Interpolation with quintic Powell-Sabin splines

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MSC : Primary : 65D05, Secondary : 65D07, 41A05

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Abstract

We discuss local interpolation schemes for C^2 -continuous quintic Powell-Sabin splines represented with a normalized B-spline basis. As part of the construction of the interpolation rules, we make use of tensor algebra and blossoming to describe and manipulate polynomials in Bernstein-Bézier representation.

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1 Introduction

The construction of interpolating functions through scattered data is a common problem in many application areas. This problem can be addressed by working with smooth spline functions defined on a triangulation where its vertices are located at the data points.

A number of authors considered C^2 -continuous quintic spline functions over triangulations with Powell-Sabin refinement (see [1, 8, 9, 10, 13]). Each triangle in such a triangulation is split into six subtriangles. A normalized basis for particular quintic Powell-Sabin splines was developed in [18]. The locally supported basis functions form a convex partition of unity, i.e., they are nonnegative and they sum up to one. This B-spline representation allows a natural definition of control polynomials and control points, which are useful for geometric modelling. A spline in such a representation can also be evaluated in a stable way, using only a sequence of simple convex combinations. A similar basis was developed for quadratic Powell-Sabin splines in [4] and for reduced cubic Clough-Tocher splines in [19].

In this paper we construct interpolation rules for quintic Powell-Sabin splines in this normalized B-spline representation. The interpolation requires data at the vertices of the triangulation and at the Powell-Sabin split points. For the computation of the rules, we make use of tensor algebra to describe and manipulate blossoms of polynomials in Bernstein-Bézier form. Higher-order tensors are a natural generalization of matrices. We also consider interpolation based on several reduced spline spaces. These reduced spline interpolants only need data at the vertices of the triangulation. Interpolation and approximation schemes based on quadratic Powell-Sabin splines can be found in [5, 11, 14, 15] and shape-preserving schemes in [12, 16, 20, 21].

The paper is organized as follows. In Section 2 we recall some relevant concepts of tensor algebra. Section 3 is devoted to Bernstein-Bézier polynomials on triangles, and we point out the relation to tensors. In Section 4 we define the quintic Powell-Sabin splines and we describe the construction of the normalized basis. Section 5 describes the interpolation rules for the quintic Powell-Sabin splines, and in Section 6 we discuss some reduced spline spaces. Finally, a numerical example is presented in Section 7.

2 Tensor algebra

In this section we introduce some basic definitions of multilinear algebra. A tensor, also known as a multiway array or a multidimensional matrix, is a higher-order generalization of a vector (first order tensor) and a matrix (second order tensor). Tensors are multilinear mappings over a set of vector spaces. The order of tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ is N . An element of $\mathcal{A}$ is denoted by $a_{i_1 i_2 \dots i_N}$ where $1 \leq i_n \leq I_n$. The n -mode vectors of $\mathcal{A}$ are the I_n -dimensional vectors obtained from $\mathcal{A}$ by varying the index i_n while keeping the other indices fixed. In this terminology, column vectors of a matrix are referred to as 1-mode vectors and row vectors as 2-mode vectors.

A tensor can be represented in matrix form by unfolding the tensor along a single dimension. The matrix unfolding $A_{(n)} \in \mathbb{R}^{I_n \times (I_{n+1} \dots I_N I_1 \dots I_{n-1})}$ of a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N}$ is a matrix composed of all the n -mode vectors of the tensor. It contains the element $a_{i_1 i_2 \dots i_N}$ at the position with row number i_n and column number equal to

$$1 + \sum_{j=1}^{n-1} (i_j - 1) \prod_{k=j+1}^{n-1} I_k + \sum_{j=n+1}^N (i_j - 1) \prod_{k=j+1}^N I_k \prod_{k=1}^{n-1} I_k.$$

These unfoldings are illustrated in Figure 1 for a tensor of order three.

A generalization of the product of two matrices is the product of a tensor and a matrix. The n -mode product of a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N}$ and a matrix $U \in \mathbb{R}^{J_n \times I_n}$ is defined as the tensor $\mathcal{B} \in \mathbb{R}^{I_1 \times \dots \times I_{n-1} \times J_n \times I_{n+1} \times \dots \times I_N}$ of which the elements are given by

$$b_{i_1 \dots i_{n-1} j_n i_{n+1} \dots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 \dots i_{n-1} i_n i_{n+1} \dots i_N} u_{j_n i_n}. \quad (2.1)$$

The product is denoted by

$$\mathcal{B} = \mathcal{A} \times_n U, \quad (2.2)$$

and, using matrix unfolding it is equal to

$$B_{(n)} = U A_{(n)}. \quad (2.3)$$

More details on tensor algebra can be found in [3].

3 Bivariate polynomials on triangles

3.1 Barycentric coordinates and the Bernstein-Bézier representation of polynomials

Let $\mathcal{T}(V_1, V_2, V_3)$ be a non-degenerate triangle. Any point P in the plane of the triangle can be uniquely expressed in terms of the barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ with respect to $\mathcal{T}$, where $\tau_1 + \tau_2 + \tau_3 = 1$. Given two points P_1 and P_2 in the plane of the triangle, the barycentric direction $\delta = (\delta_1, \delta_2, \delta_3)$ of the vector $P_2 - P_1$ with respect to $\mathcal{T}$ is defined as the difference of the barycentric coordinates of both points. If the Euclidean distance $\|P_2 - P_1\|_2 = 1$, then δ is called a unit barycentric direction. A barycentric direction δ always satisfies $\delta_1 + \delta_2 + \delta_3 = 0$. Let (x_i, y_i) be the Cartesian coordinates of vertex V_i , $i = 1, 2, 3$, then the unit barycentric directions along the x and y direction are found as

$$\delta^x = (y_2 - y_3, y_3 - y_1, y_1 - y_2)/E, \quad (3.1a)$$

$$\delta^y = (x_3 - x_2, x_1 - x_3, x_2 - x_1)/E, \quad (3.1b)$$

Figure 1: Unfolding of the $(I_1 \times I_2 \times I_3)$ -tensor $\mathcal{A}$ to (a) the $(I_1 \times I_2 I_3)$ -matrix $A_{(1)}$, (b) the $(I_2 \times I_3 I_1)$ -matrix $A_{(2)}$ and (c) the $(I_3 \times I_1 I_2)$ -matrix $A_{(3)}$ with $I_1 = I_2 = I_3 = 3$.

with

$$E = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Let Π_d denote the linear space of bivariate polynomials of total degree less than or equal to d . Any polynomial $p_d \in \Pi_d$ on triangle $\mathcal{T}$ has a unique Bernstein-Bézier representation

$$p_d(\tau) = \sum_{i+j+k=d} b_{ijk} B_{ijk}^d(\tau). \quad (3.2)$$

Here,

$$B_{ijk}^d(\tau) = \frac{d!}{i!j!k!} \tau_1^i \tau_2^j \tau_3^k \quad (3.3)$$

are the Bernstein polynomials of degree d . The coefficients b_{ijk} are called Bézier ordinates. The Bézier domain points ξ_{ijk} are defined as the points with barycentric coordinates $(\frac{i}{d}, \frac{j}{d}, \frac{k}{d})$. By associating the Bézier ordinates b_{ijk} with the Bézier domain points ξ_{ijk} , one can display the Bernstein-Bézier representation schematically as in Figure 2 for the case $d = 3$. We refer to [6] for more details.

Figure 2: Schematic representation of the Bézier ordinates of a cubic bivariate polynomial.

