H. Grassmann's Contribution to Whitehead's Foundations of Logic and Mathematics

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version not corrected

1 Introduction

The title could sound presumptuous to a reader familiar with Gert Schubring's volume [Schubring 1996] devoted to Hermann Grassmann. There, we learn from Ivor Grattan-Guinness and from Volker Peckhaus, that, if Herman Grassmann contributed to logic at all, his contribution was not as substantial as that of his brother, Robert.

Moreover, although Herman did influence Whitehead's views on mathematics, this cannot be obvious if one restricts Whitehead's work on the foundations mathematics to the work produced during his years of collaboration with Bertram Russell. This period was only one step in the development of Whitehead. Indeed, there is a single Whitehead and there is a unity in his thought that covers all his work. Moreover, this unity is deeply rooted in 19th century cultural and intellectual and scientific context.

The foundations referred to in the title thus extend far beyond the foundational issues that appeared toward the end of the century. They include questions raised by complex and hypercomplex numbers, the new algebras and the new geometries. Moreover, various perspectives and interests related to education, to ethics and to philosophy of nature that were shared by both Grassmann and Whitehead are not foreign to these questions. This is what we intend to suggest in the next few pages.

2 A. N. Whitehead's Treatise on Universal Algebra

Alfred North Whitehead (1861-1947), the logician, author with Bertrand Russell of the *Principia Mathematica*, was also a well-known natural philosopher and, in the last part of his life in Harvard, a famous philosopher. Before that, he had been a mathematician in Cambridge where, from the end of the 1880's, Grassmann's *Ausdehnungslehre* had been part of his teaching.

In 1898, he published a *Treatise on Universal Algebra with applications* [Whitehead 1898]. Volume one of this work was also the first book essentially devoted to Grassmann to be published in Britain. The preface explains Whitehead's goal of investigating all systems of symbolical reasoning related to ordinary algebra, including Grassmann's algebra, Hamilton's quaternions, Boole symbolic Logic and Peirce's associative algebras. His approach relied on a "*generalized conception of space* [...] *in the belief that the properties and operations involved*" can lead to "*a uniform method of interpretation of the various algebras*" [Whitehead 1898, V]. These would so appear as symbolical system and analysis engine of reasoning with respect to the abstract notion of space.

The style and presentation of Herman Grassmann's *lineale Ausdehnungslehre* in 1844 [A1], the theory of linear form, i.e. "*the collection of elements into which a generating* *element is transformed by a continuous evolution"*[Grassmann 1995, 47], the first part of his *Wissenschaft der extensiven Grösse*, had been a major obstacle to the reception of the work. Introduced in its full title as a new mathematical discipline, A1 started with general and foundational considerations, more philosophical than mathematical, that were certainly not on the mind of practitioners of mainstream mathematics. Indeed, as Victor Schlegel remarks in the preface of his *System der Raumlehre* [1872], at the time of publication, imaginary and impossible quantities were still considered with suspicion, non-Euclidean geometries were not unanimously received and mathematicians were not ready to show interest in the *n*-dimensional *lineale Ausdehnungslehre* and its unusual operations on unusual objects.

This failure convinced the author of rewriting the A1 entirely, stripping it of its misunderstood philosophical content and form of presentation while, at the same time, completing the originally projected second part. This work, *Die Ausdehnungslehre, Vollständig und in strenger Form*, [A2], came out in 1862.

Following this publication, Grassmann's theory attracted some attention, but certainly not to the point that he himself had wished, i.e., not that a few recipes but that the domain itself treated in the book be taken into consideration.

These difficulties being known, one could wonder why Whitehead did also begin, in the first Book of his *Treatise*, with the same long and verbose introduction on the foundations of its subject. He had an ambitious program in mind, that of showing the unity of idea of the new algebras. Indeed, in the new algebras that do no longer consider quantities and numbers, he sees "*bodies of propositions, inter-related by deductive reasoning, and based upon conventional definitions which are generalization of fundamental conceptions*"[Whitehead 1898, VIII]. This is the beginning of a science that applies to almost everything. He thus sees the ideal of mathematics in the construction of a calculus that supports the reasoning in all compartments of thought or experience, such that "*all serious thought which is not philosophy, or inductive reasoning, or imaginative literature, shall be mathematics developed by means of a calculus*" [ibd.]. This thus requires a definition of a calculus and of its object.

Following a presentation of the algebra of symbolic logic, the rest of the volume is devoted to Grassmann's algebra as applied to space in descriptive geometries, linear and metric, Euclidean and non-Euclidean.

