Maximizing dissipation in a turbulent shear flow by optimal control of its initial state

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Turbulent shear flows are known to evolve self-similar at sufficiently high Reynolds numbers. In this regime, flow properties remain constant when normalized with local large-scale reference values. Dissipation of turbulent energy occurs at small scales. Thus, controls which increase dissipation levels in a shear flow may lose their effectiveness once the shear flow attains self-similar behavior governed by large-scale motion. In the current work, we investigate the optimal control of kinetic-energy dissipation in a temporally evolving turbulent mixing layer, with controls acting on its initial condition. Gradient-based optimization is used, relying on direct numerical simulations, and adjoint formulations for the determination of the gradients. Focus is on long optimization time windows T up to 160 nondimensional time units. First we investigate the optimal controls which maximize the total energy dissipation over a simulation time window. For increasing T , optimal controls are found to become also optimal for preceding time horizons. Results indicate that these controls facilitate a fast transition of the mixing layer to a self-similar state, which occurs at nondimensional time $t \approx 160$ –200. We find that the transition Reynolds number at which self-similarity emerges is a factor of 4–5 lower than transition Reynolds numbers reported for uncontrolled temporal mixing layers. Second, we investigate the maximization of the rate of dissipation at the end of the simulation time window. It is observed that dissipation rate can be increased up to two times higher than self-similar dissipation levels. However, at long optimization time windows this leads to a delay of transition to self-similarity. We further find that optimization is strongly nonconvex, and for the second case, we identify two local optima, which are further discussed in terms of differences in coherent structures emerging during the mixing-layer evolution. © 2011 American Institute of Physics. [doi:10.1063/1.3579487]

I. INTRODUCTION

Free shear flows appear in many industrial and environmental applications. Controlling the dissipation in such systems can be important to, e.g., smear out velocity differences, increase mixing, control reaction rates in reacting shear flows, reduce noise production, etc. For sufficiently high Reynolds numbers, free shear flows attain a self-similar state,^{1–4} characterized by constant profiles in time or space for mean velocity, Reynolds stresses, production and dissipation of turbulent kinetic energy, etc., when appropriately normalized with the large-scale flow properties, such as, e.g., the shear-layer thickness. Dissipation is a small-scale turbulent phenomenon, but in case of high Reynolds number self-similar shear flows, it is governed by the large-scale production of turbulent kinetic energy. The turbulent small-scale dissipation range is in that case in quasi-equilibrium. Therefore, flow control which increases dissipation levels in a shear flow may lose its effectiveness once the shear flow attains self-similar behavior governed by large-scale motion. In the current work, we investigate the optimal control of dissipation in turbulent shear flows and the transition to self-similarity based on direct numerical simulations. For reasons

of computational resources, we focus on the temporal mixing layer. As a consequence of this choice, the “controls” in the current work correspond to optimal perturbations of the mixing layer’s initial condition (as further detailed below). Hence, the optimal control problem considered in the current study differs from traditional open-loop or feed-back systems using control laws in time.

Growing computational resources and the development of massively parallel computer platforms have allowed for the combination of optimal control with direct numerical simulations (DNS) in recent years. Pioneering work, combining the theory of optimal control of partial differential equations, with transient three-dimensional Navier–Stokes simulations in the turbulent regime (i.e., using DNS), was performed in the context of drag reduction of turbulent boundary layers by Bewley *et al.*⁵ Lately, a lot of efforts have focused on the area of drag reduction,⁶ or bluff bodies,⁷ and the reduction of flow noise.⁸ In the studies of Bewley *et al.*⁵ and Wei *et al.*,⁸ controls are optimized using gradient-based algorithms in combination with adjoint-based determination of the gradient of the optimization cost functional. In the current work, we rely on a similar approach, tailored for a temporal mixing layer, and the optimal control of its initial condition.

Since the pioneering work of Brown and Roshko,⁹ the mixing layer has been studied intensively, and the mecha-

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nisms responsible for its transition to turbulence are well documented nowadays. In the 1990s, direct numerical simulations of the temporal mixing layer became feasible, and many simulation studies illustrated the roll-up of the Kelvin–Helmholtz instability, and the subsequent pairing and break-up processes which lead to a fully turbulent flow.^{3,4,10–13} The temporal framework was particularly attractive for its geometrical simplicity, allowing the use of efficient Fourier discretization, etc., contributing to the feasibility of these simulations on the computer systems of those days. In the current work, mixing-layer optimization requires $\mathcal{O}(10^3)$ flow simulations (including adjoint-based gradient estimations), dictating our choice for the temporal framework in view of current-day high-performance systems.

We consider the optimal control of the initial condition of a temporal mixing layer, i.e., the initial condition is constructed as a combination of a hyperbolic-tangent stream-wise velocity profile, and a set of spatially distributed perturbations with zero mean, which correspond to our controls. The controls in this temporal framework could be imagined to be connected to a set of spatially distributed actuators on an upstream splitter plate in a corresponding spatial framework. However, in the current work, it is not our intention to elaborate a rigorous connection between the optimized temporal results and a mixing layer in a spatial framework. Instead, we focus on the optimal control of dissipation in the mixing layer's transition and its effects on the ensuing self-similar region.

The optimization of the initial condition of temporal mixing layers has been studied extensively based on the linearized Navier–Stokes equations, mainly focusing on modal (eigenvalue) analysis^{14–17} (the use of nonmodal stability theory has gained a lot of attention in more recent years,^{18,19} but to our best knowledge, has not been used to study the temporal mixing layer). These studies mainly focus on the early growth rate of perturbations on an initial mean-flow condition and their further transition to turbulence. The current work focuses on optimization of the initial condition based on the nonlinear Navier–Stokes equations, with optimization cost functionals which strongly depend on the nonlinear flow evolution at longer time horizons. The optimization is performed to simulation time horizons well into the turbulent range with $\text{Re}_\lambda \approx 40$.

The first case that is considered (case 1) is the maximization of the total kinetic energy dissipated over a simulation time window $[0, T]$. Different time horizons T are considered, up to $T=160$ nondimensional time units, and two different energy levels for the controls are included. For increasing T , optimal controls are found to become also optimal for preceding time horizons. Results indicate that these controls facilitate a fast transition of the mixing layer to a self-similar state, which occurs at nondimensional time $t \approx 160$ – 200 , and conclusions are independent of the energy levels used for the controls. In all these cases, we find that the maximum dissipation rate appears well before the onset of self-similarity during the early transition of the mixing layer (around $t=70$).

Therefore, we attempt in the second case (case 2) to maximize the dissipation rate at the time horizon T for time

horizons up to $T=160$. For this case, it is observed that dissipation levels can be controlled up to levels which are a factor of 2.5 higher than dissipation levels in the self-similar region. For long optimization time windows this leads to a delay of transition to self-similarity.

We found the optimization of controls in the temporal mixing layer to be strongly nonconvex, such that local optima may exist. In earlier work focusing on mixing-layer transition for low time horizons ($T \leq 40$), it was observed that this can lead to small inconsistencies between optima obtained for different time horizons:²⁰ in some cases, optimal perturbations for $T_1 < T_2$ are found to lead to lower cost functionals at $t=T_2$ than perturbations optimized for T_2 . To avoid this type of situations, we slightly adapt the optimization procedure in the current work using a simple continuation method in the space of optimal solutions. Nevertheless, for the second optimization case (maximizing dissipation rate), two strongly distinct local optima are found, depending on the starting point of the procedure. Differences between these optima are discussed, and coherent structures emerging during the mixing-layer evolution are shown.

The paper is further organized as follows. First, in Sec. II, the optimization problem, optimization technique, and computational setup are described. The optimization results are discussed in Secs. III and IV for case 1 and case 2, respectively. Conclusions are presented in Sec. V.

II. OPTIMIZATION CASE AND COMPUTATIONAL SETUP

A. Optimization problem

We optimize a temporal turbulent mixing layer by searching for an optimal set of initial perturbations which are imposed on a hyperbolic-tangent stream-wise mean velocity profile. In the current section, this optimization problem is defined in terms of cost functionals and constraints.