3.2 The blossom of polynomials

The blossom (or polar form) of $p_d(\tau)$ is denoted by $\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d)$, and it is completely characterized by the following three properties (see [17]):

1. The blossom is symmetric with respect to any permutation π of its d arguments,

$$\mathcal{P}_d(\tau^1, \tau^2, \dots, \tau^d) = \mathcal{P}_d(\pi(\tau^1, \tau^2, \dots, \tau^d)).$$

2. The blossom is multi-affine, i.e. affine in each of its d arguments,

$$\mathcal{P}_d(a\tau + b\sigma, \tau^2, \dots, \tau^d) = a\mathcal{P}_d(\tau, \tau^2, \dots, \tau^d) + b\mathcal{P}_d(\sigma, \tau^2, \dots, \tau^d), \quad a + b = 1.$$

3. The restriction of the blossom to the diagonal gives the polynomial p_d ,

$$p_d(\tau) = \mathcal{P}_d(\tau, \tau, \dots, \tau).$$

We consider the case $d = 3$ in more detail, as we are particularly interested in it further on. Let $\tau^l = (\tau_1^l, \tau_2^l, \tau_3^l)$, $l = 1, 2, 3$, then the blossom of p_3 on domain triangle $\mathcal{T}(V_1, V_2, V_3)$ is given by

$$\begin{aligned} \mathcal{P}_3(\tau^1, \tau^2, \tau^3) = & b_{300} (\tau_1^1 \tau_1^2 \tau_1^3) + b_{030} (\tau_2^1 \tau_2^2 \tau_2^3) + b_{003} (\tau_3^1 \tau_3^2 \tau_3^3) \\ & + b_{210} (\tau_2^1 \tau_1^2 \tau_1^3 + \tau_1^1 \tau_2^2 \tau_1^3 + \tau_1^1 \tau_1^2 \tau_2^3) + b_{201} (\tau_3^1 \tau_1^2 \tau_1^3 + \tau_1^1 \tau_3^2 \tau_1^3 + \tau_1^1 \tau_1^2 \tau_3^3) \\ & + b_{021} (\tau_3^1 \tau_2^2 \tau_2^3 + \tau_2^1 \tau_3^2 \tau_2^3 + \tau_2^1 \tau_2^2 \tau_3^3) + b_{120} (\tau_1^1 \tau_2^2 \tau_2^3 + \tau_2^1 \tau_1^2 \tau_2^3 + \tau_2^1 \tau_2^2 \tau_1^3) \\ & + b_{102} (\tau_1^1 \tau_3^2 \tau_3^3 + \tau_3^1 \tau_1^2 \tau_3^3 + \tau_3^1 \tau_3^2 \tau_1^3) + b_{012} (\tau_2^1 \tau_3^2 \tau_3^3 + \tau_3^1 \tau_2^2 \tau_3^3 + \tau_3^1 \tau_3^2 \tau_2^3) \\ & + b_{111} (\tau_1^1 \tau_2^2 \tau_3^3 + \tau_1^1 \tau_3^2 \tau_2^3 + \tau_2^1 \tau_3^2 \tau_1^3 + \tau_2^1 \tau_1^2 \tau_3^3 + \tau_3^1 \tau_1^2 \tau_2^3 + \tau_3^1 \tau_2^2 \tau_1^3), \end{aligned} \quad (3.4)$$

with b_{ijk} , $i + j + k = 3$, the Bézier ordinates of p_3 on $\mathcal{T}$. The multi-affine de Casteljau algorithm can be used to evaluate the blossom.

Subdivision of a polynomial (3.2) can be compactly described through blossoming. One can use the blossom to find the Bernstein-Bézier representation of polynomial p_3 with respect to another (finer) triangle. Consider a second triangle, where its vertices W_l , $l = 1, 2, 3$, have $\sigma^l = (\sigma_1^l, \sigma_2^l, \sigma_3^l)$

as barycentric coordinates with respect to the former triangle $\mathcal{T}(V_1, V_2, V_3)$. The Bézier ordinates d_{ijk} of p_3 on the new triangle are obtained using (3.4) as

$$\begin{aligned} d_{300} &= \mathcal{P}_3(\sigma^1, \sigma^1, \sigma^1), & d_{210} &= \mathcal{P}_3(\sigma^1, \sigma^1, \sigma^2), & d_{201} &= \mathcal{P}_3(\sigma^1, \sigma^1, \sigma^3), \\ d_{030} &= \mathcal{P}_3(\sigma^2, \sigma^2, \sigma^2), & d_{021} &= \mathcal{P}_3(\sigma^2, \sigma^2, \sigma^3), & d_{120} &= \mathcal{P}_3(\sigma^1, \sigma^2, \sigma^2), \\ d_{003} &= \mathcal{P}_3(\sigma^3, \sigma^3, \sigma^3), & d_{102} &= \mathcal{P}_3(\sigma^1, \sigma^3, \sigma^3), & d_{012} &= \mathcal{P}_3(\sigma^2, \sigma^3, \sigma^3), \\ d_{111} &= \mathcal{P}_3(\sigma^1, \sigma^2, \sigma^3). \end{aligned} \quad (3.5)$$

Blossoming can also be used to describe directional derivatives of a polynomial (3.2). The r -th order directional derivative of polynomial p_d with respect to the unit barycentric directions δ^i , $i = 1, \dots, r$, on triangle $\mathcal{T}(V_1, V_2, V_3)$ can be compactly expressed as

$$D_{\delta^1, \dots, \delta^r}^r p_d(\tau) = \frac{d!}{(d-r)!} \mathcal{P}_d(\tau, \dots, \tau, \delta^1, \dots, \delta^r). \quad (3.6)$$

Here, the r barycentric directions δ^i and $(d-r)$ times the barycentric coordinates τ are taken as the arguments of the blossom.

3.3 A representation based on tensors

The blossom of a polynomial in Bernstein-Bézier representation can be described using tensor algebra. In this section we focus on the case $d = 3$, but the approach can be generalized to any degree d .

We define the symmetric $(3 \times 3 \times 3)$ -tensor $\mathcal{B}$ by its matrix unfolding $B_{(1)}$ as

$$B_{(1)} = \left[\begin{array}{ccc|ccc|ccc} b_{300} & b_{210} & b_{201} & b_{210} & b_{120} & b_{111} & b_{201} & b_{111} & b_{102} \\ b_{210} & b_{120} & b_{111} & b_{120} & b_{030} & b_{021} & b_{111} & b_{021} & b_{012} \\ b_{201} & b_{111} & b_{102} & b_{111} & b_{021} & b_{012} & b_{102} & b_{012} & b_{003} \end{array} \right]. \quad (3.7)$$

This tensor consists of the Bézier ordinates b_{ijk} , $i + j + k = 3$, of the polynomial $p_3(\tau)$ in (3.2) defined on domain triangle $\mathcal{T}(V_1, V_2, V_3)$. Note that the three matrix unfoldings of $\mathcal{B}$ are equal, i.e., $B_{(1)} = B_{(2)} = B_{(3)}$. The barycentric coordinates $\tau^l = (\tau_1^l, \tau_2^l, \tau_3^l)$, $l = 1, 2, 3$, with respect to $\mathcal{T}$ can be arranged into the vectors $T_l = [\tau_1^l \ \tau_2^l \ \tau_3^l]$, $l = 1, 2, 3$. The blossom in (3.4) can then be written as the product of a tensor and three vectors:

$$\mathcal{P}_3(\tau^1, \tau^2, \tau^3) = \mathcal{B} \times_1 T_1 \times_2 T_2 \times_3 T_3. \quad (3.8)$$

It follows that a polynomial p_d in Bernstein-Bézier form (3.2) can be written as

$$p_d(\tau^1) = \mathcal{B} \times_1 T_1 \times_2 T_1 \times_3 T_1. \quad (3.9)$$

Let the $(3 \times 3 \times 3)$ -tensor $\mathcal{D}$ be defined in a similar way as $\mathcal{B}$ in (3.7) using the Bézier ordinates d_{ijk} , $i + j + k = 3$, and let the matrix S be defined as

$$S = \left[\begin{array}{ccc} \sigma_1^1 & \sigma_2^1 & \sigma_3^1 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1^3 & \sigma_2^3 & \sigma_3^3 \end{array} \right]. \quad (3.10)$$

