Improved termination analysis of CHR using self-sustainability analysis

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Abstract. In the past few years, several successful approaches to termination analysis of Constraint Handling Rules (CHR) have been proposed. In parallel to these developments, for termination analysis of Logic Programs (LP), recent work has shown that a stronger focus on the analysis of the cycles in the strongly connected components (SCC) of the program is very beneficial, both for precision and efficiency of the analysis.

In this paper we investigate the benefit of using the cycles of the SCCs of CHR programs for termination analysis. It is a non-trivial task to define the notion of a cycle for a CHR program. We introduce the notion of a self-sustaining set of CHR rules and show that it provides a natural counterpart for the notion of a cycle in LP. We prove that non-self-sustainability of an SCC in a CHR program entails termination for all queries to that SCC. Then, we provide an efficient way to prove that an SCC of a CHR program is non-self-sustainable, providing an additional, new way of proving termination of (part of) the program.

We integrate these ideas into the CHR termination analyser CHRisTA and demonstrate by means of experiments that this extension significantly improves both the efficiency and the performance of the analyser.

1 Introduction

Termination analysis techniques for Logic Programs often make use of dependency analysis. In some approaches (e.g. [5]), this is done in order to detect (mutually) recursive predicates. For non-recursive predicates, no termination proof is needed, so that termination conditions are only expressed and verified for the recursive ones. In other approaches (e.g. [11], [17]) the termination conditions are explicitly expressed in terms of cycles in the strongly connected components (SCCs) of the dependency graph. It was shown in [17] that such an approach is both efficient and precise. On the benchmark of the termination analysis competition (see [22]) it outperformed competing systems.

Recently, in [23] and [12], we adapted termination analysis techniques from Logic Programming (LP) to Constraint Handling Rules (CHR) [1, 8] and developed the CHR termination analyser CHRisTA [13]. These techniques and system are based on recursion, rather than on cycles in SCCs.

The main motivation that led to the current work is that we wanted to adapt the approach based on SCCs to CHR, with the hope of improving efficiency and precision. It turned out that there were considerable complications to achieve this, caused by the fact that for CHR a useful notion of “cycle” is hard to define.

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Consider for example the CHR program, P , consisting of the CHR rule,

$$R @ a, b \Leftrightarrow a.$$

For each application of R , two constraints, a and b , are removed, while a new constraint, a , is added. Since R adds constraints that are required to fire R , the application of R depends on itself. An SCC analysis on the dependency graph of P would therefore consider R to be cyclic, while in fact, for every application of R , a constraint b is removed and never added again. Therefore, at some point in a computation of P , the b constraints will be depleted and thus P will terminate.

A more accurate analysis of cycles in CHR could therefore still rely on an SCC analysis, but could also verify for each SCC whether there exists a dependency for each head of the rules of a component, provided for by the component itself.

Unfortunately, such an approach is still not very accurate. Consider for example the CHR program, P , consisting of the rule,

$$R @ a, a \Leftrightarrow a.$$

For R , which is part of an SCC, both heads are provided for from within the component itself. However, for every application of R , two constraints a are removed, while only one constraint a is added. Repeated application of R will therefore result in depletion of these a constraints, and thus P will terminate.

Therefore, in order for an SCC in CHR to be cyclic, we must guarantee that the component is *self-sustainable*. That is, when the rules of an SCC are applied, they must provide the constraints to fire the same rules again.

In fact, in definite LP, an SCC of the dependency graph of a definite LP program is self-sustainable by default. The reason for this is that definite LP is a single-headed language. Therefore, each cycle in the dependency graph of a definite LP program—as described by the SCCs of the program—corresponds to rule applications that have the potential to be repeated.

In CHR, to characterise the notion of a self-sustainable SCC, we need to identify the multisets of rule applications of an SCC that can be repeated. Consider for example the CHR program, P :

$$R_1 @ a, a \Leftrightarrow b, b. \quad R_2 @ b \Leftrightarrow a.$$

As can be verified, it is non-terminating. That is, when R_1 is applied once, it adds two b constraints, allowing R_2 to be applied twice. By applying R_2 twice, the input required for application of R_1 is provided for, thus resulting in a loop.

In this paper, we introduce the notion of *self-sustainability* for a set of CHR rules. It provides a rather precise approximation of cyclic behaviour in CHR. We characterise self-sustainability for a set of rules by means of a system of linear inequalities. Each solution to this system identifies a multiset based on the given rules, such that, if it is *traversed*—all rules in the multiset are applied once—it provides all required constraints to potentially traverse the multiset again.

Unfortunately, we cannot use this concept directly as a basis for termination analysis, similar to [11] for LP. The reason is that for any cyclic SCC in a CHR program, the system has infinitely many solutions. In [15], we describe a rather inefficient approach to cope with this problem, resulting in little gain in precision.

Alternatively, we can use the concept of self-sustainability in a “negative” way. If we can show that an SCC in a CHR program is *not* self-sustainable —the system has no solution— then this proves that the rules involved in that SCC cannot be part of an infinite computation. Therefore, we can ignore these rules in the termination analysis. By integrating this idea in the CHRisTA analyser, it turns out that precision and efficiency of the analysis is improved.

Our approach will be restricted to the abstract CHR semantics. It is however widely applicable as most other CHR semantics [18] are instances of the abstract CHR semantics. That is, if an SCC of a CHR program is not self-sustainable for the abstract semantics, it cannot be self-sustainable for any more refined semantics. Note that, under the abstract semantics, the two different kinds of rules —*simplification* and *propagation* rules— of a CHR program behave as *simplification* rules. Hence, we discuss our approach in terms of simplification only.