Conventionally, an optimization problem is formulated in terms of the minimization of a cost functional $\mathcal{J}$. We consider cost functionals $\mathcal{J}$ of the form

$$\mathcal{J}[\mathbf{q}(\boldsymbol{\phi})] = \int_{\Omega} \mathcal{J}[\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi})] d\mathbf{x}, \quad (1)$$

with $\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi})$ a turbulent flow solution $\mathbf{q}=[\mathbf{u}, p]$ at time horizon T and defined on the spatial domain Ω , with $\mathbf{u}$ the velocity field, and p the pressure. The flow solution $\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi})$ follows from the evolution of the mixing layer with initial condition

$$\mathbf{u}(\mathbf{x}, 0, \boldsymbol{\phi}) = \tanh(x_3) \mathbf{e}_1 + \boldsymbol{\phi}(\mathbf{x}), \quad (2)$$

with $\boldsymbol{\phi}(\mathbf{x})$ the optimization controls (i.e., the initial perturbations of the mixing layer) and $\mathbf{e}_1$ the unit vector in the stream-wise direction x_1 .

The optimization problem is conventionally formulated as minimization

$$\min_{\boldsymbol{\phi} \in L^2(\Omega)} \mathcal{J}[\mathbf{q}(\boldsymbol{\phi})], \quad (3)$$

further subject to three additional constraints. First of all, the problem is a partial differential equation (PDE)-constrained

optimization problem, since the evolution of the flow $\mathbf{q}(\mathbf{x}, t, \boldsymbol{\phi})$ is governed by the incompressible Navier–Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad (4)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{\text{Re}} \nabla^2 \mathbf{u} = 0, \quad (5)$$

with Re the Reynolds number, and with initial conditions at $t=0$ given by Eq. (2). The controls $\boldsymbol{\phi}$ are subject to two further constraints, i.e.,

$$\nabla \cdot \boldsymbol{\phi}(\mathbf{x}) = 0, \quad (6)$$

$$\frac{1}{2\Omega} \int_{\Omega} \boldsymbol{\phi} \cdot \boldsymbol{\phi} d\mathbf{x} = E_0 \frac{(\Delta U)^2}{8}. \quad (7)$$

Constraint [Eq. (6)] corresponds with the requirement that the initial velocity field $\mathbf{u}(\mathbf{x}, 0, \boldsymbol{\phi})$ satisfies continuity [cf. Eqs. (2) and (4)]. Further, Eq. (7) keeps the total energy per unit volume of the perturbations fixed at a predefined constant level $E_0(\Delta U)^2/8$, where the energy of the perturbations E_0 is expressed relative to the mean-flow kinetic energy per volume $(\Delta U)^2/8$ (for an infinite box; with $\Delta U=2$ being the velocity difference between the upper and the lower streams of the mixing layer).

As discussed in Sec. I, we focus on two cost functionals in the current study. The first cost functional maximizes the total dissipated kinetic energy up to the time horizon. Since the volume-averaged dissipation $\varepsilon=2\nu\mathcal{E}$, with the volume-averaged enstrophy defined as

$$\mathcal{E}(T, \boldsymbol{\phi}) = \frac{1}{2\Omega} \int_{\Omega} \boldsymbol{\omega}(\mathbf{x}, T, \boldsymbol{\phi}) \cdot \boldsymbol{\omega}(\mathbf{x}, T, \boldsymbol{\phi}) d\mathbf{x}, \quad (8)$$

and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, we can now define a cost functional which maximizes the volume-averaged dissipation of energy over a time window $[0, T]$ as

$$\mathcal{J}_{\text{DE}} = -\frac{\nu}{\Omega} \int_0^T \int_{\Omega} \boldsymbol{\omega}(\mathbf{x}, t, \boldsymbol{\phi}) \cdot \boldsymbol{\omega}(\mathbf{x}, t, \boldsymbol{\phi}) d\mathbf{x} dt, \quad (9)$$

$$= - \left[(1 + E_0)(\Delta U)^2/8 - \frac{1}{2\Omega} \int_{\Omega} \mathbf{u}(\mathbf{x}, T, \boldsymbol{\phi}) \cdot \mathbf{u}(\mathbf{x}, T, \boldsymbol{\phi}) d\mathbf{x} \right]. \quad (10)$$

In Eq. (10) the total dissipated energy is expressed as the difference between the total energy at $t=0$ and the energy at $t=T$, and we use this formulation of the cost functional in the optimization procedure.

A cost functional that maximizes the dissipation rate at the time horizon T is straightforwardly expressed as

$$\mathcal{J}_{\text{DR}} = -\frac{\nu}{\Omega} \int_{\Omega} \boldsymbol{\omega}(\mathbf{x}, T, \boldsymbol{\phi}) \cdot \boldsymbol{\omega}(\mathbf{x}, T, \boldsymbol{\phi}) d\mathbf{x}. \quad (11)$$

For the sake of convenience, we will denote optimal perturbations for a time horizon T using $\boldsymbol{\phi}_T$ (e.g., for $T=60$ we

write $\boldsymbol{\phi}_{60}$). We will evaluate the optimal perturbations by looking at the mixing layer's evolution of total dissipated energy in time defined as

$$\Delta E(t, \boldsymbol{\phi}_T) = \frac{\nu}{\Omega} \int_0^t \int_{\Omega} \boldsymbol{\omega}(\mathbf{x}, t', \boldsymbol{\phi}) \cdot \boldsymbol{\omega}(\mathbf{x}, t', \boldsymbol{\phi}) d\mathbf{x} dt'. \quad (12)$$

Likewise, we will monitor the dissipation rate by evaluating the volume-averaged enstrophy in time, i.e., $\mathcal{E}(t, \boldsymbol{\phi}_T)$, using its definition [Eq. (8)]. It is clear that these properties are related to the optimization cost functionals as $\Delta E(T, \boldsymbol{\phi}_T) = -\mathcal{J}_{\text{DE}}$ and $\mathcal{E}(T, \boldsymbol{\phi}_T) = -\mathcal{J}_{\text{DR}}/(2\nu)$.

B. Optimization technique

A gradient-based optimization algorithm is employed for the optimization, similar to the techniques used in the work of, e.g., Bewley *et al.*,⁵ Wei and Freund,⁸ and also documented in Delpont *et al.*:²⁰ it relies on the Polak–Ribière conjugate-gradient method^{21–23} combined with the Brent line-search algorithm.^{21,24} The method is a search method with a double iteration loop: at a point $\boldsymbol{\phi}^{(k)}$ in the outer iteration cycle, the conjugate-gradient direction is determined; the next point in the outer iteration $\boldsymbol{\phi}^{(k+1)}$ is then the optimum of the cost functional along this search direction. To find the optimum along the search direction, an iterative line-search algorithm is employed (i.e., the inner iteration cycle). First, an interval which contains a minimum is bracketed using an algorithm based on the mnbrak algorithm;²¹ subsequently, the Brent algorithm^{21,24} is used to iteratively narrow down the interval. For cost-functional evaluations at each point of the inner iteration cycle, separate Navier–Stokes simulations are required. The determination of the conjugate-gradient direction in the outer cycle requires the gradient $\mathcal{G}^{(k)}$ of the cost functional at the point $\boldsymbol{\phi}^{(k)}$.