The subdivision rules in (3.5) can then be compactly written as

$$\mathcal{D} = \mathcal{B} \times_1 S \times_2 S \times_3 S. \quad (3.11)$$

We now consider the following Hermite interpolation problem: find the Bézier ordinates b_{ijk} , $i + j + k = 3$, of the polynomial $p_3(x, y)$ that satisfies

$$\frac{\partial^{a+b}}{\partial x^a \partial y^b} p_3(P) = h_{x^a y^b}, \quad a \geq 0, b \geq 0, a + b \leq 3, \quad (3.12)$$

for a given point P and a given set of $h_{x^a y^b}$ -values. We could determine the ten unknowns b_{ijk} by formulating and solving a linear system of ten equations. Alternatively, we use the following tensor approach. Let

$$\hat{h}_{x^a y^b} = \frac{(3 - a - b)!}{6} h_{x^a y^b}, \quad (3.13)$$

we define the $(3 \times 3 \times 3)$ -tensor $\mathcal{H}$ by its matrix unfolding $H_{(1)}$ as

$$H_{(1)} = \begin{bmatrix} \hat{h} & \hat{h}_x & \hat{h}_y & \hat{h}_x & \hat{h}_{x^2} & \hat{h}_{xy} & \hat{h}_y & \hat{h}_{xy} & \hat{h}_{y^2} \\ \hat{h}_x & \hat{h}_{x^2} & \hat{h}_{xy} & \hat{h}_{x^2} & \hat{h}_{x^3} & \hat{h}_{x^2 y} & \hat{h}_{xy} & \hat{h}_{x^2 y} & \hat{h}_{xy^2} \\ \hat{h}_y & \hat{h}_{xy} & \hat{h}_{y^2} & \hat{h}_{xy} & \hat{h}_{x^2 y} & \hat{h}_{xy^2} & \hat{h}_{y^2} & \hat{h}_{xy^2} & \hat{h}_{y^3} \end{bmatrix}. \quad (3.14)$$

Assume the point P has barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ with respect to the domain triangle. We combine these barycentric coordinates together with the unit barycentric directions along the x and y direction, i.e. (3.1), into the matrix T as

$$T = \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \\ \delta_1^x & \delta_2^x & \delta_3^x \\ \delta_1^y & \delta_2^y & \delta_3^y \end{bmatrix}. \quad (3.15)$$

From (3.6) we then obtain that interpolation problem (3.12) can be reformulated as

$$\mathcal{H} = \mathcal{B} \times_1 T \times_2 T \times_3 T. \quad (3.16)$$

The solution is found as

$$\mathcal{B} = \mathcal{H} \times_1 T^{-1} \times_2 T^{-1} \times_3 T^{-1}. \quad (3.17)$$

Instead of solving a linear system of ten equations, by using the tensor approach we only need to compute the inverse of a (3×3) -matrix. We now compute the matrix T^{-1} . Let (x_P, y_P) be the Cartesian coordinates of the point P . From the definition of barycentric coordinates and the expressions (3.1), we derive the relation

$$\begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \\ \delta_1^x & \delta_2^x & \delta_3^x \\ \delta_1^y & \delta_2^y & \delta_3^y \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \begin{bmatrix} x_P & y_P & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We then obtain that

$$T^{-1} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} x_P & y_P & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & x_1 - x_P & y_1 - y_P \\ 1 & x_2 - x_P & y_2 - y_P \\ 1 & x_3 - x_P & y_3 - y_P \end{bmatrix}. \quad (3.18)$$

By using tensors of order d , one can apply the same technique to solve the Hermite interpolation problem similar to (3.12) for polynomials with arbitrary degree d .

4 Quintic Powell-Sabin splines

4.1 The PS5-spline space

Given a simply connected subset $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega$, let Δ be a conforming triangulation of Ω . Let n_v , n_t and n_e be the number of vertices, triangles and edges in Δ , respectively.

A Powell-Sabin (PS-) refinement Δ^* of Δ partitions each triangle $\mathcal{T}_j \in \Delta$ into six smaller triangles in the following way:

1. Choose an interior point Z_j in each triangle $\mathcal{T}_j$, so that if two triangles $\mathcal{T}_i$ and $\mathcal{T}_j$ have a common edge, then the line joining Z_i and Z_j intersects the common edge at a point R_{ij} between its vertices.
2. Join each point Z_j to the vertices of $\mathcal{T}_j$.
3. For each edge of the triangle $\mathcal{T}_j$
 - (a) which is common to a triangle $\mathcal{T}_i$: join Z_j to the intersection point R_{ij} of that edge and the line $Z_i - Z_j$.
 - (b) which belongs to the boundary $\partial\Omega$: join Z_j to an arbitrary point R_{ij} on that edge.

The obtained subtriangles are denoted as $\mathcal{T}^* \in \Delta^*$. The space of piecewise quintic polynomials on Δ^* with global C^2 -continuity is denoted as

$$S_5^2(\Delta^*) = \{s \in C^2(\Omega) : s|_{\mathcal{T}^*} \in \Pi_5, \mathcal{T}^* \in \Delta^*\}. \quad (4.1)$$

We consider a particular subspace of $S_5^2(\Delta^*)$ with additional smoothness around some vertices and edges. Let $\mathcal{V} = \{V_i\}_{i=1}^{n_v}$ be the set of vertices in Δ , let $\mathcal{Z}^* = \{Z_i\}_{i=1}^{n_t}$ be the set of split points in Δ^* , and let $\mathcal{E}^*$ be the set of all edges in Δ^* that connect a split point Z_i to a point R_{ij} . The quintic Powell-Sabin (PS5-) spline space is defined as

$$\hat{S}_5^2(\Delta^*) = \{s \in S_5^2(\Delta^*) : s \in C^3(W), W \in (\mathcal{V} \cup \mathcal{Z}^*); s \in C^3(e), e \in \mathcal{E}^*\}. \quad (4.2)$$

Here, $C^\mu(W)$ means that the polynomials on triangles in Δ^* sharing the vertex W have common derivatives up to order μ at that vertex. Analogously, $C^\mu(e)$ means that the polynomials on triangles in Δ^* sharing the edge e have common derivatives up to order μ along that edge. In [10] it is shown that the dimension of the space $\hat{S}_5^2(\Delta^*)$ equals $10n_v + n_t$. The following interpolation problem can then be considered: there exists a unique spline $s(x, y) \in \hat{S}_5^2(\Delta^*)$ such that

$$\frac{\partial^{a+b}}{\partial x^a \partial y^b} s(V_l) = f_{x^a y^b, l}, \quad l = 1, \dots, n_v, \quad a \geq 0, b \geq 0, a + b \leq 3, \quad (4.3a)$$

and

$$s(Z_m) = g_m, \quad m = 1, \dots, n_t, \quad (4.3b)$$

for any given set of $f_{x^a y^b, l}$ -values and g_m -values.

4.2 A normalized B-spline representation

A procedure for the construction of a normalized basis for $\hat{S}_5^2(\Delta^*)$ was developed in [18]. Every PS5-spline can be represented as

$$s(x, y) = \sum_{i=1}^{n_v} \sum_{j=1}^{10} c_{i,j}^v B_{i,j}^v(x, y) + \sum_{k=1}^{n_t} c_k^t B_k^t(x, y). \quad (4.4)$$

We call $B_{i,j}^v(x, y)$ and $B_k^t(x, y)$ a B-spline with respect to vertex V_i and triangle $\mathcal{T}_k$, respectively. The basis functions $B_{i,j}^v(x, y)$ and $B_k^t(x, y)$ have a local support and they form a convex partition of unity, i.e.,

$$B_{i,j}^v(x, y) \geq 0, \quad B_k^t(x, y) \geq 0, \quad \text{and} \quad \sum_{i=1}^{n_v} \sum_{j=1}^{10} B_{i,j}^v(x, y) + \sum_{k=1}^{n_t} B_k^t(x, y) = 1. \quad (4.5)$$

Let M_i be the molecule (also called 1-ring) of vertex V_i , defined as the union of all triangles in Δ that contain V_i . In order to construct the B-splines, we first define for each vertex V_i ten linearly independent decuplets $\{\alpha_{i,j}^{ab}, 0 \leq a + b \leq 3\}$, $j = 1, \dots, 10$, as follows:

1. For each vertex $V_i \in \Delta$, identify the corresponding PS5-points. They are defined as

$$S_{il} = \frac{2}{5}V_i + \frac{3}{5}V_l, \quad (4.6)$$

for all vertices V_l that are situated at the boundary of the molecule M_i of V_i . The vertex V_i itself is also a PS5-point.