Whitehead borrowed his title from Sylvester and several reviewers complained that the content was not universal algebra. Nevertheless, it has been argued elsewhere that the title is quite appropriate to the content and to the intention of the book. [Riche 2007] Whitehead's flavor of extension theory will not be considered here. It is a variation of A1 and A2. More may be learned from authors contributing these proceedings, David Hestenes, Alvin Swimmer and John Brown or others who did write specifically on Whitehead's geometric algebra, for example, Henry George Forder [1960] or Stephen Blake [2005].

3 A picture of A. N. Whitehead by D. Emmett

We may rely on the authority of the British philosopher Dorothy Emmett (1904-2000) [Emmet 1948], a former student of Whitehead, to summarize in a few words the leading ideas in her teacher's cosmology, his so-called "Process Philosophy".

Throughout Whitehead's work, Emmett sees two sides: "*an interest in formal schemes of logical relations, built on the model of a scheme of mathematical postulates, and an interest in the concrete many-sidedness of experience*." (266) Whitehead "*seeks the forms in the facts*" of a world of events seen as process from which the formal patterns can be abstracted. Whitehead held not only that events could be related by abstractions "*but also that there can be a general scheme of formal relationships defining the logical structure of all that exists.*" [Emmet 1948, 274]

Emmett thus quotes that startling remark of Whitehead in his last paper: "*We must end with my first love -Symbolic Logic. When in a distant future the subject has expanded, so as to examine patterns depending on connections other than those of space, number and quantity -when this expansion has occurred, I suggest that Symbolic Logic, that is to say, the symbolic examination of pattern with the use of real variables, will become the foundation of aesthetics. From that stage, it will proceed to conquer ethics and theology.*" [Whitehead, quoted in Emmet 1948, 274]

One may add a few specific themes or concepts to complete this extremely short characterisation of Whitehead: permanence amid change; abstract scheme and concrete experience; uniform and universal relatedness in the general scheme, that is, its logical structure and the process of the world supplying it with empirical value. The ambiguity in "value" then adds an ethical dimension to the assignation of empirically observable or real content to the logical variables of the scheme.

4 Structure and method: From Leibniz to the Grassmanns and A. N. Whitehead

These characteristics can be used as guidelines through the all work of Whitehead in search of an abstract and general structure explaining all that exists. They are not foreign to H. Grassmann himself, and actually, to the whole Grassmann family. These were inherited from Leibniz.

4.1 Leibniz's thesis

In 1666, G. Leibniz submitted a nine pages "*Disputatio Arithmetica de Complexionibus*" [Leibniz 1666a] to the Philosophy Faculty of the Academy of Leipzig. This short piece is extracted from the thesis he defended shortly after, his "*De Arte Combinatoria*". At the time, the idea of a Mathesis Universalis, a universal science, was popular and, among mathematicians and scientists, the famous German Jesuit, Athanase Kircher, author of an "*Ars Magna Sciendi sive Combinatoria*", was a source of inspiration. It is there that Leibniz found his "*ars inveniendi*", his combinatory theory that he sees as universally applicable and which allowed him to go further than the current mathematical method based on analysis and synthesis, i.e., deconstruction and reconstruction, and to reach his art of invention and judgement.

4.1.1 Divisibility

In two words, Leibniz starts from the metaphysical category of Being and its affections. These are either, quality, quantity, or relation; they are respectively absolute and relative modes of Being, i.e., Being with respect to itself, to its parts and to other Beings.

Relation is either a union, i.e., it relates the parts in a unit, thus forming a whole, or it is a *convenientia*, i.e., some gathering of elements. From the union, unity as well as number can be abstracted. Quantity is thus the number of parts; and in all things, quantity and number coincide. Nevertheless, the relation or the ratio of quantity to number remains extrinsic as long as the number of parts is not known. This is, according to Leibniz, where the *Analytica Speciosa* originates. It was mainly developed by Descartes and later collected by van Schoten and Bartholinus in their *Elements of Mathesis Universalis*. Analysis is thus the doctrine of ratios and proportions or of unknown quantity while arithmetics is that of known quantity or numbers.

Numbers having that universal property of being in some way incorporated in any union of Beings, they belong to metaphysics -as long as metaphysics is accepted as the doctrine that concerns all kinds of Beings. Indeed, the Mathesis is no longer a single discipline but several disciplines whose particular objects are treated independently of their particular quantity, and in a unified way. For example, arithmetics and analysis concern Quantity in Beings, geometry concerns Quantity in bodies, their spatial extension.