Finally, note that a CHR program may contain propagation rules and that these rules do not remove the constraints on which they fire. Under the abstract CHR semantics, propagation rules are infinitely applicable on the same combination of constraints. Represented as a simplification rule, they explicitly replace the constraints on which they fire and thus are considered self-sustainable by default. Our approach can therefore only prove non-self-sustainability of an SCC without propagation. In Section 6, we provide intuitions regarding self-sustainability under the theoretical CHR semantics [8]. This refinement of the abstract CHR semantics additionally considers a *fire-once policy* on propagation rules to overcome the infinite applicability of propagation rules.

Overview of the paper. In Section 2, we discuss the syntax and semantics of abstract CHR. Section 3 defines CHR dependency graphs for an SCC analysis and CHR nets —a more accurate description of the dependencies of a CHR program— for a self-sustainability analysis. In Section 4, we formalise the notion of self-sustainability of an SCC (Section 4.1). Then, we present a test for verifying that an SCC is not self-sustainable (Section 4.2). In Section 5, we evaluate our approach and discuss CHR programs that we can only prove terminating using the test for non-self-sustainability. Finally, in Section 6, we conclude the paper.

2 Preliminaries

We assume familiarity with LP and its main results [2, 9]. By L , we denote the first order language underlying a CHR program P . By $Term_P$ and $Atom_P$, we denote, respectively, the sets of terms and atoms constructible from L .

2.1 The syntax of abstract CHR

The rules of a CHR program act on first-order atoms, called *constraints*.

Definition 1 (constraint). Constraints, $c(t_1, \dots, t_n)$ ($n \geq 0$), are first-order predicates applied to terms, t_i ($1 \leq i \leq n$). We distinguish between two kinds of constraints: CHR constraints are user-defined and are solved by the CHR rules of a CHR program; built-in constraints are pre-defined and are solved by a constraint theory (CT) defined in the host language. $\square$

Constraints are kept in a *constraint store*, of which the behaviour corresponds to a *multiset* (or bag) of constraints. Therefore, to formalise the syntax and semantics of CHR, we first recall multiset theory [4,6].

A *multiset* is a tuple, $M_S = \langle S, m_S \rangle$, where S is a set, called the *underlying set*, and m_S a *multiplicity function*, mapping the elements, e , of S to natural numbers, $m_S(e) \in \mathbb{N}_0$, representing the number of occurrences of e in M_S . Like any function, m_S may be represented as a set, $\{(s, m_S(s)) : s \in S\}$. An example of a multiset is $\langle \{a, b\}, \{(a, 2), (b, 1)\} \rangle$. We introduce the alternative notation to represent this multiset as $\llbracket a, a, b \rrbracket$.

If a *universe* U in which the elements of S must live is specified, the definition of a multiset can be simplified to a *multiset indicator function*. Let U be a universe for a multiset $M_S = \langle S, m_S \rangle$. Then, we define the *multiset indicator function*, $\mu_S : U \rightarrow \mathbb{N}$, of M_S w.r.t. U as: $\mu_S(u) = m_S(u)$ if $u \in S$ and $\mu_S(u) = 0$ if $u \notin S$. Since it is often more favourable to discuss multisets in terms of some universe, we can —as an abuse of notation— denote a multiset $M_S = \langle S, m_S \rangle$ also by $\langle U, \mu_S \rangle$, or μ_S if its universe is clear from context.

Furthermore, we will need the notion of a *multisubset* in the constraint store since the rules of a CHR program will be applied on multisubsets of constraints. Let U be a universe for the multisets M_{S_1} and M_{S_2} . Then, multiset M_{S_1} is a *multisubset* of multiset M_{S_2} , denoted $M_{S_1} \sqsubseteq M_{S_2}$, iff the multiset indicator functions, μ_{S_1} and μ_{S_2} , w.r.t. U , of M_{S_1} and M_{S_2} respectively, are such that $\mu_{S_1}(u) \leq \mu_{S_2}(u)$, for all u in U . Associated to $\sqsubseteq$, we define *strict multisubsets*: $M_{S_1} \subset M_{S_2} \leftrightarrow M_{S_1} \sqsubseteq M_{S_2} \wedge M_{S_2} \not\sqsubseteq M_{S_1}$.

We are now ready to recall the syntax of *simplification rules*.

Definition 2 (abstract CHR program). *An abstract CHR program is a finite set of simplification rules, R_i .*

A simplification rule,

$$R_i @ H_1^i, \dots, H_{n_i}^i \Leftrightarrow G_1^i, \dots, G_{k_i}^i \mid B_1^i, \dots, B_{l_i}^i, C_1^i, \dots, C_{m_i}^i.$$

is applicable on a multiset of CHR constraints, matching with the head, $\llbracket H_1^i, \dots, H_{n_i}^i \rrbracket$, of the rule, such that the guard, $\llbracket G_1^i, \dots, G_{k_i}^i \rrbracket$, is satisfiable. Upon application, the multiset of constraints matching with the head is replaced by an appropriate instance of the multiset of built-in and CHR constraints, $\llbracket B_1^i, \dots, B_{l_i}^i, C_1^i, \dots, C_{m_i}^i \rrbracket$, from the body of the rule.

Note that the guard, $\llbracket G_1^i, \dots, G_{k_i}^i \rrbracket$, of a CHR rule may only consist of built-in constraints. Also note that naming a rule by “rulename @” is optional. $\square$

The next example program is an introductory example to CHR(Prolog).

