Construction of normalized B-splines for a family of smooth spline spaces over Powell-Sabin triangulations

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We construct a suitable B-spline representation for a family of bivariate spline functions with smoothness $r \geq 1$ and polynomial degree $3r - 1$. They are defined on a triangulation with Powell-Sabin refinement. The basis functions have a local support, they are nonnegative and they form a partition of unity. The construction involves the determination of triangles that must contain a specific set of points. We further consider a number of CAGD applications. We show how to define control points and tangent control polynomials (of degree $2r - 1$), and we consider the efficient and stable computation of the Bernstein-Bézier form of such splines.

Keywords : smooth Powell-Sabin splines, normalized B-splines, macro-elements, control points, control polynomials, Bernstein-Bézier form

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Abstract

We construct a suitable B-spline representation for a family of bivariate spline functions with smoothness $r \geq 1$ and polynomial degree $3r - 1$. They are defined on a triangulation with Powell-Sabin refinement. The basis functions have a local support, they are nonnegative and they form a partition of unity. The construction involves the determination of triangles that must contain a specific set of points. We further consider a number of CAGD applications. We show how to define control points and tangent control polynomials (of degree $2r - 1$), and we consider the efficient and stable computation of the Bernstein-Bézier form of such splines.

Keywords: smooth Powell-Sabin splines, normalized B-splines, macro-elements, control points, control polynomials, Bernstein-Bézier form

1 Introduction

A number of authors studied the construction of smooth (finite element) spline spaces defined on arbitrary triangulations. Such a spline space usually has the following properties:

- a certain order of global smoothness, and possibly local supersmoothness;
- a dimension which can be simply expressed in terms of geometrically interesting characteristics of the triangulation (like the number of vertices, edges and/or triangles);
- a linearly independent and stable basis;
- full approximation power.

We refer to [3], and references quoted therein, for an overview of such spline spaces. For the construction of smooth splines with a low polynomial degree one often considers triangulations with a particular macro-structure. Each triangle in the triangulation is then split into a number of subtriangles. The Clough-Tocher split (into three subtriangles) and the Powell-Sabin split (into six subtriangles) are commonly used splits. Splines defined on such refined triangulations are referred to as Clough-Tocher splines and Powell-Sabin splines, respectively.

For Powell-Sabin splines with smoothness $r = 1$ and polynomial degree $d = 2$, Dierckx [1] has developed an interesting normalized B-spline representation, in which the locally supported basis functions form a convex partition of unity. The spline coefficients in this representation have a geometric meaning, and allow a natural definition of control triangles which are a useful tool for computer aided geometric design. This normalized B-spline representation has been effective in a

wide range of application areas (see, e.g., [4, 5, 13, 14]). Recently, a basis with similar properties has been constructed for Powell-Sabin splines with smoothness $r = 2$ and degree $d = 5$, see [11], and for (reduced) Clough-Tocher splines with smoothness $r = 1$ and degree $d = 3$, see [12]. In this paper, we will generalize this construction of a normalized spline basis towards a family of spline spaces with general smoothness $r \geq 1$. These spline spaces are defined on triangulations with Powell-Sabin refinement, and have a polynomial degree $d = 3r - 1$.

The paper is organized as follows. In Section 2 we recall some preliminary concepts of bivariate polynomials in Bernstein-Bézier form defined on triangles. In the next section we define a particular family of smooth spline spaces over triangulations of Powell-Sabin's type. Section 4 is devoted to the construction of a normalized basis for these spline spaces. The construction is based on determining a set of triangles that must contain a specific set of points. In Section 5 we define control points and control polynomials which can be useful for modelling purposes. We also show how the Bernstein-Bézier form of such a spline can be computed using only convex combinations. This leads to a stable algorithm for the evaluation of such a spline in its normalized B-spline representation. In Section 6 we consider a family of reduced spline spaces. Finally, we end with some concluding remarks in Section 7.

2 Bivariate polynomials on triangles

Let $\mathbb{P}_d$ denote the linear space of bivariate polynomials of total degree less than or equal to d . In this section we recall the Bernstein-Bézier representation of a polynomial $p_d \in \mathbb{P}_d$, and some properties of the blossom of such a polynomial.

2.1 Bernstein-Bézier representation of polynomials

Let $\mathcal{T}(V_1, V_2, V_3)$ be a non-degenerate triangle. Any point P in the plane of the triangle can be uniquely expressed in terms of the barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ with respect to $\mathcal{T}$, such that

$$P = \sum_{i=1}^3 \tau_i V_i, \quad \text{and} \quad \tau_1 + \tau_2 + \tau_3 = 1. \quad (2.1)$$

Given two points P^1 and P^2 , the barycentric coordinates $\delta = (\delta_0, \delta_1, \delta_3)$ of the vector $P^2 - P^1$ with respect to $\mathcal{T}$ are defined as the difference of the barycentric coordinates of both points. If the Euclidean distance $\|P^2 - P^1\|_2 = 1$, then δ is called a unit barycentric direction.

Any polynomial $p_d \in \mathbb{P}_d$ on triangle $\mathcal{T}$ has a unique Bernstein-Bézier representation

$$p_d(P) = p_d(\tau) = \sum_{i_1+i_2+i_3=d} b_{i_1 i_2 i_3} B_{i_1 i_2 i_3}^d(\tau). \quad (2.2)$$

Here,

$$B_{i_1 i_2 i_3}^d(\tau) = \frac{d!}{i_1! i_2! i_3!} (\tau_1)^{i_1} (\tau_2)^{i_2} (\tau_3)^{i_3} \quad (2.3)$$

are the Bernstein basis polynomials of degree d . The coefficients $b_{i_1 i_2 i_3}$ are called Bézier ordinates. The Bézier domain points $\xi_{i_1 i_2 i_3}$ are defined as the points with barycentric coordinates $(\frac{i_1}{d}, \frac{i_2}{d}, \frac{i_3}{d})$. By associating the Bézier ordinates $b_{i_1 i_2 i_3}$ with the Bézier domain points $\xi_{i_1 i_2 i_3}$, one can display the Bernstein-Bézier representation schematically. The piecewise linear interpolant of the Bézier control points, defined as $\mathbf{b}_{i_1 i_2 i_3} = (\xi_{i_1 i_2 i_3}, b_{i_1 i_2 i_3})$, is called the Bézier control net. This control net is tangent to the polynomial surface at the three vertices.

The disk D_r of radius r around vertex V_1 of $\mathcal{T}$ is the set of domain points defined by

$$D_r(V_1) = \{\xi_{i_1 i_2 i_3} : i_1 \geq d - r\}. \quad (2.4)$$

Given a triangulation Δ , then the disk $D_r(V_1)$ in Δ is defined as the set of all domain points in (2.4) for each triangle in Δ having V_1 as a vertex. The row E_r at distance r parallel to edge $e_{12} = \langle V_1, V_2 \rangle$ in $\mathcal{T}$ is the set of domain points defined by

$$E_r(e_{12}) = \{\xi_{i_1 i_2 i_3} : i_3 = r\}. \quad (2.5)$$

In the sequel, if we refer to a Bézier ordinate in a disk or on a row, then we actually mean a Bézier ordinate whose corresponding domain point is in that location.

Smoothness conditions are often required between adjoining polynomial patches. Consider a second polynomial $\tilde{p}_d$ defined on the domain triangle $\tilde{\mathcal{T}}(V_4, V_3, V_2)$. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the barycentric coordinates of point V_4 with respect to triangle $\mathcal{T}(V_1, V_2, V_3)$. In order to get C^μ -continuity between p_d and $\tilde{p}_d$ across the edge $V_2 - V_3$, the Bézier ordinates $\tilde{b}_{i_1 i_2 i_3}$ of $\tilde{p}_d$ must satisfy the conditions

$$\tilde{b}_{m, i_2, i_3} = \sum_{j_1 + j_2 + j_3 = m} b_{j_1, i_3 + j_2, i_2 + j_3} B_{j_1 j_2 j_3}^m(\sigma), \quad (2.6)$$

for each $m = 0, \dots, \mu$, and $i_2 + i_3 = d - m$. We refer to [2, 3] for more details.

2.2 The blossom of polynomials

The blossom (or polar form) of a polynomial $p_d(\tau)$, denoted by $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d)$, is completely characterized by the following three properties (see, e.g., [7, 10] for more details):

Property 2.1. *The blossom is symmetric with respect to any permutation π of its d arguments,*

$$\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d) = \mathcal{P}_d(\pi(\tau^1, \tau^2, \dots, \tau^d)).$$

Property 2.2. *The blossom is multi-affine, i.e. affine in each of its d arguments. For example,*

$$\mathcal{P}_d(a\tau + b\sigma, \tau^2, \dots, \tau^d) = a\mathcal{P}_d(\tau, \tau^2, \dots, \tau^d) + b\mathcal{P}_d(\sigma, \tau^2, \dots, \tau^d),$$

for $a + b = 1$.

Property 2.3. *The restriction of the blossom to the diagonal gives the polynomial p_d ,*

$$p_d(\tau) = \mathcal{P}_d(\tau, \tau, \dots, \tau).$$

Let $\tau^l = (\tau_1^l, \tau_2^l, \tau_3^l)$ for $l = 1, \dots, d$. One can verify that the blossom of p_d can be written as

$$\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d) = \sum_{i_1 + i_2 + i_3 = d} b_{i_1 i_2 i_3} \sum_{\pi \in \Pi_{i_1 i_2 i_3}^d} \prod_{l=1}^d \tau_{\pi(l)}^l, \quad (2.7)$$

where $b_{i_1 i_2 i_3}$ are the Bézier ordinates of p_d given in (2.2), and $\Pi_{i_1 i_2 i_3}^d$ is the set of all unique permutations of

$$\left(\underbrace{1, \dots, 1}_{i_1 \text{ times}}, \underbrace{2, \dots, 2}_{i_2 \text{ times}}, \underbrace{3, \dots, 3}_{i_3 \text{ times}} \right). \quad (2.8)$$

Note that multi-set (2.8) has $\frac{d!}{i_1! i_2! i_3!}$ unique permutations, and each $\pi \in \Pi_{i_1 i_2 i_3}^d$ consists of d elements.

The multi-affine de Casteljau algorithm can be used for the evaluation of the blossom. The de Casteljau algorithm computes $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d)$ as follows:

$$\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d) = b_{0,0,0}^{[d]}, \quad (2.9a)$$

where

$$b_{i_1, i_2, i_3}^{[0]} = b_{i_1 i_2 i_3}, \quad (2.9b)$$

for $i_1 + i_2 + i_3 = d$, and

$$b_{i_1, i_2, i_3}^{[m]} = \tau_1^m b_{i_1+1, i_2, i_3}^{[m-1]} + \tau_2^m b_{i_1, i_2+1, i_3}^{[m-1]} + \tau_3^m b_{i_1, i_2, i_3+1}^{[m-1]}, \quad (2.9c)$$

for $i_1 + i_2 + i_3 = d - m$, and $m = 1, \dots, d$. This algorithm allows us to evaluate the blossom at points inside the domain triangle in a stable way, using only convex combinations.

One can also use the blossom to find the Bernstein-Bézier form of a polynomial p_d with respect to another (usually finer) triangle. Let $\mathcal{T}(V_1, V_2, V_3)$ be a given triangle, and let $b_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = d$, be the Bézier ordinates of p_d with respect to $\mathcal{T}$. Consider a second triangle, whose vertices W_l , $l = 1, 2, 3$, have $\sigma^l = (\sigma_1^l, \sigma_2^l, \sigma_3^l)$ as barycentric coordinates with respect to the former triangle $\mathcal{T}$. The Bézier ordinates $d_{i_1 i_2 i_3}$ of p_d related to the new triangle can be compactly written as

$$d_{i_1 i_2 i_3} = \mathcal{P}_d \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}} \right), \quad (2.10)$$

for all $i_1 + i_2 + i_3 = d$. If the new triangle is a smaller triangle inside the original domain triangle $\mathcal{T}$, then expression (2.10) is a convex combination of the original ordinates, for any choice $i_1 + i_2 + i_3 = d$. In this way the blossom can be used for subdividing the polynomial onto a triangular partition of $\mathcal{T}$.

Blossoming can also be used to describe directional derivatives of a polynomial (2.2). The m -th order (mixed) directional derivative of polynomial p_d with respect to the unit barycentric directions δ^i , $i = 1, \dots, m$, related to triangle $\mathcal{T}$ can be compactly expressed as

$$D_{\delta^1, \dots, \delta^m}^m p_d(\tau) = \frac{d!}{(d-m)!} \mathcal{P}_d \left(\underbrace{\tau, \tau, \dots, \tau}_{d-m \text{ times}}, \delta^1, \dots, \delta^m \right), \quad (2.11)$$

where the m barycentric directions δ^i and $(d-m)$ times the barycentric coordinates τ are taken as the arguments of the blossom.

We end this section by formulating three theorems.