To compute the gradient of the cost functional, we need to evaluate the sensitivity of $\mathcal{J}$ at a given point $\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi})$ to small changes $\delta\boldsymbol{\phi}$ of the control parameters $\boldsymbol{\phi}$, i.e., the initial perturbations to the mixing layer. The sensitivity is calculated as the Gateau-differential of $\mathcal{J}$ in the point $\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi})$,²⁵

$$\delta\mathcal{J}(\boldsymbol{\phi}; \delta\boldsymbol{\phi}) \equiv \left. \frac{d}{d\alpha} \mathcal{J}[\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi} + \alpha\delta\boldsymbol{\phi})] \right|_{\alpha=0}, \quad (13)$$

$$= \int_{\Omega} \frac{\partial \mathcal{J}[\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi})]}{\partial \mathbf{q}} \cdot \delta\mathbf{q} d\mathbf{x}, \quad (14)$$

with $\delta\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi}, \delta\boldsymbol{\phi})$ the sensitivity of the flow solution at the time horizon to a change of $\boldsymbol{\phi}$ in the direction $\delta\boldsymbol{\phi}$. In the current case, $\boldsymbol{\phi}$ contains a large number of degrees of freedom, and the determination $\delta\mathbf{q}$ for every possible $\delta\boldsymbol{\phi}$ becomes prohibitive. This problem is circumvented by solving an adjoint problem,^{5,26–29}

$$\nabla \cdot \mathbf{u}^* = 0, \quad (15)$$

$$-\frac{\partial \mathbf{u}^*}{\partial t} - \mathbf{u} \cdot \nabla \mathbf{u}^* - \mathbf{u} \cdot (\nabla \mathbf{u}^*)^T - \nabla p^* - \frac{1}{\text{Re}} \nabla^2 \mathbf{u}^* = 0. \quad (16)$$

For a temporal mixing layer, periodic boundary conditions are used in the stream-wise and span-wise directions, and

FIG. 1. Optimization procedure for a set of optimizations toward different time horizons: (a) as used in previous work (Ref. 20) and (b) currently used.

symmetric boundary conditions in the normal direction.²⁰ Further, the initial condition for the adjoint equations, which is given at time T , is $\mathbf{u}^*(\mathbf{x}, T, \boldsymbol{\phi}) = \partial J[\mathbf{u}(\mathbf{x}, T, \boldsymbol{\phi})] / \partial \mathbf{u}$, and the gradient of the cost functional is expressed as²⁰

$$\delta \mathcal{J}[\mathbf{q}(\mathbf{x}, T, \boldsymbol{\phi}), \delta \boldsymbol{\phi}] = \int_{\Omega} \mathbf{u}^*(\mathbf{x}, 0, \boldsymbol{\phi}) \cdot \delta \boldsymbol{\phi} d\mathbf{x}. \quad (17)$$

To take the two constraints [Eqs. (6) and (7)] on the control parameters into account, the parameter elimination²³ and the gradient-projection method²² are used, respectively. The parameter elimination method is used for the linear continuity constraint $\nabla \cdot \boldsymbol{\phi}(\mathbf{x}) = 0$. It reduces the possible perturbations $\boldsymbol{\phi}$ on the mixing layer to a subset of linear independent parameters by taking into account the linear constraints imposed by the discretized continuity equations. This subset of linear independent parameters is then optimized using the conjugate-gradient method. The gradient-projection method is used to enforce the energy constraint. The conjugate-gradient search direction is projected onto the nonlinear energy-constraint surface, such that the constraint is satisfied at all steps during the optimization process. More details can be found in Ref. 20.

For nonconvex optimization, gradient-based procedures may lead to local optima which depend on the starting point of the optimization procedure. In Delpont *et al.*,²⁰ it was found that local optima emerge with a dependence on the optimization time horizon T that is inconsistent. For instance, when optimization is performed to $\mathcal{J}_{\text{DE}}$, it is expected for two different optimization time horizons $T_2 > T_1$ that $\Delta E(T_2, \boldsymbol{\phi}_{T_2}) \geq \Delta E(T_2, \boldsymbol{\phi}_{T_1})$ (cf. Sec. II A). However, in Delpont *et al.*²⁰ we found that this is not always the case while using the conjugate-gradient procedure starting from the same initial guess for the controls (i.e., this was observed for $T_1 = 20$, $T_2 = 40$, and $\mathcal{J}_{\text{KE}}$). Also at the outset of the current study, we were faced with this type of local optima when starting the conjugate-gradient iterations from the same starting point $\boldsymbol{\phi}_A$. Therefore, we designed a different optimization procedure, which tries to avoid inconsistent local optima. The procedure is outlined in the two following paragraphs.

In order to get a more consistent characterization of optima as a function of the time horizon T , we slightly changed the optimization approach, as schematically shown in Fig. 1. Instead of starting all optimizations for different time horizons directly from $\boldsymbol{\phi}_A$, we only start the optimization for the

lowest time horizon ($T=60$) from $\boldsymbol{\phi}_A$. Subsequently, the optimal parameter set $\boldsymbol{\phi}_{60}$ is used as a starting point for the optimization to the next time horizon ($T=80$), and so forth, until the last time horizon $T=160$ (cf. Fig. 1). In this way, $\Delta E(T_2, \boldsymbol{\phi}_{T_2}) \geq \Delta E(T_2, \boldsymbol{\phi}_{T_1})$ is guaranteed when $\mathcal{J}_{\text{DE}}$ is minimized (with T_1 and T_2 as two subsequent time horizons and $T_2 > T_1$), and $\mathcal{E}(T_2, \boldsymbol{\phi}_{T_2}) \geq \mathcal{E}(T_2, \boldsymbol{\phi}_{T_1})$ in case $\mathcal{J}_{\text{DR}}$. This procedure corresponds to a simple continuation approach in the space of optimal solutions. If we presume that $\boldsymbol{\phi}_T$ depends continuously on T , this is nothing more than a first-order extrapolation of $\boldsymbol{\phi}$, which we use as a starting point for the conjugate-gradient method. In the current work, higher-order continuation approaches were not further investigated.

Notwithstanding the procedure described above, the optimization problem remains nonconvex, and local optima may still exist. In particular, for optimization toward maximum dissipation of energy at the time horizon (case 2), we identified two distinct optima (cf. Sec. IV below for a further discussion on these optima). These optima were found by trying a number of different initial starting points; the two starting points $\boldsymbol{\phi}_A(\mathbf{x})$ and $\boldsymbol{\phi}_B(\mathbf{x})$ which led us to these local optima are briefly discussed in Sec. II C.

C. Case setup and discretization

We consider a temporal mixing layer in a box with periodic boundary conditions in stream-wise (x_1) and span-wise (x_2) directions, and symmetry conditions in the normal direction (x_3). The initial stream-wise mean velocity corresponds to a tanh-profile (cf. Sec. II A), which imposes a velocity difference between the upper and lower flows of $\Delta U = 2$, and an initial vorticity thickness $\delta_\omega = 2$ [with $\delta_\omega \equiv \Delta U / (d\langle u_1 \rangle / dx_3)|_{\text{max}}$]. In the current work, all results are normalized using $\delta_\omega/2$ and $\Delta U/2$ as reference length scale and reference velocity scale. The Reynolds number, based on half of the vorticity thickness, and half of the velocity difference, corresponds to 50, which is similar to the Reynolds number in the study of Vreman *et al.*¹³ The box size in stream-wise direction corresponds to $L_1 = 123.2 = 8\lambda_1$, with $\lambda_1 = 15.4$ corresponding with the most unstable wavelength following from linear stability theory. In span-wise direction, the box size $L_2 = 74.04 = 8\lambda_2$ with $\lambda_2 = 0.6\lambda_1$. Here, λ_2 is the wavelength of the most unstable span-wise perturbation of rollers originating from the Kelvin–Helmholtz instability in the stream-wise direction.¹⁶ Finally, the box size in the nor-

mal direction, $L_3=240$, is selected sufficiently large to exclude interactions between the mixing region in the center of the domain and the symmetry conditions at the boundaries.

For the discretization of both the standard and adjoint Navier–Stokes equations, a mixed pseudospectral finite-volume code is employed.^{20,30} In the two periodic directions (stream-wise and span-wise), a pseudospectral Fourier discretization is used. Dealiasing of the nonlinear terms is performed using the two thirds dealiasing rule.³¹ The normal direction is discretized using a fourth-order energy-conserving staggered finite-volume discretization,³² and the locations of normal velocities u_3 are shifted in the normal direction by half a cell compared to p , u_1 , and u_2 . Continuity is retained by solving a Poisson equation for the pressure using a direct solver.

