on the left side has no relation to the uncertainty on a wheel on the right side. An alternative is to define several control boxes for different areas of the model, but this creates discontinuities and, more importantly, the fixed structure of the control box, i.e. in the form of a parallelepiped, makes it impossible to delimit an arbitrary subpart of the model.
Below we propose a more flexible method to define interval uncertainty, where the user can freely define intervals at arbitrarily selected nodes. The intervals at other points are then defined via interpolation, as outlined in section 5. In using interpolation to define the uncertainty at these other points, the concept of distance plays a crucial role, as is motivated in the next section.

4 THE CONCEPT OF DISTANCE IN FINITE ELEMENT MODELS

To specify how close two points in space are, one needs to define a distance measure. The most common distance measure is, of course, the Euclidean distance between two vectors, which is defined as $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ for given vectors $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_n\}$.
However, we argue that this distance measure is not suitable in the context of finite element models. To illustrate this, we consider a cast component, which is studied in more detail in section 6 as a case study. For now it is sufficient to state that the cast component under study is an axisymmetric object, implying that we can equally well limit the study to an arbitrary 2D cross-section of it, as shown in Fig. 4. The model of the considered part of the cast component is described with the finite element method, containing 208 elements.
Now consider the two points indicated by A and B in the figure. According to Euclidean distance, the distance between A and B equals the length of the line connecting them, denoted as path 1. However, none of the points on this line belong to the topology. On the other hand, if a variable is analyzed that varies over the physical object under study, it varies *only* over the topology. Thus if one wants to analyze and plot the variation of such a variable when going from A to B, one shall prefer to plot the values of this variable for the points lying on path 2. Path 1 is without meaning for the considered variable. This also implies that the shortest path between A and B should be defined as path 2.
This motivates us to measure distance between two points of a finite element model as the smallest length of all paths P, for which it holds that P connects both points and for which all points belonging to P also belong to the model under study.
However, this definition cannot be used directly in practice, since a finite element model can take virtually any form. A

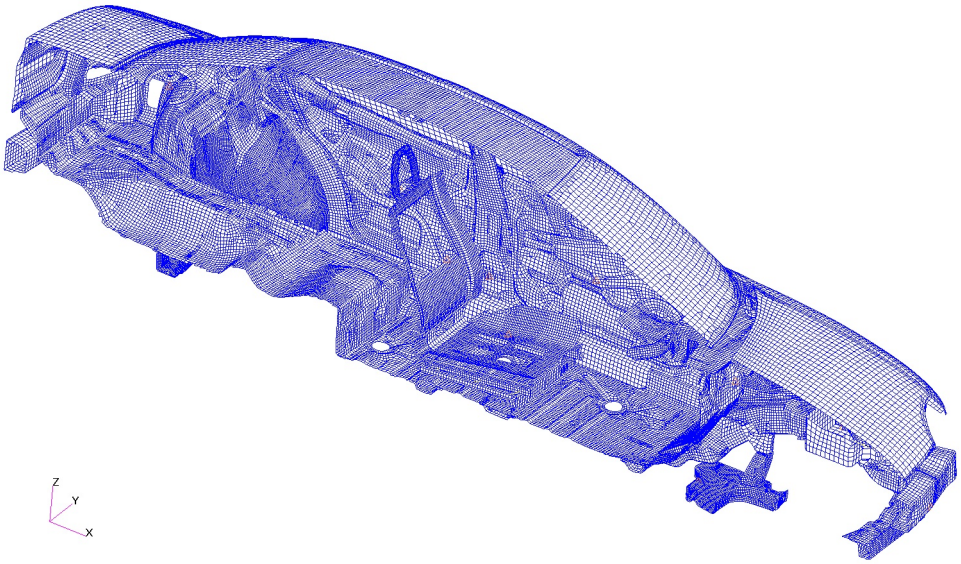
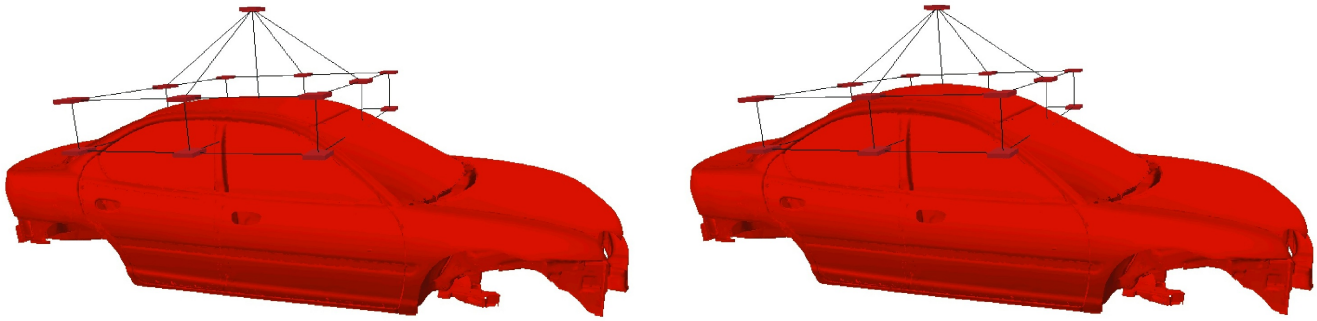


Figure 1: A section of the trimmed car FE model



(a) Roof nodal coordinates at nominal value

(b) Roof nodal coordinates at maximal value

Figure 2: Car FE model

simple, but useful approximation in the context of finite element models is to define the length of the path connecting two nodes or two elements in terms of the number of elements through which the considered path runs. Thus if two elements are neighboring elements, the distance between them is defined as, for example, 1. If a certain element can be reached from another element via one other element, the distance between them is 2, and so on. If distance is measured between points, the same reasoning applies.
Notice that this approximation has some deficiencies, e.g. that the size of the elements that are part of the finite element model is not necessarily constant over the model. It is easily seen that the size of the elements influences the distance, since, for example, a square element can be divided into two triangle elements, all other things being equal, changing the distance between certain elements, although nothing fundamental has changed. Further research is needed to define the length of a path in a more sophisticated manner. For now the practical definition just given is used, i.e. the distance between two nodes or elements in a finite element model is the smallest number of elements that has to be crossed in going from one node or element to the other. Although simple, this definition makes certainly sense and it has the advantage of being intuitive.