Theorem 2.1. Let $\tau^l = (\tau_1^l, \tau_2^l, \tau_3^l)$, $l = 1, \dots, d$, be barycentric coordinates with respect to a triangle $\mathcal{T}_1$, and let $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d)$ be the blossom of a given polynomial p_d defined on $\mathcal{T}_1$. The Bézier ordinates of p_d are denoted by $b_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = d$. If these ordinates can be expressed in terms of a blossom $\mathcal{Q}_d$ related to another triangle $\mathcal{T}_2$,

$$b_{i_1 i_2 i_3} = \mathcal{Q}_d \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}} \right), \quad (2.12)$$

where $\sigma^l = (\sigma_1^l, \sigma_2^l, \sigma_3^l)$, $l = 1, 2, 3$, are given barycentric coordinates with respect to $\mathcal{T}_2$, then

$$\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d) = \mathcal{Q}_d \left(\sum_{l=1}^3 \tau_l^1 \sigma^l, \sum_{l=1}^3 \tau_l^2 \sigma^l, \dots, \sum_{l=1}^3 \tau_l^d \sigma^l \right). \quad (2.13)$$

Proof. The blossom $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d)$ can be evaluated using the multi-affine de Casteljau algorithm. By (2.9b) and (2.12) we have

$$b_{i_1, i_2, i_3}^{[0]} = \mathcal{Q}_d \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}} \right),$$

for all $i_1 + i_2 + i_3 = d$. After one step in the de Casteljau algorithm we obtain

$$b_{i_1, i_2, i_3}^{[1]} = \tau_1^1 b_{i_1+1, i_2, i_3}^{[0]} + \tau_2^1 b_{i_1, i_2+1, i_3}^{[0]} + \tau_3^1 b_{i_1, i_2, i_3+1}^{[0]},$$

for all $i_1 + i_2 + i_3 = d - 1$, and using the multi-affinity and symmetry property of blossom $\mathcal{Q}_d$ this can be written as

$$b_{i_1, i_2, i_3}^{[1]} = \mathcal{Q}_d \left(\tau_1^1 \sigma^1 + \tau_2^1 \sigma^2 + \tau_3^1 \sigma^3, \underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}} \right).$$

Repeating the same argument for each step in the de Casteljau algorithm, we get

$$b_{i_1, i_2, i_3}^{[m]} = \mathcal{Q}_d \left(\sum_{l=1}^3 \tau_l^1 \sigma^l, \dots, \sum_{l=1}^3 \tau_l^m \sigma^l, \underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}} \right),$$

for $i_1 + i_2 + i_3 = d - m$ and $m = 1, \dots, d$. Equality (2.13) follows from the last step in the de Casteljau algorithm, $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d) = b_{0,0,0}^{[d]}$. $\square$

Theorem 2.2. Let $\tau^l = (\tau_1^l, \tau_2^l, \tau_3^l)$, $l = 1, \dots, d$, be barycentric coordinates with respect to a triangle $\mathcal{T}_1$, and let $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d)$ be the blossom of a given polynomial p_d defined on $\mathcal{T}_1$. The Bézier ordinates of p_d are denoted by b_{i_1, i_2, i_3} , $i_1 + i_2 + i_3 = d$. Suppose that some of these ordinates can be expressed in terms of a blossom $\mathcal{Q}_d$ related to another triangle $\mathcal{T}_2$:

$$b_{i_1+m, i_2, i_3} = \mathcal{Q}_d \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}}, \rho^1, \rho^2, \dots, \rho^m \right), \quad (2.14)$$

for a given $m < d$ and all $i_1 + i_2 + i_3 = d - m$. Here, $\sigma^l = (\sigma_1^l, \sigma_2^l, \sigma_3^l)$, $l = 1, 2, 3$, and $\rho^l = (\rho_1^l, \rho_2^l, \rho_3^l)$, $l = 1, \dots, m$, are given barycentric coordinates with respect to $\mathcal{T}_2$. Then,

$$\mathcal{P}_d \left(\tau^1, \dots, \tau^{d-m}, \underbrace{e^1, \dots, e^1}_m \right) = \mathcal{Q}_d \left(\sum_{l=1}^3 \tau_l^1 \sigma^l, \dots, \sum_{l=1}^3 \tau_l^{d-m} \sigma^l, \rho^1, \dots, \rho^m \right), \quad (2.15)$$

with $e^1 = (1, 0, 0)$.

Proof. We evaluate blossom $\mathcal{P}_d$ in (2.15) with the aid of the multi-affine de Casteljau algorithm. Because of the symmetry property of the blossom, we are allowed to use the barycentric coordinates e^1 in the first m steps of the algorithm. Then, by (2.9b)–(2.9c) and (2.14) we obtain

$$\begin{aligned} b_{i_1, i_2, i_3}^{[m]} &= b_{i_1+m, i_2, i_3}^{[0]} \\ &= \mathcal{Q}_d \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}}, \rho^1, \dots, \rho^m \right), \end{aligned}$$

for all $i_1 + i_2 + i_3 = d - m$. We can proceed as in the proof of Theorem 2.1 for the remaining barycentric coordinates τ^l , $l = 1, \dots, d - m$. The next step in the de Casteljau algorithm gives us for each $i_1 + i_2 + i_3 = d - m - 1$,

$$\begin{aligned} b_{i_1, i_2, i_3}^{[m+1]} &= \tau_1^1 b_{i_1+m+1, i_2, i_3}^{[0]} + \tau_2^1 b_{i_1+m, i_2+1, i_3}^{[0]} + \tau_3^1 b_{i_1+m, i_2, i_3+1}^{[0]} \\ &= \mathcal{Q}_d \left(\sum_{l=1}^3 \tau_l^1 \sigma^l, \underbrace{\sigma^1, \dots, \sigma^1}_{i_1 \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 \text{ times}}, \rho^1, \dots, \rho^m \right), \end{aligned}$$

using the multi-affinity and symmetry property of the blossom. Repeating this argument for each τ^l , $l = 2, \dots, d - m$, we obtain the equality (2.15). $\square$

Theorem 2.3. Let p_{d_1} be a polynomial of degree d_1 defined on the triangle $\mathcal{T}_1(V_1, V_2, V_3)$, and let p_{d_2} be a polynomial of degree d_2 defined on the triangle $\mathcal{T}_2(V_1, W_2, W_3)$, where $d_2 \leq d_1$ and

$$W_2 = \frac{d_1 - d_2}{d_1} V_1 + \frac{d_2}{d_1} V_2, \quad W_3 = \frac{d_1 - d_2}{d_1} V_1 + \frac{d_2}{d_1} V_3.$$

The Bézier ordinates of p_{d_1} are denoted by $b_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = d_1$, and the Bézier ordinates of p_{d_2} are denoted by $d_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = d_2$. If

$$D_x^a D_y^b p_{d_1}(V_1) = \frac{\binom{d_1}{a+b}}{\binom{d_2}{a+b}} \left(\frac{d_2}{d_1} \right)^{a+b} D_x^a D_y^b p_{d_2}(V_1), \quad (2.16)$$

for a given value $0 \leq \mu \leq d_2$ and all $0 \leq a + b \leq \mu$, then

$$b_{d_1-\mu+i_1, i_2, i_3} = d_{d_2-\mu+i_1, i_2, i_3}, \quad (2.17)$$

for all $i_1 + i_2 + i_3 = \mu$.

Proof. Let δ^x and δ^y be the unit barycentric directions with respect to triangle $\mathcal{T}_1$ in the x - and y -direction. Let ϵ^x and ϵ^y be the unit barycentric directions with respect to triangle $\mathcal{T}_2$ in the x - and y -direction. Because of the definition of the triangles, we know that

$$\epsilon^x = \frac{d_1}{d_2} \delta^x, \quad \epsilon^y = \frac{d_1}{d_2} \delta^y. \quad (2.18)$$

The barycentric coordinates of V_1 are identical with respect to both triangles, and they are denoted by $e^1 = (1, 0, 0)$. Let $\mathcal{P}_{d_1}$ and $\mathcal{P}_{d_2}$ be the blossoms of polynomials p_{d_1} and p_{d_2} , respectively. By (2.11) we have

$$D_x^a D_y^b p_{d_1}(V_1) = \frac{d_1!}{(d_1 - a - b)!} \mathcal{P}_{d_1} \left(\underbrace{e^1, e^1, \dots, e^1}_{(d_1 - a - b) \text{ times}}, \underbrace{\delta^x, \dots, \delta^x}_a, \underbrace{\delta^y, \dots, \delta^y}_b \right),$$

and by using (2.18) we also have

$$D_x^a D_y^b p_{d_2}(V_1) = \frac{d_2!}{(d_2 - a - b)!} \left(\frac{d_1}{d_2} \right)^{a+b} \mathcal{P}_{d_2} \left(\underbrace{e^1, e^1, \dots, e^1}_{(d_2 - a - b) \text{ times}}, \underbrace{\delta^x, \dots, \delta^x}_a, \underbrace{\delta^y, \dots, \delta^y}_b \right).$$

Because of (2.16), it follows that for all $0 \leq a + b \leq \mu$

$$\begin{aligned} \mathcal{P}_{d_1} \left(\underbrace{e^1, e^1, \dots, e^1}_{(d_1 - \mu) \text{ times}}, \underbrace{e^1, e^1, \dots, e^1}_{(\mu - a - b) \text{ times}}, \underbrace{\delta^x, \dots, \delta^x}_a \text{ times}, \underbrace{\delta^y, \dots, \delta^y}_b \text{ times} \right) \\ = \mathcal{P}_{d_2} \left(\underbrace{e^1, e^1, \dots, e^1}_{(d_2 - \mu) \text{ times}}, \underbrace{e^1, e^1, \dots, e^1}_{(\mu - a - b) \text{ times}}, \underbrace{\delta^x, \dots, \delta^x}_a \text{ times}, \underbrace{\delta^y, \dots, \delta^y}_b \text{ times} \right), \end{aligned}$$

and using (2.7) we get for all $0 \leq a + b \leq \mu$

$$\sum_{i_1 + i_2 + i_3 = \mu} (b_{d_1 - \mu + i_1, i_2, i_3} - d_{d_2 - \mu + i_1, i_2, i_3}) \sum_{\pi \in \Pi_{i_1 i_2 i_3}^\mu} \prod_{l=1}^{\mu - a - b} e_{\pi(l)}^1 \prod_{l=\mu - a - b + 1}^{\mu - b} \delta_{\pi(l)}^x \prod_{l=\mu - b + 1}^{\mu} \delta_{\pi(l)}^y = 0.$$

We have $\binom{\mu+2}{2}$ linearly independent constraints involving only the Bézier ordinates $b_{d_1 - \mu + i_1, i_2, i_3}$ and $d_{d_2 - \mu + i_1, i_2, i_3}$ with $i_1 + i_2 + i_3 = \mu$. This linear system implies (2.17). $\square$

3 A family of smooth spline spaces over PS-triangulations

Consider a simply connected subset $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Let Δ be a conforming triangulation Δ of Ω , where no triangle contains a vertex different from its own three vertices. Let n_v , n_t and n_e be the number of vertices, triangles and edges in Δ , respectively.

A Powell-Sabin (PS-) refinement Δ^* of the triangulation Δ partitions each triangle $\mathcal{T}_j \in \Delta$ into six smaller triangles in the following way [1, 6]:

1. Choose an interior point Z_j in each triangle $\mathcal{T}_j$, so that if two triangles $\mathcal{T}_i$ and $\mathcal{T}_j$ have a common edge, then the line joining Z_i and Z_j intersects the common edge at a point between its vertices. For instance, one can take Z_j as the incentre of $\mathcal{T}_j$.
2. Join each point Z_j to the vertices of $\mathcal{T}_j$.
3. For each edge of the triangle $\mathcal{T}_j$
 - (a) which is common to a triangle $\mathcal{T}_i$: join Z_j to the intersection point of that edge and the line $Z_i - Z_j$.
 - (b) which belongs to the boundary $\partial\Omega$: join Z_j to an arbitrary point on that edge.

The obtained subtriangles are denoted by $\mathcal{T}^* \in \Delta^*$.

Given $0 \leq r < d$, the spline space of degree d on Δ^* with global C^r -continuity is denoted by

$$S_d^r(\Delta^*) = \{s \in C^r(\Omega) : s|_{\mathcal{T}^*} \in \mathbb{P}_d, \mathcal{T}^* \in \Delta^*\}. \quad (3.1)$$

We consider a particular subspace of $S_d^r(\Delta^*)$ with additional smoothness around some vertices and edges. Let $\mathcal{V} = \{V_i\}_{i=1}^{n_v}$ be the set of vertices in Δ , let $\mathcal{Z}^* = \{Z_i\}_{i=1}^{n_t}$ be the set of triangle split points in Δ^* , and let $\mathcal{E}^*$ be the set of all edges in Δ^* that connect a triangle split point Z_i to an edge split point. Given $0 \leq r \leq \rho < d$, the space $S_d^{r,\rho}(\Delta^*)$ of supersmooth splines on Δ^* is defined by

$$S_d^{r,\rho}(\Delta^*) = \{s \in S_d^r(\Delta^*) : s \in C^\rho(W), W \in (\mathcal{V} \cup \mathcal{Z}^*); s \in C^\rho(e), e \in \mathcal{E}^*\}. \quad (3.2)$$

Here, $C^\rho(W)$ means that the polynomials on triangles in Δ^* sharing the vertex W have common derivatives up to order ρ at that vertex. Analogously, $C^\rho(e)$ means that the polynomials on triangles in Δ^* sharing the edge e have common derivatives up to order ρ along that edge.

Figure 1: A Powell-Sabin split of a triangle $\mathcal{T}_k(V_1, V_2, V_3)$ drawn with dashed lines.