For the Navier–Stokes simulation, the time integration is performed by a classical four-stage fourth-order accurate Runge–Kutta time integration. Time steps are restricted by setting both the convective and viscous Courant–Friedrichs–Lewy numbers to 0.2. For the adjoint simulation, the same time scheme is used but now backward in time. The adjoint flow is solved at the same points in time as in the accompanying nonlinear simulations. This enables the adjoint simulations to use the solution of the preceding Navier–Stokes equations $\mathbf{u}(\mathbf{x}, t_n, \boldsymbol{\phi})$ without interpolation. The solution $\mathbf{u}(\mathbf{x}, t_n, \boldsymbol{\phi})$ is stored onto disk during the preceding simulation at every grid point and every time step. At the beginning of every Runge–Kutta time step of the adjoint system, the nonlinear solution $\mathbf{u}(\mathbf{x}, t_n, \boldsymbol{\phi})$ at step t_n is read from the disk, but subsequently kept constant during the Runge–Kutta stages.

All simulations are performed on a $128 \times 128 \times 256$ mesh (dealiasing of the nonlinear terms is performed on $192 \times 192 \times 256$). In the normal direction, the grid is stretched with a grid-spacing $\Delta x_3 \sim |x_3|^{1/4}$ (this particular stretching is selected based on the observation that, for self-similar mixing layers, the Kolmogorov scale $\eta \sim \theta^{1/4}$, with θ the momentum thickness of the layer).

During optimization, we have to ensure that the perturbation field $\boldsymbol{\phi}$ remains sufficiently smooth and well-resolved on the computational grid. In the theory on optimal control of partial differential equations, this is ensured by selecting a sufficiently smooth functional space to which the controls should belong. Elaboration of the gradient, which is the Riesz representation of the derivative (e.g., cf. Ref. 33, p. 98), then leads in practice to the construction of the gradient of the cost functional based on smoothed Gateau differentials (e.g., for a Sobolev space H^1 this would correspond to an elliptic smoothing operator³⁴). In the absence of smoothing, it is well documented that optimal controls may become very irregular and “unphysical.”²⁸ In the current work, we approach the issue of constructing a smooth gradient in a more pragmatic way, which does not require the inversion of an elliptic PDE operator. First of all, in the normal direction (x_3), a Gaussian filter was used to filter the gradient, with filter width $10\Delta x_3$, such that $\boldsymbol{\phi}$ remains well-resolved on the computational grid (see Ref. 20 for more details on consistency and convergence of the methodology). In the stream-wise and span-wise directions, we elected to restrict the ini-

tial perturbations $\boldsymbol{\phi}$ to wavelengths larger or equal to $1/16$ th of the box size, which is equivalent to applying a sharp cut-off filter on the gradients. Hence, $\boldsymbol{\phi}$ contains Fourier modes with wave numbers $(k_1, k_2) = (\alpha 2\pi/L_1, \beta 2\pi/L_2)$ with $\alpha, \beta = 0, \pm 1, \dots, \pm 16$ (the mode $\alpha = \beta = 0$ is excluded). In this way, the optimal controls remain sufficiently smooth and well-resolved in x_1 and x_2 directions. We verified that the eventual controls from our optimization have negligible energy in the vicinity of the Fourier cut-off frequency. Finally, taking all smoothing operations into account, we end up with about $\mathcal{O}(10^5)$ degrees of freedom in the controls, which are to be optimized.

Finally, as previously mentioned in Sec. II B, two starting points for the optimization $\boldsymbol{\phi}^{(0)}$ are employed, $\boldsymbol{\phi}_A(\mathbf{x})$ and $\boldsymbol{\phi}_B(\mathbf{x})$. In Fig. 2 the mixing-layer evolution starting from these two initial conditions is displayed. It is appreciated that these initial perturbations (using $E_0 = 10^{-4}$) lead to distinctly different evolutions of the temporal mixing layer, both when evaluating ΔE and $\mathcal{E}$, or when looking at the coherent structures using the λ_2 criterion. The construction of the perturbations $\boldsymbol{\phi}_A(\mathbf{x})$ and $\boldsymbol{\phi}_B(\mathbf{x})$ themselves is rather *ad hoc*, but is briefly added here for the sake of completeness.

The first field $\boldsymbol{\phi}_A(\mathbf{x})$ is constructed similarly to the initial conditions reported by Moser and Rogers¹² and Vreman *et al.*¹³ The perturbations are a superposition of two-dimensional modes [having mode numbers $(\alpha, 0)$, $\alpha = \mp 1 \dots \mp 16$] and three-dimensional modes [having mode numbers (α, β) , $\beta \neq 0$]. All two-dimensional modes use the most unstable eigenfunction in the normal direction for that particular wavelength, following from a linear stability analysis. The phases of the modes are set to zero. For the three-dimensional modes, a Gaussian profile (with $\sigma = 3.5$) is used in the normal direction for u_2 and u_3 (the profile for u_1 follows from u_2 and u_3 using the continuity equation). The total energy is evenly distributed over all modes.

The second set of initial perturbations $\boldsymbol{\phi}_B(\mathbf{x})$ is composed of broadband random noise for 20% of its kinetic energy and of low-pass filtered broadband random noise for the remaining 80% of its total kinetic energy. The noise was generated with the standard pseudorandom number generator of FORTRAN. It imposed the noise field in physical space. Subsequently, the noise was made divergence-free. The low-pass filter employed is a Gaussian filter. Its width is selected such that the transfer function of the filter has a value of 0.5 at the most unstable wave number (8,0) obtained from linear stability theory.

III. CASE 1: MAXIMIZE DISSIPATED KINETIC ENERGY

In the current section, the first case is discussed. In Sec. III A, convergence is discussed, results are presented, and prevalent flow features of the optimal solutions are shown. Next, in Sec. III B, optimal solutions are evaluated in terms of their self-similar evolution for long simulation times. In Sec. III C, we further discuss the consequences related to the maximization of total dissipated energy for long time windows.

FIG. 2. (Color online) Mixing-layer evolution starting from two different initial perturbations ϕ_A and ϕ_B (with $E_0=10^{-4}$). [(a) and (b)] λ_2 visualization (Ref. 35) of the velocity field at time $t=120$ (colored online with the distance from the mixing center plane); [—(in gray)]: profiles for $\langle u_1 \rangle$. (c) Evolution of the dissipated energy ΔE and (d) enstrophy $\mathcal{E}$ for initial conditions (—) ϕ_A and (---) ϕ_B .

A. Optimization results

Optimization results for the maximization of total dissipated total kinetic energy (i.e., based on $\mathcal{J}_{DE}$), and using ϕ_A as an initial guess, are now presented in Fig. 3 for both energy levels $E_0=10^{-5}$ and 10^{-4} . In Figs. 3(a) and 3(c), the convergence history as a function of the number of conjugate-gradient (outer) iterations is presented. We use a convergence criterion based on a relative improvement of the cost functional of 10^{-10} normalized with the total kinetic energy at the time horizon. As shown in Figs. 3(a) and 3(c), not all cases are converged up to this criterion; instead, some were stopped after 200 iterations in order to limit the computational costs (i.e., we recall that one conjugate-gradient iteration requires one adjoint Navier–Stokes simulation for the gradient calculation and about eight Navier–Stokes simulations for the line-search algorithm).

The evolution of the total dissipated energy $\Delta E(t, \phi_T)$ is displayed for different time horizons in Figs. 3(b) and 3(d), respectively, for $E_0=10^{-5}$ and 10^{-4} . A large increase of the total dissipated energy for the optimized cases is appreciated

when compared to the evolution with ϕ_A (also displayed in the figures). Optimizations starting from different initial guesses (e.g., ϕ_B) do not lead to better optima for $\mathcal{J}_{DE}$ and are not further discussed here.

When looking at Figs. 3(b) and 3(d), the first important observation is that, for $T_2 > T_1$, the corresponding optimal energy levels $\Delta E(T_1, \phi_{T_2}) \approx \Delta E(T_1, \phi_{T_1})$. Hence, for the time horizons investigated here, the optimal solution for a time horizon T is also optimal for all time horizons smaller than T , in particular, we find that ϕ_{160} is optimal for $60 \leq T \leq 160$.