Example 1 (Primes). The program below implements the Sieve of Eratosthenes for finding the prime numbers up to a given number. User-defined *prime/1* constraints are used to hold the values of the positive integers for which we derive whether they are prime. User-defined *primes/1* constraints are used to query the program and to generate the *prime/1* constraints for prime evaluation.

$$\begin{aligned}
R_1 @ \text{primes}(2) &\Leftrightarrow \text{prime}(2). \\
R_2 @ \text{primes}(N) &\Leftrightarrow N > 2 \mid Np \text{ is } N - 1, \text{prime}(N), \text{primes}(Np). \\
R_3 @ \text{prime}(M), \text{prime}(N) &\Leftrightarrow \text{div}(M, N) \mid \text{prime}(M).
\end{aligned}$$

R_2 generates the numbers for prime evaluation top-down. It replaces a $\text{primes}(n)$ constraint, matching with the head $\text{primes}(N)$, by the constraints $Np \text{ is } n - 1$, $\text{prime}(n)$ and $\text{primes}(Np)$, if $CT \models n > 2$. R_1 removes a $\text{primes}(2)$ constraint, replacing it with the final number for prime evaluation, $\text{prime}(2)$. R_3 implements the sieve. For any two matching constraints, $\text{prime}(m)$ and $\text{prime}(n)$, such that m divides n , we replace both $\text{prime}/1$ constraints by $\text{prime}(m)$. $\square$

2.2 The semantics of abstract CHR

The abstract CHR semantics is defined as a state transition system.

Definition 3 (abstract CHR state). *An abstract CHR state is a multiset, Q , of constraints (with universe Atom_P), called a constraint store.* $\square$

To define the transitions on CHR states, as given by the CHR rules of the program, we introduce the multiset operator *join*, adding multisets together. Let M_A and M_B be multisets with universe U , and let u be an element of U . Then, the multiset indicator function of the *join* $M_C = M_A \uplus M_B$ is given by the sum, $\mu_C(u) = \mu_A(u) + \mu_B(u)$, of the multiset indicator functions of M_A and M_B .

The CHR transition relation represents the consecutive CHR states for a CHR program P and constraint theory CT .

Definition 4 (abstract CHR transition relation). *Let P be a CHR program and CT a constraint theory for the built-in constraints. Let θ represent the substitutions corresponding to the bindings generated when resolving built-in constraints by CT , and let σ represent substitutions for the variables in the head of the rule as a result of matching. Then, the transition relation, $\rightarrow_P$, for P , where Q is a CHR state, is defined by:*

1. **A solve transition:**
*if $Q = \llbracket b \rrbracket \uplus Q'$, with b a built-in constraint, and $CT \models b\theta$,
then, $Q \rightarrow_P Q'\theta$.*
2. **Simplification:**
*if $(R_i @ H_1^i, \dots, H_{n_i}^i \Leftrightarrow G_1^i, \dots, G_{k_i}^i \mid B_1^i, \dots, B_{l_i}^i, C_1^i, \dots, C_{m_i}^i) \in P$,
and if $Q = \llbracket h_1, \dots, h_{n_i} \rrbracket \uplus Q'$ and
 $\exists \sigma\theta : CT \models (h_1 = H_1^i\sigma) \wedge \dots \wedge (h_{n_i} = H_{n_i}^i\sigma) \wedge (G_1^i, \dots, G_{k_i}^i)\sigma\theta$,
then, $Q \rightarrow_P (\llbracket B_1^i, \dots, B_{l_i}^i, C_1^i, \dots, C_{m_i}^i \rrbracket \uplus Q')\sigma\theta$.*

Rules in CHR are non-deterministically applied until exhaustion, thus until no more transitions are possible. Rule application is a committed choice. Built-in constraints are assumed to return an answer in finite time and cannot introduce new CHR constraints. If built-ins cannot be solved by CT , the program fails. $\square$

By an *initial CHR state* or *query state* for a CHR program P , we mean any abstract CHR state for P . By a *final CHR state* or *answer state* for P , we mean a CHR state Q , such that no CHR state Q' exists for which $(Q, Q') \in \rightarrow_P$.

Example 2 (Primes continued). Executing the program from Example 1 on an initial CHR state $Q_0 = \llbracket \text{primes}(7) \rrbracket$, we obtain by application of R_2 the next state $Q_1 = \llbracket Np \text{ is } 7-1, \text{prime}(7), \text{primes}(Np) \rrbracket$. Solving the built-in constraint, $Np \text{ is } 7-1$, binds 6 to Np , yielding the state, $Q_2 = \llbracket \text{prime}(7), \text{primes}(6) \rrbracket$. Proceeding in this way, we obtain the state $Q_{10} = \llbracket \text{prime}(3), \text{primes}(2), \text{prime}(4), \text{prime}(5), \text{prime}(6), \text{prime}(7) \rrbracket$, on which R_2 is no longer applicable. Then, by application of R_1 , we obtain $Q_{11} = \llbracket \text{prime}(2), \text{prime}(3), \text{prime}(4), \text{prime}(5), \text{prime}(6), \text{prime}(7) \rrbracket$.

Now only R_3 is applicable, replacing two $\text{prime}/1$ constraints by the constraint dividing the other. For example, we can remove $\text{prime}(4)$ since $\text{prime}(2)$ divides it. A next state could therefore be $Q_{12} = \llbracket \text{prime}(2), \text{prime}(3), \text{prime}(5), \text{prime}(6), \text{prime}(7) \rrbracket$. For $\text{prime}(6)$ there are two prime numbers that divide it. One possibility is that $\text{prime}(3)$ is used, and obtain $Q_{13} = \llbracket \text{prime}(3), \text{prime}(2), \text{prime}(5), \text{prime}(7) \rrbracket$.

Since on Q_{13} no more rules are applicable, we reach an answer state. This state holds all prime numbers up to 7. $\square$

3 The CHR dependency graph and CHR net

In CHR, the head constraints of a rule represent the constraints *required* for rule application. The body CHR constraints represent the constraints *newly available* after rule application. To represent these sets, we introduce *abstract CHR constraints*.