4.1.2 Combinatorics

Leibniz' combinatorics is based on the notion of "*complexions*", i.e., the decomposition of a whole into its smaller parts and their combinations, the parts varying with respect to each other and with respect to the whole. These variations are of two sorts: the combinations ("*complexions*") and the positions (*situs*) of the parts. Although they both belong to metaphysics, the quantity of variation belong to number and arithmetic and the variation of position belongs to the figures. In the "*situs*", the positions of the parts with respect to the whole can be absolute, this introduces order. Or they can be relative to the other parts and this introduces neighbourhood.

These general ideas are then systematically defined precisely. Obviously, the complexions amount to the usual n to k combinations, $C(n,k)$ of n elements or, as Leibniz write them, the *com2/3/4/nations*, the rows of Pascal's triangle or some permutations of elements.

Soon after the defense, in a letter to Christian Daum, Leibniz admitted that it is to please the audience rather than to follow his own ideas that he mentioned Kircher and Lulle and entitled his work *de Arte Combinatoria*. Because, "*it is not so much the variation of conjunctions of things but the combination of these same things by the method of permutation of positions that matters. Moreover, in the operation of combination, we can decide which are the most useful and the best transpositions* [...] *Here is the key to all sciences whose precepts contain the Universal Logic of discovery.*" [Leibniz 1666b, 5]

As Leibniz wrote in his thesis, his *analysis situs*, his calculus of invention, different from algebra in its operations and use of symbols, explains the *situs* immediately and accurately. This is the geometric calculus that H. Grassmann will develop.

With this new calculus, Leibniz discovered the operational character of thought and the premises of the notion of group as it will be developed later by Joseph-Louis Lagrange and Evarist Galois in their treatment of algebraic equations on the basis of the invariance of permutations of their roots. One may remark here that it is in groups, their form and invariance that Henry Poincaré saw the essence of mathematical reasoning.

4.2 What Hermann did learn from his father Justus

The publication of some of Leibniz's works during the 17th century renewed the interest for his ideas and for his combinatorics. It is obvious with Christian Wolf, Gottfried Ploucquet and Johann Heinrich Lambert, but more so with Carl Friedrich Hindenburg in his *Beschreibung einer ganz neuen Art,...*, [1776], with Heinrich August Töpfer in *Combinatorische Analytik und Theorie der Dimensionszeichen in Parallele gestellt* [1793], or in Christian August Semler's *Versuch über die combinatorische Methode, ein Beitrag zur angewandten Logik und allgemeinen Methodik*, [1811].

4.2.1 From crystallonomy

Of course, this 'neuen Art' influenced Justus Grassmann, the father of Hermann and Robert. In his *Zur physischen Krystallonomie und geometrischen Combinationslehre* [1829], or in his *Combinatorische Entwicklung der Krystallgestalten* [1833], he devised simpler methods and redicovered independently earlier ones. Indeed, the respective positions of the faces and cleavage planes of a crystal were usually determined by sets of specific equations corresponding to the planes of a specific crystal. J. Grassmann considered normals to the surfaces, planes and cleavages, of the crystal and their projection on a circumscribing sphere. Doing so, the analytical approach was replaced by spherical trigonometry, an approach wich does not make any reference to the specific crystal it is intended to represent. This method will be advertised much later by William H. Miller in his *On the Crystallographic Method of Grassmann, and on its employment in the investigation of the general geometric properties of Crystals* [1868].

Obviously, combinatorics and geometrical methods introduced some improvement in crystallography, in the classification of the various forms of crystals and in their structural theory. In the A1, as an application of his own method, Hermann will show that given three edges (or boundary planes) of a crystal not in one plane, every other edge (or boundary planes) can be represented by a rational linear combination of the given ones.

4.2.2 From philosophy

Justus Grassmann also developed the philosophical interests of his sons. Jakob Friedrich Fries, author of *Die mathematische Naturphilosophie nach philosophischer Methode bearbeitet: Ein Versuch* [1822], who discusses the issue of the foundations of mathematics, pure and applied, may have been important but not as much as Friedrich Schleiermacher.

We know from A. C. Lewis [Lewis 1996] that Schleiermacher's *Dialektik*

[Schleiermacher 1839] had impressed the Grassmann brothers. Here, we want to insist on Schleiermacher's method and on his ethics, because *Dialektik* is the continuation of his earlier *Grundlinien*, a critique of ethics of the times [Schleiermacher 1803], and because his philosophy, in many respects, sounds rather "Whiteheadian". A single quotation from the last pages of the *Grundlinien* may suffice to give the tone and to clarify what we mean here: "*The way we represent the nature of any science to ourselves depends on the nature of our own moral consciousness, so that, in general, the true idea of a system of human knowledge without which no science can be complete nor true, depends also on* [...] *the full consciousness of the highest laws and of the true nature of humanity*" [Schleiermacher 1803, 352].

