Purely finitely additive measures are non-constructible objects*

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April 13, 2010

Abstract. The existence of a purely finitely additive measure cannot be proved in Zermelo-Frankel set theory if the use of the Axiom of Choice is disallowed.

Key words. Finitely additive probabilities; Charges; Axiom of choice; Constructivism.

1 Introduction

A finitely additive measure μ on $\mathbb{N}$ assigns to each subset of $\mathbb{N}$ a nonnegative real number and assigns to the union of disjoint sets the sum of their numbers. The measure μ is said to be countably additive if the measure of a countable union of pairwise disjoint sets is equal to the sum of the measures of those sets. The finitely additive measure ν is dominated by μ (and we write $\nu \leq \mu$) is for each subset S of $\mathbb{N}$, we have $\nu(S) \leq \mu(S)$. The finitely additive measure μ is said to be purely finitely additive if the inequalities $0 \leq \nu \leq \mu$ with ν countably additive imply that $\nu = 0$. From Yosida and Hewitt (1952) and Rao (1958) we know that each finitely additive measure uniquely decomposes as the sum of a countably additive and a purely additive measure. Typically, a purely finitely additive measure is obtained by means of Hahn-Banach's theorem or by means of a free ultrafilter.[1] Let us describe the second route.

*This note extends the result obtained in "The uniform distributions puzzle" (2007, 2009) to arbitrary purely finitely additive measures. I thank professors J.B. Kadane, Philippe Mongin, Stephen Portnoy, K.P.S. Bhaskara Rao, and T. Seidenfeld for their comments on these earlier versions.

[1]A free ultrafilter on the set $\mathbb{N}$ of natural numbers is a collection $\mathcal{U}$ of subsets of $\mathbb{N}$ such that (i) $\mathbb{N} \in \mathcal{U}$ and $\varnothing \notin \mathcal{U}$, (ii) if $A \subseteq B$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$, (iii) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, and (iv) for each $A \subseteq \mathbb{N}$, either A or $\mathbb{N} - A$ belongs to $\mathcal{U}$. The existence of a free ultrafilter follows from Zorn's Lemma.

A free ultrafilter $\mathcal{F}$ on $\mathbb{N}$ defines a limit on X. Consider a sequence x in X and all of its limit points. Each limit point is the limit of a subsequence. There is only one limit point with a converging subsequence $x_{i_1}, x_{i_2}, \ldots, x_{i_t}, \ldots$ for which the set $\{i_1, i_2, \ldots, i_t, \ldots\}$ of indices belongs to $\mathcal{F}$. Define $\lim_{\mathcal{F}}(x) = \lim_{t\to\infty} x_{i_t}$. Due to the intersection property of $\mathcal{F}$, we have $\lim_{\mathcal{F}}(x+y) = \lim_{\mathcal{F}}(x) + \lim_{\mathcal{F}}(y)$ for each x and y in X. The ultrafilter-based-limit $\lim_{\mathcal{F}}$ defines a finitely additive measure:

$$\mu_{\mathcal{F}}(S) \;=\; \lim_{\mathcal{F}} s_t \qquad \text{with } s_t = \frac{\#(S \cap \{1, 2, \ldots, t\})}{t},$$

and S a subset of $\mathbb{N}$. If the sequence $s_1, s_2, \ldots, s_t, \ldots$ has only one accumulation point, then $\mu_{\mathcal{F}}(S)$ coincides with 'the' limit of this sequence and is known as the natural density of S. For example, the set of even numbers has a natural density equal to .5; the set of all multiples of 20 has a natural density equal to .05. Not every subset of $\mathbb{N}$ has a natural density. For example, the set

$$S_1 \;=\; \{1, 10, 11, \ldots, 19, 100, 101, \ldots, 199, 1000, 1001, \ldots\}$$

of all natural numbers having their first digit equal to 1 has no natural density. The measure $\mu_{\mathcal{F}}(S_1)$ depends upon the particular (non-constructible!) ultrafilter $\mathcal{F}$ and can take any value between 1/9 and 5/9.[2]

Both routes to obtain purely finitely additive measures (Hahn-Banach's theorem and a free ultrafilter) rely upon AC and involve non-constructive methods. Obviously, one cannot conclude from this that purely finitely additive measures are non-constructible objects. The knowledge that non-constructive methods can be used to obtain a purely finitely additive measure, does not answer the question whether a purely finitely additive measure can be obtained without recurse to non-constructive methods.

This note shows that the existence of a purely finitely additive measure on $\mathbb{N}$ entails the existence of a non-Ramsey set. From Mathias (1977) we know that a non-Ramsey set is a non-constructible object.

The next section touches the notion of constructivism and recalls the Axiom of Choice and the Axiom of Dependent choice. Section 3 states and proves the main result. The result extends to purely finitely measures on $\mathbb{R}$.