Definition 5 (abstract CHR constraint). An abstract CHR constraint is a pair $\mathcal{C} = (C, B)$, where C is a CHR constraint, and B a conjunction of built-in constraints. The denotation of an abstract CHR constraint is given by a mapping $\wp : (C, B) \mapsto \{C\sigma \mid \exists \theta : CT \models B\sigma\theta\}$.

In a CHR program P , we distinguish between two types of abstract CHR constraints. An abstract input constraint, $in_j^i = (H_j^i, G^i)$, represents the CHR constraints that can be used to match the head H_j^i of a CHR rule R_i , without invalidating the guards $G^i = G_1^i \wedge \dots \wedge G_{k_i}^i$ of R_i . An abstract output constraint, $out_j^i = (C_j^i, G^i)$, represents the CHR constraints that become newly available after rule application, and are related to the body CHR constraints C_j^i of R_i . $\square$

By $\mathcal{In}_{R_i}$ and $\mathcal{Out}_{R_i}$, we represent the sets of abstract input and output constraints of a rule R_i . By $\mathcal{In}_P$ and $\mathcal{Out}_P$, we represent the abstract input and output constraints of a CHR program P .

Example 3 (abstract CHR constraints of Primes). We revisit the Primes program from Example 1, and derive:

- $\mathcal{In}_P = \mathcal{In}_{R_1} \cup \mathcal{In}_{R_2} \cup \mathcal{In}_{R_3} = \{in_1^1, in_1^2, in_1^3, in_2^3\}$, where

$$\begin{aligned} in_1^1 &= (\text{primes}(2), \text{true}) & in_1^2 &= (\text{primes}(N), N > 2) \\ in_1^3 &= (\text{prime}(M), \text{div}(M, N)) & in_2^3 &= (\text{prime}(N), \text{div}(M, N)) \end{aligned}$$
 - $\mathcal{Out}_P = \mathcal{Out}_{R_1} \cup \mathcal{Out}_{R_2} \cup \mathcal{Out}_{R_3} = \{out_1^1, out_1^2, out_2^2, out_1^3\}$, where

$$\begin{aligned} out_1^1 &= (\text{prime}(2), \text{true}) & out_1^2 &= (\text{prime}(N), N > 2) \\ out_2^2 &= (\text{primes}(Np), N > 2) & out_1^3 &= (\text{prime}(M), \text{div}(M, N)) \end{aligned}$$
- $\square$

The rules of a CHR program P relate abstract inputs to abstract outputs. We call this relation the *rule transition relation* of P .

Definition 6 (rule transition relation). A rule transition of an abstract CHR program, P , is an ordered pair $T_i = (\mathcal{In}_{R_i}, \mathcal{Out}_{R_i})$, relating the set of abstract input constraints $\mathcal{In}_{R_i} = \{in_1^i, \dots, in_{n_i}^i\}$ of $R_i \in P$ to the set of abstract output constraints $\mathcal{Out}_{R_i} = \{out_1^i, \dots, out_{m_i}^i\}$ of R_i . The rule transition relation $\mathcal{T}_P = \{T_i \mid R_i \in P\}$ of P is the set of rule transitions of P . $\square$

Example 4 (rule transition relation of Primes). The Primes program P from Example 1 defines three rule transitions:

$$\begin{aligned} - \mathcal{T} &= \{T_1, T_2, T_3\}, \text{ where} \\ T_1 &= (\{in_1^1\}, \{out_1^1\}) & T_2 &= (\{in_1^2\}, \{out_1^2, out_2^2\}) \\ T_3 &= (\{in_1^3, in_2^3\}, \{out_1^3\}) \end{aligned} \quad \square$$

Abstract output constraints relate to abstract input constraints by a *match transition relation*. This second kind of relation is the result of a dependency analysis between abstract constraints, relating the constraints newly available after rule application to constraints required for rule application.

Definition 7 (match transition relation). A match transition of an abstract CHR program P is an ordered pair $M_{(i,j,k,l)} = (out_j^i, in_l^k)$, relating an output $out_j^i = (C_j^i, G^i)$ of $\mathcal{Out}_P$ to an input $in_l^k = (C_l^k, G^k)$ of $\mathcal{In}_P$ such that $\exists \theta : CT \models (C_j^i = C_l^k \wedge G^i \wedge G^k)\theta$. The match transition relation $\mathcal{M}_P$ is the set of all match transitions $M_{(i,j,k,l)}$ in P . $\square$

Note that a match transition exists for an abstract output and input if the intersection of their denotation is non-empty, i.e. $\wp(out_j^i) \cap \wp(in_l^k) \neq \emptyset$.

Example 5 (match transition relation of Primes). The Primes program, P , from Example 1 defines the match transition relation,

$$\mathcal{M}_P = \{M_{(1,1,3,1)}, M_{(1,1,3,2)}, M_{(2,1,3,1)}, M_{(2,1,3,2)}, M_{(2,2,1,1)}, M_{(2,2,2,1)}, M_{(3,1,3,1)}, M_{(3,1,3,2)}\}. \quad \square$$

As in LP, the dependency graph of a CHR program is a directed graph where the nodes represent rules, and the directed arcs dependencies between these rules.

Definition 8 (CHR dependency graph). A CHR dependency graph, $\mathcal{D}_P$, of an abstract CHR program P is an ordered tuple $\langle T, D \rangle$ of nodes T , one for each transition in the rule transition relation, $\mathcal{T}_P$, of P , and directed arcs D , one for each ordered pair of transitions between which a match transition exists in the match transition relation, $\mathcal{M}_P$, of P . $\square$

For an SCC analysis, a CHR dependency graph is sufficient, however, for self-sustainability we rely in the next sections on a CHR net instead.