2. For each vertex V_i , find a triangle $t_i(Q_{i,1}^v, Q_{i,2}^v, Q_{i,3}^v)$ that contains all the PS5-points of V_i . Denote its vertices by $Q_{i,j}^v = (X_{i,j}^v, Y_{i,j}^v)$. The triangles t_i , $i = 1, \dots, n_v$, are called PS5-triangles. We remark that the PS5-triangles are not uniquely defined. In [18] an optimization strategy was proposed to select triangles of minimal area.
3. Consider the ten Bernstein polynomials (3.3) of degree three defined on PS5-triangle t_i . Each decuplet $\{\alpha_{i,j}^{ab}, 0 \leq a + b \leq 3\}$, $j = 1, \dots, 10$, is related to the function and derivative values up to order three of one of these cubic Bernstein polynomials evaluated at the point V_i :

$$\begin{aligned} \alpha_{i,1}^{ab} &= \frac{20}{(5-a-b)(4-a-b)} \left(\frac{3}{5}\right)^{a+b} \frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{300}^3(V_i), \\ \alpha_{i,2}^{ab} &= \frac{20}{(5-a-b)(4-a-b)} \left(\frac{3}{5}\right)^{a+b} \frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{030}^3(V_i), \\ \alpha_{i,3}^{ab} &= \frac{20}{(5-a-b)(4-a-b)} \left(\frac{3}{5}\right)^{a+b} \frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{003}^3(V_i), \\ \alpha_{i,4}^{ab} &= \frac{20}{(5-a-b)(4-a-b)} \left(\frac{3}{5}\right)^{a+b} \frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{210}^3(V_i), \\ &\dots, \\ \alpha_{i,10}^{ab} &= \frac{20}{(5-a-b)(4-a-b)} \left(\frac{3}{5}\right)^{a+b} \frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{111}^3(V_i), \end{aligned} \quad (4.7)$$

for all $0 \leq a + b \leq 3$.

A triangle $\mathcal{T}(V_1, V_2, V_3)$ with its PS-refinement is shown in Figure 3. Figure 4 depicts a PS5-triangle t_1 containing the PS5-points V_1 , S_{12} and S_{13} on the triangle $\mathcal{T}$.

The basis functions $B_{i,j}^v(x, y)$ and $B_k^t(x, y)$ are then constructed as follows:

1. The B-spline $B_{i,j}^v(x, y)$ with respect to vertex V_i is defined as the unique solution of interpolation problem (4.3) with all $f_{x^a y^b, l} = 0$, except for $l = i$, where $f_{x^a y^b, i} = \alpha_{i,j}^{ab}$, and with all $g_m = 0$, except for any m such that $\mathcal{T}_m \in M_i$, where $g_m = \beta_{i,j}^m \neq 0$. The B-spline is zero outside the molecule of vertex V_i . The values $\alpha_{i,j}^{ab}$ are determined by the PS5-triangle

Figure 3: A Powell-Sabin split of a triangle $\mathcal{T}_k(V_1, V_2, V_3)$ drawn with dashed lines.

t_i , see (4.7). The values $\beta_{i,j}^m$ are specified with the aid of the Bernstein-Bézier representation of the B-spline. We consider the macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$, as shown in Figure 3. On each of the six subtriangles, the spline is a quintic polynomial that can be represented in its Bernstein-Bézier formulation, i.e. with $d = 5$ in equations (3.2) and (3.3). The corresponding Bézier ordinates are schematically represented in Figure 5(a). Non-zero Bézier ordinates are denoted by filled bullets $\bullet$, and zero ordinates are denoted by open bullets $\circ$. Because of the C^3 -smoothness at vertex V_1 , the Bézier ordinates in the 3-disk around vertex V_1 are completely determined by the values $\alpha_{i,j}^{ab}$. The Bézier ordinates in the 3-disk around the split point Z_k are found by subdivision of a single cubic polynomial p_3 defined on the triangle with vertices

$$P_1 = \frac{3}{5}V_1 + \frac{2}{5}Z_k, \quad P_2 = \frac{3}{5}V_2 + \frac{2}{5}Z_k, \quad P_3 = \frac{3}{5}V_3 + \frac{2}{5}Z_k. \quad (4.8)$$

The Bernstein-Bézier representation of this polynomial p_3 has Bézier ordinates that are all zero, except for the three ordinates in the neighbourhood of P_1 . Their values are also determined by $\alpha_{i,j}^{ab}$. The values of the remaining non-zero Bézier ordinates in Figure 5(a) can be computed from the C^3 -smoothness conditions across the interior edges $Z_k - R_{12}$ and $Z_k - R_{31}$. The value of $\beta_{i,j}^k$ is obtained by evaluating the cubic polynomial p_3 at split point Z_k .

2. The B-spline $B_k^t(x, y)$ with respect to triangle $\mathcal{T}_k$ is defined as the unique solution of interpolation problem (4.3) with all $f_{x^a y^b, l} = 0$ and with all $g_m = 0$, except for $m = k$, where $g_k = \beta_k \neq 0$. It is easy to prove that such a B-spline vanishes outside $\mathcal{T}_k$. The value β_k is determined with the aid of the Bernstein-Bézier representation of the B-spline. We consider again the triangle $\mathcal{T}_k(V_1, V_2, V_3)$ shown in Figure 3. The corresponding Bézier ordinates are schematically represented in Figure 5(b). Most of these ordinates are zero. The Bézier ordinates in the 3-disk around the split point Z_k are found by subdivision of a cubic polynomial p_3 defined on the triangle with vertices (4.8). The Bézier ordinates in the Bernstein-Bézier representation of this polynomial p_3 are all zero, except for the central ordinate $b_{111} = 1$. The value of β_k is obtained by evaluating p_3 at split point Z_k .

For more details we refer to [18]. Since each B-spline $B_{i,j}^v(x, y)$, $j = 1, \dots, 10$, with respect to vertex V_i is related to a Bernstein polynomial of degree three, see (4.7), the corresponding coefficients $c_{i,j}^v$ of the PS5-spline in (4.4) can be represented schematically as in Figure 6 with respect to PS5-triangle t_i . The cubic polynomial defined on t_i with the coefficients $c_{i,j}^v$ as Bézier ordinates is called the control polynomial with respect to vertex V_i , and it is denoted by $T_i(x, y)$. This control polynomial is tangent to the PS5-spline $s(x, y)$ at vertex V_i .

Figure 4: A PS5-triangle $t_1(Q_{1,1}^v, Q_{1,2}^v, Q_{1,3}^v)$ of vertex V_1 containing the PS5-points V_1 , S_{12} and S_{13} on triangle $\mathcal{T}(V_1, V_2, V_3)$, together with the schematic representation of the Bézier ordinates $e_{1,j}$, $j = 1, \dots, 10$, of the subdivided control polynomial $T_1(x, y)$ onto the triangle with as vertices the points V_1 , S_{12} and S_{13} .

5 PS5-spline interpolation

5.1 Interpolation at the vertices

In this section we will show how to satisfy the interpolation conditions (4.3a) at the vertices. We first arrange the spline coefficients $c_{i,j}^v$, $j = 1, \dots, 10$, associated with vertex $V_i = (x_i, y_i)$ into a $(3 \times 3 \times 3)$ -tensor $\mathcal{C}$. Its matrix unfolding $C_{(1)}$ is given by

$$C_{(1)} = \begin{bmatrix} c_{i,1}^v & c_{i,4}^v & c_{i,5}^v & c_{i,4}^v & c_{i,7}^v & c_{i,10}^v & c_{i,5}^v & c_{i,10}^v & c_{i,8}^v \\ c_{i,4}^v & c_{i,7}^v & c_{i,10}^v & c_{i,7}^v & c_{i,2}^v & c_{i,6}^v & c_{i,10}^v & c_{i,6}^v & c_{i,9}^v \\ c_{i,5}^v & c_{i,10}^v & c_{i,8}^v & c_{i,10}^v & c_{i,6}^v & c_{i,9}^v & c_{i,8}^v & c_{i,9}^v & c_{i,3}^v \end{bmatrix}. \quad (5.1)$$

The barycentric coordinates of V_i with respect to its PS5-triangle t_i are denoted by $(\gamma_{i,1}, \gamma_{i,2}, \gamma_{i,3})$. The unit barycentric directions along the x and y direction with respect to t_i are denoted by $(\gamma_{i,1}^x, \gamma_{i,2}^x, \gamma_{i,3}^x)$ and $(\gamma_{i,1}^y, \gamma_{i,2}^y, \gamma_{i,3}^y)$, respectively. They can be arranged into the matrix G as

$$G = \begin{bmatrix} \gamma_{i,1} & \gamma_{i,2} & \gamma_{i,3} \\ \gamma_{i,1}^x & \gamma_{i,2}^x & \gamma_{i,3}^x \\ \gamma_{i,1}^y & \gamma_{i,2}^y & \gamma_{i,3}^y \end{bmatrix}. \quad (5.2)$$

By the definition of t_i and by (3.18) we have

$$G^{-1} = \begin{bmatrix} 1 & X_{i,1}^v - x_i & Y_{i,1}^v - y_i \\ 1 & X_{i,2}^v - x_i & Y_{i,2}^v - y_i \\ 1 & X_{i,3}^v - x_i & Y_{i,3}^v - y_i \end{bmatrix}. \quad (5.3)$$

Figure 5: Schematic representation of the Bézier ordinates of (a) a B-spline with respect to a vertex and (b) a B-spline with respect to a triangle. Non-zero Bézier ordinates are denoted by filled bullets $\bullet$, and zero ordinates are denoted by open bullets $\circ$.