In this paper we focus on spline spaces $S_d^{r,\rho}(\Delta^*)$ of arbitrary smoothness $r \geq 1$, and

$$d = 3r - 1, \quad \rho = 2r - 1. \quad (3.3)$$

Such a space will be referred to as Powell-Sabin spline space of smoothness r , or PS r -spline space. It will be denoted by $S^r(\Delta^*)$. A spline $s(x, y) \in S^r(\Delta^*)$ can be defined by means of the following Hermite interpolation problem.

Theorem 3.1. *There exists a unique spline $s(x, y) \in S^r(\Delta^*)$ satisfying*

$$D_x^a D_y^b s(V_l) = f_{x^a y^b, l}, \quad 0 \leq a + b \leq 2r - 1, \quad l = 1, \dots, n_v, \quad (3.4a)$$

and

$$D_x^a D_y^b s(Z_m) = g_{x^a y^b, m}, \quad 0 \leq a + b \leq r - 2, \quad m = 1, \dots, n_t, \quad (3.4b)$$

for a given set of $f_{x^a y^b, l}$ -values and $g_{x^a y^b, m}$ -values. A PS r -spline is uniquely defined by means of its derivatives up to order $2r - 1$ at the n_v vertices V_l in Δ and its derivatives up to order $r - 2$ at the n_t triangle split points Z_m in Δ^* .

Proof. We give an explicit construction of the spline $s(x, y) \in S^r(\Delta^*)$ satisfying (3.4) in terms of its Bernstein-Bézier representation. We begin by considering a single macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$ in Δ , as shown in Figure 1. On each of the six subtriangles, the spline is a polynomial of degree $3r - 1$ that can be represented in its Bernstein-Bézier formulation, i.e. with $d = 3r - 1$ in equations (2.2) and (2.3). The Bernstein-Bézier representation of $s(x, y)$ on macro-triangle $\mathcal{T}_k$ is illustrated in Figure 2 for $r = 4$. We now show that the Bézier ordinates in this representation on $\mathcal{T}_k$ are uniquely determined by the interpolation conditions (3.4) and the spline smoothness conditions (3.2).

From the Bernstein-Bézier theory and the C^{2r-1} -smoothness at vertex V_l , we know that the set of derivatives (3.4a) at V_l uniquely specifies the Bézier ordinates corresponding to the domain points in the disk $D_{2r-1}(V_l)$ for each $l = 1, 2, 3$. These three disks are shaded in Figure 2, and their domain points are indicated by the bullets $\circ$ and $\bullet$.

By the C^{2r-1} -smoothness at triangle split point Z_k we can regard the Bézier ordinates of s in the disk $D_{2r-1}(Z_k)$ as the ordinates (after subdivision) of a single polynomial p of degree $2r - 1$ defined on the triangle $\tilde{\mathcal{T}}_k(W_1, W_2, W_3)$ where

$$W_i = \frac{2r-1}{3r-1} V_i + \frac{r}{3r-1} Z_k, \quad (3.5)$$

Figure 2: Schematic representation of the Bézier ordinates of a spline $s \in S^r$ (with $r = 4$) on the macro-triangle $\mathcal{T}_k$ shown in Figure 1. The points indicated by filled bullets $\bullet$ and filled triangles $\blacktriangle$ form a minimal determining set.

for $i = 1, 2, 3$. The disk $D_{2r-1}(Z_k)$ is also shaded in Figure 2, and the corresponding domain points are indicated by the triangles $\triangle$ and $\blacktriangle$. Considering only domain points in this disk, we note that the Bézier ordinates related to domain points in $D_{r-1}(W_i)$, $i = 1, 2, 3$, are already determined by our construction. This can be verified in Figure 2 by considering the overlap of the shaded regions. The ordinates related to domain points in $D_{r-2}(Z_k)$ are specified by the interpolation conditions (3.4b). This gives us a total of

$$3 \binom{r+1}{2} + \binom{r}{2} = \binom{2r+1}{2} \quad (3.6)$$

independent constraints. They completely specify polynomial p , which has degree $2r-1$ and whose number of degrees of freedom is equal to (3.6).

For each edge $e_{ij} = \langle V_i, V_j \rangle$ of $\mathcal{T}_k$, the remaining Bézier ordinates on $\mathcal{T}_k$ associated with domain points on the rows $0, 1, \dots, r-1$ parallel to e_{ij} are found by using the C^{2r-1} -smoothness conditions across the interior edge $\langle R_{ij}, Z_k \rangle$, with R_{ij} the split point on edge e_{ij} , see Figure 1. These domain points are indicated in Figure 2 by squares $\square$. Fix $0 \leq m \leq r-1$, then the central $4r-1$ Bézier ordinates on row E_m parallel to e_{12} can be regarded as the Bézier ordinates (after subdivision) of a single onedimensional polynomial $\hat{p}$ of degree $2r-1$ defined over the line segment between the points $\hat{W}_1$ and $\hat{W}_2$,

$$\hat{W}_j = \frac{2r-1}{3r-1} V_j + \frac{r-m}{3r-1} R_{12} + \frac{m}{3r-1} Z_k, \quad (3.7)$$

$j = 1, 2$. Of these $4r-1$ Bézier ordinates, we know that $2r$ ordinates are already determined by the interpolation conditions (3.4a) at V_1 and V_2 . This implies that polynomial $\hat{p}$ of degree $2r-1$ is completely specified as well.

By the above construction, we can uniquely determine the Bézier ordinates on each macro-triangle $\mathcal{T}_k \in \Delta$ separately. It remains to check that the C^r -smoothness conditions across the common edge of any pair of neighbouring macro-triangles in Δ are satisfied. These conditions, see (2.6), involve

the Bézier ordinates corresponding to domain points in the rows $0, 1, \dots, r$ parallel to that edge. Let $\mathcal{T}_1^*(V_1, R_{12}, Z)$, $\mathcal{T}_2^*(V_2, R_{12}, Z)$, $\mathcal{T}_3^*(V_1, R_{12}, \tilde{Z})$ and $\mathcal{T}_4^*(V_2, R_{12}, \tilde{Z})$ be the four subtriangles in Δ^* sharing edge $\langle V_1, V_2 \rangle$. By the construction of the PS-triangulation, we know that R_{12} is the intersection point between the lines $\langle V_1, V_2 \rangle$ and $\langle Z, \tilde{Z} \rangle$. Let $b_{i_1 i_2 i_3}^j$ be the Bézier ordinates related to subtriangle $\mathcal{T}_j^*$, $i_1 + i_2 + i_3 = 3r - 1$ and $j = 1, 2, 3, 4$. Let $\alpha_n^{i_3}$ be such that the C^r -smoothness conditions across $\langle V_1, R_{12} \rangle$ and $\langle V_2, R_{12} \rangle$ are given by

$$b_{i_1 i_2 i_3}^3 = \sum_{n=0}^{i_3} \alpha_n^{i_3} b_{i_1, 3r-1-n-i_1, n}^1, \quad (3.8a)$$

$$b_{i_1 i_2 i_3}^4 = \sum_{n=0}^{i_3} \alpha_n^{i_3} b_{i_1, 3r-1-n-i_1, n}^2, \quad (3.8b)$$

for any $0 \leq i_3 \leq r$ and $i_1 + i_2 + i_3 = 3r - 1$. Note that the weights $\alpha_n^{i_3}$ are the same in (3.8a) and (3.8b). Because of (3.4a) and the C^{2r-1} -smoothness at the vertices V_1 and V_2 , the conditions in (3.8) with $i_1 \geq r$ are automatically satisfied. These conditions connect Bézier ordinates inside disk $D_{2r-1}(V_1)$ or disk $D_{2r-1}(V_2)$. We now show that the conditions in (3.8a) are also satisfied for $i_1 < r$. Because of the C^{2r-1} -smoothness across edge $\langle R_{12}, Z \rangle$ we can represent $b_{i_1 i_2 i_3}^1$ for any $0 \leq i_1 < r$, $0 \leq i_3 \leq r$ and $i_1 + i_2 + i_3 = 3r - 1$ by

$$b_{i_1 i_2 i_3}^1 = \sum_{m=0}^{r-1} \beta_m^{i_1} b_{r+m, 2r-1-m-i_3, i_3}^1 + \sum_{m=0}^{r-1} \gamma_m^{i_1} b_{r+m, 2r-1-m-i_3, i_3}^2, \quad (3.9)$$

with particular weights $\beta_m^{i_1}$ and $\gamma_m^{i_1}$. Analogously, because of the C^{2r-1} -smoothness across edge $\langle R_{12}, \tilde{Z} \rangle$ we can represent $b_{i_1 i_2 i_3}^3$ for any $0 \leq i_1 < r$, $0 \leq i_3 \leq r$ and $i_1 + i_2 + i_3 = 3r - 1$ by

$$b_{i_1 i_2 i_3}^3 = \sum_{m=0}^{r-1} \beta_m^{i_1} b_{r+m, 2r-1-m-i_3, i_3}^3 + \sum_{m=0}^{r-1} \gamma_m^{i_1} b_{r+m, 2r-1-m-i_3, i_3}^4, \quad (3.10)$$

with the same weights $\beta_m^{i_1}$ and $\gamma_m^{i_1}$ as in (3.9). We already know that the conditions in (3.8) hold for $i_1 \geq r$. Thus, combining (3.8)–(3.10), we get for $i_1 < r$

$$\begin{aligned} b_{i_1 i_2 i_3}^3 &= \sum_{m=0}^{r-1} \beta_m^{i_1} \left(\sum_{n=0}^{i_3} \alpha_n^{i_3} b_{r+m, 2r-1-n-m, n}^1 \right) + \sum_{m=0}^{r-1} \gamma_m^{i_1} \left(\sum_{n=0}^{i_3} \alpha_n^{i_3} b_{r+m, 2r-1-n-m, n}^2 \right) \\ &= \sum_{n=0}^{i_3} \alpha_n^{i_3} \left(\sum_{m=0}^{r-1} \beta_m^{i_1} b_{r+m, 2r-1-n-m, n}^1 + \sum_{m=0}^{r-1} \gamma_m^{i_1} b_{r+m, 2r-1-n-m, n}^2 \right) \\ &= \sum_{n=0}^{i_3} \alpha_n^{i_3} b_{i_1, 3r-1-n-i_1, n}^1. \end{aligned}$$

Using a similar argument, we can also prove that the conditions in (3.8b) are satisfied for $i_1 < r$.

We can conclude that the above construction of the Bernstein-Bézier representation of $s(x, y)$ shows that conditions (3.4) are sufficient and necessary to specify (all Bézier ordinates of) a Powell-Sabin spline $s(x, y)$ uniquely on Δ^* . $\square$

Remark 3.1. By the constructive nature of the proof of Theorem 3.1, one can easily define a stable local minimal determining set of Bézier domain points. A minimal determining set is a minimal subset of Bézier domain points ξ_i , such that if the corresponding Bézier ordinates b_i in the Bernstein-Bézier representation of a spline $s \in S^r$ are equal to zero then the spline s must be zero (see [3]). Such a minimal determining set is illustrated in Figure 2. The points in this set are indicated by filled bullets $\bullet$ and filled triangles $\blacktriangle$.

The dimension of the space $S^r(\Delta^*)$ follows immediately from Theorem 3.1.

Theorem 3.2. *The dimension of $S^r(\Delta^*)$ is given by*

$$\dim S^r(\Delta^*) = r(2r+1)n_v + \frac{r(r-1)}{2}n_t. \quad (3.11)$$

Proof. By Theorem 3.1 we know that the spline space $S^r(\Delta^*)$ has $\binom{2r+1}{2}$ degrees of freedom for each vertex V_l and $\binom{r}{2}$ degrees of freedom for each triangle split point Z_m . $\square$

The spline space $S^r(\Delta^*)$ has also an optimal approximation order. Let $\|\cdot\|_{L_p}$ be the L_p -norm on Ω , and let $|\cdot|_{W_p^k}$ be the usual semi-norm in the Sobolev space $W_p^k(\Omega)$, i.e.,

$$|v|_{W_p^k} = \left(\sum_{a+b=k} \|D_x^a D_y^b v\|_{L_p}^p \right)^{1/p}. \quad (3.12)$$

Theorem 3.3. *For every $0 \leq k \leq d := 3r - 1$, there exists an interpolation operator $\mathcal{I}^r : W_p^{k+1}(\Omega) \rightarrow S^r(\Delta^*)$ such that*

$$\|D_x^a D_y^b (f - \mathcal{I}^r f)\|_{L_p} \leq K |\Delta|^{k+1-a-b} |f|_{W_p^{k+1}}, \quad (3.13)$$

for every function $f \in W_p^{k+1}(\Omega)$, with $0 \leq a+b \leq k$ and $|\Delta|$ the mesh size of Δ . If Ω is convex then the constant K depends only on r, p, k and on the smallest angle in Δ^* . If Ω is non-convex, it also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω .

Proof. Using the techniques provided in Chapters 5 and 8 of [3], the result follows immediately from the construction of a stable local minimal determining set (see Remark 3.1). $\square$

The spline space $S^r(\Delta^*)$ with $r = 1$ ($d = 2, \rho = 1$) is the well-known quadratic Powell-Sabin spline space, see [1, 6, 9], and the space with $r = 2$ ($d = 5, \rho = 3$) is the quintic Powell-Sabin spline space constructed in [11].