Definition 9 (CHR net). A CHR net, $\mathcal{N}_P$, of an abstract CHR program, P , is a quadruple $\langle \mathcal{In}_P, \mathcal{Out}_P, \mathcal{T}_P, \mathcal{M}_P \rangle$. Here, respectively, $\mathcal{In}_P$ and $\mathcal{Out}_P$ are the abstract input and output constraints of P , and $\mathcal{T}_P$ and $\mathcal{M}_P$ the rule and match transition relations of P . $\square$

Note that a CHR net corresponds to a bipartite hypergraph. We illustrate both notions in Figure 1 for the Primes program of Example 1.

Fig. 1. Dependency graph and CHR net for Primes

4 Self-sustainable SCCs of a CHR program

To derive the SCCs of a CHR program, we perform an SCC analysis on the dependency graph of the CHR program. This is similar to such an analysis in LP and can be done efficiently using Tarjan's algorithm [21].

Example 6 (SCCs of Primes). Based on $\mathcal{D}_P = (\{T_1, T_2, T_3\}, \{(T_1, T_3), (T_2, T_1), (T_2, T_2), (T_2, T_3), (T_3, T_3)\})$, the dependency graph of Primes of Figure 1, we derive two SCCs, $\{T_2\}$ and $\{T_3\}$. $\square$

4.1 Self-sustainable SCCs

Consider again the non-terminating example program from the Introduction:

$$R_1 @ a, a \Leftrightarrow b, b. \quad R_2 @ b \Leftrightarrow a.$$

Its dependency graph and CHR net are shown in Figure 2. As can be verified, in Figure 2, $in_1^1 = in_2^1 = out_1^2 = (a, true)$ and $out_1^1 = out_2^1 = in_1^2 = (b, true)$. For the example, there is only one SCC: $\{T_1, T_2\}$.

Fig. 2. Dependency graph and CHR net for the a - b -example

What we want to characterise—in the notion of self-sustainability for such an SCC—is whether we can duplicate some of the transitions of this SCC, such that for the resulting multiset of transitions and its *extended CHR net*, it is so that a multisubset of all $M_{(i,j,k,l)}$ match transitions exists, that maps a multisubset of all the out_q^p nodes *onto all of* the in_s^r nodes. Returning to our example, consider

the multiset $\llbracket T_1, T_2, T_2 \rrbracket$ based on $\{T_1, T_2\}$. In Figure 3, we expand the CHR net of Figure 2 to represent all match transitions for this multiset.

It is clear that from the multiset of 8 match transitions in Figure 3, we can select 4 transitions, for example $\llbracket M_{(1,1,2,1)}, M_{(1,2,2,1)}, M_{(2,1,1,1)}, M_{(2,1,1,2)} \rrbracket$, such that these transitions define a function from a multisubset of all out_q^p nodes onto all in_s^r nodes. This function is represented in Figure 3 in the thick arrows.

Fig. 3. Expanded CHR net for $\llbracket T_1, T_2, T_2 \rrbracket$

Note that in this example, the multisubsets of out_q^p nodes on which the function is defined is equal to the entire multiset, $\llbracket out_1^1, out_2^1, out_1^2, out_2^2 \rrbracket$, of out_q^p nodes. In general, this is not necessary: out_q^p nodes are allowed to be “unused” by in_s^r nodes. It suffices that a CHR constraint is produced for each in_s^r node.

We generalise the construction in the example above in the following definition. In this definition, we start off from a set of CHR rules, C , and its CHR net, $\mathcal{N}_C = \langle In_C, Out_C, \mathcal{T}_C, \mathcal{M}_C \rangle$. In the definition we will consider a multiset with universe the rule transitions $\mathcal{T}_C$, and will denote it by $\mu_{\mathcal{T}}$. Note that, given such a multiset $\mu_{\mathcal{T}}$, there are associated multisets μ_{In} and μ_{Out} , with universes In_C and Out_C , respectively. If a transition $T_i \in \mathcal{T}_C$ occurs n_i times in $\mu_{\mathcal{T}}$, then all its in_j^i and out_l^i abstract constraints occur n_i times in μ_{In} and μ_{Out} , respectively.

Given $\mu_{\mathcal{T}}$, there is also an associated multiset $\mu_{\mathcal{M}}$, with universe $\mathcal{M}_C$. If transitions T_i and T_k occur respectively n_i and n_k times in $\mu_{\mathcal{T}}$, then all $M(i, j, k, l) \in \mathcal{M}_C$ occur $n_i \times n_k$ times in $\mu_{\mathcal{M}}$.

Definition 10 (self-sustainability of a set of CHR rules). *Let C be a set of simplification rules and $\mathcal{N}_C = \langle In_C, Out_C, \mathcal{T}_C, \mathcal{M}_C \rangle$ the CHR net of C . The set C is self-sustainable iff there exist*

- a non-empty multiset $\mu_{\mathcal{T}}$, with the associated multisets μ_{In} , μ_{Out} and $\mu_{\mathcal{M}}$,
- a multiset $\mu'_{Out} \subseteq \mu_{Out}$, and
- a multiset $\mu'_{\mathcal{M}} \subseteq \mu_{\mathcal{M}}$,

such that $\mu'_{\mathcal{M}}$ defines a function from μ'_{Out} onto μ_{In} . □

Before we illustrate this concept on our running example, Primes, we provide an alternative, numeric characterisation.