Let

$$\hat{f}_{x^a y^b} = \frac{(5-a-b)!}{120} \left(\frac{5}{3}\right)^{a+b} f_{x^a y^b, i}, \quad (5.4)$$

we define the $(3 \times 3 \times 3)$ -tensor $\mathcal{F}$ by its matrix unfolding $F_{(1)}$ as

$$F_{(1)} = \begin{bmatrix} \hat{f} & \hat{f}_x & \hat{f}_y & \hat{f}_x & \hat{f}_x^2 & \hat{f}_{xy} & \hat{f}_y & \hat{f}_{xy} & \hat{f}_y^2 \\ \hat{f}_x & \hat{f}_x^2 & \hat{f}_{xy} & \hat{f}_x^2 & \hat{f}_x^3 & \hat{f}_{x^2 y} & \hat{f}_{xy} & \hat{f}_{x^2 y} & \hat{f}_{xy^2} \\ \hat{f}_y & \hat{f}_{xy} & \hat{f}_y^2 & \hat{f}_{xy} & \hat{f}_{x^2 y} & \hat{f}_{xy^2} & \hat{f}_y^2 & \hat{f}_{xy^2} & \hat{f}_y^3 \end{bmatrix}. \quad (5.5)$$

Combining (3.12)–(3.16) with (4.7), it follows that interpolation problem (4.3a) at vertex V_i can be reformulated as

$$\mathcal{F} = \mathcal{C} \times_1 G \times_2 G \times_3 G. \quad (5.6)$$

Figure 6: Schematic representation of the B-spline coefficients $c_{i,j}^v$, $j = 1, \dots, 10$, with respect to PS5-triangle $t_i(Q_{i,1}^v, Q_{i,2}^v, Q_{i,3}^v)$.

Thus,

$$\mathcal{C} = \mathcal{F} \times_1 G^{-1} \times_2 G^{-1} \times_3 G^{-1}. \quad (5.7)$$

The values of the coefficients $c_{i,j}^v$ can then be computed using (2.1), (5.1), (5.3), (5.5) and (5.7). For a compact notation of these expressions, we make use of a function $R_i(P_1, P_2, P_3)$ with three points $P_l = (x_{P_l}, y_{P_l})$, $l = 1, 2, 3$, as arguments. This function is defined as

$$\begin{aligned} R_i(P_1, P_2, P_3) = & f_i \\ & + \frac{1}{3} ((x_{P_1} - x_i) + (x_{P_2} - x_i) + (x_{P_3} - x_i)) f_{x,i} \\ & + \frac{1}{3} ((y_{P_1} - y_i) + (y_{P_2} - y_i) + (y_{P_3} - y_i)) f_{y,i} \\ & + \frac{5}{36} ((x_{P_1} - x_i)(x_{P_2} - x_i) + (x_{P_2} - x_i)(x_{P_3} - x_i) + (x_{P_3} - x_i)(x_{P_1} - x_i)) f_{x^2,i} \\ & + \frac{5}{36} ((x_{P_1} - x_i)(y_{P_2} - y_i) + (x_{P_2} - x_i)(y_{P_3} - y_i) + (x_{P_3} - x_i)(y_{P_1} - y_i) \\ & \quad + (y_{P_1} - y_i)(x_{P_2} - x_i) + (y_{P_2} - y_i)(x_{P_3} - x_i) + (y_{P_3} - y_i)(x_{P_1} - x_i)) f_{xy,i} \\ & + \frac{5}{36} ((y_{P_1} - y_i)(y_{P_2} - y_i) + (y_{P_2} - y_i)(y_{P_3} - y_i) + (y_{P_3} - y_i)(y_{P_1} - y_i)) f_{y^2,i} \\ & + \frac{25}{324} (x_{P_1} - x_i)(x_{P_2} - x_i)(x_{P_3} - x_i) f_{x^3,i} \\ & + \frac{25}{324} ((x_{P_1} - x_i)(x_{P_2} - x_i)(y_{P_3} - y_i) + (y_{P_1} - y_i)(x_{P_2} - x_i)(x_{P_3} - x_i) \\ & \quad + (x_{P_1} - x_i)(y_{P_2} - y_i)(x_{P_3} - x_i)) f_{x^2y,i} \\ & + \frac{25}{324} ((y_{P_1} - y_i)(y_{P_2} - y_i)(x_{P_3} - x_i) + (x_{P_1} - x_i)(y_{P_2} - y_i)(y_{P_3} - y_i) \\ & \quad + (y_{P_1} - y_i)(x_{P_2} - x_i)(y_{P_3} - y_i)) f_{xy^2,i} \\ & + \frac{25}{324} (y_{P_1} - y_i)(y_{P_2} - y_i)(y_{P_3} - y_i) f_{y^3,i}. \end{aligned} \quad (5.8)$$

We then arrive at the following theorem.

Figure 7: Schematic representation of the Bézier ordinates of a PS5-spline.

Theorem 5.1. *The PS5-spline $s(x, y)$ in (4.4), with the coefficients $c_{i,j}^v$, $j = 1, \dots, 10$, corresponding to vertex V_i chosen as*

$$\begin{aligned}
 c_{i,1}^v &= R_i(Q_{i,1}^v, Q_{i,1}^v, Q_{i,1}^v), & c_{i,2}^v &= R_i(Q_{i,2}^v, Q_{i,2}^v, Q_{i,2}^v), & c_{i,3}^v &= R_i(Q_{i,3}^v, Q_{i,3}^v, Q_{i,3}^v), \\
 c_{i,4}^v &= R_i(Q_{i,1}^v, Q_{i,1}^v, Q_{i,2}^v), & c_{i,5}^v &= R_i(Q_{i,1}^v, Q_{i,1}^v, Q_{i,3}^v), & c_{i,6}^v &= R_i(Q_{i,2}^v, Q_{i,2}^v, Q_{i,3}^v), \\
 c_{i,7}^v &= R_i(Q_{i,2}^v, Q_{i,2}^v, Q_{i,1}^v), & c_{i,8}^v &= R_i(Q_{i,3}^v, Q_{i,3}^v, Q_{i,1}^v), & c_{i,9}^v &= R_i(Q_{i,3}^v, Q_{i,3}^v, Q_{i,2}^v), \\
 c_{i,10}^v &= R_i(Q_{i,1}^v, Q_{i,2}^v, Q_{i,3}^v),
 \end{aligned} \tag{5.9}$$

satisfies interpolation problem (4.3a) at vertex V_i .

5.2 Interpolation at the split points

To compute the value of a PS5-spline at a split point Z_k we make use of the Bernstein-Bézier representation. We consider the macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$, as shown in Figure 3, and we assume that the points indicated in the figure have the following barycentric coordinates:

$$\begin{aligned}
 V_1 &= (1, 0, 0), & V_2 &= (0, 1, 0), & V_3 &= (0, 0, 1), & Z_k &= (z_1, z_2, z_3), \\
 R_{12} &= (\lambda_{12}, \lambda_{21}, 0), & R_{23} &= (0, \lambda_{23}, \lambda_{32}), & R_{31} &= (\lambda_{13}, 0, \lambda_{31}).
 \end{aligned} \tag{5.10}$$

The Bézier ordinates are schematically represented in Figure 7. In [18] it is shown how these ordinates can be computed from the spline coefficients $c_{i,j}^v$ and c_k^t in a stable way, using a sequence of convex combinations. We will briefly review the computation steps without going into the details. We will use the same notation as in [18] for the representation of all (intermediate) ordinates.