5 INTERPOLATION

5.1 The use of interpolation in finite element models

The most common form of interpolation assumes that positions in space are given, denoted as $\mathbf{x} = \{x_1, \ldots, x_n\}$. The value of a function f at some points $\mathbf{x_1}, \ldots, \mathbf{x_m}$ is known, and the interpolation problem is then defined as the problem of finding a good approximation for the values at other points by using these known values.
A popular interpolation method is polynomial interpolation. If $n = 1$ the value of f at some unknown point x is approximated by $\sum_{i=0}^{N} a_i x^i$, where the coefficients a_i are determined from the known values of f at the points $x_1, \ldots, x_m$. Throughout the paper we assume, for convenience, that $f(\mathbf{x}) \in \mathbb{R}$.
Interpolation is a useful tool in modeling uncertainty with the interval method in the context of finite element models, and this in two respects. First, it ensures user-friendliness. Instead of requiring from the user to define the uncertainty interval for each element of the model, it suffices that the user provides an interval for an adequate number of elements. The intervals at other elements can then be defined via interpolation. Secondly, thanks to interpolation it is not necessary

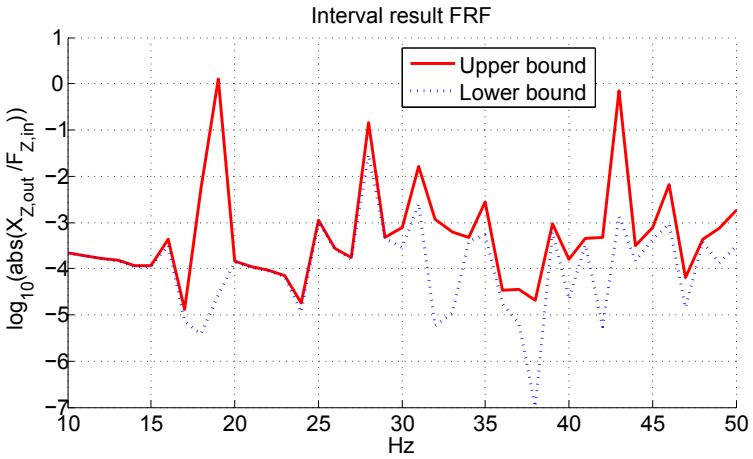


Figure 3: Bounds on the frequency response function of the car FE model

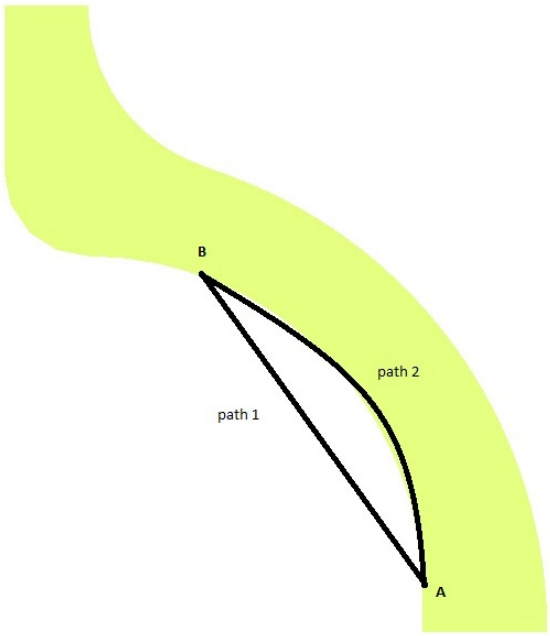


Figure 4: Distance in finite element models

that the user knows detailed information about the uncertainty that is present in the model. Interpolation can be used to approximate the missing information, since it can be used to define the uncertainty intervals at elements for which no suitable interval is known, based on the uncertainty intervals at the other, 'known', elements.

5.2 Relation between distance and interpolation

We now consider the interpolation problem in more detail and analyze its relation to the concept of distance. Polynomial interpolation is taken as the archetype of interpolation. In this way we avoid to have to do the analysis in a too general manner, which would lead us too far outside the scope of this paper. For simplicity we assume that the points at which f is known are real, denoted as $x_1,\ldots,x_m$.
Given the value of f at the known points $x_1,\ldots,x_m$, the problem is then to approximate the value of f at unknown points $S\setminus\{x_1,\ldots,x_m\}$ with a function of the form $\sum_{i=0}^{N} a_i x^i$, with $S\subseteq\mathbb{R}$, typically an interval.
This problem can be solved as follows. Define the following matrix:

$$A=\begin{pmatrix} 1 & x_1 & \ldots & x_1^N \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_m & \ldots & x_m^N \end{pmatrix} \tag{1}$$

and the vectors

$$v=\begin{pmatrix} a_0 \\ a_1 \\ \ldots \\ a_N \end{pmatrix}, \quad p=\begin{pmatrix} f(x_1) \\ f(x_2) \\ \ldots \\ f(x_m) \end{pmatrix} \tag{2}$$

Determining the coefficients a_i from the values at the known points is then equivalent to solving the equation $Av = p$. It is assumed that $m > N$, to ensure that the system is not underdetermined, i.e. to ensure that the number of known points is large enough to well determine the coefficients. If $m > N+1$ an exact solution is not guaranteed, since in this case A is not invertible. This follows easily from one of the fundamental theorems of linear algebra which, in this case, reads as $\dim Ker(A) + \dim range(A) = N+1$, with 'dim' referring to the dimension of the considered subspace, $Ker(A) = \{y \in \mathbb{R}^{N+1} \,|\, Ay = 0\}$ and $range(A) = \{Ay \,|\, y \in \mathbb{R}^{N+1}\}$. Since $\dim Ker(A) \geq 0$, it follows that $\dim range(A) \leq N+1$. Since $p \in \mathbb{R}^m$, it is thus clear that the existence of a v for which $Av = p$, is not assured.
To find the best approximate solution, one has to define first the meaning of the term 'best'. This starts by extending the vector spaces $(\mathbb{R}^{N+1}, \mathbb{R})$ and $(\mathbb{R}^m, \mathbb{R})$ with an inner product $< .,. >$ such that $< v_1, v_2 > \in \mathbb{R}$ with $v_1, v_2 \in \mathbb{R}^{N+1}$ or $v_1, v_2 \in \mathbb{R}^m$ and such that the inner product obeys the axioms of symmetry, linearity and positive definiteness. Given an inner product, the norm of a vector v is then defined as $||v|| = \sqrt{< v, v >}$. From this norm, the definition of the distance d between two vectors v_1 and v_2 follows: $d(v_1, v_2) = ||v_1 - v_2||$.
The best approximate solution v to the above equation thus minimizes $d(Av, p) = ||Av - p||$ or, equivalently, minimizes $||Av - p||^2 =< Av - p, Av - p >$. It is easily shown that the vector $u \in range(A)$ that is closest to p is given by the orthogonal projection of p on $range(A)$. Thus it follows that $Av = u$.
Now, denote with $range(A)^\perp$ the orthogonal complement of $range(A)$. Then it is well known that there exists a unique $u^\perp \in range(A)^\perp$ such that $p = u + u^\perp$ and $< u, u^\perp >= 0$.
To find v we consider an arbitrary $y \in \mathbb{R}^{n+1}$. Then $Ay \in range(A)$ and thus, since it holds that $u^\perp \in range(A)^\perp$, we find that $< u^\perp, Ay >= 0$. This gives the following equivalences:

$$\begin{aligned} & < u^\perp, Ay > &=& \ 0 \\ \Leftrightarrow \quad & < p - u, Ay > &=& \ 0 \\ \Leftrightarrow \quad & < p - Av, Ay > &=& \ 0 \\ \Leftrightarrow \quad & < A^T(p - Av), y > &=& \ 0 \end{aligned}$$

where the last line follows from the definition of the adjoint of a matrix which, in case this matrix represents a mapping between real vector spaces, equals the transpose of it. The last line holds for all y, thus also for $y = A^T(p - Av)$, implying that $\left|\left|A^T(p - Av)\right|\right| = 0$. By the positive definiteness of the inner product it follows that $A^T(p - Av) = 0$. Thus v can be found from the new equation $A^T Av = A^T p$. If $\dim Ker(A) = 0$, then $A^T A$ is invertible, and there is a unique solution for v given by $v = (A^T A)^{-1} A^T p$.

These considerations show that the solution to the interpolation problem is dependent on the definition of a distance measure, although this is not immediately obvious. Indeed, we are not used to relate the interpolation problem to the concept of distance. In practice, the interpolation problem is simply solved by first constructing the matrix A, given the known points, as in Eq. (1), and defining the vector p, as in Eq. (2). The best solution for the coefficients a_i, as represented by the vector v, is then given by $v = (A^T A)^{-1} A^T p$, provided that $\dim Ker(A) = 0$. Thus it seems that no distance measure is involved.
However, in defining the matrix A as in Eq. (1) and solving the interpolation problem as $v = (A^T A)^{-1} A^T p$ one chooses the Euclidean distance implicitly as distance measure. First of all, the fact that the adjoint of A equals its transpose, for a linear mapping between real vector spaces, only holds if A is expressed with respect to an orthonormal basis. Such a basis can only be defined after choosing an inner product. Furthermore, A can only be taken as in Eq. (1) if the coefficient vectors of these basis vectors equal the basis vectors themselves. This is only true if the basis is chosen as $\{e_1, \ldots, e_{N+1}\}$ with $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$, the 1 being at the ith place. These basis vectors are orthonormal with respect to the Euclidean inner product, defined as $< (v_1, \ldots, v_{N+1})^T, (w_1, \ldots, w_{N+1})^T >= \sum_i v_i w_i$ with $v_i, w_i \in \mathbb{R}$. In short, defining A as in Eq. (1) and solving the interpolation problem as $v = (A^T A)^{-1} A^T p$ implies choosing the Euclidean distance as distance measure.

Of course, one is free to choose any other inner product than the Euclidean one, but this creates the problem of finding orthonormal basis vectors with respect to this inner product. Another problem is the need for an analytical expression for the inner product. Such an expression is missing for the distance we defined in section 4. The solution to this problem is to avoid using an interpolation method that defines values in terms of locations in space. Instead, we can define the values at unknown points in terms of the *distance* to the known points, thereby using directly our definition of distance measure from section 4.
To be more specific, we remember that polynomial interpolation determines the value at an unknown point x as $\sum_i a_i x^i$. More generally we can write that the value at x is determined as $g(x, f(x_1), \ldots, f(x_m)) = h(x, x_1, \ldots, x_m)$ for suitable functions g and h. Our purpose is then to do interpolation using a function $h(d(x, x_1), \ldots, d(x, x_m))$, where d represents any distance measure. Interpolating in this way allows to choose any suitable distance measure without the need to define an inner product in an analytical way.
In section 5.3 we describe two existing, useful examples of interpolation functions that are explicitly based on the notion of distance.

5.3 Distance-based interpolation

5.3.1 Inverse distance weighting

Inverse distance weighting (IDW) interpolation defines $h(d(x,x_1),\ldots,d(x,x_m))$ as

$$h(d(x,x_1),\ldots,d(x,x_m)) = \sum_{i=1}^{m} \frac{d(x,x_i)^{-p}}{\sum_{j=1}^{m} d(x,x_j)^{-p}} f(x_i) \tag{3}$$

with $p \geq 0$, and with $x, x_1, \ldots, x_m$ denoting real numbers or vectors in Euclidean space. However, the bold notation used in section 5.1 to denote vectors is dropped from now on, for convenience.
Thus at each unknown point the value is determined as a weighted combination of the known values, where the weights decrease with increasing distance. Having chosen a value for p and a definition for the distance measure d, the values at the unknown points are uniquely determined from the known values.
The advantage of the inverse distance weighting method is that it is intuitive and easy to implement.

5.3.2 Radial basis functions

A radial basis function (RBF) is a function of the form $\phi(d(x,c))$ where c represents some fixed point that determines the location of the radial basis function, also called the center. In this paper we focus on the Gaussian radial basis function, probably the most commonly used type of radial basis function: $\phi(x,c) = \mathrm{e}^{-\beta d(x,c)^2}$, with $\beta \geq 0$.
Radial basis functions can be seen as a compromise between interpolation methods like polynomial interpolation, i.e. using a function $h(x,x_1,\ldots,x_m)$, and interpolation that is purely distance based, i.e. using a function $h(d(x,x_1),\ldots,d(x,x_m))$. In this case, it is not the distance between an unknown point and each of the known points that is determined, but the distance between a given unknown point and each of the centers of the radial basis functions. Thus one uses a function $h(x_1,\ldots,x_m,d(x,c_1),\ldots,d(x,c_k))$ to do the interpolation. More concretely, the value at an unknown point x is determined as $\sum_{i=1}^{k} w_i \mathrm{e}^{-\beta d(x,c_i)^2}$, where the w_i are determined using the known values $f(x_1),\ldots,f(x_m)$.
The weights w_i can be determined in the same way as for polynomial interpolation, provided that β and $c_1,\ldots,c_k$ are chosen beforehand, by defining

$$A = \begin{pmatrix} \mathrm{e}^{-\beta d(x_1,c_1)^2} & \ldots & \mathrm{e}^{-\beta d(x_1,c_k)^2} \\ \ldots & \ldots & \ldots \\ \mathrm{e}^{-\beta d(x_m,c_1)^2} & \ldots & \mathrm{e}^{-\beta d(x_m,c_k)^2} \end{pmatrix}, \quad v = \begin{pmatrix} w_1 \\ \ldots \\ w_k \end{pmatrix}, \quad p = \begin{pmatrix} f(x_1) \\ \ldots \\ f(x_m) \end{pmatrix} \tag{4}$$

Care should be taken that $m \geq k$, and preferably $m > k$, to avoid the problem of overfitting. This problems occurs when the interpolation function is a very good representation at the known points, but a poor representation at the unknown points. In other words: the function describes the noise rather than the underlying relationship, which often happens when the number of parameters is high relative to the number of observations. This can be more rigorously described in terms of the well-known bias-variance trade-off, see e.g. (Briscoe and Feldman, 2011).
The parameters $\beta, c_1, \ldots, c_k$ can also be optimized instead of being chosen beforehand, but in this case one has to solve a set of nonlinear equations. The technique of neural networks can be used for this purpose, see e.g. (Maglogiannis et al., 2008) and (Idri, Zakrani and Zahi, 2010).