4 Normalized Powell-Sabin B-splines

In this section we look for a suitable B-spline representation of $s(x, y) \in S^r(\Delta^*)$,

$$s(x, y) = \sum_{i=1}^{n_v} \sum_{j_1+j_2+j_3=2r-1} c_{i,j_1 j_2 j_3}^{v,r} B_{i,j_1 j_2 j_3}^{v,r}(x, y) + \sum_{k=1}^{n_t} \sum_{j_1+j_2+j_3=r-2} c_{k,j_1 j_2 j_3}^{t,r} B_{k,j_1 j_2 j_3}^{t,r}(x, y), \quad (4.1)$$

in which the basis functions $B_{i,j_1 j_2 j_3}^{v,r}(x, y)$ and $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$ have a local support and form a convex partition of unity, i.e., they are nonnegative and sum up to one. We will call $B_{i,j_1 j_2 j_3}^{v,r}(x, y)$ and $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$ a Powell-Sabin B-spline with respect to vertex V_i and triangle $\mathcal{T}_k$, respectively. We first describe the construction of the Powell-Sabin B-splines, and then we show some properties.

4.1 Construction of the basis

Let M_i be the molecule (also called 1-ring or cell) of vertex V_i , defined as the union of all triangles in Δ that contain V_i . Let N^v and N^t be defined by

$$N^v = \binom{2r+1}{2}, \quad N^t = \binom{r}{2}. \quad (4.2)$$

With each vertex V_i we associate N^v linearly independent sets $A_{i,j_1j_2j_3}$ consisting of N^v elements, i.e.,

$$A_{i,j_1j_2j_3} = \{\alpha_{i,j_1j_2j_3}^{ab}, \quad 0 \leq a+b \leq 2r-1\}, \quad (4.3)$$

for all $j_1 + j_2 + j_3 = 2r-1$. With each triangle $\mathcal{T}_k$ we associate N^t linearly independent sets $B_{k,j_1j_2j_3}$ consisting of N^t elements, i.e.,

$$B_{k,j_1j_2j_3} = \{\beta_{k,j_1j_2j_3}^{ab}, \quad 0 \leq a+b \leq r-2\}, \quad (4.4)$$

for all $j_1 + j_2 + j_3 = r-2$. These sets are constructed as follows.

1. For each vertex V_i in Δ , identify the so-called PSr-points. They are defined as

$$S_{il} = \frac{r}{3r-1} V_i + \frac{2r-1}{3r-1} V_l, \quad (4.5)$$

for all vertices V_l that are situated at the boundary of the molecule M_i of V_i . The vertex V_i itself is also a PSr-point.

2. For each vertex V_i , find a triangle $t_i(Q_{i,1}, Q_{i,2}, Q_{i,3})$ that contains all the PSr-points of V_i . Denote its vertices by $Q_{i,j} = (X_{i,j}, Y_{i,j})$. The triangles t_i , $i = 1, \dots, n_v$, will be called PSr-triangles.
3. For each vertex V_i , consider the N^v Bernstein basis polynomials (2.3) of degree $2r-1$ defined on PSr-triangle t_i . Each set $A_{i,j_1j_2j_3}$, $j_1 + j_2 + j_3 = 2r-1$, is then related to the function and derivative values up to order $2r-1$ of one of these Bernstein basis polynomials evaluated at the point V_i :

$$\alpha_{i,j_1j_2j_3}^{ab} = \frac{\binom{3r-1}{a+b}}{\binom{2r-1}{a+b}} \left(\frac{2r-1}{3r-1} \right)^{a+b} D_x^a D_y^b B_{j_1j_2j_3}^{2r-1}(V_i), \quad (4.6)$$

for all $0 \leq a+b \leq 2r-1$.

4. For each triangle $\mathcal{T}_k(V_1, V_2, V_3)$, consider the N^v Bernstein basis polynomials (2.3) of degree $2r-1$ defined on the triangle with vertices W_i , $i = 1, 2, 3$, as given in (3.5). Each set $B_{k,j_1j_2j_3}$, $j_1 + j_2 + j_3 = r-2$, is then related to the function and derivative values up to order $r-2$ of one of these Bernstein basis polynomials evaluated at the split point Z_k of macro-triangle $\mathcal{T}_k$:

$$\beta_{k,j_1j_2j_3}^{ab} = \frac{\binom{3r-1}{a+b}}{\binom{2r-1}{a+b}} \left(\frac{2r-1}{3r-1} \right)^{a+b} D_x^a D_y^b B_{r-1-j_1, r-1-j_2, r-1-j_3}^{2r-1}(Z_k), \quad (4.7)$$

for all $0 \leq a+b \leq r-2$.

Figure 3: Schematic representation of the Bézier ordinates of (a) a PS r B-spline with respect to a vertex and (b) a PS r B-spline with respect to a triangle, for $r = 4$, on the macro-triangle $\mathcal{T}_k$ shown in Figure 1. Bézier ordinates that are known to be zero are indicated by open bullets $\circ$; other ordinates are indicated by filled bullets $\bullet$. The Bernstein-Bézier representation of the polynomial p used for the computation of the ordinates corresponding to disk $D_{2r-1}(Z_k)$ is also given.

We remark that the PS_r -triangles are not uniquely defined. An optimization strategy can be used to select triangles of minimal area, as has been suggested in the case $r = 1$ and $r = 2$, see [1, 11]. Constructing such triangles of minimal area is equivalent to solving a quadratic programming problem.

The basis functions $B_{i,j_1j_2j_3}^{v,r}(x, y)$ and $B_{k,j_1j_2j_3}^{t,r}(x, y)$ are then constructed in the following way:

1. The B-spline $B_{i,j_1j_2j_3}^{v,r}(x, y)$ with respect to vertex V_i is defined as the unique solution of interpolation problem (3.4) with all $f_{x^a y^b, l} = 0$, except for $l = i$, where $f_{x^a y^b, i} = \alpha_{i,j_1j_2j_3}^{ab}$, and with all $g_{x^a y^b, m} = 0$, except for any m such that $\mathcal{T}_m \in M_i$, where $g_{x^a y^b, m} = \gamma_{i,m,j_1j_2j_3}^{ab}$. The values $\alpha_{i,j_1j_2j_3}^{ab}$ are determined by the PS_r -triangle t_i , see (4.6). The values $\gamma_{i,m,j_1j_2j_3}^{ab}$ are specified with the aid of the Bernstein-Bézier representation of the B-spline as follows.

We consider a macro-triangle $\mathcal{T}_m(V_1, V_2, V_3)$ in the molecule M_1 of V_1 , as illustrated in Figure 1, and the B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ associated with vertex V_1 . The corresponding Bézier ordinates are schematically represented in Figure 3(a) for $r = 4$. Bézier ordinates, that are known to be zero, are indicated by open bullets $\circ$ in the figure. The other ordinates are indicated by filled bullets $\bullet$. Because of the C^{2r-1} -smoothness at vertex V_1 , the Bézier ordinates in the disk $D_{2r-1}(V_1)$ are completely determined by the set of values $A_{1,j_1j_2j_3}$. The Bézier ordinates in the disk $D_{2r-1}(Z_m)$ can be regarded as ordinates (after subdivision) of a single polynomial p of degree $2r - 1$ defined on the triangle with vertices W_i , $i = 1, 2, 3$, given in (3.5). The polynomial p is defined on this triangle by means of its Bernstein-Bézier representation. Let $b_{i_1i_2i_3}$, $i_1 + i_2 + i_3 = 2r - 1$, be the Bézier ordinates in this representation. All ordinates $b_{i_1i_2i_3}$ with $i_1 < r$ are taken to be zero. The values of the ordinates $b_{i_1i_2i_3}$ with $i_1 \geq r$ are uniquely determined by the (subdivision) relation between them and the Bézier ordinates in the disk $D_{2r-1}(V_1)$ of the B-spline (which are specified by the set $A_{1,j_1j_2j_3}$). The zero and nonzero ordinates $b_{i_1i_2i_3}$ are illustrated on the triangle shown on the right in Figure 3(a).

The value of $\gamma_{1,m,j_1j_2j_3}^{ab}$ related to triangle $\mathcal{T}_m$ is then obtained by evaluating the polynomial p of degree $2r - 1$ and its derivatives at split point Z_m :

$$\gamma_{1,m,j_1j_2j_3}^{ab} = \frac{\binom{3r-1}{a+b}}{\binom{2r-1}{a+b}} \left(\frac{2r-1}{3r-1} \right)^{a+b} D_x^a D_y^b p(Z_m). \quad (4.8)$$

2. The B-spline $B_{k,j_1j_2j_3}^{t,r}(x, y)$ with respect to triangle $\mathcal{T}_k$ is defined as the unique solution of interpolation problem (3.4) with all $f_{x^a y^b, l} = 0$ and with all $g_{x^a y^b, m} = 0$, except for $m = k$, where $g_{x^a y^b, k} = \beta_{k,j_1j_2j_3}^{ab}$. The values $\beta_{k,j_1j_2j_3}^{ab}$ are defined by (4.7).

To illustrate the Bernstein-Bézier representation of this B-spline, we consider again the triangle $\mathcal{T}_k(V_1, V_2, V_3)$ shown in Figure 1. The corresponding Bézier ordinates are schematically represented in Figure 3(b) for $r = 4$. Most of these ordinates are zero. The Bézier ordinates in the disk $D_{2r-1}(Z_k)$ are found by subdivision of a polynomial p of degree $2r - 1$ defined on the triangle with vertices W_i , $i = 1, 2, 3$, given in (3.5). In the Bernstein-Bézier representation of p on this triangle, the Bézier ordinates in the neighbourhood of W_i are zero. The corresponding domain points are highlighted on the triangle shown on the right in Figure 3(b).

In Section 5.2 we will provide a more elegant and practical way to compute the Bernstein-Bézier representation of a Powell-Sabin spline $s \in S^r(\Delta^*)$ in representation (4.1), and thus also of a normalized B-spline.

4.2 Some properties of the Powell-Sabin B-splines

In this section we mainly focus on the Powell-Sabin B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ with respect to vertex V_1 , and the B-spline $B_{k,j_1j_2j_3}^{t,r}(x, y)$ with respect to triangle $\mathcal{T}_k(V_1, V_2, V_3)$, as shown in Figure 1. We assume that the points indicated in this figure have the following barycentric coordinates with respect to $\mathcal{T}_k$:

$$\begin{aligned} V_1 &= (1, 0, 0), & V_2 &= (0, 1, 0), & V_3 &= (0, 0, 1), & Z_k &= (z_1, z_2, z_3), \\ R_{12} &= (\lambda_{12}, \lambda_{21}, 0), & R_{23} &= (0, \lambda_{23}, \lambda_{32}), & R_{31} &= (\lambda_{13}, 0, \lambda_{31}). \end{aligned} \quad (4.9)$$

We first formulate some lemmata with regard to the Bernstein-Bézier representation of the B-splines.

Lemma 4.1. *The Bézier ordinates corresponding to domain points in the disk $D_{2r-1}(V_1)$ of a Powell-Sabin B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ in Bernstein-Bézier form can be regarded as the Bézier ordinates (after subdivision) of the Bernstein basis polynomial $B_{j_1j_2j_3}^{2r-1}(x, y)$ of degree $2r - 1$ defined on the PSr-triangle t_1 of vertex V_1 . These Bézier ordinates are nonnegative.*

Proof. From definition (4.6) and Theorem 2.3 (with $d_1 = 3r - 1$ and $\mu = d_2 = 2r - 1$), it follows that the Bézier ordinates in the disk $D_{2r-1}(V_1)$ of the Powell-Sabin B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ can be regarded as the Bézier ordinates (after subdivision) of a Bernstein basis polynomial defined on the PSr-triangle.

We now prove that these ordinates are nonnegative. To this end, we first show that the domain points in the disk $D_{2r-1}(V_1)$ are contained in the PSr-triangle t_1 of vertex V_1 . Let $\mathcal{T}_k(V_1, V_2, V_3)$ be a macro-triangle in the molecule M_1 of V_1 , as indicated in Figure 1. The domain points inside disk $D_{2r-1}(V_1)$ in $\mathcal{T}_k$ are defined by

$$\tau_1 V_1 + \tau_2 R_{12} + \tau_3 Z_k, \quad \text{or} \quad \tau_1 V_1 + \tau_2 R_{13} + \tau_3 Z_k,$$

for some $\frac{r}{3r-1} \leq \tau_1$, $0 \leq \tau_2$, $0 \leq \tau_3$, $\tau_1 + \tau_2 + \tau_3 = 1$. By the definition of the PSr-triangle t_1 we know that it contains the triangle spanned by the PSr-points V_1 , S_{12} and S_{13} , see (4.5). It is easy to see that the considered domain points are all inside this triangle. It follows that the barycentric coordinates of these points with respect to the PSr-triangle are nonnegative. We can use a similar argument for all other triangles in the molecule M_1 of V_1 .