The goal of Schleiermacher in the *Grundlinien* was to furnish ethics with a scientific foundation that would give it a degree of certainty equal to that of arithmetic or geometry. Having missed his goal, he looked for a method that could provide the expected basis. Among the methods he contemplates, the heuristic method is the method that will be echoed in the introduction of the A1, in the derivation and the exposition of the notions of pure mathematics and of extension theory. Actually, this method amounts to the construction of knowledge, mathematical or else, in what we have considered earlier as a structural way, from the combination of elementary parts.

In the process of building up knowledge, the opposition of reason and nature determines the existence of two main sciences: the sciences of the mind and the sciences of nature. They both have a speculative and an empirical side. In the sciences of reason, these are ethics and history. Ethics is constituted of the good, the virtue and the duty. Virtue is the union of nature and reason; seen objectively, this union of nature and reason is the good; and the duty is action directed toward the understanding of nature.

Any knowledge is rooted in the Mannigfaltigkeit of appearences and in the unity of the world. *Dialektik* that supervises and organizes the process of knowledge also insures its unity. Of course, didactics, one of the Grassmanns as well as Whitehead's major interests, is included in that process. Knowledge is now part of our moral duties; being virtuous will make us good and secure the unity of the world.

4.3 Parenthesis on prices

Proof of the renewed interest for Leibniz, the Jablonowskische Gesellschaft der Wissenschaften of Leipzig whose vocation was to stimulate the cooperation of European researchers with scientific questions and prices, proposed in 1845, on the occasion of the celebration the next year of the 200th anniversary of Leibniz's birthday, the following topic: "to evaluate and to develop the geometric calculus invented by Leibniz or to construct one equivalent".

That celebration was also the occasion to inaugurate the Royal Society of Sciences of Saxony. At the end of the ceremony, it was announced that the price of 48 ducats was attributed to Hermann Grassmann, school teacher in Stettin, for solving the problem. The reviewer of his Preisschrift, the *Geometric Analyse*, for the Archiv der Mathematik und Physik, (1847), recommended the essay to his readers, noting that it is related to the A1, and that it could not be understood without an understanding of the later.

Coincidence? Perhaps. If H. Grassmann was the lucky (and only) candidate for the Jablonowski price, A. N. Whitehead will be the unlucky candidate of the second Lobachewski Price in 1900. According to the price regulations, a draw would decide in case several works submitted were found of equal value. Even though the report of R. Ball on Whitehead's *Treatise* is far from being impressive (and even accurate), it happened that the commission of the "Société Physico-Mathématique de Kasan" found of equal scientific merit two of the three submissions, those of Wilhelm Killing and of A. N. Whitehead. Space permitting, it would be interesting to speculate on this equivalence. A preference for one or the other would indeed be difficult to justify given the highest value of the two proposals. In any cases, the draw decided that the Lobatchewski price was thus luckily decerned to W. Killing for his *Einführung in die Grundlagen der Geometrie* [1893-98] and his other memoirs relative to non-Euclidean geometry and to the theory of transformation groups.

4.4 The new geometries

B. Riemann's Habilitationschrift of 1854, *On the hypotheses which lie at the bases of geometry*, was published posthumously in 1867 [cf. Riemann 1854] and translated into English by W. Clifford in 1873 [Riemann 1873]. Any reader of Riemann and Grassmann will easily notice the close relationship of the first with ideas of A1. To some extent, Riemann adds to Grassmann an empirical dimension and a measure theory. According to F. Klein, Riemann's main goal had always been to find a mathematical form in which to represent, in a unique way, the laws to which the external world complies.

In the 1870s, reporting in *Nature* on some ideas of W. Clifford advertising a fourth dimension of space, the mathematician J. J. Sylvester prompted a debate on the precise meaning of space as a form of intuition and a priori concept in the philosophy of Kant, a philosophy which was used as a shield by the defenders of the traditional Euclidean tri-dimensional space of our everyday experiences.

This sort of debate on geometry was part of the intellectual environment of the young Whitehead. Moreover, his domains of research and, later, of teaching, directly confronted him with the question of space and time: applied mathematics and algebra, the algebras of Grassmann or those of Hamilton that had permitted and boosted the recent developments of physics with the calculus of vectors. In following Grassmann and incorporating his foundations in the first book of his *Treatise*, Whitehead was inheriting a Kantian legacy passed down to Grassmann by Schleiermacher and Schopenhauer.