6 CASE STUDY 1

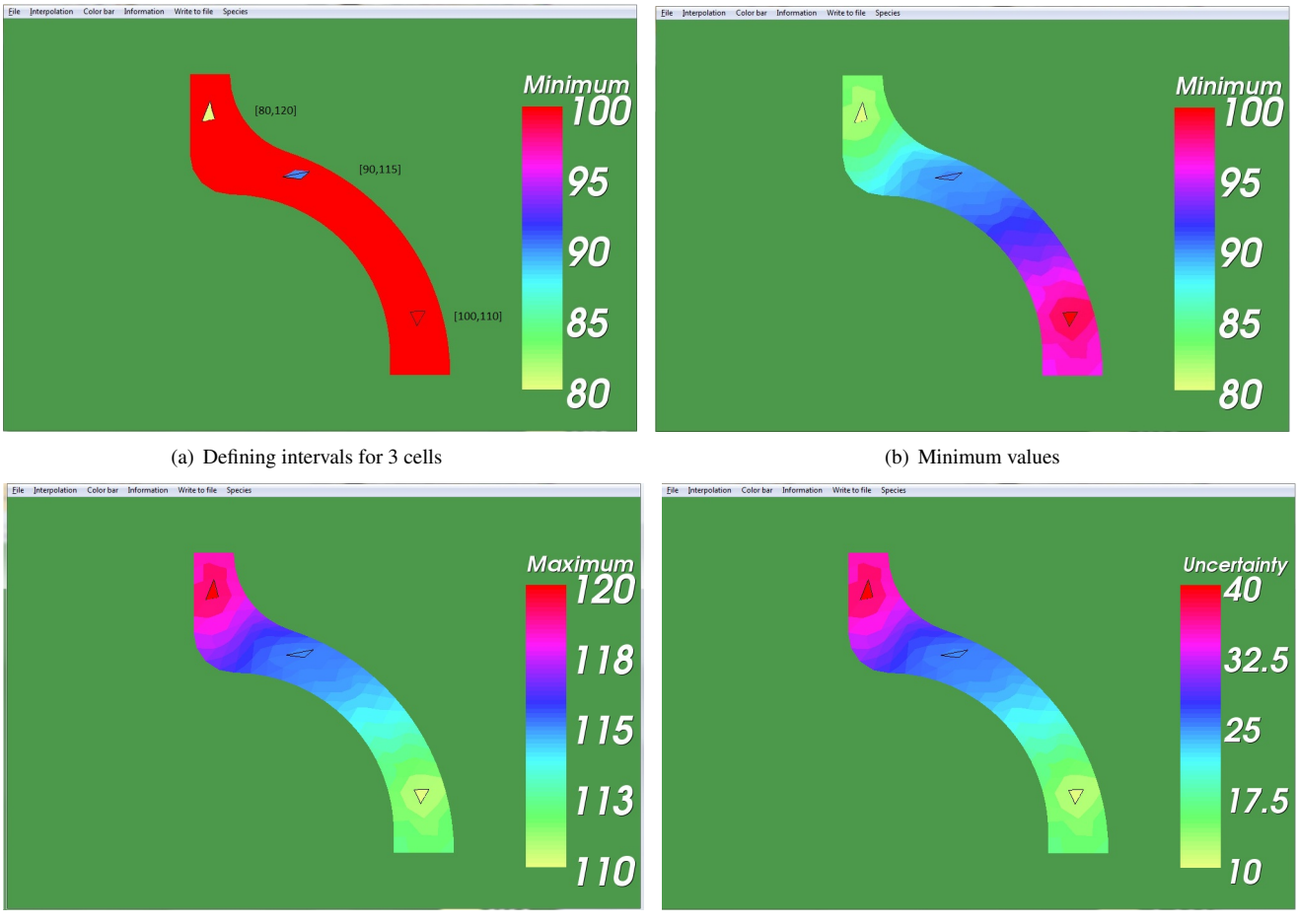
6.1 The physical object

We consider the part of a cast component, introduced in section 4. During manufacturing of this part of the cast component, the cooling process does not manifest itself uniformly over the cast component. This creates uncertainty in the values of the material parameters at each point of the cast component. In this case study we consider Young's modulus, where we know that for the part of the cast component at hand the nominal value is 110 GPa.
It's crucial to do an analysis of the uncertainty of Young's modulus before manufacturing, since this can provide invaluable insight to guide the manufacturing, thereby avoiding unpleasant surprises, e.g. that the structure breaks down due to the fact that the engineer only found it necessary to take the nominal value into account.
We assume that the user is able to determine a suitable interval for the values of the variable of interest in some elements. This is of course a very weak restriction, since it simply implies that the user knows a minimum and maximum bound for the considered variable in at least some elements. The knowledge about these bounds can come from background knowledge or from measurements taken at specific locations, in the same way as measuring the load at some locations of a cantilever beam.
The 'known' elements are defined as those elements for which the user has determined an interval. The 'unknown' elements are all other elements. For simplicity, all intervals are supposed to be closed intervals.