Any Bernstein basis polynomial is nonnegative on its domain triangle. Its blossom $\mathcal{B}_{2r-1}(\tau^1, \dots, \tau^{2r-1})$ is nonnegative when the barycentric coordinates τ^l , $l = 1, \dots, 2r - 1$, are nonnegative. Applying subdivision rules (2.10), we then obtain that the Bézier ordinates of the Powell-Sabin B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ in the disk $D_{2r-1}(V_1)$ are nonnegative as well. $\square$

Lemma 4.2. *Let $\mathcal{T}_k(V_1, V_2, V_3)$ be a triangle in the molecule of vertex V_1 . The Bézier ordinates corresponding to domain points in the disk $D_{2r-1}(Z_k)$ of a Powell-Sabin B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ related to vertex V_1 in Bernstein-Bézier form can be regarded as the Bézier ordinates (after subdivision) of a nonnegative polynomial p of degree $2r - 1$ defined over the triangle with vertices W_i , $i = 1, 2, 3$, as given in (3.5). These Bézier ordinates are nonnegative.*

Proof. From the construction of B-spline $B_{1,j_1j_2j_3}^{v,r}(x, y)$ we know that its Bézier ordinates corresponding to domain points in the disk $D_{2r-1}(Z_k)$ can be regarded as the Bézier ordinates (after subdivision) of a polynomial p of degree $2r - 1$. Let $b_{i_1i_2i_3}$, $i_1 + i_2 + i_3 = 2r - 1$, be the Bézier ordinates of this polynomial in Bernstein-Bézier representation on triangle $\tilde{\mathcal{T}}_k(W_1, W_2, W_3)$. In this representation only the $\binom{r+1}{2}$ Bézier ordinates in the neighbourhood of W_1 are nonzero (see the

triangle on the right in Figure 3(a)). The values of these nonzero ordinates are uniquely determined by the set $A_{1,j_1j_2j_3}$. We also consider the Bernstein-Bézier form of this polynomial on the subtriangle $\hat{T}_k(W_1, T_{12}, Z_k)$, where

$$T_{12} = \frac{2r-1}{3r-1} R_{12} + \frac{r}{3r-1} Z_k = \lambda_{12} W_1 + \lambda_{21} W_2.$$

Let $d_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = 2r - 1$, be the Bézier ordinates of p related to this subtriangle $\hat{T}_k$. By Lemma 4.1 we know that a subset of these ordinates in the neighbourhood of W_1 can be computed by subdividing a particular Bernstein basis polynomial of degree $2r - 1$ defined over PSr-triangle t_1 . Let $\mathcal{B}_{2r-1}$ be the blossom of this Bernstein basis polynomial, and let $\sigma^l = (\sigma_1^l, \sigma_2^l, \sigma_3^l)$, $l = 1, 2, 3$, be the barycentric coordinates with respect to t_1 of the points V_1 , U_{12} and U_{13} , respectively, where

$$U_{12} = \frac{r}{3r-1} V_1 + \frac{2r-1}{3r-1} R_{12}, \quad U_{13} = \frac{r}{3r-1} V_1 + \frac{2r-1}{3r-1} Z_k.$$

Then, the ordinates $d_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = 2r - 1$ and $i_1 \geq r$, can be written as

$$d_{i_1 i_2 i_3} = \mathcal{B}_{2r-1} \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 - r \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{i_3 + r \text{ times}} \right). \quad (4.10)$$

Let $\varrho^l = (\varrho_1^l, \varrho_2^l, \varrho_3^l)$, $l = 1, 2, 3$, be the barycentric coordinates of the points W_l with respect to subtriangle $\hat{T}_k(W_1, T_{12}, Z_k)$. Using (4.9), we have

$$\varrho^1 = (1, 0, 0), \quad \varrho^2 = \left(\frac{-\lambda_{12}}{\lambda_{21}}, \frac{1}{\lambda_{21}}, 0 \right), \quad \varrho^3 = \left(\frac{z_2 \lambda_{12}}{z_3 \lambda_{21}} - \frac{z_1}{z_3}, -\frac{z_2}{z_3 \lambda_{21}}, \frac{1}{z_3} \right). \quad (4.11)$$

Let $\mathcal{P}_{2r-1}$ be the blossom of polynomial p with respect to $\hat{T}_k$. Then, by (2.10), we can write the Bézier ordinates $b_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = 2r - 1$, of p with respect to $\hat{T}_k$ in terms of the ordinates $d_{i_1 i_2 i_3}$:

$$b_{i_1 i_2 i_3} = \mathcal{P}_{2r-1} \left(\underbrace{\varrho^1, \dots, \varrho^1}_{i_1 \text{ times}}, \underbrace{\varrho^2, \dots, \varrho^2}_{i_2 \text{ times}}, \underbrace{\varrho^3, \dots, \varrho^3}_{i_3 \text{ times}} \right). \quad (4.12)$$

Because $\varrho^1 = (1, 0, 0)$, only the ordinates $d_{i_1 i_2 i_3}$ with $i_1 \geq r$ are involved in the expressions (4.12) of the ordinates $b_{i_1 i_2 i_3}$ with $i_1 \geq r$. Using Theorem 2.2 together with (4.10) and (4.12), we obtain that for $i_1 \geq r$

$$b_{i_1 i_2 i_3} = \mathcal{B}_{2r-1} \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 - r \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_r, \underbrace{\sum_{l=1}^2 \varrho_l^2 \sigma^l, \dots, \sum_{l=1}^2 \varrho_l^2 \sigma^l}_{i_2 \text{ times}}, \underbrace{\sum_{l=1}^3 \varrho_l^3 \sigma^l, \dots, \sum_{l=1}^3 \varrho_l^3 \sigma^l}_{i_3 \text{ times}} \right). \quad (4.13)$$

By the definition of σ^l and ϱ^l , we find after some elementary calculations that $\sum_{l=1}^2 \varrho_l^2 \sigma^l$ are the barycentric coordinates with respect to PSr-triangle t_1 of the PSr-point

$$\varrho_1^2 V_1 + \varrho_2^2 U_{12} = \frac{-\lambda_{12}}{\lambda_{21}} V_1 + \frac{1}{\lambda_{21}} U_{12} = \frac{r}{3r-1} V_1 + \frac{2r-1}{3r-1} V_2 = S_{12},$$

and that $\sum_{l=1}^3 \varrho_l^3 \sigma^l$ are the barycentric coordinates of the PSr-point

$$\varrho_1^3 V_1 + \varrho_2^3 U_{12} + \varrho_3^3 U_{13} = \left(\frac{z_2 \lambda_{12}}{z_3 \lambda_{21}} - \frac{z_1}{z_3} \right) V_1 - \frac{z_2}{z_3 \lambda_{21}} U_{12} + \frac{1}{z_3} U_{13} = \frac{r}{3r-1} V_1 + \frac{2r-1}{3r-1} V_3 = S_{13}.$$

By the construction of PS r -triangle t_1 , we know that t_1 contains the PS r -points V_1 , S_{12} and S_{13} . One can also easily see that the points U_{12} and U_{13} are inside t_1 . Thus, the barycentric coordinates of these points with respect to t_1 are all nonnegative. Each Bernstein basis polynomial is nonnegative on its domain triangle, and its blossom $\mathcal{B}_{2r-1}(\tau^1, \dots, \tau^{2r-1})$ is nonnegative whenever the barycentric coordinates τ^l , $l = 1, \dots, 2r-1$, are nonnegative. From (4.13) it follows that the Bézier ordinates $b_{i_1 i_2 i_3}$ ($i_1 \geq r$) of polynomial p are nonnegative when PS r -triangle t_1 contains the points V_1 , S_{12} and S_{13} . The remaining $b_{i_1 i_2 i_3}$ ($i_1 < r$) are known to be zero. Thus, polynomial p is nonnegative on $\tilde{\mathcal{T}}_k$.

Using subdivision rules (2.10), we can conclude that the Bézier ordinates of the Powell-Sabin B-spline $B_{1,j_1 j_2 j_3}^{v,r}(x, y)$ in the disk $D_{2r-1}(Z_k)$ are nonnegative as well. $\square$

Lemma 4.3. *Let $\mathcal{T}_k(V_1, V_2, V_3)$ be a macro-triangle in the molecule of vertex V_1 , and consider the Bernstein-Bézier representation of a Powell-Sabin B-spline $B_{1,j_1 j_2 j_3}^{v,r}(x, y)$ related to vertex V_1 on $\mathcal{T}_k$. For a given $0 \leq m \leq r$, let E_m be the row of domain points at distance m parallel to the edges $\langle V_1, R_{12} \rangle \cup \langle R_{12}, V_2 \rangle$. Then, the Bézier ordinates of the B-spline corresponding to the central $4r-1$ domain points on E_m can be regarded as the Bézier ordinates (after subdivision) of a onedimensional nonnegative polynomial $\hat{p}$ of degree $2r-1$ defined over the line segment between the points $\hat{W}_1$ and $\hat{W}_2$, given in (3.7). These Bézier ordinates are nonnegative.*

Proof. We use a similar line of argument as in the proof of Lemma 4.2. From the properties of Bernstein-Bézier polynomials and from the C^{2r-1} -smoothness of the B-spline $B_{1,j_1 j_2 j_3}^{v,r}(x, y)$ across the interior edge $\langle R_{12}, Z_k \rangle$, it follows that the Bézier ordinates of the B-spline corresponding to the central $4r-1$ domain points on E_m can be regarded as the Bézier ordinates (after subdivision) of a onedimensional polynomial $\hat{p}$ of degree $2r-1$. Let $\hat{b}_{i_1 i_2}$, $i_1 + i_2 = 2r-1$, be the Bézier ordinates of this polynomial in Bernstein-Bézier form given on the line segment $\langle \hat{W}_1, \hat{W}_2 \rangle$. By the construction of the B-spline, only the ordinates $\hat{b}_{i_1 i_2}$ with $i_1 \geq r$ in the neighbourhood of $\hat{W}_1$ are nonzero. The values of these nonzero ordinates are uniquely determined by the set $A_{1,j_1 j_2 j_3}$. We also consider the Bernstein-Bézier form of this polynomial on the smaller line segment $\langle \hat{W}_1, \hat{T}_{12} \rangle$, where

$$\hat{T}_{12} = \frac{3r-1-m}{3r-1} R_{12} + \frac{m}{3r-1} Z_k = \lambda_{12} \hat{W}_1 + \lambda_{21} \hat{W}_2.$$

Let $\hat{d}_{i_1 i_2}$, $i_1 + i_2 = 2r-1$, be the Bézier ordinates of $\hat{p}$ related to this smaller line segment. Similarly to (4.10), we can show that ordinates $\hat{d}_{i_1 i_2}$ with $i_1 \geq r$ can be expressed in terms of the blossom of a particular Bernstein basis polynomial of degree $2r-1$ defined over PS r -triangle t_1 , i.e.,

$$\hat{d}_{i_1 i_2} = \mathcal{B}_{2r-1} \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 - r \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{i_2 + r - m \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{m \text{ times}} \right). \quad (4.14)$$

Let $\hat{\mathcal{P}}_{2r-1}$ be the blossom of polynomial $\hat{p}$ with respect to line segment $\langle \hat{W}_1, \hat{T}_{12} \rangle$. Using (2.10) and (4.11) we can write the ordinates $\hat{b}_{i_1 i_2}$, $i_1 + i_2 = 2r-1$, as

$$\hat{b}_{i_1 i_2} = \hat{\mathcal{P}}_{2r-1} \left(\underbrace{\varrho^1, \dots, \varrho^1}_{i_1 \text{ times}}, \underbrace{\varrho^2, \dots, \varrho^2}_{i_2 \text{ times}} \right). \quad (4.15)$$

Only the ordinates $\hat{d}_{i_1 i_2}$ with $i_1 \geq r$ are involved in the expressions (4.15) of the ordinates $\hat{b}_{i_1 i_2}$ with $i_1 \geq r$. Similarly to (4.13), we can show that

$$\hat{b}_{i_1 i_2} = \mathcal{B}_{2r-1} \left(\underbrace{\sigma^1, \dots, \sigma^1}_{i_1 - r \text{ times}}, \underbrace{\sigma^2, \dots, \sigma^2}_{r - m \text{ times}}, \underbrace{\sigma^3, \dots, \sigma^3}_{m \text{ times}}, \underbrace{\sum_{l=1}^2 \varrho_l^2 \sigma^l, \dots, \sum_{l=1}^2 \varrho_l^2 \sigma^l}_{i_2 \text{ times}} \right), \quad (4.16)$$

for $i_1 \geq r$. In the proof of Lemma 4.2, we have already shown that $\sum_{l=1}^2 \varrho_l^2 \sigma^l$ are the barycentric coordinates of PSr-point S_{12} with respect to PSr-triangle t_1 . Thus, all arguments of the blossom in (4.16) are nonnegative. It follows that all Bézier ordinates $\hat{b}_{i_1 i_2}$ of polynomial $\hat{p}$ are nonnegative. $\square$

Lemma 4.4. *The Bézier ordinates corresponding to domain points in the disk $D_{2r-1}(Z_k)$ of a B-spline $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$ related to triangle $\mathcal{T}_k$ in Bernstein-Bézier form can be regarded as the Bézier ordinates (after subdivision) of the Bernstein basis polynomial $B_{r-1-j_1, r-1-j_2, r-1-j_3}^{2r-1}(x, y)$ of degree $2r-1$ defined on the triangle with vertices W_i , $i = 1, 2, 3$, as given in (3.5). These Bézier ordinates are nonnegative.*

Proof. From definition (4.7) and Theorem 2.3 (with $d_1 = 3r-1$, $d_2 = 2r-1$ and $\mu = r-2$) it follows that the Bézier ordinates corresponding to domain points in the disk $D_{r-2}(Z_k)$ are equal in the Bernstein-Bézier representation of both the B-spline $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$ and the considered Bernstein basis polynomial of degree $2r-1$. We also know that

$$D_x^a D_y^b B_{r-1-j_1, r-1-j_2, r-1-j_3}^{2r-1}(V_i) = 0, \quad (4.17)$$

for all $0 \leq a+b \leq r-1$ and $i = 1, 2, 3$. From the definition of the Powell-Sabin B-spline and (4.17), it follows that the Bézier ordinates corresponding to domain points in the three disks $D_{r-1}(W_i)$, $i = 1, 2, 3$, are equal to zero in both Bernstein-Bézier representations.