Proposition 1 (numeric characterisation self-sustainability). *Let C be a set of simplification rules and $\mathcal{N}_C = \langle In_C, Out_C, \mathcal{T}_C, \mathcal{M}_C \rangle$ its CHR net. The set C is self-sustainable iff there exist multisets $\mu_{\mathcal{T}}$, with associated multisets μ_{In} ,*

$\mu_{\mathcal{O}ut}, \mu_{\mathcal{M}}, \mu'_{\mathcal{O}ut} \sqsubseteq \mu_{\mathcal{O}ut}$ and $\mu'_{\mathcal{M}} \sqsubseteq \mu_{\mathcal{M}}$, such that $\sum_{i:T_i \in \mathcal{T}_C} \mu_{\mathcal{T}}(T_i) \geq 1$ and such that

$$\begin{aligned} \forall out_j^i \in \mathcal{O}ut_C : \quad & \sum_{k,l:M_{(i,j,k,l)} \in \mathcal{M}_C} \mu'_{\mathcal{M}}(M_{(i,j,k,l)}) \leq \mu_{\mathcal{T}}(T_i), \text{ and} \\ \forall in_j^i \in \mathcal{I}n_C : \quad & \sum_{k,l:M_{(k,l,i,j)} \in \mathcal{M}_C} \mu'_{\mathcal{M}}(M_{(k,l,i,j)}) = \mu_{\mathcal{T}}(T_i). \end{aligned} \quad \square$$

Proof. The first inequality expresses that $\mu_{\mathcal{T}}$ is non-empty. The second kind of inequalities —one for each out_j^i in $\mathcal{O}ut_C$ — state that for an abstract output out_j^i , the number of match transitions $M_{(i,j,k,l)}$ in $\mu'_{\mathcal{M}}$ that have (i,j) as their first two arguments does not exceed the number of occurrences of T_i in $\mu_{\mathcal{T}}$. This corresponds to stating that $\mu'_{\mathcal{M}}$ can be regarded as a function defined on a multisubset $\mu'_{\mathcal{O}ut}$ of $\mu_{\mathcal{O}ut}$.

The third kind of inequalities —one for each in_j^i in $\mathcal{I}n_C$ — state that for an abstract input constraint in_j^i , the number of match transitions $M_{(k,l,i,j)}$ in $\mu'_{\mathcal{M}}$ that have (i,j) as their last two arguments is exactly the number of occurrences of T_i in $\mu_{\mathcal{T}}$. This means that the function defined by $\mu'_{\mathcal{M}}$ is onto $\mu_{\mathcal{I}n}$. $\square$

Example 7 (self-sustainable SCCs for Primes). Consider the SCCs, $C_1 = \{T_2\}$ and $C_2 = \{T_3\}$, (see Example 6) of the Primes program, P , of Example 1. While the first component of the program is self-sustainable, the second is not. That is, at some point in a computation of C_2 , the *prime/1* constraints to fire the rule will be depleted. Consider their CHR nets,

$$\begin{aligned} \mathcal{N}_{C_1} &= \langle \{in_1^2\}, \{out_1^2, out_2^2\}, \{T_2\}, \{M_{(2,2,2,1)}\} \rangle \text{ and} \\ \mathcal{N}_{C_2} &= \langle \{in_1^3, in_2^3\}, \{out_1^3\}, \{T_3\}, \{M_{(3,1,3,1)}, M_{(3,1,3,2)}\} \rangle. \end{aligned}$$

Let us denote $\mu'_{\mathcal{M}}(M_{(i,j,k,l)})$ as $m_{(i,j,k,l)}$ and $\mu_{\mathcal{T}}(T_i)$ as t_i , where all $m_{(i,j,k,l)}$ and t_i are natural numbers. Then, we can characterise self-sustainability of C_1 and C_2 , respectively, by the systems of linear inequalities,

$$\begin{aligned} t_2 \geq 1 \quad & m_{(2,2,2,1)} \leq t_2 \quad m_{(2,2,2,1)} = t_2 \quad \text{and} \\ t_3 \geq 1 \quad & m_{(3,1,3,1)} + m_{(3,1,3,2)} \leq t_3 \quad m_{(3,1,3,1)} = t_3 \quad m_{(3,1,3,2)} = t_3. \end{aligned}$$

If a solution to these systems exists, the underlying component is self-sustainable. C_1 is clearly self-sustainable, while C_2 is not. $\square$

Several comments with respect to Definition 10 and Proposition 1 are in order. First, the statement that $\mu'_{\mathcal{M}}$ defines a function from $\mu'_{\mathcal{O}ut}$ onto $\mu_{\mathcal{I}n}$ is imprecise. In fact, $\mu'_{\mathcal{M}}$ defines a set of functions from $\mu'_{\mathcal{O}ut}$ onto $\mu_{\mathcal{I}n}$. This is because $\mu'_{\mathcal{O}ut}$ and $\mu_{\mathcal{I}n}$ are multisets, that may contain elements more than once.

Therefore, if in_j^i or out_l^k occur multiple times in $\mu_{\mathcal{I}n}$, respectively $\mu'_{\mathcal{O}ut}$, it is unclear which mapping is defined by an element $M_{(k,l,i,j)}$ of $\mu'_{\mathcal{M}}$. Looking back at the *a-b*-example of Figure 3, there are four different functions in this graph, all corresponding to the multiset $\mu'_{\mathcal{M}} = \llbracket M_{(1,1,2,1)}, M_{(1,2,2,1)}, M_{(2,1,1,1)}, M_{(2,1,1,2)} \rrbracket$. The thick lines in the figure represent one of these, but selecting other arcs, with the same labels, produces the other three.

Apart from this, for a fixed multiset $\mu_{\mathcal{T}}$, there can be several mappings $\mu'_{\mathcal{M}}$ that map a multiset $\mu'_{\mathcal{O}ut}$ onto $\mu_{\mathcal{I}n}$. This is because there can be different abstract constraints out_l^k in $\mu_{\mathcal{O}ut}$ that all have a match transition to some abstract constraint in_j^i . This may give a number of alternative candidates for $\mu'_{\mathcal{M}}$.