6.2 Modeling the uncertainty

The intervals for the unknown elements are then determined via interpolation. Here we illustrate the use of the inverse distance weighting method, see section 5.3.1. The distance measure is chosen as specified in section 4. The parameter p is simply chosen as $p = 1$.
The method we implemented requires that the user defines initially a temporary interval for Young's modulus that applies to all cells. We have chosen the interval $[100, 120]$, meaning that at this initial stage it is assumed that for each cell Young's

modulus is between 100 and 120.
Now we select three cells, giving as input the intervals $[90, 115]$, $[80, 120]$ and $[100, 110]$. The result of supplying this input is shown in Fig. 5(a). The intervals for the other cells are then defined via the inverse distance weighting method. The result of this interpolation, for the minimum values, is shown in Fig. 5(b), for the maximum values in Fig. 5(c) and for the interval in Fig. 5(d).
These visualizations provide the user insight into the distribution of the minimum and maximum values, and of the uncertainty. For example, it is clear that the top of the considered part of the cast component has the largest uncertainty for Young's modulus.
If new information becomes available, it is very easy to update the visualization by selecting other cells. For example in Fig. 6(a) and Fig. 6(b) the interval $[75, 110]$ is defined for a fourth cell. The figure shows the result for the uncertainty and the minimum after a new interpolation has been done.
Of course, for models in 3D and for a much higher number of known cells, it is much more interesting to analyze the resulting distribution of minimum, maximum and uncertainty. However, a theoretical study of the inferences one can make from a given distribution of the uncertainty is outside the scope of this paper. What is demonstrated here is that the value of Young's modulus indeed varies in accordance with the shape of the physical object, and not with the coordinates of the locations in space, as would be the case with the Euclidean distance measure. Thus we have shown that the distance measure defined in section 4 is much more consistent when applied to finite element models than traditional distance measures, that have their use when applied to points in space, such as the Euclidean distance.



(a) Defining intervals for 3 cells

(b) Minimum values

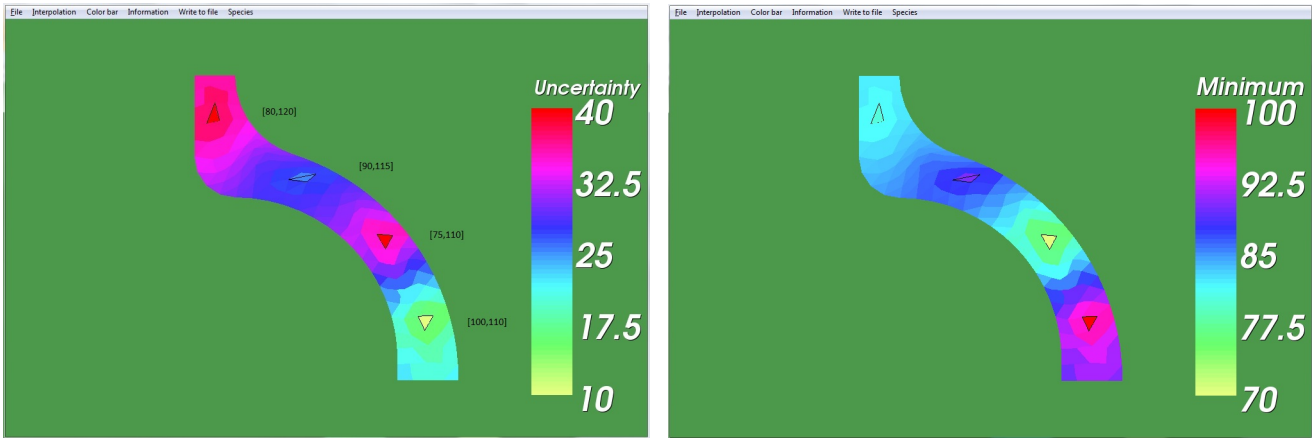
(c) Maximum values

(d) Uncertainty

Figure 5: Interpolation

6.3 Realizations

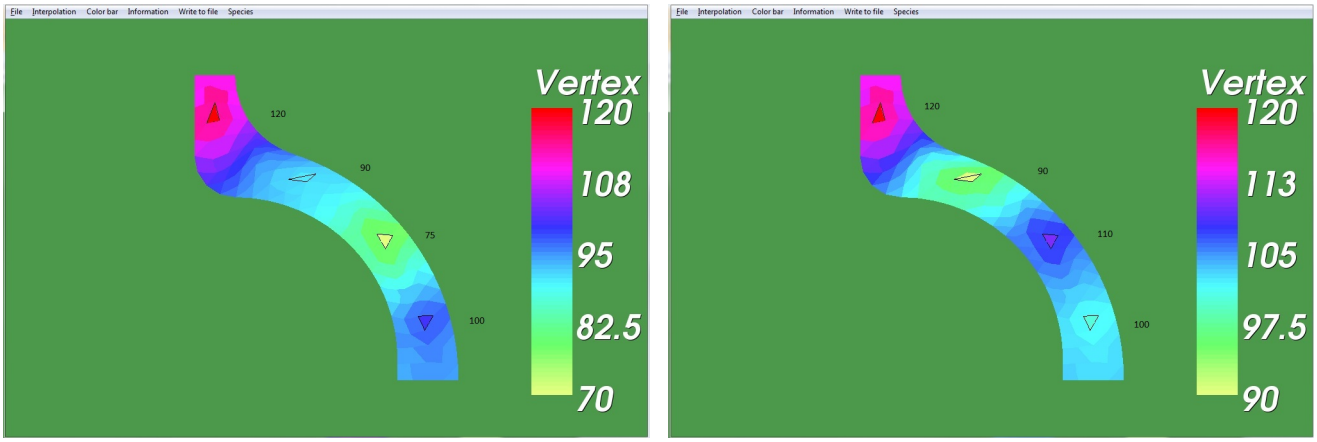
It is instructive to analyze the visualization under the hypothesis that in the known cells specific values are realized, i.e. let Young's modulus take specific values in each of the known cells, perform an interpolation, and visualize the result. Such realizations are commonly called combinations on the vertex of the interval uncertainty.
In Fig. 7(a) we set the value of Young's modulus in each known cell at its minimum value, except for the top cell, for which the value is set at the maximum. An interpolation is done and the result is visualized. In Fig. 7(b) the value of Young's modulus for the third cell, reading from top to bottom, is changed from its minimum to its maximum, while the values at the other cells are unchanged. The visualization of such realizations is very useful to answer 'what happens if...' questions, for example 'what happens if in this region Young's modulus is very high, while in the other regions it is very



(a) Uncertainty (b) Minimum

Figure 6: Adding a fourth known cell

low?'.



(a) (b)

Figure 7: Visualizing specific realizations

7 CASE STUDY 2

In this case study we do some computer experiments to compare the inverse distance weighting method and the radial basis functions method. We consider some functions $f : \mathbb{R}^2 \to \mathbb{R}$ over the domain $[0,2] \times [0,2]$ to be approximated. Some points $K = \{x_1, \ldots, x_m\}$ in the domain are assumed to be known, i.e. it is known that the values at these points are $f(x_1), \ldots, f(x_m)$, while the values at other points are determined via interpolation. For practical applications, it is necessary to consider only a finite subset of the above domain. We consider the subset $D = \{0, 0.01, 0.02, \ldots, 0.09, 2\} \times \{0, 0.01, 0.02, \ldots, 0.09, 2\}$. The unknown points are thus $U = D \setminus K$. The value assigned by the either the inverse distance weighting method or the radial basis function method to an unknown point $u \in U$ is written as $p(u)$. The real, but unknown, value is $f(u)$. The error in the approximation is defined as $E = 1/|U| \sum_{u \in U} (f(u) - p(u))^2$ with $|U|$ denoting the number of elements in U.