To further analyze this, it is useful to focus on the maximum dissipated total kinetic energy for each time horizon $\Delta E(T, \phi_T)$ as a function of the time horizon T . This is not anymore the evaluation of the time evolution of one optimum [as, e.g., expressed by $\Delta E(t, \phi_T)$], but rather the evolution of the optima as a function of the optimization time horizon, evaluated at time $t=T$. Figure 4 shows that $\Delta E(T, \phi_T)$ is a monotonic function that increases with T .

FIG. 3. Optimization results for $\mathcal{J}_{DE}$ starting from ϕ_A . [(a) and (c)] Convergence history for the different time horizons [with $\Delta \mathcal{J}_{DE} = \mathcal{J}_{DE}(it) - \mathcal{J}_{\text{final}}$] normalized with the kinetic energy at the time horizon when the final parameters are used $|\mathbf{u}(T)|_{\text{final}}$. [(b) and (d)] Evolution of the total dissipated energy $\Delta E(t, \phi_T)$ as a function of time for different time horizons. Symbols in (a)–(d): $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$). Additional lines in (b) and (d): evolution of the total dissipated energy for initial condition ϕ_A (—) and ϕ_B (---); vertical lines are drawn at the different optimization time horizons.

Further, $\Delta E(T, \phi_T) = \Delta E(T, \phi_{160})$, with $T=160$ the longest optimization time horizon included in the current study. $\Delta E(t, \phi_T)$ may also be expressed as

$$\Delta E(t, \phi_T) = 2\nu \int_0^t \mathcal{E}(t', \phi_T) dt', \quad (18)$$

with $\mathcal{E}(t, \phi_T)$ the volume-averaged enstrophy evolution starting from the $\mathcal{J}_{DE}$ -optimal perturbations ϕ_T . Therefore, $\Delta E(T_1, \phi_{T_2}) \approx \Delta E(T_1, \phi_{T_1})$ for $T_2 > T_1$ requires that the integrals of $\mathcal{E}(t, \phi_{T_1})$ and $\mathcal{E}(t, \phi_{T_2})$ are equal when integrated

FIG. 4. Maximal total dissipated energy as a function of time horizon T for flows optimized toward $\mathcal{J}_{DE}$. (— $\diamond$) $E=10^{-5}$ and (— $\square$) $E=10^{-4}$.

from 0 to T_1 . This is illustrated in Fig. 5(a), where the enstrophy evolution of flows optimized to $\mathcal{J}_{DE}$ (and $E_0=10^{-4}$) is shown. In this figure, it is now observed for all optima (except for $T=60$ and 80) that the enstrophy evolutions collapse with a peak around $t \approx 70$. Moreover, the longer the optimization time horizons, the longer optimal solutions manage to sustain a high enstrophy level after the peak (i.e., for $T=80$, the enstrophy drastically drops down starting at $t \approx 70$, for $T=100$ this occurs at $t \approx 90$, for $T=120$ at $t=110$, etc.).

Further, we also display for the different optimization cases the evolution of the turbulent kinetic energy in Fig. 5(b). It is appreciated that the turbulent kinetic energy increases significantly compared to its initial level, i.e., 0.5×10^{-4} initially compared to, e.g., 0.015 at $t=60$. The latter energy level corresponds to turbulent fluctuations with an amplitude of approximately 10% of the reference velocity ($\Delta U/2$). Taking into account that all optimization time horizons at which cost-functional evaluations are defined are larger (or equal) than $t=60$, this clearly illustrates that the current optimization is in the nonlinear regime, with amplitudes which are too large to rely on linear methods.¹⁸

When comparing $\Delta E(T, \phi_T)$ for $E_0=10^{-5}$ and $E_0=10^{-4}$ in Fig. 4, we observe a straightforward effect of the control energy: more energy in the initial perturbations yields higher

FIG. 5. Evolution of the enstrophy (a) and the turbulent kinetic energy (b) in time for flows optimized toward $\mathcal{J}_{DE}$: $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$) for $E_0=10^{-4}$.

dissipated energy for all optimization time horizons. Apart from this, we did not observe any substantial differences with respect to the observations discussed above. When looking at the shape of turbulent structures which emerge in the evolution of the mixing layer, small differences are observed, as discussed next.

The flow structures of the shear layer optimized to $\mathcal{J}_{DE}$ with time horizon $T=160$ are now visualized in Fig. 6 using the λ_2 criterion.³⁵ The structures observed for other optimization time horizons are very similar and are not further shown here. In Fig. 6 (right), the coherent structures at the time horizon $t=160$ are displayed. It is appreciated that the

FIG. 6. (Color online) λ_2 visualization of the velocity field for parameters optimized toward $\mathcal{J}_{DE}$ (colored online with the distance from the mixing center plane); profile $\langle u_1 \rangle$ (—); visualization of $t=30$ (left) and $t=160$ (right).

FIG. 7. Energy distribution over the modes of the optimal initial perturbations (at $t=0$) optimized to time horizon $T=160$ and colored by $\log[E_{\text{mode}}/(E_0 \Delta U^2/8)]$. Energy is integrated over x_3 .

shear layer is fully turbulent, dominated by small-scale turbulent structures. In Fig. 6, λ_2 results are also shown at $t=30$ to evaluate the coherent structures in the early transition phase of the shear layer. The evaluation of λ_2 at $t=0$ does not yield any meaningful results.

It is appreciated in Fig. 6 (left) that the coherent structures at $t=30$ for $E_0=10^{-5}$ form a diamond-shaped vortex pattern. Figure 7, displaying the energy distribution of the optimal perturbations (at $t=0$ and with optimization time horizon $T=160$), illustrates that these diamonds originate from a high initial energy concentration in the $(6, \pm 2)$ mode. This mode has a wavelength $(8\lambda_1/6, 8\lambda_2/2)$ (with $\lambda_1=15.4$ and $\lambda_2=0.6\lambda_1$, cf. Sec. II C). The coherent structures at $t=30$ for $E_0=10^{-4}$ look similar, but with some additional features imposed on top of the smooth diamond shapes observed for $E_0=10^{-5}$. Also in Fig. 7, it is observed for $E_0=10^{-4}$ that the energy is mainly concentrated at the same modes.

Similar diamond-shaped vortex structures have also been observed in experiment of a forced mixing layer by Nygaard and Glezer (Fig. 5 in Ref. 36). They tested the impact of a span-wise array of surface heaters, placed on a splitter plate, on the subsequent flow evolution. With an adequate phase difference between the heaters, diamond-shaped structures are obtained. Collis *et al.*³⁷ showed that these structures may also be observed in an incompressible temporal mixing layer when a pair of oblique disturbances is used for the initial condition, which is similar to our observation. They also showed that this flow behavior increases mixing compared to standard rib/roller flows.³⁷

It is also interesting to compare the current diamond-shaped structures to the vortex patterns emerging in classical transition to turbulence reported for mixing layers. In the latter case, random perturbations trigger the Kelvin–Helmholtz instability (with wavelength $\lambda_1=15.4$ in the current study), which grows exponentially at first, and subsequently saturates into two-dimensional Kelvin–Helmholtz rollers. These two-dimensional vortex structures may pair into larger rollers, but in the presence of three-dimensional disturbances they are also subject to secondary three-dimensional instabilities, which lead to the formation of stream-wise vorticity and transition to turbulence. The two main secondary instabilities are a core-centered elliptic insta-

bility, with most unstable wavelength in the x_2 direction around^{16,17} $0.5\lambda_1-0.7\lambda_1$, and a braid-centered hyperbolic instability, with most unstable wavelength (in x_2) around¹⁷ $0.2\lambda_1$. Our current, optimized perturbations [dominated by $(6, \pm 2)$ modes] clearly follow a different transition path, and the dominant wavelengths are much larger, i.e., by 30% in the stream-wise direction (compared to the Kelvin–Helmholtz wavelength) and by a factor of 4–12 in the span-wise direction (compared to wavelengths of the core-centered and braid-centered instabilities, respectively).