Following a similar reasoning as used in the proof of Theorem 3.1, we can conclude that the Bézier ordinates in the disk $D_{2r-1}(Z_k)$ of the Powell-Sabin B-spline $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$ can be regarded as Bézier ordinates (after subdivision) of the considered Bernstein basis polynomial of degree $2r-1$. From the subdivision rules (2.10) and the nonnegativity of a Bernstein basis polynomial over its domain triangle, it follows that the considered Bézier ordinates are nonnegative. $\square$

We now show that the Powell-Sabin B-splines have a local support and form a convex partition of unity.

Property 4.1. *The B-splines $B_{i,j_1 j_2 j_3}^{v,r}(x, y)$, $i = 1, \dots, n_v$, $j_1 + j_2 + j_3 = 2r-1$, and the B-splines $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$, $k = 1, \dots, n_t$, $j_1 + j_2 + j_3 = r-2$, have a local support.*

Proof. By their definition, it is easy to see that a B-spline $B_{i,j_1 j_2 j_3}^{v,r}$ is zero outside the molecule M_i of vertex V_i , and that a B-spline $B_{k,j_1 j_2 j_3}^{t,r}$ vanishes outside triangle $\mathcal{T}_k$. $\square$

Property 4.2. *The B-splines $B_{i,j_1 j_2 j_3}^{v,r}(x, y)$, $i = 1, \dots, n_v$, $j_1 + j_2 + j_3 = 2r-1$, and the B-splines $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$, $k = 1, \dots, n_t$, $j_1 + j_2 + j_3 = r-2$, form a partition of unity on Ω , i.e.,*

$$\sum_{i=1}^{n_v} \sum_{j_1+j_2+j_3=2r-1} B_{i,j_1 j_2 j_3}^{v,r}(x, y) + \sum_{k=1}^{n_t} \sum_{j_1+j_2+j_3=r-2} B_{k,j_1 j_2 j_3}^{t,r}(x, y) = 1, \quad (4.18)$$

for all $(x, y) \in \Omega$.

Proof. We consider a particular macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$ in Δ and we show that the equality (4.18) holds on this triangle. The same argument can be used for the other triangles in Δ . We focus on the Bernstein-Bézier representation of the Powell-Sabin spline in the left-hand side of (4.18) and we verify that all Bézier ordinates in this representation (see Figure 2) are equal to one. If all Bézier ordinates are equal to one, then the spline is also equal to one.

From the definition of the Powell-Sabin B-splines it follows that only the N^v basis functions related to vertex V_i have function and derivative values at V_i that are not all zero. We also know that the Bernstein basis polynomials used in definition (4.6) form a partition of unity on PSr-triangle t_i . We then immediately obtain that

$$\sum_{j_1+j_2+j_3=2r-1} \alpha_{i,j_1j_2j_3}^{00} = 1, \quad (4.19a)$$

and

$$\sum_{j_1+j_2+j_3=2r-1} \alpha_{i,j_1j_2j_3}^{ab} = 0, \quad 1 \leq a+b \leq 2r-1, \quad (4.19b)$$

for $i = 1, 2, 3$. It follows that all Bézier ordinates in the three disks $D_{2r-1}(V_i)$, $i = 1, 2, 3$, are equal to one.

From the construction of the Powell-Sabin spline we know that the Bézier ordinates in the disk $D_{2r-1}(Z_k)$ around split point Z_k can be regarded as Bézier ordinates after subdivision of a single polynomial p of degree $2r-1$ defined on the triangle $\tilde{\mathcal{T}}_k$ with vertices W_i , $i = 1, 2, 3$, given in (3.5). Let $b_{i_1i_2i_3}$, $i_1 + i_2 + i_3 = 2r-1$, be the Bézier ordinates in the Bernstein-Bézier representation of this polynomial p on $\tilde{\mathcal{T}}_k$. By the construction of the Powell-Sabin spline, the values of $b_{i_1i_2i_3}$ in the three disks $D_{r-1}(W_i)$, $i = 1, 2, 3$, are specified by the values of $\alpha_{i,j_1j_2j_3}^{ab}$ in the sets $A_{i,j_1j_2j_3}$, $i = 1, 2, 3$ and $j_1 + j_2 + j_3 = 2r-1$. Because of (4.19), the ordinates $b_{i_1i_2i_3}$, for each $i_1 \geq r$ or $i_2 \geq r$ or $i_3 \geq r$, must be equal to one. We also know that the Powell-Sabin B-splines with respect to the vertices do not influence the values of the other ordinates $b_{i_1i_2i_3}$ with $i_1 < r$ and $i_2 < r$ and $i_3 < r$. Lemma 4.4 implies that the Powell-Sabin B-splines (in Bernstein-Bézier representation) with respect to triangle $\mathcal{T}_k$ can be locally regarded as a Bernstein basis polynomial of degree $2r-1$ defined on the smaller triangle $\tilde{\mathcal{T}}_k$. By taking the sum over these Bernstein basis polynomials we obtain that also the ordinates $b_{i_1i_2i_3}$, with $i_1 < r$ and $i_2 < r$ and $i_3 < r$, of polynomial p must be equal to one. Since all Bézier ordinates of p are equal to one, it follows that the Bernstein-Bézier representation of the Powell-Sabin spline in the left-hand side of (4.18) has Bézier ordinates in the disk $D_{2r-1}(Z_k)$ equal to one.

For each edge $e_{ij} = \langle V_i, V_j \rangle$ of $\mathcal{T}_k$, the remaining Bézier ordinates on the rows $0, \dots, r-1$ parallel to e_{ij} are determined by means of the C^{2r-1} -smoothness conditions across the edges $\langle R_{ij}, Z_k \rangle$. Because we already know that the Bézier ordinates in the three disks $D_{2r-1}(V_i)$ are equal to one, these remaining Bézier ordinates must also be equal to one. $\square$

Property 4.3. *The B-splines $B_{i,j_1j_2j_3}^{v,r}(x, y)$, $i = 1, \dots, n_v$, $j_1 + j_2 + j_3 = 2r-1$, and the B-splines $B_{k,j_1j_2j_3}^{t,r}(x, y)$, $k = 1, \dots, n_t$, $j_1 + j_2 + j_3 = r-2$, are nonnegative.*

Proof. We first prove that a B-spline $B_{i,j_1j_2j_3}^{v,r}(x, y)$ with respect to vertex V_i is nonnegative. Because of the local support we only have to show nonnegativity over the molecule M_i of vertex V_i . For the sake of simplicity, let $i = 1$ and let $\mathcal{T}_k(V_1, V_2, V_3)$ be a triangle in the molecule M_1 . We consider the Bernstein-Bézier representation of the B-spline on macro-triangle $\mathcal{T}_k$, see Figure 3(a). By Lemma 4.1 we know that the Bézier ordinates in the disk $D_{2r-1}(V_1)$ are nonnegative. Lemma 4.2 implies that the ordinates in the disk $D_{2r-1}(Z_k)$ are nonnegative. From Lemma 4.3 it follows that the Bézier ordinates on the rows $0, 1, \dots, r-1$ parallel to edge $\langle V_1, V_2 \rangle$ are nonnegative, and in an analogous way one can prove that the Bézier ordinates on the rows $0, 1, \dots, r-1$ parallel to edge $\langle V_1, V_3 \rangle$ are also nonnegative. By the construction of the B-spline, the remaining Bézier ordinates are zero. Since all its Bézier ordinates are nonnegative, the B-spline $B_{i,j_1j_2j_3}^{v,r}(x, y)$ is also nonnegative on $\mathcal{T}_k$. A similar argument can be used for all other triangles in the molecule M_1 .

We now prove the nonnegativity of a B-spline $B_{k,j_1j_2j_3}^{t,r}(x, y)$ with respect to triangle $\mathcal{T}_k(V_1, V_2, V_3)$. By the local support of the B-spline, it is sufficient to show nonnegativity over $\mathcal{T}_k$. We consider

the Bernstein-Bézier representation of the B-spline on macro-triangle $\mathcal{T}_k$, see Figure 3(b). By the construction of the spline, the spline has only nonzero Bézier ordinates corresponding to domain points in the disk $D_{2r-1}(Z_k)$. By Lemma 4.4 these Bézier ordinates can be regarded as the Bézier ordinates after subdivision of a particular Bernstein basis polynomial of degree $2r-1$. Since these Bézier ordinates are nonnegative, the B-spline $B_{k,j_1 j_2 j_3}^{t,r}(x, y)$ is nonnegative on $\mathcal{T}_k$ as well. $\square$

5 Applications

In this section we first show how control polynomials and control points can be defined. Then, a stable computation of the Bernstein-Bézier representation of a PSr-spline is given.

5.1 Control polynomials and control points

Each Powell-Sabin B-spline $B_{i,j_1 j_2 j_3}^{v,r}(x, y)$, $j_1 + j_2 + j_3 = 2r-1$, with respect to vertex V_i is related to a Bernstein basis polynomial of degree $2r-1$, see (4.6). The corresponding spline coefficients $c_{i,j_1 j_2 j_3}^{v,r}$, $j_1 + j_2 + j_3 = 2r-1$, of the Powell-Sabin spline in representation (4.1) can be regarded as the Bézier ordinates of a polynomial of degree $2r-1$ defined on PSr-triangle t_i . This polynomial will be called the control polynomial with respect to vertex V_i , and will be denoted by $T_i(x, y)$:

$$T_i(x, y) = \sum_{j_1+j_2+j_3=2r-1} c_{i,j_1 j_2 j_3}^{v,r} B_{j_1 j_2 j_3}^{2r-1}(x, y), \quad (5.1)$$

with $B_{j_1 j_2 j_3}^{2r-1}(x, y)$ the Bernstein basis polynomials (2.3) of degree $2r-1$ defined on PSr-triangle t_i . By the local support of the B-splines and by definitions (4.6) and (5.1), we have the following relation between a PSr-spline $s(x, y)$ and its control polynomial $T_i(x, y)$ at vertex V_i :

$$\begin{aligned} D_x^a D_y^b s(V_i) &= \sum_{j_1+j_2+j_3=2r-1} c_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{ab} \\ &= \frac{\binom{3r-1}{a+b}}{\binom{2r-1}{a+b}} \left(\frac{2r-1}{3r-1} \right)^{a+b} \sum_{j_1+j_2+j_3=2r-1} c_{i,j_1 j_2 j_3}^{v,r} D_x^a D_y^b B_{j_1 j_2 j_3}^{2r-1}(V_i) \\ &= \frac{\binom{3r-1}{a+b}}{\binom{2r-1}{a+b}} \left(\frac{2r-1}{3r-1} \right)^{a+b} D_x^a D_y^b T_i(V_i). \end{aligned} \quad (5.2)$$

Property 5.1. *The control polynomial $T_i(x, y)$ is tangent to the Powell-Sabin spline $s(x, y)$ at vertex V_i .*

Proof. Using (5.2), we have

$$s(V_i) = T_i(V_i), \quad D_x s(V_i) = D_x T_i(V_i), \quad D_y s(V_i) = D_y T_i(V_i),$$

implying that $T_i(x, y)$ is tangent to the spline at V_i . $\square$

Let $Q_{i,j_1 j_2 j_3}^{v,r} = (X_{i,j_1 j_2 j_3}^{v,r}, Y_{i,j_1 j_2 j_3}^{v,r})$ and $Q_{k,j_1 j_2 j_3}^{t,r} = (X_{k,j_1 j_2 j_3}^{t,r}, Y_{k,j_1 j_2 j_3}^{t,r})$ be points satisfying

$$\sum_{i=1}^{n_v} \sum_{j_1+j_2+j_3=2r-1} X_{i,j_1 j_2 j_3}^{v,r} B_{i,j_1 j_2 j_3}^{v,r}(x, y) + \sum_{k=1}^{n_t} \sum_{j_1+j_2+j_3=r-2} X_{k,j_1 j_2 j_3}^{t,r} B_{k,j_1 j_2 j_3}^{t,r}(x, y) = x, \quad (5.3a)$$

and

$$\sum_{i=1}^{n_v} \sum_{j_1+j_2+j_3=2r-1} Y_{i,j_1 j_2 j_3}^{v,r} B_{i,j_1 j_2 j_3}^{v,r}(x, y) + \sum_{k=1}^{n_t} \sum_{j_1+j_2+j_3=r-2} Y_{k,j_1 j_2 j_3}^{t,r} B_{k,j_1 j_2 j_3}^{t,r}(x, y) = y. \quad (5.3b)$$

We then define the PSr-control points as

$$\mathbf{c}_{i,j_1 j_2 j_3}^{v,r} = (X_{i,j_1 j_2 j_3}^{v,r}, Y_{i,j_1 j_2 j_3}^{v,r}, c_{i,j_1 j_2 j_3}^{v,r}), \quad (5.4a)$$

$$\mathbf{c}_{k,j_1 j_2 j_3}^{t,r} = (X_{k,j_1 j_2 j_3}^{t,r}, Y_{k,j_1 j_2 j_3}^{t,r}, c_{k,j_1 j_2 j_3}^{t,r}), \quad (5.4b)$$

with $c_{i,j_1 j_2 j_3}^{v,r}$ and $c_{k,j_1 j_2 j_3}^{t,r}$ the B-spline coefficients in representation (4.1). Since the Powell-Sabin spline basis forms a convex partition of unity, it follows that the graph of spline (4.1) lies inside the convex hull of the control points (5.4).