Since the use of the distance measure defined in section 4 has been motivated higher, its practical use being demonstrated in the first case study, the experiments below use the Euclidean distance measure, since this distance measure is easier to calculate. Furthermore, since the domain in the current case study is convex, there is no fundamental difference between the Euclidean measure and the distance measure defined in section 4.

Concerning the radial basis function method, the parameter β and the centers are chosen beforehand, since the main focus of the paper is to demonstrate the use of distance based interpolation methods in the context of finite element models, and not to do an extensive comparison between such interpolation methods. A more detailed, and more honest, comparison necessitates making β and the centers part of the optimization process, thereby requiring to solve a system of nonlinear equations. Defining β and the centers beforehand allows us to use basic linear algebra to find the optimal weights, as outlined in sections 5.2 and 5.3.2. Nevertheless, even by defining β and the centers beforehand we are able to derive some meaningful hypotheses concerning the strengths and weaknesses of both interpolation methods (see section 7.5).

7.1 Experiment 1

We consider the function $f(x,y) = x+y$. We choose 5 known points in an arbitrary way: $K = \{(0.2,0.2),(0.3,0.9),(0.4,1.8),(1.5,0),(1.5,1.75)\}$. The function is plotted in Fig. 8, with the known points indicated by black dots. Three RBF's are chosen, with centers $c_1 = (0.75,1), c_2 = (1.25,1.5), c_3 = (1.75,2)$.
The errors E, as defined above, for both interpolation methods are shown in Fig. 9. The errors are plotted for varying p for inverse distance weighting, and for varying β for the radial basis function method. It is clearly seen that the error reaches a minimum for both methods: for the IDW method this minimum is somewhere between 2 and 3, while for the RBF method there are several minima at very small, although larger than zero, β. It is also noticed that the minimum error for the RBF method is 0, while the minimum error for the IDW method is around 0.08.

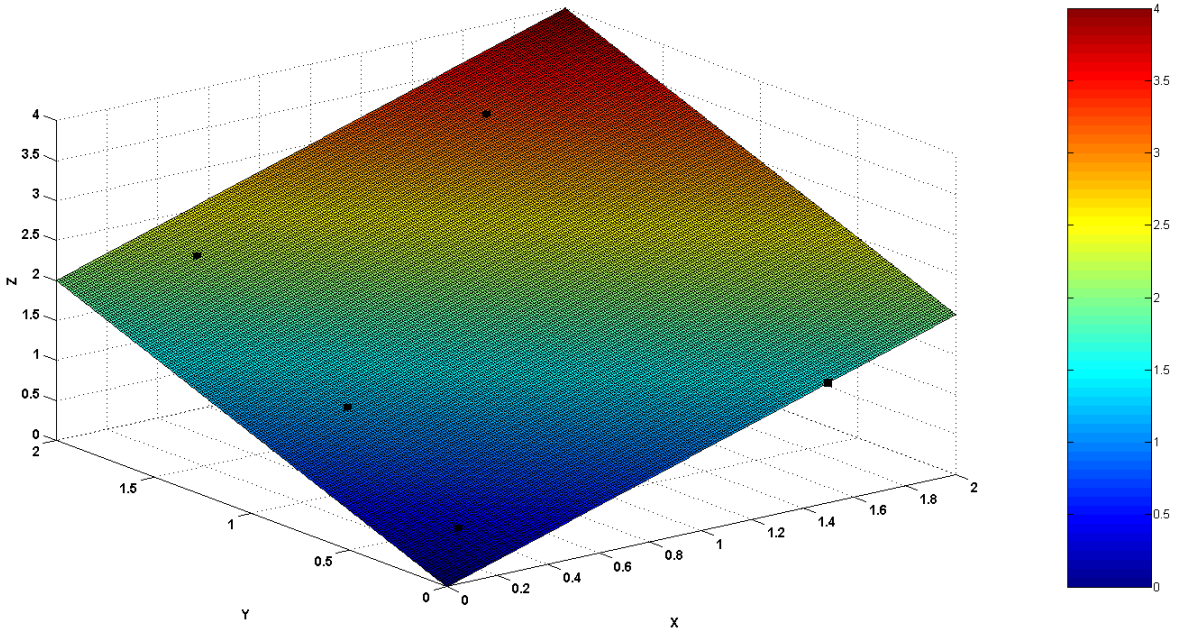
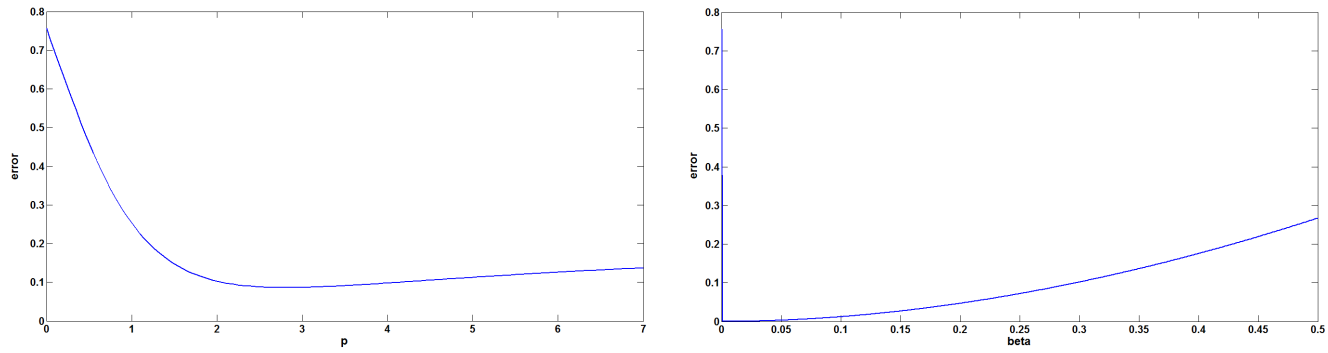