Based on the observations in Figs. 6 and 7, we investigate how many modes are essential for the optimal evolution (leading to maximal dissipated energy) of the mixing layer. To this end, for $E_0=10^{-5}$ initial conditions are constructed by only taking the first n dominant modes from Fig. 7 (based on the energy per mode), and we monitor the total energy dissipation for different values of n . For $n=2$ [only modes $(6,2)$ and $(6,-2)$], it is found that the total dissipation reaches 66% of the optimal dissipation. For $n=4$, the total energy dissipated corresponds to 74% of the optimal value. For $n=8$, we find a total dissipation that is only slightly higher (75%). Hence, using only four modes, up to 75% of the energy dissipation realized by optimal control may be recovered.

B. Self-similarity

Here, the evolution of the optimized mixing layers for longer simulation times is monitored beyond the time horizon for which the initial controls are optimized. For all cases, self-similarity is observed.

For temporal mixing layers, self-similarity is characterized by linear growth of the momentum thickness ($\theta = \int [1/4 - (\langle u_1 \rangle / \Delta U)^2] dx_3$) in time. Also, turbulent kinetic energy grows linearly in time, and the ratio between the total production and the total dissipation of turbulent kinetic energy is constant. Small turbulent scales in the flow are in equilibrium, which can, e.g., be verified by looking at the balance between enstrophy production and dissipation. Various other diagnostics can be investigated, such as the collapse of mean velocity profiles, and Reynolds stresses when appropriately scaled with θ and ΔU , etc. More details are, e.g., found in Refs. 3, 4, 38, and 39.

FIG. 8. Evolution of the momentum thickness growth rate in time for flows optimized toward $\mathcal{J}_{DE}$, $E_0=10^{-4}$. $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$).

In Fig. 8, the growth rate $r_\theta \equiv d\theta/dt/\Delta U$ of the mixing-layer momentum thickness is displayed for $E_0=10^{-4}$ and flow optimized to $\mathcal{J}_{DE}$. It is observed that optimized results roughly reach a constant growth rate for $t \geq 160$ with values between 0.007 and 0.01. This range is a bit lower than the value of 0.014 obtained in the study of Rogers and Moser.³

In Fig. 9(a), the ratio between production and dissipation of turbulent kinetic energy at the mixing-layer center is shown. They are, respectively, defined as (see, e.g., Ref. 39)

$$\mathcal{P} = -\frac{1}{L_1 L_2} \int \left(u'_1 u'_3 \frac{\partial \langle u_1 \rangle}{\partial x_3} \right) \Big|_{x_3=0} dx_1 dx_2 \quad (19)$$

and

$$\mathcal{D} = \frac{1}{L_1 L_2} \int 2\nu(s:s)_{x_3=0} dx_1 dx_2, \quad (20)$$

with $s = [\nabla \mathbf{u}' + (\nabla \mathbf{u}')^T]/2$ the shear-stress tensor related to the fluctuating velocity $\mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$ (and $\langle \mathbf{u} \rangle$ the velocity averaged over x_1 - x_2 planes). For all cases, we observe a ratio $\mathcal{P}/\mathcal{D} \approx 1.35$, evaluated in the mixing layer's self-similar region. This corresponds well to the ratio of 1.4, which was observed by Rogers and Moser.³

(a) $\mathcal{P}/\mathcal{D}$

To investigate the equilibrium of the small-scale turbulence, the ratio of enstrophy production to enstrophy dissipation is displayed in Fig. 9(b). Production and dissipation of enstrophy are defined by (cf. Ref. 39)

$$\mathcal{P}_\varepsilon = \frac{1}{\Omega} \int_{\Omega} (\boldsymbol{\omega} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\omega} d\mathbf{x} \quad (21)$$

and

$$\mathcal{D}_\varepsilon = \frac{1}{\Omega} \int_{\Omega} \nu (\nabla \boldsymbol{\omega} : \nabla \boldsymbol{\omega}) d\mathbf{x}. \quad (22)$$

It is appreciated that a ratio of one is reached in all cases, and we observe (not shown here) a constant enstrophy level of 3.7×10^{-3} for all cases; this value is closely corresponding to the values reported in Balaras *et al.*⁴ and Rogers and Moser.³ The Taylor-Reynolds number in our optimized cases, measured in the mixing center plane, ranges between $Re_\lambda \approx 40$ (at $t=160$) and $Re_\lambda \approx 55$ (at $t=400$).

The time at which the self-similar region starts is difficult to identify precisely. We look at the emergence of a constant growth rate, at the collapse of normal Reynolds-stress profiles (not shown here), and at a constant $\mathcal{P}/\mathcal{D}$ ratio, and find that self-similarity starts at $t \approx 160$ – 200 for all cases.

C. Discussion

Results in Figs. 8 and 9 show self-similarity for all optimized cases when the simulations are sufficiently continued beyond the optimization time horizon, i.e., at $t=200$. These observations are further discussed here.

The Reynolds number based on momentum thickness $Re_\theta \equiv \theta \Delta U / \nu$ at which a self-similar state emerges is significantly lower than the values reported in earlier studies. We find in our optimized cases $Re_\theta \approx 350$ at the onset of self-similarity (evaluated at $t=200$), which should be compared to $Re_\theta \approx 1900$ found by Balaras *et al.*⁴ and $Re_\theta \approx 1440$ found by Rogers and Moser.³ Hence, in the current study, Reynolds numbers at the start of the self-similar region are a factor of

(b) $\mathcal{P}_\varepsilon/\mathcal{D}_\varepsilon$

FIG. 9. (a) Evolution of $\mathcal{P}/\mathcal{D}$ in the center of the layer in time. Horizontal line at 1.35. (b) Evolution of the rate of enstrophy production to enstrophy dissipation in time. Horizontal line at 1.0. [(a) and (b)] Flows optimized toward $\mathcal{J}_{DE}$ with $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$), $E_0=10^{-4}$.

4–5 lower. The Taylor–Reynolds number at the onset of self-similarity corresponds to $Re_\lambda \approx 40$. Consequently, compared to uncontrolled initial conditions, optimization to $\mathcal{J}_{DE}$ leads to a significant speedup of the transition of the mixing layer to a self-similar state, which is reached at much lower Reynolds numbers.

In Sec. III A, it was observed that the optimal controls ϕ_{160} are also optimal for $60 \leq T \leq 160$. Moreover, the shear-layer evolution of this optimum displays a fast transition to a self-similar regime at Reynolds numbers which are significantly lower than previously reported. This suggests that optimal control maximizing the total dissipation leads to a fast transition to self-similarity.

When looking at the evolution of dissipation in time of the optimal solutions, we find that they all peak around $t = 70$ (cf. Fig. 5). Afterward, the level of dissipation decreases and eventually falls back in the self-similar region (starting at $t \approx 160$ –200) to a constant level which is the same for all cases, and which also corresponds to enstrophy levels of other uncontrolled self-similar mixing layers.^{3,4} Based on these elements, we surmise that control actions which optimize total dissipated energy do not influence the self-similar region. This also implies that for $T \rightarrow \infty$ this type of controls may become ineffective, i.e., the transition part which can be controlled becomes negligible in the total integral.

Obviously, the optima ϕ_T discussed here may correspond to a branch of local optima, i.e., other optima (which we did not find) might exist which do not comply with the above observations. Moreover, limitations on computational resources did not allow optimizations with $T > 160$, such that time horizons extending significantly into the self-similar region are currently not feasible. Therefore, in a next step to investigate control of dissipation in a shear layer, we consider a second optimization case, which maximizes the dissipation rate at the time horizon (or, equivalently, the enstrophy). For longer time horizons (up to $T = 160$), the “optimal” level of enstrophy compared to the self-similar level is elaborated, and effects of the controls on the transition to self-similarity are quantified. This is elaborated in Sec. IV.

IV. CASE 2: MAXIMIZE DISSIPATION RATE AT TIME HORIZON

Maximization of the kinetic-energy dissipation rate at the time horizon is now considered. In Sec. IV A, optimization results are presented. Subsequently, in Sec. IV B, a discussion is presented, including the effects of optimal controls on the transition to self-similarity.

A. Optimization results

As in Sec. III, optimization is performed for energy levels $E_0 = 10^{-4}$ and $E_0 = 10^{-5}$ starting from point ϕ_A . Next to that, for one case ($\mathcal{J}_{DR}$ and $E_0 = 10^{-4}$), the iterative procedure is started from a different starting point ϕ_B , again using six different time horizons, leading to a different branch of optima, further discussed below.