Property 5.2. Let $Q_{i,1}$, $Q_{i,2}$ and $Q_{i,3}$ be the three corner points of PSr-triangle t_i corresponding to vertex V_i . Let $\mathcal{T}_k(V_1, V_2, V_3)$ be a macro-triangle in Δ , containing the points W_l , $l = 1, 2, 3$, as defined in (3.5). If the points $Q_{i,j_1 j_2 j_3}^{v,r}$ related to V_i are taken as

$$Q_{i,j_1 j_2 j_3}^{v,r} = \frac{j_1}{2r-1} Q_{i,1} + \frac{j_2}{2r-1} Q_{i,2} + \frac{j_3}{2r-1} Q_{i,3}, \quad (5.5)$$

for all $j_1 + j_2 + j_3 = 2r - 1$, and the points $Q_{k,j_1 j_2 j_3}^{t,r}$ related to $\mathcal{T}_k$ are taken as

$$Q_{k,j_1 j_2 j_3}^{t,r} = \frac{r-1-j_1}{2r-1} W_1 + \frac{r-1-j_2}{2r-1} W_2 + \frac{r-1-j_3}{2r-1} W_3, \quad (5.6)$$

for all $j_1 + j_2 + j_3 = r - 2$, then they satisfy (5.3).

Proof. We denote the spline in the left-hand side of (5.3a) by $s^x(x, y)$, and the spline in the left-hand side of (5.3b) by $s^y(x, y)$. We consider a particular macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$ in Δ and we show that the equalities in (5.3) hold on this triangle. A polynomial in Bernstein-Bézier form (2.2), where the Bézier ordinates are taken as the x -coordinates (or y -coordinates) of the corresponding Bézier domain points, is equal to the plane $z = x$ (or $z = y$). This is a necessary and sufficient condition. We now verify that the Bézier control points in the Bernstein-Bézier representation of $s^x(x, y)$ lie on the plane $z = x$, and that the Bézier control points in the Bernstein-Bézier representation of $s^y(x, y)$ lie on the plane $z = y$. It then follows that $s^x(x, y) = x$ and $s^y(x, y) = y$.

By the choice (5.5) for $Q_{i,j_1 j_2 j_3}^{v,r}$ and by relation (4.6) at vertex $V_i = (x_i, y_i)$, we obtain

$$\sum_{j_1+j_2+j_3=2r-1} X_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{00} = x_i, \quad \sum_{j_1+j_2+j_3=2r-1} Y_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{00} = y_i, \quad (5.7a)$$

$$\sum_{j_1+j_2+j_3=2r-1} X_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{10} = 1, \quad \sum_{j_1+j_2+j_3=2r-1} Y_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{10} = 0, \quad (5.7b)$$

$$\sum_{j_1+j_2+j_3=2r-1} X_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{01} = 0, \quad \sum_{j_1+j_2+j_3=2r-1} Y_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{01} = 1, \quad (5.7c)$$

and

$$\sum_{j_1+j_2+j_3=2r-1} X_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{ab} = 0, \quad \sum_{j_1+j_2+j_3=2r-1} Y_{i,j_1 j_2 j_3}^{v,r} \alpha_{i,j_1 j_2 j_3}^{ab} = 0, \quad (5.7d)$$

for $2 \leq a + b \leq 2r - 1$. It follows that all Bézier control points of $s^x(x, y)$ in the three disks $D_{2r-1}(V_i)$, $i = 1, 2, 3$, are on the plane $z = x$, and that all Bézier control points of $s^y(x, y)$ in the three disks $D_{2r-1}(V_i)$ are on the plane $z = y$.

By the C^{2r-1} -smoothness of a Powell-Sabin spline of smoothness r at split point Z_k , we know that its Bézier control points in the disk $D_{2r-1}(Z_k)$ can be regarded as the Bézier control points after subdivision of a single polynomial p of degree $2r-1$ defined on the triangle $\tilde{\mathcal{T}}_k$ with vertices W_i , $i = 1, 2, 3$, given in (3.5). These polynomials corresponding to $s^x(x, y)$ and $s^y(x, y)$ are denoted by p^x and p^y , respectively. Let $\mathbf{b}_{i_1 i_2 i_3}^x$, $i_1 + i_2 + i_3 = 2r-1$, be the Bézier control points of polynomial p^x on triangle $\tilde{\mathcal{T}}_k$. From the construction of a PS r -spline and from (5.7), it follows that the control points $\mathbf{b}_{i_1 i_2 i_3}^x$, for each $i_1 \geq r$ or $i_2 \geq r$ or $i_3 \geq r$, must lie on the plane $z = x$. We know that the Powell-Sabin B-splines with respect to the vertices cannot influence the values of the other control points $\mathbf{b}_{i_1 i_2 i_3}^x$ with $i_1 < r$ and $i_2 < r$ and $i_3 < r$. Lemma 4.4 implies that the Powell-Sabin B-splines (in Bernstein-Bézier representation) with respect to triangle $\mathcal{T}_k$ can be locally regarded as a Bernstein basis polynomial of degree $2r-1$ defined on the smaller triangle $\tilde{\mathcal{T}}_k$. From the choice (5.6) for $Q_{k,j_1 j_2 j_3}^{t,r}$ it then follows that also the control points $\mathbf{b}_{i_1 i_2 i_3}^x$, with $i_1 < r$ and $i_2 < r$ and $i_3 < r$, lie on the plane $z = x$. Analogously, we can show that the Bézier control points $\mathbf{b}_{i_1 i_2 i_3}^y$ of polynomial p^y lie on the plane $z = y$. Thus, the Bézier control points of $s^x(x, y)$ and $s^y(x, y)$ in the disk $D_{2r-1}(Z_k)$ lie on the planes $z = x$ and $z = y$, respectively.

By the C^{2r-1} -smoothness of the splines $s^x(x, y)$ and $s^y(x, y)$ across the interior edges $\langle R_{ij}, Z_k \rangle$, one can easily see that the remaining Bézier control points on the rows $0, \dots, r-1$ parallel to the edges $\langle V_i, V_j \rangle$ of $\mathcal{T}_k$ must also lie on the planes $z = x$ and $z = y$, respectively. $\square$

We note that $\mathbf{c}_{i,j_1 j_2 j_3}^{v,r}$, $j_1 + j_2 + j_3 = 2r-1$, related to vertex V_i are the Bézier control points of the control polynomial $T_i(x, y)$. They can be organized in a triangular Bézier control net. The control points $\mathbf{c}_{k,j_1 j_2 j_3}^{t,r}$, $j_1 + j_2 + j_3 = r-2$, related to triangle $\mathcal{T}_k$ can also be organized in a triangular net.

The control polynomials mimic the shape of the spline surface in the neighbourhood of their corresponding vertices. They can be used to locally model or change the shape of the spline surface in a predictable way. The control polynomials can be geometrically manipulated through the control points. Because of the local support of the B-splines, a change of a control polynomial (or control point) will only affect the spline patches related to a few triangles in the triangulation.

5.2 Bézier ordinates of a PS r -spline

The Bézier ordinates of a Powell-Sabin spline $s(x, y)$ in representation (4.1) can be computed in a stable way from its coefficients $c_{i,j_1 j_2 j_3}^{v,r}$ and $c_{k,j_1 j_2 j_3}^{t,r}$, using only convex combinations. We consider the Bernstein-Bézier representation of the spline restricted to a triangle $\mathcal{T}_k(V_1, V_2, V_3)$ as shown in Figure 1. The barycentric coordinates of the points indicated in this figure are given in (4.9).

Because of the C^{2r-1} -smoothness of the PS r -spline at vertex V_1 and the local support of the B-splines, the Bézier ordinates in the disk $D_{2r-1}(V_1)$ are completely specified by the choice of control polynomial $T_1(x, y)$ corresponding to vertex V_1 , see (5.1). In Lemma 4.1 it has been shown that the Bézier ordinates of the B-splines $B_{1,j_1 j_2 j_3}^{v,r}(x, y)$, $j_1 + j_2 + j_3 = 2r-1$, in the neighbourhood of vertex V_1 can be computed as the Bézier ordinates of the Bernstein basis polynomials of degree $2r-1$ with respect to PS r -triangle t_1 after subdivision. Hence, the Bézier ordinates of the spline $s(x, y)$ in the disk $D_{2r-1}(V_1)$ can be simply found by subdividing control polynomial $T_1(x, y)$, using (2.10), onto the two smaller triangles spanned by the four points

$$V_1, \quad \lambda_{12}V_1 + \lambda_{21}S_{12}, \quad \lambda_{13}V_1 + \lambda_{31}S_{13}, \quad z_1V_1 + z_2S_{12} + z_3S_{13}, \quad (5.8)$$

where S_{12} and S_{13} are PS r -points related to vertex V_1 , see (4.5). Analogously, one finds that the Bézier ordinates of the spline $s(x, y)$ in the disks $D_{2r-1}(V_2)$ and $D_{2r-1}(V_3)$ can be computed through subdivision of the control polynomials $T_2(x, y)$ and $T_3(x, y)$ corresponding to vertices V_2 and V_3 , respectively.

By the C^{2r-1} -smoothness of the spline $s(x, y)$ at split point Z_k , the Bézier ordinates in the disk $D_{2r-1}(Z_k)$ are specified by a single polynomial p of degree $2r-1$ defined over the triangle $\tilde{T}_k$ with the vertices W_i , $i = 1, 2, 3$, given in (3.5). Let $b_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = 2r-1$, be the Bézier ordinates of this polynomial p on $\tilde{T}_k$. From the construction of the B-splines, it follows that

$$b_{r-1-j_1, r-1-j_2, r-1-j_3} = c_{k, j_1 j_2 j_3}^{t, r}, \quad (5.9)$$

for all $j_1 + j_2 + j_3 = r-2$. The ordinates $b_{i_1 i_2 i_3}$ with $i_1 \geq r$ are completely specified by the choice of control polynomial $T_1(x, y)$. Following the same line of argument as in the proof of Lemma 4.2 to obtain (4.13), we find that these ordinates can be expressed in terms of the blossom of control polynomial $T_1(x, y)$ with as arguments a mixture of the barycentric coordinates of the PS r -points S_{12} and S_{13} and the points in (5.8). We can use the multi-affine de Casteljau algorithm, see (2.9), to evaluate this blossom. The ordinates $b_{i_1 i_2 i_3}$ with $i_2 \geq r$ or $i_3 \geq r$ can be computed in a similar way from the control polynomials $T_2(x, y)$ or $T_3(x, y)$, respectively. Then, the Bézier ordinates of the spline $s(x, y)$ in the disk $D_{2r-1}(Z_k)$ can be computed through subdivision of this polynomial p .

Given $0 \leq m \leq r-1$, let E_m be the row of domain points at distance m parallel to the edges $\langle V_1, R_{12} \rangle \cup \langle R_{12}, V_2 \rangle$. From the C^{2r-1} -smoothness of the spline $s(x, y)$ across the interior edge $\langle R_{12}, Z_k \rangle$, it follows that the Bézier ordinates of the spline corresponding to the central $4r-1$ domain points along row E_m are specified by a single onedimensional polynomial $\hat{p}$ of degree $2r-1$ defined over the line segment between the points $\hat{W}_1$ and $\hat{W}_2$, given in (3.7). Let $\hat{b}_{i_1 i_2}$, $i_1 + i_2 = 2r-1$, be the Bézier ordinates of this polynomial $\hat{p}$ on $\langle \hat{W}_1, \hat{W}_2 \rangle$. The ordinates $\hat{b}_{i_1 i_2}$ with $i_1 \geq r$ are completely specified by control polynomial $T_1(x, y)$. Following a similar reasoning as in the proof of Lemma 4.3 to obtain (4.16), we find that these ordinates can be expressed in terms of the blossom of control polynomial $T_1(x, y)$ with as arguments a mixture of the barycentric coordinates of PS r -point S_{12} and the points given in (5.8). One can use the multi-affine de Casteljau algorithm to evaluate this blossom. The other ordinates $\hat{b}_{i_1 i_2}$ with $i_2 \geq r$ can be obtained in an analogous way from the control polynomial $T_2(x, y)$. Then, the unknown Bézier ordinates of the spline $s(x, y)$ on row E_m can be computed through subdivision of this polynomial $\hat{p}$.

In a similar way one can also compute the remaining Bézier ordinates of the spline $s(x, y)$ on the rows $0, 1, \dots, r-1$ parallel to the edges $\langle V_2, V_3 \rangle$ and $\langle V_3, V_1 \rangle$ from the control polynomials $T_1(x, y)$, $T_2(x, y)$ and $T_3(x, y)$.

Only convex combinations are needed in the presented computation of all Bézier ordinates of the spline $s(x, y)$ given its B-spline coefficients. More details of this computation can be found in [11] in the case of a quintic Powell-Sabin spline with smoothness $r = 2$.