In order to briefly characterize this philosophical heritage that can be traced in Whitehead's work, a quotation from E. Cassirer may suffice. In [Cassirer 1923], relying explicitly on Grassmann, he investigated the epistemological foundations of relativity theory. There, he wrote, for example, that *"the real elements of mathematical calculus are not magnitudes but relations"* [Cassirer 1923, 99],, and also, "*the system of Grassmann* -that he calls universal geometry- *shows that the pure concept that is tied to it can receive any possible form of change in the empirical character of the perceptions*"[???] and, again, that *"the universal serial form is the means by which every order of the empirical is to be* *understood and logically mastered”* [ibd. 111].” These are ideas that will reappear in Whitehead.

4.5 Courses in Cambridge

It is in the courses of Homersham Cox in Cambridge that the student Whitehead first heard of Grassmann’s work. For example, in Cox’s [Cox 1882], he compares the non-Euclidean geometries to ordinary geometry on the basis of the parallels axiom. Relying on works of Cayley, Sylvester and Klein, he arrives at systems of forces whose properties in ordinary geometry can be deduced from the three simple rules: addition of forces is commutative, associative and transitive (when applied along a straight line). Only experience can teach us that forces combine in this way. But even if it was not the case, it would be possible to introduce, as H. Grassmann did in A1, purely geometrical quantities supposed to follow these laws.

Elsewhere, Cox [Cox 1883a] begins a paper with general remarks on the new algebras, those of Hamilton, Grassmann and Boole. These are systems of symbols whose formal rules of combination guarantee the validity of the results independently of any geometrical or arithmetical interpretation of the symbols. Here, in an original contribution, and following a geometrical method analogous to that of A1, much simpler but longer than the analytrical method, he applies theses principles to the treatment of circles, their addition and multiplication.

Finally, we may still consider Cox’s application of the theory of quaternions and of the theory of extension to space [cf. Cox 1883b]. Following Hamilton and Grassmann, the author establishes a pure algebraical calculus whose laws coincide with those of geometry. That is, algebra being seen as the calculus of one dimensional space, it is included in a two or more dimensional geometrical calculus. Next, incorporating Cayley’s and Klein’s ideas on distance into Grassmann’s calculus, he shows that distance can be introduced in three different ways, showing thus that there are three different kinds of uniform geometry: ordinary geometry, spherical geometry, and the non-Euclidean geometry.

These are some the themes that Whitehead will later consider in his own work.

4.6 Whitehead’s early geometrical works

For example, in a paper on geodesic geometry [Whitehead 1897-98], Whitehead shows that the surfaces in any non-Euclidean space can be reduced to the geodesic geometry of straight lines in a Euclidean space.

First, he makes an important clarification about the notion of “radius of space curvature” that has misled philosophers as well as many mathematicians “*into the belief that some peculiar property of flatness is to be ascribed to Euclidean space*”[Whiehead 1897-98, ???]. Indeed, Beltrami’s theorem refutes that belief because planes or other sorts of space can be represented as surfaces in that space and Euclidean geometry itself can be interpreted as geodesic geometry in a non-Euclidean space.

With Grassmann’s method, and starting from that theorem of Beltrami, Whitehead investigates the properties of surfaces in the three types of geometries which are such that their geodesic geometry is that of straight lines in a Euclidean space. Extending the theorem to non-Euclidean space, he showed that surfaces of constant curvature are the only surfaces for which geodesics can be transformed into straight lines when the surface is represented point for point on a plane.

Although the mind can contemplate the various sorts of space, our geometrical interpretation of phenomena cannot make use of the idea of a fourth dimension because we have no intuition of it. This finding will take Whitehead to develop his own approach to physics in his later work. Classical physics distinguishes space and time because it has no way of representing four-dimensional events. Einstein's theory of relativity relied on two principles, the equivalence of inertial frames and the constancy of speed of light, from which Lorentz transformations relating space to time are deduced. But the physical content of Einstein's theory can be deduced without these two principles as Whitehead showed in deducing Lorentz's transformations from the uniformity and symmetry of space-time and from the symmetry and transitivity of the transformations.

The laws of nature are deduced from experience. They must not only correspond to experience in observation, but they have to explain nature as it is perceived. This will be achieved with the help of Whitehead's method of extensive abstraction, a direct adaptation of Grassmann's theory of extension. [cf. Whitehead 1929; Ringel 2008].

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Abstract

Alfred North Whitehead, author of a *Treatise on Universal Algebra* almost entirely based on Hermann Grassmann’s *Ausdehnungslehre*, did not only advertised this work but he also incorporated some of its leading ideas into his main project, federating logic, mathematics and physics in his monumental cosmology. Here, through some historical account starting with Leibniz, we suggest some essential lines of thought underlying both works, emphasizing method and structure and their ethical connections.

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