Finally, by considering different multisets $\mu_{\mathcal{T}}$ based on $\mathcal{T}_C$, we may obtain a very large number of solutions for $\mu'_{\mathcal{M}}$ in Definition 10. In the context of the *a-b*-example, consider a multiset $\mu_{\mathcal{T}} = \llbracket T_1, T_1, T_2, T_2, T_2, T_2 \rrbracket$. From the fact that there are twice as many T_2 rules than T_1 rules, it should be intuitively clear that it again allows to find multisets $\mu'_{\mathcal{O}ut}$ and $\mu'_{\mathcal{M}}: \mu'_{\mathcal{O}ut} \rightarrow \mu_{\mathcal{I}n}$, with the latter being *onto*. Because of the increased multiplicity of the rule transitions, the number of different functions that $\mu'_{\mathcal{M}}$ represents in this context is much higher than for the previous $\mu_{\mathcal{T}}$. Moreover, it turns out that by increasing the multiplicity of the rule transitions, we can construct concrete functions associated to $\mu'_{\mathcal{M}}$ that cannot be obtained as the union of multiple concrete functions associated to a solution for a $\mu_{\mathcal{T}}$ with lower multiplicity of rule transitions (see [14]).

This observation implies that we are unable to use the notion of a self-sustainable set of rules as a direct basis for a termination analysis. Such an approach would have to identify a finite set of minimal self-sustainable cycles and then prove that all these are terminating. But since in CHR, there are in general an infinite set of minimal cycles, such an approach is not feasible.

In [15], to tackle the problem of an infinite number of minimal cycles, a new concept of minimality is introduced, based on a finite constructive set of solutions—the Hilbert basis—for the inequalities representing self-sustainability. This approach is however slow, with little gain in precision.

Fortunately, there is another way for using the notion of a self-sustaining set of rules. After determining the SCCs of a CHR program, we can verify which SCCs are not self-sustainable, and disregard such SCCs. This observation is based on the following theorems.

Theorem 1. *If a CHR program, P , is not self-sustainable, then P terminates for every query.* $\square$

Theorem 2. *Let P be a CHR program and let C be an SCC of P . If C is not self-sustainable and if $P \setminus C$ terminates for every query, then P terminates for every query.* $\square$

4.2 Non-self-sustainable SCCs of a CHR program

From here on, we refer to self-sustainable as '*selfs*' and use the matrix form: $A \times X \leq B$; to represent systems of linear inequalities.

Example 8 (system in matrix form for Primes). We revisit Example 7 and represent for C_1 and C_2 , respectively, their systems of inequalities in matrix form:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} t_2 \\ m_{(2,2,2,1)} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} t_3 \\ m_{(3,1,3,1)} \\ m_{(3,1,3,2)} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Note that the equalities of the original system are replaced by two inequalities. E.g. $m_{(2,2,2,1)} = t_2$ is replaced by $m_{(2,2,2,1)} \geq t_2$ and $m_{(2,2,2,1)} \leq t_2$. $\square$

In order to prove that a component is not *selfs*, we need to prove that no positive integer solution exists for the variables of the linear inequalities representing that it is *selfs*. Thus, we need to prove that such a system is *infeasible*.

Definition 11 (feasible). A system, $S = A \times X \leq B$, is (in)feasible iff S has (not) a solution in $\mathbb{R}^+$. $\square$

The following lemma is due to Farkas [7].

Lemma 1 (Farkas' lemma). Let $S = A \times X \leq B$ be a system of linear inequalities. Then, S is feasible iff $\forall P \geq 0 : A^T \times P \geq 0 \rightarrow B^T \times P \geq 0$. Alternatively, S is infeasible iff $\exists P \geq 0 : A^T \times P \geq 0 \wedge B^T \times P < 0$. $\square$

Note that Farkas' lemma is only applicable to the real case: if a real matrix P exists, such that $P \geq 0 \wedge A^T \times P \geq 0 \wedge B^T \times P < 0$, then, the original system, S , is infeasible. Therefore, it has no solution in $\mathbb{N} \subset \mathbb{R}^+$ and must be *non-selfs*.

Infeasibility has received much attention in linear programming (see e.g. [3]) and several approaches exist to tackle the problem. It is not in our intention to improve on these approaches. To evaluate our approach, we do however formulate a simple test, where we first represent infeasibility as a constraint problem on symbolic coefficients. That is, we introduce a symbolic matrix P' , for each infeasibility problem, of the same dimensions as P , with symbolic coefficients p_i , one for each position in the matrix. Then, we derive constraints on the symbolic coefficients, based on $P' \geq 0 \wedge A^T \times P' \geq 0 \wedge B^T \times P' < 0$.

Example 9 (infeasibility for Primes). We revisit Example 8, and formulate infeasibility of the systems of linear inequalities. That is, let P_1 for C_1 be a (4×1) -matrix of (integer) symbolic coefficients $p_0^1, p_1^1, \dots, p_3^1$, all greater or equal to 0. Then, to prove *non-selfs* for C_1 , we need to satisfy:

$$\begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \times P_1 \geq 0 \quad \text{and} \quad [0 \ 0 \ 0 \ -1] \times P_1 < 0.$$

We derive the following problem, where $\forall i \in \{0, 1, \dots, 3\} : p_i^1 \geq 0$:

$$-p_0^1 + p_1^1 - p_2^1 - p_3^1 \geq 0 \quad p_0^1 - p_1^1 + p_2^1 \geq 0 \quad -p_3^1 < 0$$