The Bézier ordinates $d_1, \dots, d_{16}$ in the neighbourhood of V_1 are completely determined by control polynomial $T_1(x, y)$. These ordinates can be found through subdivision of $T_1(x, y)$. A two-stage subdivision approach was proposed in [18]. In the first stage the Bernstein-Bézier form of the control polynomial is computed onto the triangle with the PS5-points V_1, S_{12} and S_{13} as its three vertices. The Bézier ordinates of the control polynomial on this finer triangle are denoted by $e_{1,j}$, $j = 1, \dots, 10$, as indicated in Figure 4. In the second stage the values of $d_1, \dots, d_{16}$ are computed from the ordinates $e_{1,j}$. Let $e_{2,j}$, $j = 1, \dots, 10$, be the Bézier ordinates of control polynomial $T_2(x, y)$ restricted on the triangle with vertices (V_2, S_{23}, S_{21}) , and let $e_{3,j}$, $j = 1, \dots, 10$, be the Bézier ordinates of control polynomial $T_3(x, y)$ restricted on the triangle with vertices (V_3, S_{31}, S_{32}) . The values of Bézier ordinates $d_{17}, \dots, d_{48}$ in the neighbourhood of vertices V_2 and V_3 are then computed from $e_{2,j}$ and $e_{3,j}$. The values of the Bézier ordinates $d_{49}, \dots, d_{75}$ are obtained from the C^3 -smoothness conditions of the PS5-spline across the interior edges $Z_k - R_{12}$, $Z_k - R_{23}$ and $Z_k - R_{31}$. The remaining Bézier ordinates $d_{76}, \dots, d_{91}$ are found by subdivision of a single cubic polynomial p_3 defined on the triangle with vertices (4.8). This polynomial p_3 is defined by the following ten Bézier ordinates:

$$\begin{aligned} b_{300} &= d_7, & b_{210} &= \tilde{d}_{12}, & b_{201} &= \tilde{d}_{14}, & b_{030} &= d_{23}, & b_{021} &= \tilde{d}_{28}, & b_{120} &= \tilde{d}_{30}, \\ b_{003} &= d_{39}, & b_{102} &= \tilde{d}_{44}, & b_{012} &= \tilde{d}_{46}, & b_{111} &= c_k^t. \end{aligned} \quad (5.11)$$

The values of d_7 , $\tilde{d}_{12}$ and $\tilde{d}_{14}$ are given by

$$\begin{aligned} d_7 &= z_1 (z_1 e_{1,1} + z_2 e_{1,4} + z_3 e_{1,5}) + z_2 (z_1 e_{1,4} + z_2 e_{1,7} + z_3 e_{1,10}) \\ &\quad + z_3 (z_1 e_{1,5} + z_2 e_{1,10} + z_3 e_{1,8}), \end{aligned} \quad (5.12a)$$

$$\begin{aligned} \tilde{d}_{12} &= z_1 (z_1 e_{1,4} + z_2 e_{1,7} + z_3 e_{1,10}) + z_2 (z_1 e_{1,7} + z_2 e_{1,2} + z_3 e_{1,6}) \\ &\quad + z_3 (z_1 e_{1,10} + z_2 e_{1,6} + z_3 e_{1,9}), \end{aligned} \quad (5.12b)$$

$$\begin{aligned} \tilde{d}_{14} &= z_1 (z_1 e_{1,5} + z_2 e_{1,10} + z_3 e_{1,8}) + z_2 (z_1 e_{1,10} + z_2 e_{1,6} + z_3 e_{1,9}) \\ &\quad + z_3 (z_1 e_{1,8} + z_2 e_{1,9} + z_3 e_{1,3}). \end{aligned} \quad (5.12c)$$

The values of d_{23} , $\tilde{d}_{28}$ and $\tilde{d}_{30}$ are computed from $e_{2,j}$, $j = 1, \dots, 10$, and their expressions are similar to (5.12). Likewise, the expressions for d_{39} , $\tilde{d}_{44}$ and $\tilde{d}_{46}$ are based on $e_{3,j}$, $j = 1, \dots, 10$. For more details we refer to [18].

The value of the PS5-spline $s(x, y)$ at split point Z_k can be computed by evaluating the polynomial p_3 with Bézier ordinates (5.11) at Z_k , i.e.,

$$\begin{aligned} s(Z_k) &= d_{91} = z_1^3 d_7 + 3 z_1^2 z_2 \tilde{d}_{12} + 3 z_1^2 z_3 \tilde{d}_{14} + z_2^3 d_{23} + 3 z_2^2 z_3 \tilde{d}_{28} + 3 z_2^2 z_1 \tilde{d}_{30} \\ &\quad + z_3^3 d_{39} + 3 z_3^2 z_1 \tilde{d}_{44} + 3 z_3^2 z_2 \tilde{d}_{46} + 6 z_1 z_2 z_3 c_k^t. \end{aligned} \quad (5.13)$$

The following theorem follows directly from (5.13).

Theorem 5.2. *The PS5-spline $s(x, y)$ in (4.4), with the coefficient c_k^t corresponding to triangle $\mathcal{T}_k(V_1, V_2, V_3) \in \Delta$ chosen as*

$$\begin{aligned} c_k^t &= \frac{1}{6 z_1 z_2 z_3} \left(g_k - z_1^3 d_7 - 3 z_1^2 z_2 \tilde{d}_{12} - 3 z_1^2 z_3 \tilde{d}_{14} - z_2^3 d_{23} - 3 z_2^2 z_3 \tilde{d}_{28} \right. \\ &\quad \left. - 3 z_2^2 z_1 \tilde{d}_{30} - z_3^3 d_{39} - 3 z_3^2 z_1 \tilde{d}_{44} - 3 z_3^2 z_2 \tilde{d}_{46} \right), \end{aligned} \quad (5.14)$$

satisfies interpolation problem (4.3b) at the split point Z_k in triangle $\mathcal{T}_k$.

Combining Theorems 5.1 and 5.2 gives us the full interpolation scheme for PS5-splines in representation (4.4).

6 Reduced PS5-spline interpolants

6.1 C^3 -smoothness across an interior edge

In this section we derive the restriction on the spline coefficient c_k^t such that the PS5-spline will be C^3 -continuous across the interior edge $Z_k - V_1$ inside macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$ (see Figure 3). This condition is satisfied when the values of $d_3, d_6, d_7, d_8, d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{52}, d_{55}, d_{72}, d_{75}, d_{76}, d_{84}$ and d_{85} (see Figure 7) can be regarded as Bézier ordinates of a single cubic polynomial after subdivision. We consider the cubic polynomial with Bézier ordinates f_{abc} , $a + b + c = 3$, defined on the triangle with vertices

$$P_1 = \frac{4}{5}V_1 + \frac{1}{5}Z_k, \quad P_2 = \frac{1}{5}V_1 + \frac{3}{5}V_2 + \frac{1}{5}Z_k, \quad P_3 = \frac{1}{5}V_1 + \frac{3}{5}V_3 + \frac{1}{5}Z_k. \quad (6.1)$$

In case of the required additional C^3 -smoothness, it follows that the values of $d_3, d_6, d_7, d_8, d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{52}, d_{55}, d_{72}, d_{75}, d_{76}, d_{84}$ and d_{85} can be computed from f_{abc} using the formulae (3.4)–(3.5). This gives us new expressions for these d_i in terms of f_{abc} . Comparing the new expressions with the expressions given in [18], we get the restriction on the spline coefficient c_k^t .