Figure 8: Locations of known points for experiment 1



(a) Inverse distance weighting

(b) Radial basis function

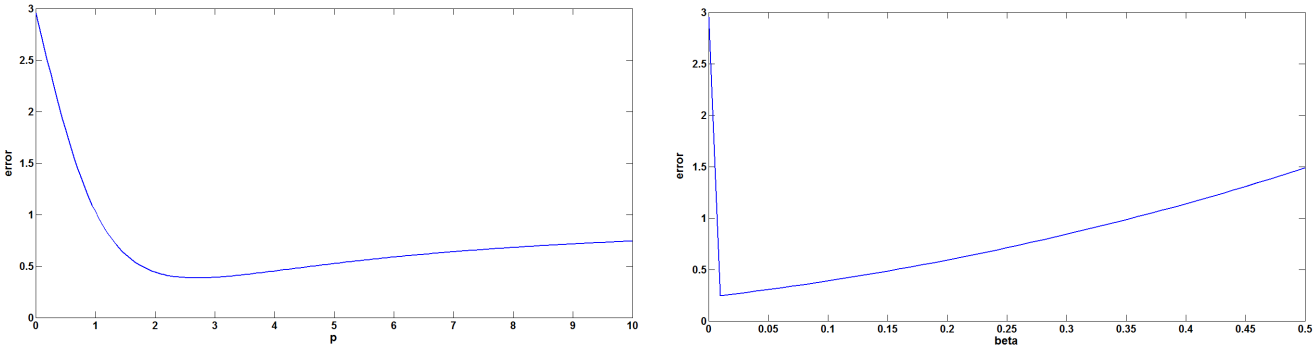
Figure 9: Errors for experiment 1, $f(x,y) = x+y$

7.2 Experiment 2

In this experiment we take $f(x,y) = x^2 + y^2$. The known points and the centers of the RBF's are the same as in the first experiment. The errors are shown in Fig. 10. The graphs are rather similar to the ones shown in Fig. 9. The minimum of the errors for the IWD method and the RBF method are approximately 0.39 and 0.25 resp.

7.3 Experiment 3

In this experiment we take $f(x,y) = x^3 + y^3$. The number of known points is now chosen as 21, with the x-coordinate varying from 0 to 2, in steps of 0.1, and the y-coordinate arbitrary chosen, so that the points are more or less uniformly distributed over the domain. The positions of the known points are shown in Fig. 11.
The errors for the IDW method are shown in Fig. 12(a). The minimum error is around 0.73. For the RBF method,



(a) Inverse distance weighting

(b) Radial basis function

Figure 10: Errors for experiment 2, $f(x,y) = x^2 + y^2$

we consider three different configurations for the positions of the RBF's. The first configuration consists of the same 3 RBF's as in the previous experiments. The corresponding errors are shown in Fig. 12(b), with the minimum error around 1.26. The second configuration defines 5 RBF's, with the x-coordinate of the positions varying in steps of 0.5 and the y-coordinate arbitrarily chosen: $c_1 = (0,0.3), c_2 = (0.5,1.8), c_3 = (1,1), c_4 = (1.5,0.4), c_5 = (2,1.5)$. The errors are shown in Fig. 12(c), the minimum error is approximately 0.0003. Finally, the third configuration consists of 10 RBF's, with the x-coordinate varying in steps of 0.2 and the y-coordinate again arbitrarily chosen: $c_1 = (0.2,0.2), c_2 = (0.4,1.6), c_3 = (0.6,0.6), c_4 = (0.8,1.4), c_5 = (1,1), c_6 = (1.2,0.2), c_7 = (1.4,1.8), c_8 = (1.6,0.4), c_9 = (1.8,1.6), c_{10} = (2,2)$. The errors are shown in Fig. 12(d), the minimum error is nearly 0.

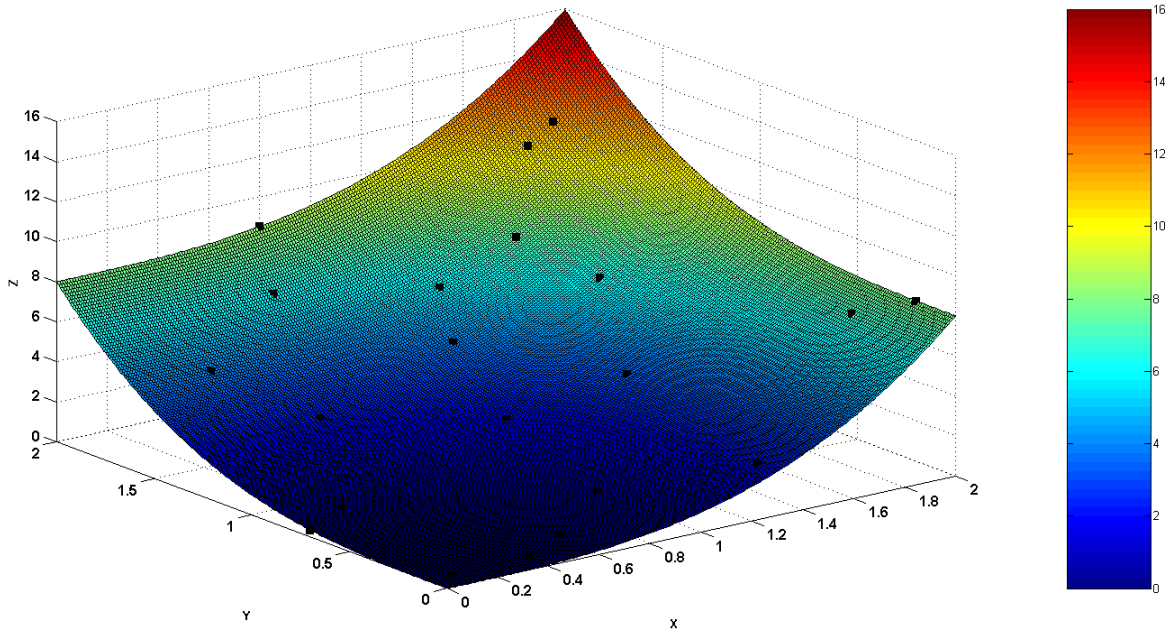
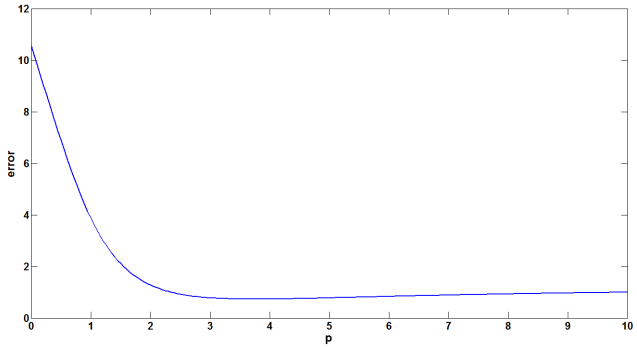