Optimization results for $\mathcal{J}_{DR}$ and $E_0 = 10^{-4}$ starting from ϕ_A are presented in Fig. 10. In Fig. 10(a), the convergence history as a function of the number of conjugate-gradient

(outer) iterations is presented. We used a convergence criterion based on a relative improvement of the cost functional of 10^{-10} . It is appreciated that optimizations to all time horizons converge well with respect to this criterion, except for the optimization to $T = 120$, which was stopped after 200 iterations for resource reasons.

First of all, in Fig. 10(b) the enstrophy evolution $\mathcal{E}(t, \phi_T)$ is displayed for the different time horizons. A large increase of the enstrophy for the optimized cases is appreciated when compared to the enstrophy evolution of ϕ_A (also displayed in the figure). For low time horizons ($T = 60$ and 80), the enstrophy evolution strongly peaks near the time horizon for which it is optimized, followed by a sharp decline of enstrophy for $t > T$. For longer time horizons ($T \geq 100$), this is not the case any more. Now, peaks around $t = 80$ are observed followed with a gradual decline up to the time horizon ($t = T$). For $t > T$, again a sharper decline in enstrophy is seen. In Fig. 11(c), results for the lower energy level $E_0 = 10^{-5}$ are also presented (convergence for all time horizons up to the relative convergence criterion of 10^{-10} is reached—not shown here). It is appreciated that also for a lower energy level of the controls, a significant improvement of the enstrophy at the optimization time horizon can be obtained compared to the unoptimized starting point ϕ_A . Next to that, the same trends may be observed as for $E_0 = 10^{-4}$.

The evolution of the enstrophy starting from the initial field ϕ_B (without any optimization) is also displayed in Fig. 10(b) [and Fig. 10(c)]. We observe that this (unoptimized) solution yields a higher level of enstrophy than the optimal solutions $\mathcal{E}(t, T)$ for $T > 120$ optimized using the initial guess ϕ_A . Consequently, for $T > 120$, the optima obtained using ϕ_A as an initial starting point are local optima.

To further investigate this, a set of optimizations runs is started from initial point ϕ_B .⁴⁰ For the sake of simple nomenclature, we refer to the set of optima coming from the initial starting point ϕ_A as “group A,” and similarly from ϕ_B as “group B.” In Fig. 11(a), the evolution of the enstrophy is shown for the optima from group B with $E_0 = 10^{-4}$. As is now appreciated, the optimization results based on ϕ_B provide a large increase of enstrophy compared to the unoptimized ϕ_B simulation and are considerably better than optima from group A for $T > 120$.

To further collect all information on the optimal control of the dissipation rate into one plot, for all cases we display in Fig. 12 the maximum enstrophy for each time horizon $\mathcal{E}(T, \phi_T)$ as a function of the time horizon T [instead of $\mathcal{E}(t, \phi_T)$]. Looking first at the maximum enstrophy using an energy level $E_0 = 10^{-4}$ for the controls, a difference between optimizations in group A and group B is noted. For $T < 100$, optimizations in group A yield the highest enstrophy $\mathcal{E}(T, \phi_T)$, while the inverse holds for $T > 100$. Overall, combining the best optima of group A and group B, we find that optimal controls lead to a maximal enstrophy larger than 8.6×10^{-3} , which is almost three times higher than the eventual equilibrium level in the self-similar region (cf. Sec. IV B for further discussion). Using the best optima from groups A and B, a slight drop in the level of maximal enstrophy (about 10%) is observed for longer time horizons compared to shorter optimization time horizons. Figure 12 also shows

FIG. 10. Optimization results for $\mathcal{J}_{\text{DR}}$ starting from ϕ_A . (a) Convergence history for the different time horizons and $E_0 = 10^{-4}$ [with $\Delta \mathcal{J}_{\text{DR}} = \mathcal{J}_{\text{DR}}(it) - \mathcal{J}_{\text{final}}$]. (b) Evolution of the enstrophy $\mathcal{E}(t, \phi_T)$ as a function of time for different time horizons and $E_0 = 10^{-4}$. (c) Evolution of $\mathcal{E}(t, \phi_T)$ for $E_0 = 10^{-5}$. Symbols in (a)–(c): $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$). Additional lines in (b) and (c): enstrophy evolution of initial condition ϕ_A (—) and ϕ_B (---); vertical lines are drawn at the different optimization time horizons.

$\mathcal{E}(T, \phi_T)$ for $E_0 = 10^{-5}$. Comparing this to the maximum enstrophy of control-energy level $E_0 = 10^{-4}$ (both from group A), we observe for short time horizons ($T < 100$) that the higher energy level realizes a higher level of enstrophy. For longer time horizons, this difference disappears, and enstrophy levels for both cases are roughly equal.

Finally, the coherent structures associated with the different solutions are elaborated in Fig. 13 using the λ_2 -criterion.³⁵ We focus on the structures during early tran-

sition at $t=40$. Results for $t=160$ are not shown here. In this case, the flow is fully turbulent, dominated by small-scale turbulent structures, and the λ_2 structures look visually quite similar to those shown in Fig. 6.

For group A, we do not observe meaningful differences between visualizations of flows optimized toward different time horizons. It is appreciated in Figs. 13(a)–13(d) that the selected plots at $T=80$ and $T=120$ are quite similar, and they are representative for the type of structures observed for

FIG. 11. Evolution of the enstrophy in time for flows optimized toward $\mathcal{J}_{\text{DR}}$ starting from ϕ_B and $E_0 = 10^{-4}$. Symbols: $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$); also shown is enstrophy evolution of initial condition ϕ_A (—) and ϕ_B (---).

FIG. 12. Maximal enstrophy as a function of time horizon T for flows optimized toward $\mathcal{J}_{\text{DR}}$. (— $\diamond$) group A, $E_0 = 10^{-4}$; (--- $\triangle$) group B, $E = 10^{-4}$; and (— $\diamond$) group A, $E = 10^{-5}$ (lines are obtained by interpolation using splines).

(a) $E_0 = 10^{-5}$, group A, $T = 80$ (b) $E_0 = 10^{-5}$, group A, $T = 120$ (c) $E_0 = 10^{-4}$, group A, $T = 80$ (d) $E_0 = 10^{-4}$, group A, $T = 120$ (e) $E_0 = 10^{-4}$, group B, $T = 80$ (f) $E_0 = 10^{-4}$, group B, $T = 120$

FIG. 13. (Color online) λ_2 visualization of the velocity field for parameters optimized toward $\mathcal{J}_{DR}$ (colored online with the distance from the mixing center plane); profile $\langle u_1 \rangle$ (—); all visualizations at $t=40$.

other time horizons. For group B, more pronounced differences are visible when the time horizon is changed. The flow structures for group B can be divided into two groups according to their time horizon: $T \leq 80$ and $T \geq 100$. The typical structures for $T \leq 80$ are presented in Fig. 13(e) using the

case $T=80$. At $t=40$, structures show a diamond-shape pattern of vorticity in the mixing plane. For $T \geq 100$, typical structures are shown in Fig. 13(f) (using $T=120$). The vorticity is now more two-dimensional, mainly distributed in span-wise vorticity. In this case, the initial energy is more

FIG. 14. (Color) (a) Evolution of the momentum thickness growth rate r_θ . (b) Evolution of $\mathcal{P}/\mathcal{D}$ in the center of the layer in time. Horizontal line at 1.35. (c) Evolution of the rate of enstrophy production to enstrophy dissipation in time. Horizontal line at 1.0. [(a)–(c)] Flows optimized toward $\mathcal{J}_{\text{DR}}$ with $T=60$ ($\diamond$), $T=80$ ($\square$), $T=100$ ($\circ$), $T=120$ (∇), $T=140$ ($\triangle$), $T=160$ ($\star$), A-set (—, empty green symbols), and B-set (---, filled symbols, $T \leq 80$ blue, $T \geq 100$ red), $E_0=10^{-4}$.

stored in two-dimensional structures, such that vortex stretching, break-up of vortices, and subsequent dissipation are “postponed” to later times compared to the flows evolving from Figs. 13(a)–13(e), such that also the enstrophy rise (e.g., observed in Fig. 11) is significantly shifted to larger time horizons.