We recall from [18] the following relations:

$$d_3 = z_1 e_{1,1} + z_2 e_{1,4} + z_3 e_{1,5}, \quad (6.2a)$$

$$\tilde{d}_6 = \frac{d_6 - \lambda_{12} d_3}{\lambda_{21}} = z_1 e_{1,4} + z_2 e_{1,7} + z_3 e_{1,10}, \quad (6.2b)$$

$$\tilde{d}_8 = \frac{d_8 - \lambda_{13} d_3}{\lambda_{31}} = z_1 e_{1,5} + z_2 e_{1,10} + z_3 e_{1,8}, \quad (6.2c)$$

$$\tilde{\tilde{d}}_{11} = \frac{d_{11} - \lambda_{12} d_6 - \lambda_{12} \lambda_{21} \tilde{d}_6}{\lambda_{21}^2} = z_1 e_{1,7} + z_2 e_{1,2} + z_3 e_{1,6}, \quad (6.2d)$$

$$\tilde{\tilde{d}}_{15} = \frac{d_{15} - \lambda_{13} d_8 - \lambda_{13} \lambda_{31} \tilde{d}_8}{\lambda_{31}^2} = z_1 e_{1,8} + z_2 e_{1,9} + z_3 e_{1,3}, \quad (6.2e)$$

$$\tilde{\tilde{d}}_{13} = \frac{d_{12} - \lambda_{12} d_7 - \lambda_{21} z_1 \tilde{d}_6 - \lambda_{21} z_2 \tilde{\tilde{d}}_{11}}{\lambda_{21} z_3} = z_1 e_{1,10} + z_2 e_{1,6} + z_3 e_{1,9}. \quad (6.2f)$$

In case of C^3 -smoothness across the interior edge $Z_k - V_1$, it is easy to verify that we must have

$$d_3 = f_{300}, \quad \tilde{d}_6 = f_{210}, \quad \tilde{d}_8 = f_{201}, \quad \tilde{\tilde{d}}_{11} = f_{120}, \quad \tilde{\tilde{d}}_{15} = f_{102}, \quad \tilde{\tilde{d}}_{13} = f_{111}. \quad (6.3)$$

We also obtain from [18] that

$$\tilde{d}_{53} = \frac{d_{52} - \lambda_{12} d_{11}}{\lambda_{21}} = \lambda_{12} (\lambda_{12} \tilde{d}_6 + \lambda_{21} \tilde{\tilde{d}}_{11}) + \lambda_{21} (\lambda_{12} \tilde{\tilde{d}}_{31} + \lambda_{21} \tilde{d}_{24}), \quad (6.4a)$$

$$\tilde{d}_{56} = \frac{d_{55} - \lambda_{12} d_{12}}{\lambda_{21}} = \lambda_{12} (z_1 \tilde{d}_6 + z_2 \tilde{\tilde{d}}_{11} + z_3 \tilde{d}_{13}) + \lambda_{21} (z_1 \tilde{\tilde{d}}_{31} + z_2 \tilde{d}_{24} + z_3 \tilde{d}_{29}), \quad (6.4b)$$

$$\tilde{d}_{71} = \frac{d_{72} - \lambda_{13} d_{15}}{\lambda_{31}} = \lambda_{13} (\lambda_{13} \tilde{d}_8 + \lambda_{31} \tilde{\tilde{d}}_{15}) + \lambda_{31} (\lambda_{13} \tilde{\tilde{d}}_{43} + \lambda_{31} \tilde{d}_{38}), \quad (6.4c)$$

$$\tilde{d}_{74} = \frac{d_{75} - \lambda_{13} d_{14}}{\lambda_{31}} = \lambda_{13} (z_1 \tilde{d}_8 + z_2 \tilde{d}_{13} + z_3 \tilde{\tilde{d}}_{15}) + \lambda_{31} (z_1 \tilde{\tilde{d}}_{43} + z_2 \tilde{d}_{45} + z_3 \tilde{d}_{38}), \quad (6.4d)$$

where

$$\tilde{d}_{24} = \frac{d_{24} - \lambda_{21} d_{19}}{\lambda_{12}} = z_1 e_{2,8} + z_2 e_{2,5} + z_3 e_{2,10}, \quad (6.5a)$$

$$\tilde{\tilde{d}}_{31} = \frac{d_{31} - \lambda_{21} d_{24} - \lambda_{21} \lambda_{12} \tilde{d}_{24}}{\lambda_{12}^2} = z_1 e_{2,3} + z_2 e_{2,8} + z_3 e_{2,9}, \quad (6.5b)$$

$$\tilde{d}_{29} = \frac{d_{30} - \lambda_{21} d_{23} - \lambda_{12} z_1 \tilde{\tilde{d}}_{31} - \lambda_{12} z_2 \tilde{d}_{24}}{\lambda_{12} z_3} = z_1 e_{2,9} + z_2 e_{2,10} + z_3 e_{2,6}. \quad (6.5c)$$

and

$$\tilde{d}_{38} = \frac{d_{38} - \lambda_{31} d_{35}}{\lambda_{13}} = z_1 e_{3,7} + z_2 e_{3,10} + z_3 e_{3,4}, \quad (6.6a)$$

$$\tilde{\tilde{d}}_{43} = \frac{d_{43} - \lambda_{31} d_{38} - \lambda_{31} \lambda_{13} \tilde{d}_{38}}{\lambda_{13}^2} = z_1 e_{3,2} + z_2 e_{3,6} + z_3 e_{3,7}, \quad (6.6b)$$

$$\tilde{d}_{45} = \frac{d_{44} - \lambda_{31} d_{39} - \lambda_{13} z_1 \tilde{\tilde{d}}_{43} - \lambda_{13} z_3 \tilde{d}_{38}}{\lambda_{13} z_2} = z_1 e_{3,6} + z_2 e_{3,9} + z_3 e_{3,10}. \quad (6.6c)$$

In case of C^3 -smoothness across $Z_k - V_1$, we find that

$$\tilde{d}_{53} = \lambda_{12} (\lambda_{12} f_{210} + \lambda_{21} f_{120}) + \lambda_{21} (\lambda_{12} f_{120} + \lambda_{21} f_{030}), \quad (6.7a)$$

$$\tilde{d}_{56} = \lambda_{12} (z_1 f_{210} + z_2 f_{120} + z_3 f_{111}) + \lambda_{21} (z_1 f_{120} + z_2 f_{030} + z_3 f_{021}), \quad (6.7b)$$

$$\tilde{d}_{71} = \lambda_{13} (\lambda_{13} f_{201} + \lambda_{31} f_{102}) + \lambda_{31} (\lambda_{13} f_{102} + \lambda_{31} f_{003}), \quad (6.7c)$$

$$\tilde{d}_{74} = \lambda_{13} (z_1 f_{201} + z_2 f_{111} + z_3 f_{102}) + \lambda_{31} (z_1 f_{102} + z_2 f_{012} + z_3 f_{003}). \quad (6.7d)$$

Combining (6.3), (6.4) and (6.7), we find the following necessary conditions for the additional C^3 -smoothness

$$\lambda_{12} \tilde{\tilde{d}}_{31} + \lambda_{21} \tilde{d}_{24} = \lambda_{12} f_{120} + \lambda_{21} f_{030}, \quad (6.8a)$$

$$z_1 \tilde{\tilde{d}}_{31} + z_2 \tilde{d}_{24} + z_3 \tilde{d}_{29} = z_1 f_{120} + z_2 f_{030} + z_3 f_{021}, \quad (6.8b)$$

$$\lambda_{13} \tilde{\tilde{d}}_{43} + \lambda_{31} \tilde{d}_{38} = \lambda_{13} f_{102} + \lambda_{31} f_{003}, \quad (6.8c)$$

$$z_1 \tilde{\tilde{d}}_{43} + z_2 \tilde{d}_{45} + z_3 \tilde{d}_{38} = z_1 f_{102} + z_2 f_{012} + z_3 f_{003}. \quad (6.8d)$$

We can reformulate (6.8) as

$$f_{030} = \tilde{d}_{24} + \frac{\lambda_{12}}{\lambda_{21}} (\tilde{\tilde{d}}_{31} - \tilde{d}_{11}), \quad (6.9a)$$

$$f_{021} = \tilde{d}_{29} + \left(\frac{z_1}{z_3} - \frac{z_2}{z_3} \frac{\lambda_{12}}{\lambda_{21}} \right) (\tilde{\tilde{d}}_{31} - \tilde{d}_{11}), \quad (6.9b)$$

$$f_{003} = \tilde{d}_{38} + \frac{\lambda_{13}}{\lambda_{31}} (\tilde{\tilde{d}}_{43} - \tilde{d}_{15}), \quad (6.9c)$$

$$f_{012} = \tilde{d}_{45} + \left(\frac{z_1}{z_2} - \frac{z_3}{z_2} \frac{\lambda_{13}}{\lambda_{31}} \right) (\tilde{\tilde{d}}_{43} - \tilde{d}_{15}). \quad (6.9d)$$