There is no solution to this problem, thus C_1 is *selfs*. For C_2 , we have

$$\begin{bmatrix} -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \times P_2 \geq 0 \quad \text{and} \quad [0 \ 0 \ 0 \ 0 \ 0 \ -1] \times P_2 < 0.$$

We derive the following problem, where $\forall i \in \{0, 1, \dots, 5\} : p_i^2 \geq 0$:

$$\begin{array}{ll} -p_0^2 + p_1^2 - p_2^2 + p_3^2 - p_4^2 - p_5^2 \geq 0 & p_0^2 - p_1^2 + p_4^2 \geq 0 \\ p_2^2 - p_3^2 + p_4^2 \geq 0 & -p_5^2 < 0 \end{array}$$

One solution is: $p_0^2 = 0, p_1^2 = 1, p_2^2 = 0, p_3^2 = 1, p_4^2 = 1$, and $p_5^2 = 1$. Therefore, C_2 is *non-selfs* and thus must be terminating. $\square$

To find a solution to these constraint problems, we can transform them to SAT problems, representing that a component is *non-selfs*, by using a transformation based on signed integer arithmetic. Thus, in this approach, we can only represent integers and not reals. This is still sufficient to represent the infeasibility problem, however, yields an incomplete approach (see Lemma 1): it is not guaranteed that a solution in the integers can be found if the original system is infeasible. Nevertheless, as it turns out, the approach works well in practice.

5 Evaluation

We have implemented the *non-selfs* test using SWI-Prolog [19] and integrated it with CHRisTA, a termination analyser for CHR(Prolog) [13]. This resulted in a new termination analyser, T-CoP [20] (Termination of CHR on top of Prolog).

CHRisTA uses CHR nets as a representation of CHR programs and implements an SCC analysis based on Tarjan’s algorithm. To solve the satisfiability problems that CHRisTA is confronted with, we represent the Diophantine constraints as a sufficient SAT problem based on signed integer arithmetic and solve it with MiniSAT2 [10]. Of course, by doing so, we place limits on the domains of variables and integers of the original problem. That is, we limit on the bit-sizes used for representing the variables and the integers of the problem. We reuse this system to prove infeasibility of the linear inequalities that we obtain for *selfs*.

Therefore, integrating the *non-selfs* test into CHRisTA involved the following steps: deriving a system of linear inequalities for *selfs* from the CHR net of an SCC, formulating infeasibility of that system, and transforming the resulting infeasibility problem to a SAT problem. Afterwards, a proof of termination of the remaining SCCs, that are possibly *selfs*, is attempted by the termination proof procedure of CHRisTA.

CHR program	3bit	sec	4bit	sec	CHRisTA (sec)	T-CoP (sec)
factorial	-	0.10	-	0.14	1.50	1.64
mean	+	0.35	+	0.58	1.26	0.58
mergesortsccl	-	0.12	-	0.20	1.44	1.64
mergesortsccl2	+	0.19	+	0.34	1.51	0.34
newcase1	+	0.41	+	0.74	timeout	0.74
newcase2	-	1.09	+	2.03	timeout	2.03
primessccl	-	0.11	-	0.19	1.67	1.86
primessccl2	+	0.18	+	0.33	1.11	0.33

Table 1. Results of *non-selfs* test on terminating SCCs.

Table 1 contains a representative set of the results that we obtained. T-CoP, as well as the full benchmark, is available online for reference [20]. Note that the considered programs of Table 1 contain only a single SCC of a size and complexity similar to the SCCs expected in real-sized CHR programs.

In Table 1, there are SCCs that are *non-selfs*, which are proven terminating (+) using at most a 4 bit representation for the variables and integers of the SAT problems. The SCCs, only proven terminating by CHRisTA, are *selfs*.

Among these programs, there are two programs on which the proof procedure of CHRisTA fails (see [12]). An example, representative for such programs, is

$\{(R_1 @ a, a \Leftrightarrow b.), (R_2 @ b \Leftrightarrow a.)\}$. To prove termination, for R_1 , the size of a must be greater or equal to b . For R_2 , a has to be strictly smaller than b .

Our new approach can handle such programs, and integrated with CHRisTA, improves the precision of the termination analysis as shown in Table 1 under “T-CoP”. From these results we may also conclude that the *non-selfs* test improves the efficiency of the analysis: If an SCC cannot be proven *non-selfs*, the overhead is acceptable; if it can be proven *non-selfs*, the gain in speed is relatively high.

Finally, note that the transformational approach of [16], combined with the technique of [17], can be used to handle programs such as newcase 1 and 2. Unfortunately, in the presence of propagation, the transformational approach of [16] cannot be applied.

6 Conclusion and future work

In this paper, we developed an approach to detect non-self-sustainability of the SCCs of a CHR program by means of a satisfiability problem. Integrating the approach with CHRisTA [13], resulted in a more efficient analyser, T-CoP. For one, the approach improves the precision of the termination analysis, being able to prove termination of a new class of CHR programs automatically. Furthermore, we improve the efficiency of the termination analyser CHRisTA (cfr. T-CoP).

Future work will consider propagation rules. For self-sustainability of propagation, we will need to assume presence of a fire-once policy. Intuitively, under such a policy, if no new combinations of constraints are ever introduced as a consequence of applying a propagation rule, such that the propagation rule can be applied again on the newly introduced combinations of constraints, then the propagation rule is non-selfs. Informally, this corresponds to verifying whether there does not exist a cycle in the CHR net of a CHR program across an abstract input constraint of the propagation rule and across some abstract output constraint *added* by a rule of the program. If so, the propagation rule can be ignored. Therefore, if we obtain by this approach an SCC that ultimately does not contain propagation, then, we can verify whether the SCC is non-selfs by the approach of this paper. Otherwise, the remaining SCC is still selfs by default.

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