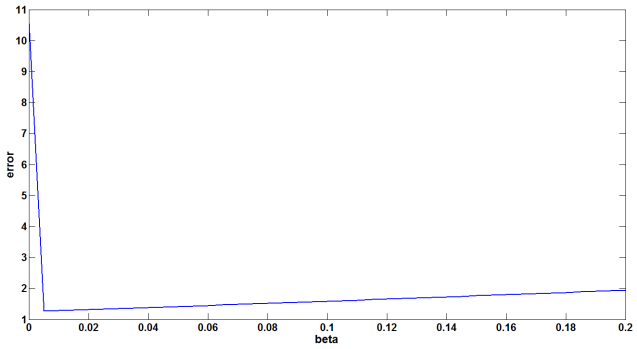
Figure 11: Locations of known points for experiment 3

7.4 Experiment 4

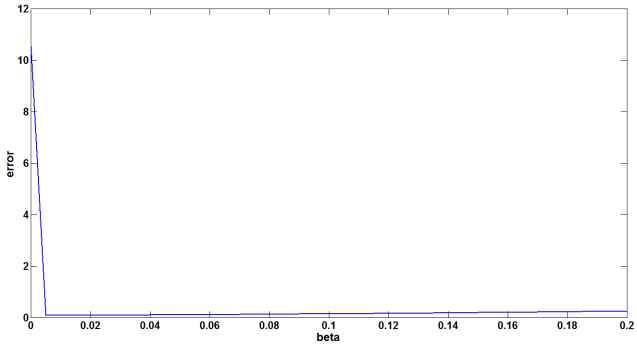
In the final experiment 50 known points are randomly chosen, distributed uniformly over the domain. The function is now chosen as $f(x,y) = x^2 + y^3$. The RBF's are chosen as in the third configuration of experiment 3. The errors are shown in Fig. 13. The minimum error for the IDW method is around 0.3, while the minimum error for the RBF is practically 0.



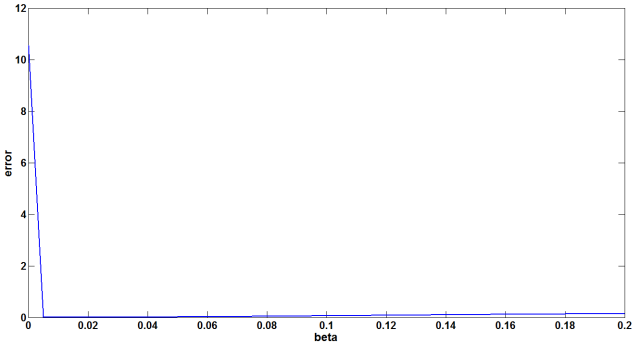
(a) Inverse distance weighting



(b) Radial basis function: 3 RBF's

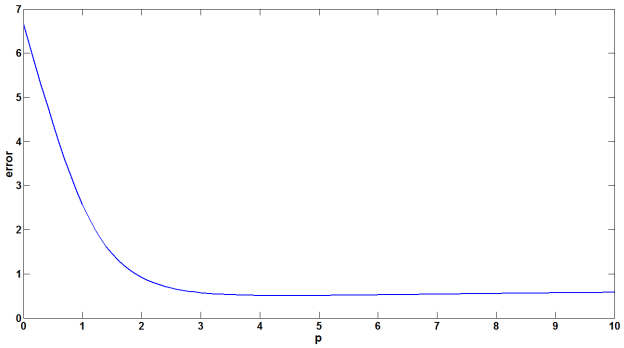


(c) Radial basis function: 5 RBF's

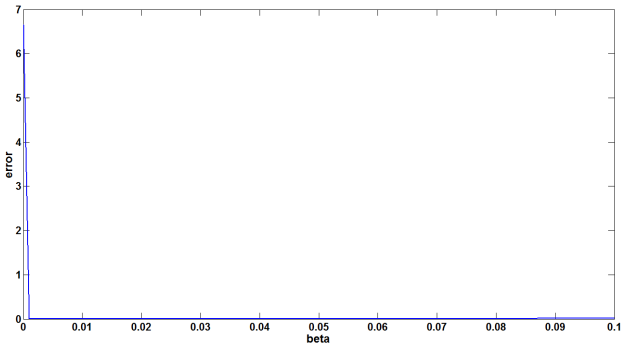


(d) Radial basis function: 10 RBF's

Figure 12: Errors for experiment 3, $f(x,y) = x^3 + y^3$



(a) Inverse distance weighting



(b) Radial basis function

Figure 13: Errors for experiment 4, $f(x,y) = x^2 + y^3$

7.5 Analysis of the experiments

Although the number of experiments is limited, some preliminary conclusions can be drawn. Of course, further research is needed to validate these conclusions.

Optimal value. For all experiments, and for both interpolation methods, there is one optimal value for the distance parameter, i.e. p for the IDW method and β for the RBF method. The optimal value for β is very small, although - of course - larger than zero. The optimal value for p is always somewhere between 2 and 4.

Comparison between IDW and RBF method. For all experiments, the minimum error for the RBF method is lower than for the IDW method, except for experiment 3 in case of the third configuration consisting of 3 RBF's. However, notice that in this case there are only 3 parameters, the three centers, while there are 21 unknown points, so that the system as defined by Eq. (4) is in fact too overdetermined. Furthermore, one can argue that the comparison to the IDW method is not fair here, and thus not meaningful, since the IDW interpolation function has 21 parameters in this case, 7 times more than the IWD interpolation function.
Thus it can be suggested that the RBF method is a more powerful tool for distance based interpolation than the IDW method.
In practice, the real function f is of course not known, and optimal values for p and β are to be determined using non-linear techniques. Here the optimal values were determined in the assumption that f is known, in order to compare both

interpolation methods. The centers of the RBF's, and even the number of RBF's, can also be considered as parameters to be optimized, instead of fixing them beforehand. Also, instead of choosing one β that applies to all RBF's, each RBF can be given its own specific β. This implies that the RBF method offers a much great flexibility than the IDW method. There is one nuance: the more parameters of the RBF method that are chosen to be optimized, the larger the number of known points should be, to avoid the problem of overfitting (see section 5.3.2).

8 CONCLUSION

We described a user-friendly method that allows the user to analyze uncertainty in an environment where little is known about the involved uncertainty. More specifically, no suitable model for the uncertainty is known, and the information about the uncertainty either comes from measurements at specific points of the model or from some background knowledge, possibly under the form of linguistic terms.
The uncertainty is defined at particular nodes or elements of a finite element model via the interval or fuzzy logic method. Via interpolation the uncertainty is defined over the complete model. We showed that traditional interpolation techniques, especially polynomial interpolation, are suitable when distance is interpreted in Euclidean terms, but that for finite element models this distance measure is not consistent. This motivates the choice for distance-based interpolation, that is explicitly depended on the chosen distance measure. In this paper we defined a simple, but useful distance measure in the context of finite element models. Two case studies were presented: the first one demonstrates the practical use of distance-based interpolation, the second one compares two existing distance-based interpolation methods.

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