Finally, we deduce from [18] that

$$\tilde{d}_{77} = \frac{d_{76} - \lambda_{12} d_{13}}{\lambda_{21}} = z_1 (z_1 \tilde{d}_6 + z_2 \tilde{\tilde{d}}_{11} + z_3 \tilde{d}_{13}) + z_2 (z_1 \tilde{\tilde{d}}_{31} + z_2 \tilde{d}_{24} + z_3 \tilde{d}_{29}) + z_3 c_k^t. \quad (6.10)$$

The additional C^3 -smoothness across $Z_k - V_1$ implies that

$$\begin{aligned} \tilde{d}_{77} = & z_1 (z_1 f_{210} + z_2 f_{120} + z_3 f_{111}) + z_2 (z_1 f_{120} + z_2 f_{030} + z_3 f_{021}) \\ & + z_3 (z_1 f_{111} + z_2 f_{021} + z_3 f_{012}), \end{aligned} \quad (6.11)$$

resulting in the following condition on the spline coefficient c_k^t :

$$c_k^t = z_1 f_{111} + z_2 f_{021} + z_3 f_{012}. \quad (6.12)$$

Filling (6.2), (6.3), (6.5)–(6.6) and (6.9) into (6.12), we get an expression for c_k^t in terms of $e_{i,j}$, $i = 1, 2, 3$ and $j = 1, \dots, 10$. Since the ordinates $e_{i,j}$ can be computed through subdivision from the three control polynomials $T_i(x, y)$ corresponding to the three vertices V_i , $i = 1, 2, 3$, we have an expression for c_k^t in terms of the vertex coefficients $c_{i,j}^v$, $i = 1, 2, 3$ and $j = 1, \dots, 10$.

A similar procedure can be used to derive the restrictions on c_k^t for additional C^3 -smoothness across the other interior edges $Z_k - V_2$ or $Z_k - V_3$.

6.2 Reduced PS5-splines

In this section we consider some reduced PS5-spline spaces that preserve full approximation order. The B-spline coefficients c_k^t , $k = 1, \dots, n_t$, in representation (4.4) are determined in a specific way such that the obtained PS5-splines are still able to reproduce quintic polynomials.

A first approach is requiring additional C^3 -smoothness across an interior edge for all macro-triangles in the triangulation. Such a reduced PS5-spline space was described in [1]. For each macro-triangle $\mathcal{T}_k(V_1, V_2, V_3)$ in Δ , there are three possible choices to impose the additional smoothness, i.e., across the edge $Z_k - V_1$, $Z_k - V_2$ or $Z_k - V_3$. In case of $Z_k - V_1$, we get the following theorem.

Theorem 6.1. *The PS5-spline $s(x, y)$ in (4.4), with the coefficient c_k^t corresponding to triangle $\mathcal{T}_k(V_1, V_2, V_3) \in \Delta$ chosen as*

$$\begin{aligned} c_k^t = & z_1 \tilde{d}_{13} + z_2 \tilde{d}_{29} + z_3 \tilde{d}_{45} + \frac{z_2}{z_3} \left(z_1 - z_2 \frac{\lambda_{12}}{\lambda_{21}} \right) (\tilde{d}_{31} - \tilde{d}_{11}) \\ & + \frac{z_3}{z_2} \left(z_1 - z_3 \frac{\lambda_{13}}{\lambda_{31}} \right) (\tilde{d}_{43} - \tilde{d}_{15}), \end{aligned} \quad (6.13)$$

reproduces quintic polynomials.

Proof. From (6.3), (6.9) and (6.12), it follows that the spline $s(x, y)$ is C^3 -continuous across interior edge $Z_k - V_1$. From [1] we know that such a spline reproduces quintic polynomials. $\square$

The next theorem gives a symmetric condition on the coefficient c_k^t . This reduced spline space was developed in [18].

Theorem 6.2. *The PS5-spline $s(x, y)$ in (4.4), with the coefficient c_k^t corresponding to triangle $\mathcal{T}_k(V_1, V_2, V_3) \in \Delta$ chosen as*

$$\begin{aligned} c_k^t = & z_1 (z_1 e_{1,10} + z_2 e_{1,6} + z_3 e_{1,9}) + z_2 (z_1 e_{2,9} + z_2 e_{2,10} + z_3 e_{2,6}) \\ & + z_3 (z_1 e_{3,6} + z_2 e_{3,9} + z_3 e_{3,10}), \end{aligned} \quad (6.14)$$

reproduces quintic polynomials.

We now consider reduced PS5-splines defined on a uniform PS-triangulation Δ^* , where the barycentric coordinates in (5.10) satisfy

$$z_1 = z_2 = z_3 = 1/3 \quad \text{and} \quad \lambda_{12} = \lambda_{21} = \lambda_{23} = \lambda_{32} = \lambda_{31} = \lambda_{13} = 1/2 \quad (6.15)$$

on all triangles.

Theorem 6.3. *Let Δ^* be a uniform PS-refinement of a uniform triangulation Δ , satisfying (6.15), then the reduced PS5-splines described in Theorems 6.1 and 6.2 are equivalent on Δ^* . Moreover, these splines are locally C^3 -continuous on each macro-triangle in Δ .*

Proof. The first three terms in (6.13) are equivalent to (6.14). The remaining terms in (6.13) are zero because of the uniformity of the PS-triangulation. The local C^3 -continuity on each macro-triangle follows from the symmetry of formula (6.14). $\square$

The reduced spline space in Theorem 6.3 was also considered in [8] for the uniform three-direction mesh.

Theorem 5.1 can be used to construct an interpolation scheme based on the reduced PS5-splines described in Theorems 6.1 and 6.2. These reduced spline interpolants have the advantage that only data are needed located at the vertices of the triangulation.

7 A numerical example

For a practical example we consider the function

$$\begin{aligned} f(x, y) = & \frac{3}{4} \exp\left(-\frac{(9x-2)^2 + (9y-2)^2}{4}\right) + \frac{3}{4} \exp\left(-\frac{(9x+1)^2}{49} - \frac{9y+1}{10}\right) \\ & + \frac{1}{2} \exp\left(-\frac{(9x-7)^2 + (9y-3)^2}{4}\right) - \frac{1}{5} \exp(-(9x-4)^2 - (9y-7)^2), \end{aligned} \quad (7.1)$$

on the domain $\Omega = [0, 1] \times [0, 1]$. This function is known as Franke's test function (see, e.g., [2, 7]). Let $\{V_i\}_{i=1}^{n_v}$ be the set of 25 data points given in Figure 8. The figure shows the corresponding Delaunay triangulation with a particular PS-refinement. The dimension of the general PS5-spline space on this mesh is 282, and the dimension of the reduced spline space is 250. The data $f_{x^a y^b, l}$ and g_m in (4.3) are sampled from the function $f(x, y)$.

To measure the accuracy of a spline fit s , we have computed the maximum error and the mean error on a 400×400 uniform grid G on Ω , i.e.,

$$E_{max} = \max_{(x,y) \in G} |f(x, y) - s(x, y)|, \quad E_{mean} = \frac{\sum_{(x,y) \in G} |f(x, y) - s(x, y)|}{400 \times 400}.$$

The PS5-spline interpolant, defined by Theorems 5.1 and 5.2 on the mesh shown in Figure 8, has a maximum error equal to 0.01018, and its mean error is 0.00040. The reduced PS5-spline interpolant, defined by Theorems 5.1 and 6.2, has a maximum error of 0.01503 and a mean error of 0.00060. This reduced PS5-spline interpolant is depicted in Figure 9.

Figure 8: A triangulation with PS-refinement.

Figure 9: A reduced PS5-spline (with the triangular mesh lines) and its contour plot. The spline is defined on the mesh in Figure 8, and it interpolates the function (7.1) at the vertices.

8 Concluding remarks

In this paper we provided local interpolation rules for quintic Powell-Sabin splines represented with a normalized B-spline basis. The general interpolation scheme uses data at the vertices of the triangulation and at the Powell-Sabin split points. We also discussed interpolation based on several reduced spline spaces. These reduced spline interpolants only need data at the vertices of the triangulation.

The presented interpolation schemes make use of derivative information (up to order three) at the vertices. When no derivative information is available, these interpolation schemes can be incorporated into a two-stage method. In the first stage the needed derivative information is then locally estimated from the data, e.g., by doing local least squares with polynomials or radial basis functions.

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