Once the Bernstein-Bézier representation of the spline is known, evaluation or differentiation of the spline within each of the six subtriangles can then be performed using the de Casteljau algorithm (see, e.g., [2]).

6 Reduced spline spaces

In this section we propose a further reduction of the Powell-Sabin spline space (3.2)–(3.3). We will derive specific choices of the B-spline coefficients $c_{k, j_1 j_2 j_3}^{t, r}$, $k = 1, \dots, n_t$ and $j_1 + j_2 + j_3 = r-2$, in representation (4.1), such that the resulting Powell-Sabin splines are still able to reproduce exactly polynomials up to degree $3r-1$. In this way, their optimal approximation order will be retained with less degrees of freedom.

We consider a single macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$ in Δ , on which we define a polynomial $q(x, y)$ of degree $3r-1$ in Bernstein-Bézier form (2.2). Its Bézier ordinates are denoted by $d_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = 3r-1$. We now derive conditions on the B-spline coefficients of a Powell-Sabin spline $s(x, y) \in S^r(\Delta^*)$ to be equal to this polynomial $q(x, y)$ on $\mathcal{T}_k$. To this end, we consider the

restriction of control polynomial $T_1(x, y)$ onto the triangle with the PS r -points V_1 , S_{12} and S_{13} as its three vertices. We denote the Bézier ordinates of the control polynomial on this finer triangle by $e_{1,j_1j_2j_3}$, $j_1 + j_2 + j_3 = 2r - 1$. Using (2.9) and (2.10), these ordinates can be easily computed from the spline coefficients $c_{1,j_1j_2j_3}^{v,r}$ related to vertex V_1 . In this computation we only use convex combinations. From the definitions of PS r -spline $s(x, y)$ and control polynomial $T_1(x, y)$, it then follows that $s(x, y)$ interpolates polynomial $q(x, y)$ and its derivatives up to order $2r - 1$ at vertex V_1 , i.e.,

$$D_x^a D_y^b s(V_1) = D_x^a D_y^b q(V_1), \quad (6.1)$$

for all $0 \leq a + b \leq 2r - 1$, if

$$e_{1,j_1,j_2,j_3} = d_{r+j_1,j_2,j_3}, \quad (6.2)$$

for all $j_1 + j_2 + j_3 = 2r - 1$. We obtain similar interpolation conditions at the vertices V_2 and V_3 . Let $e_{2,j_1j_2j_3}$, $j_1 + j_2 + j_3 = 2r - 1$, be the Bézier ordinates of control polynomial $T_2(x, y)$ restricted on the triangle with vertices (V_2, S_{23}, S_{21}) , and let $e_{3,j_1j_2j_3}$, $j_1 + j_2 + j_3 = 2r - 1$, be the Bézier ordinates of control polynomial $T_3(x, y)$ restricted on the triangle with vertices (V_3, S_{31}, S_{32}) . The spline $s(x, y)$ interpolates $q(x, y)$ and its derivatives up to order $2r - 1$ at the vertices V_2 and V_3 , if for all $j_1 + j_2 + j_3 = 2r - 1$,

$$e_{2,j_1,j_2,j_3} = d_{j_3,r+j_1,j_2}, \quad (6.3)$$

and

$$e_{3,j_1,j_2,j_3} = d_{j_2,j_3,r+j_1}, \quad (6.4)$$

respectively.

Referring to Figure 2, the Bézier ordinates of spline $s(x, y)$ in the disk $D_{2r-1}(Z_k)$ can be computed as the Bézier ordinates after subdivision of a polynomial $p(x, y)$ of degree $2r - 1$ defined on the triangle with vertices W_i , $i = 1, 2, 3$, given in (3.5). The spline $s(x, y)$ has the same function value and derivative values up to order $2r - 1$ at split point Z_k as polynomial $q(x, y)$, i.e.,

$$D_x^a D_y^b s(Z_k) = D_x^a D_y^b q(Z_k), \quad (6.5)$$

for all $0 \leq a + b \leq 2r - 1$, if the Bézier ordinates $b_{i_1i_2i_3}$ of $p(x, y)$ satisfy

$$b_{i_1i_2i_3} = b_{i_1,i_2,i_3}^{[r]}, \quad (6.6a)$$

for $i_1 + i_2 + i_3 = 2r - 1$. Here, the values of $b_{i_1,i_2,i_3}^{[r]}$ are obtained from $d_{i_1i_2i_3}$ by applying r steps of the de Casteljau algorithm, i.e.,

$$b_{i_1,i_2,i_3}^{[0]} = d_{i_1i_2i_3}, \quad (6.6b)$$

for $i_1 + i_2 + i_3 = 3r - 1$, and

$$b_{i_1,i_2,i_3}^{[m]} = z_1 b_{i_1+1,i_2,i_3}^{[m-1]} + z_2 b_{i_1,i_2+1,i_3}^{[m-1]} + z_3 b_{i_1,i_2,i_3+1}^{[m-1]}, \quad (6.6c)$$

for $i_1 + i_2 + i_3 = 3r - m - 1$, and $m = 1, \dots, r$. This can be shown as follows. Let $\mathcal{P}$ and $\mathcal{Q}$ be the blossoms of polynomials p and q , respectively. By (6.6), we have

$$\mathcal{P}(\tau^1, \tau^2, \dots, \tau^{2r-1}) = \mathcal{Q}\left(\underbrace{z, z, \dots, z}_{r \text{ times}}, \tau^1, \tau^2, \dots, \tau^{2r-1}\right), \quad (6.7)$$

with $z = (z_1, z_2, z_3)$ the barycentric coordinates of split point Z_k with respect to $\mathcal{T}_k$. Equality (6.7) can be easily verified by applying the multi-affine de Casteljau algorithm to both blossoms. From the subdivision property of blossoms (see (2.10)) and (6.7), it follows that the Bézier ordinates in the disk $D_{2r-1}(Z_k)$ of a spline $s(x, y)$ satisfying (6.6) can be regarded as Bézier ordinates after subdivision of the polynomial $q(x, y)$. This implies (6.5).

From (5.9) we know that the spline coefficients $c_{k,j_1j_2j_3}^{t,r}$, $j_1 + j_2 + j_3 = r - 2$, are equal to a specific subset of Bézier ordinates of polynomial $p(x, y)$. Using (6.6) we get the constraints

$$c_{k,j_1j_2j_3}^{t,r} = b_{r-1-j_1, r-1-j_2, r-1-j_3}^{[r]}, \quad (6.8)$$

for all $j_1 + j_2 + j_3 = r - 2$. From (6.6) and (6.8) it follows that a coefficient $c_{k,j_1j_2j_3}^{t,r}$ is only influenced by the ordinates $d_{i_1i_2i_3}$ of polynomial $q(x, y)$, where

$$\begin{aligned} r - 1 - j_1 &\leq i_1 \leq 2r - 1 - j_1, & \text{and} \\ r - 1 - j_2 &\leq i_2 \leq 2r - 1 - j_2, & \text{and} \\ r - 1 - j_3 &\leq i_3 \leq 2r - 1 - j_3. \end{aligned}$$

Because $j_1 + j_2 + j_3 = r - 2$, it follows that at most the ordinates $d_{i_1i_2i_3}$, with

$$1 \leq i_1 \leq 2r - 1, \quad \text{and} \quad 1 \leq i_2 \leq 2r - 1, \quad \text{and} \quad 1 \leq i_3 \leq 2r - 1,$$

are involved in any expression (6.8). Inspired by interpolation conditions (6.2)–(6.4) at the vertices, we can set the values $d_{i_1i_2i_3}$ in (6.6b) as follows.

- If $i_1 \leq 2r - 1$, $i_2 \leq r - 1$ and $i_3 \leq r - 1$, then

$$d_{i_1i_2i_3} = e_{1,i_1-r,i_2,i_3}. \quad (6.9a)$$

- If $i_1 \leq r - 1$, $i_2 \leq 2r - 1$ and $i_3 \leq r - 1$, then

$$d_{i_1i_2i_3} = e_{2,i_2-r,i_3,i_1}. \quad (6.9b)$$

- If $i_1 \leq r - 1$, $i_2 \leq r - 1$ and $i_3 \leq 2r - 1$, then

$$d_{i_1i_2i_3} = e_{3,i_3-r,i_1,i_2}. \quad (6.9c)$$

- If $i_1 \geq r$, $i_2 \geq r$ and $i_3 \geq 1$, then

$$d_{i_1i_2i_3} = \tau_1 e_{1,i_1-r,i_2,i_3} + \tau_2 e_{2,i_2-r,i_3,i_1}, \quad (6.9d)$$

given some values $\tau_1 \geq 0, \tau_2 \geq 0$ and $\tau_1 + \tau_2 = 1$.

- If $i_1 \geq r$, $i_2 \geq 1$ and $i_3 \geq r$, then

$$d_{i_1i_2i_3} = \tau_1 e_{1,i_1-r,i_2,i_3} + \tau_3 e_{3,i_3-r,i_1,i_2}, \quad (6.9e)$$

given some values $\tau_1 \geq 0, \tau_3 \geq 0$ and $\tau_1 + \tau_3 = 1$.

- If $i_1 \geq 1$, $i_2 \geq r$ and $i_3 \geq r$, then

$$d_{i_1i_2i_3} = \tau_2 e_{2,i_2-r,i_3,i_1} + \tau_3 e_{3,i_3-r,i_1,i_2}, \quad (6.9f)$$

given some values $\tau_2 \geq 0, \tau_3 \geq 0$ and $\tau_2 + \tau_3 = 1$.

If the spline coefficients $c_{k,j_1j_2j_3}^{t,r}$ are chosen as in (6.8) with (6.6) and (6.9), then the corresponding Powell-Sabin spline $s(x, y)$ will be able to reproduce polynomials of degree $3r - 1$.

Note that only convex combinations are used in (6.6) and (6.9). The values of e_{1,j_1,j_2,j_3} , e_{2,j_1,j_2,j_3} and e_{3,j_1,j_2,j_3} can be found by subdivision of the control triangles. Thus, the proposed choice of the B-spline coefficients $c_{k,j_1j_2j_3}^{t,r}$ can be computed from the other B-spline coefficients $c_{i,j_1j_2j_3}^{v,r}$ through a sequence of convex combinations. It follows that the graph of such a reduced Powell-Sabin spline lies inside the convex hull of the control points $\mathbf{c}_{i,j_1j_2j_3}^{v,r}$.

A spline $s(x, y)$ in such a reduced spline space $S_{red}^r(\Delta^*)$ is uniquely defined by the following interpolation problem:

$$D_x^a D_y^b s(V_l) = f_{x^a y^b, l}, \quad 0 \leq a + b \leq 2r - 1, \quad l = 1, \dots, n_v, \quad (6.10)$$

for a given set of $f_{x^a y^b, l}$ -values. Thus, the dimension of $S_{red}^r(\Delta^*)$ equals $r(2r + 1)n_v$.

7 Concluding remarks

In this paper we have presented a family of bivariate (finite element) spline spaces with smoothness $r \geq 1$ and polynomial degree $d = 3r - 1$. They are defined on a triangulation with Powell-Sabin refinement. We have also constructed a set of basis functions with some useful properties for computer aided geometric design. The spline spaces constructed in this paper are not of minimal polynomial degree for a given smoothness $r \geq 3$, see e.g. [3]. Yet, they have several other interesting properties.

1. PS r -splines can be uniquely defined by means of interpolation scheme (3.4) requiring function and derivative information at the vertices and split points. There is no need to know normal derivatives at certain points along edges. The reduced interpolation scheme (6.10) only requires function and derivative information at the vertices of the original triangulation.
2. PS r -splines can be represented in a normalized B-spline representation. The B-splines have a local support, they are nonnegative, and they form a partition of unity. The construction of the basis also allows the definition of control points and control polynomials, which are tangent to the spline surface. They may be useful for modelling purposes.
3. The Bernstein-Bézier form of such a spline can be computed from the PS r -spline coefficients using only convex combinations. In combination with the de Casteljau algorithm, this gives us a stable method to evaluate a PS r -spline.

Sablonnière constructed in [8] similar spline spaces with the same combination of smoothness r and polynomial degree d . If the triangulation consists of a single triangle, then these spline spaces are identical to the ones presented in this paper. Yet, the construction slightly differs when the triangulation has more triangles. Sablonnière proposed an interpolation scheme requiring partial derivatives at the vertices and triangle split points, and also normal derivatives at points along the edges. He also suggested a family of spline spaces with reduced dimension.

The condition that each PS r -triangle must contain the PS r -points (4.5) is a sufficient condition for the nonnegativity of the basis functions. It is not a necessary condition: in case of quadratic Powell-Sabin splines ($r = 1$, $d = 2$), Dierckx [1] has proposed another set of points, closer to the vertex.

For the reduced Powell-Sabin spline space presented in Section 6 one can also construct a normalized basis, such that its basis functions have a local support and form a convex partition of unity. Such a basis could be constructed in a similar way as the basis described in Section 4 for the full PS r -spline space. A reduced basis function is defined by setting in the spline representation (4.1), with $c_{k,j_1j_2j_3}^{t,r}$ chosen as in Section 6, a single coefficient $c_{i,j_1j_2j_3}^{v,r}$ to be one and all other coefficients $c_{i,j_1j_2j_3}^{v,r}$ to be zero.

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