B. Discussion

We now turn back to the discussion on control of dissipation and effects on the ensuing self-similar region. By combining the best optima from group A and group B in Sec. IV A, we found that the dissipation rate may be controlled up to the time horizon with an enstrophy level at the time horizon which is almost three times higher than the level of enstrophy observed in the self-similar region of case 1.

To evaluate self-similarity, the time evolution of the momentum thickness, the ratios $\mathcal{P}/\mathcal{D}$ and $\mathcal{P}_\varepsilon/\mathcal{D}_\varepsilon$ [cf. the definitions in Eqs. (19)–(22)] are displayed in Fig. 14 for all optimal solutions from case 2 with control energy $E_0=10^{-4}$. We first focus on the peaks in Figs. 14(a)–14(c). It is appreciated that the optimal solutions from case 2 roughly fall apart into two sets. The first set consists of all optima from group A and optima from group B with $T \leq 80$. The second set contains optima from group B with $T \geq 100$. For the momentum thickness growth rate [Fig. 14(a)] and the enstrophy growth rate [Fig. 14(b)], the first set has maxima earlier in

time, respectively, at $t \approx 50$ and $t \approx 33$, compared to $t \approx 80$ and $t \approx 50$ for the second set (see also Table I). For the ratio $\mathcal{P}/\mathcal{D}$ in Fig. 9(c), peaks are at the same location in time, but have a different width. This partition into two sets also corresponds with the difference in coherent structures (more three-dimensional versus more two-dimensional) observed at the end of Sec. IV A for all these cases.

These results for case 2 are now compared to results for optimization toward maximum dissipated energy [case 1, cf. Figs. 8, 9(a), and 9(b)]. Peaks in r_θ , $\mathcal{P}/\mathcal{D}$, and $\mathcal{P}_\varepsilon/\mathcal{D}_\varepsilon$ are listed in Table I. It is appreciated for case 2 that peaks shift to later times compared to case 1. Clearly, the high dissipation rates at the time horizon dictated by the cost functional used for case 2 are realized by the optimal controls via a shift in

TABLE I. Location in time of the peak in momentum thickness growth rate r_θ , rate of production to dissipation of turbulent kinetic energy $\mathcal{P}/\mathcal{D}$, and rate of production to dissipation of enstrophy $\mathcal{P}_\varepsilon/\mathcal{D}_\varepsilon$ evaluated for different optima.

	r_θ	$\mathcal{P}/\mathcal{D}$	$\mathcal{P}_\varepsilon/\mathcal{D}_\varepsilon$
Case 1 ($\mathcal{J}_{\text{DE}}$)	30	8	25
Case 2 ($\mathcal{J}_{\text{DR}}$) group A, group B with $T \leq 80$	50	11	33
Case 2 ($\mathcal{J}_{\text{DR}}$) group B with $T \geq 100$	80	11	50

FIG. 15. Evolution in time of the profile $0.5\langle u'u' + v'v' + w'w' \rangle / \Delta U^2$ in function of the scaled normal coordinate $\xi = x_3 / \theta$. Flows optimized to $\mathcal{J}_{DE}$ (upper part figure) and $\mathcal{J}_{DR}$ (lower part figure) for $T=140$ (left) and $T=160$ (right). Energy of the controls $E_0 = 10^{-4}$.

time of the peak in enstrophy growth rate. These shifts also have a large impact on the onset of self-similarity.

Figure 14(a) shows that all optima in case 2 are still evolving, and even up to $t=400$ do not reach self-similarity. Hence, the onset of self-similarity for case 2 starts significantly later than observed in Sec. III B for case 1. To further illustrate this, normalized profiles of turbulent kinetic energy are shown in Fig. 15 for $\mathcal{J}_{DE}$ and $\mathcal{J}_{DR}$ (group B) at $T=140$ and 160. The figure shows that the profiles of case 1 ($\mathcal{J}_{DE}$) become self-similar from $t \approx 160$ –200 onward; the normalized turbulent kinetic-energy profiles for case 2 ($\mathcal{J}_{DR}$) are still changing up to $t=400$. Thus, optimization in case 2 shows that dissipation rates may be increased considerably via dedicated optimal controls, but as a result the transition to self-similarity shifts further back in time.

V. CONCLUSION

In the current work, the optimal control of dissipation in a turbulent mixing layer is elaborated. Controls correspond to a set of perturbations on a tangent-hyperbolic mean velocity profile used to initialize the mixing layer in a temporal framework. A suite of DNS-based optimizations is performed with different optimization time windows ($T=60, 80, \dots, 160$), including two different energy levels for the controls $E_0=10^{-4}$ and $E_0=10^{-5}$, and studying two different cost functionals which are relevant for the dissipation of kinetic energy in a shear layer. Also, the effect of optimal controls (aiming to influence dissipation) on the transition of the mixing layer toward a self-similar state is investigated.

In a first case (case 1), optimal control is used to maximize the total dissipated energy in the mixing layer. It is observed that the optimal control for the largest time window ($T=160$) is optimal for any of the time windows ($60 \leq T \leq 160$) considered in this study. This suggests that, for sufficiently large time windows in the transition zone, optimal controls are found to become independent of the time window (i.e., by using the controls found at the largest time horizon for all lower time horizons). Results further indicate that these controls facilitate a fast transition of the mixing layer to self-similarity. The coherent turbulent structures during early transition display a diamond-shaped pattern similar

to, e.g., patterns observed in experiments in a forced mixing layer by Nygaard and Glezer.³⁶ We further found that an approximation of the optimal control with only a few modes (i.e., 4–8) leads to a control which realizes 75% of the optimal dissipated kinetic energy.

We evaluated (for case 1) the mixing-layer evolution for simulation times beyond the optimization time horizon, and found that optimization of the dissipated energy fastens up the evolution to a self-similar state. The onset of self-similarity is observed at a Reynolds number which is a factor of 4–5 lower than observed in the mixing-layer studies by Rogers and Moser³ and Balaras *et al.*⁴ The level of total enstrophy observed in our work is the same, and the ratio of production and dissipation of turbulent kinetic energy in our optimized cases (i.e., $\mathcal{P}/\mathcal{D} \approx 1.35$) also corresponded well with values reported in Rogers and Moser.³

Since optimal controls for maximization of dissipated energy become independent of the time horizon, and since these controls lead to a fast transition to self-similarity where the dissipation levels are constant (and also at the same level as observed in uncontrolled mixing layers), we surmise that control actions which optimize total dissipated energy do not influence the self-similar region. This also implies that for $T \rightarrow \infty$ this type of controls may become ineffective. To further investigate this, we considered a second case (case 2) in which the dissipation rate at the time horizon is maximized.

For case 2, we observed that it was possible to increase the dissipation rate at the time horizon up to levels which are almost three times higher than the self-similar level of dissipation. We found two different branches of optima depending on the initial guess used to start up the optimization procedure. The first group (A) of optimal solutions is best for smaller time horizons $T \leq 100$ but yields inferior optima for $T > 100$. Optima from the second group (B) are best for $T > 100$. The difference between optima good for $T \leq 100$ and optima for $T > 100$ was identified by looking at coherent structures during the initial transition of the mixing layer. For large time horizons ($T \geq 100$) it is observed that vorticity is initially more allocated in two-dimensional modes, postponing the development of enstrophy toward the higher optimization time horizons.

Focusing for case 2 on group B and $T=160$ (after which self-similarity emerges in the corresponding case 1 optimum), we also observe an enstrophy level at the time horizon which is significantly higher than the self-similar level. In comparison to optima from case 1 (optimizing total dissipated energy), this leads to a shift to later times—in our simulation with more than 200 nondimensional time units—of the onset of self-similarity. In the self-similar region, the same levels of enstrophy, etc., are detected as in case 1.

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- ⁴⁰A procedure similar to the one discussed in Sec. II B is used, but using optimization to $T=140$ as first point in the series.