2 Constructivism

The Axiom of Choice (AC) postulates for each nonempty family $\mathcal{D}$ of nonempty sets the existence of a function f such that $f(S) \in S$ for each set S in the family $\mathcal{D}$. The function f is referred to as a choice function. AC does not provide an explicit way to construct such a choice function and provoked considerable criticism in the aftermath of Zermelo's

[2] In this example, $\liminf(s_t)$ is the limit of the sequence 1/9, 11/99, 111/999, ... and is equal to 1/9; $\limsup(s_t)$ is the limit of the sequence 1, 11/19, 111/199, ... and is equal to 5/9.

formulation in 1904.[3] Among the applications of AC, we mention Zorn's Lemma, the theorem of Hahn-Banach, and the existence of free ultrafilters. AC also implies a number of paradoxes such as the decomposition of a sphere into a sphere of smaller size, and the existence of a non-measurable set of real numbers. The nonconstructive character of AC is further revealed by Dianonescu (1975) who showed that AC implies the law of the excluded middle.[4] Constructive mathematics rejects the law of the excluded middle and hence rejects AC. On the other hand, the Axiom of Dependent Choice (DC) is generally accepted by constructivists (Beeson, 1988, p. 42). Let S be a nonempty set and let R be a binary relation in S such that for each a in S there is a b in S with $(a, b) \in R$. Then, DC postulates the existence of a sequence $(a_1, a_2, \ldots, a_n, \ldots)$ of elements in S such that $(a_k, a_{k+1}) \in R$ for each $k = 1, 2, \ldots$.

The nonconstructive object used in this note is known as a non-Ramsey set. Let I be an infinite set and let n be a positive integer. Let $[I]^n$ collect all the subsets of I with exactly n elements. Ramsey (1928) shows that for each subset S of $[I]^n$, there exists an infinite set $J \subset I$ such that either $[J]^n \subset S$ or $[J]^n \cap S = \varnothing$. When n is replaced by countable infinity, then Ramsey's theorem fails. There exists a subset S of $[I]^\infty$ such that for each infinite subset J of I the class $[J]^\infty$ intersects S and its complement $[I]^\infty - S$ as well. Such a set S is said to be non-Ramsey. Mathias (1977) showed that the existence of non-Ramsey sets does not follow from ZF (without AC).[5]

3 Finitely additive measures

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathcal{F}$ be the field of all subsets of $\mathbb{N}$. A finitely additive probability is a map $\mu : \mathcal{F} \to \mathbb{R}^+$ that satisfies and the condition

$$\mu\Big(A_1 \cup A_2 \cup \ldots \cup A_n\Big) \;=\; \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n),$$

where the sets $A_1, A_2, \ldots, A_n$ are disjoint, for all finite n. We now state the main result of this note.

Theorem. The existence of a finitely additive measure μ on $\mathbb{N}$ that attaches zero probability to each natural number entails the existence of a non-Ramsey set.

Proof. Rescale the measure μ such that $\mu(\mathbb{N}) = 1$. We use some additional notation. For two natural numbers $i > j$, let $[\,i, j[$ denote the set $\{i, i{+}1, \ldots, j{-}1\}$. Furthermore, to each

[3] AC is (i) consistent and (ii) independent: (i) AC can be added to the Zermelo-Fraenkel axioms of set theory (ZF) without yielding a contradiction, and (ii) AC is not a theorem of ZF (Fraenkel et al, 1973).

[4] The law of the excluded middle states the truth of 'P or not-P' for each proposition P and can be used to claim the existence of certain objects without any hint to its construction. For example, the real number $c = \sqrt{2}^{\sqrt{2}}$ *either* is rational (in which case one sets $a = b = \sqrt{2}$) *or* is not rational (in which case one sets $a = c$ and $b = \sqrt{2}$). Conclude the *existence* of irrational numbers a and b for which a^b is rational.

[5] More precisely, Solovay (1970) proposed a model in which ZF and DC are true and in which AC fails. Mathias showed that in this Solovay-model a non-Ramsey set does not exist. Hence, the existence of a non-Ramsey set is independent of ZF + DC.

infinite set $A \subseteq \mathbb{N}$ we connect a set, denoted by A_0, as follows. Let $A = \{n_0, n_1, \ldots, n_k, \ldots\}$ with $n_k < n_{k+1}$ for each k, then $A_0 = [\,n_0, n_1[\, \cup [\,n_2, n_3[\, \cup \ \ldots \ \cup \ [\,n_{2k}, n_{2k+1}[\, \cup \ \ldots$.

Now, let μ satisfy the requirements listed in the theorem. We show that

$$S = \{A \subseteq \mathbb{N} \,|\, \mu(A_0) > 0.5\}$$

is a non-Ramsey set. It is sufficient to show that each infinite set $A = \{n_0, n_1, \ldots, n_k, \ldots\}$ includes an infinite subset B such either A or B belongs to S (the 'either-or' being exclusive). We distinguish three cases.

Case 1. $A \notin S$, in particular $\mu(A_0) < 0.5$. Let $B = A - \{n_0\}$. Then, $[\,0, n_0[\, \cup A_0 \cup B_0 = \mathbb{N}$. Since $\mu([\,0, n_0[) = 0$, $\mu(A_0) < 0.5$, and $\mu(\mathbb{N}) = 1$; we obtain that $\mu(B_0) > 0.5$. Therefore, $B \subseteq A$ and $B \in S$.

Case 2. $A \notin S$, in particular $\mu(A_0) = 0.5$. Let $B = \{n_0, n_3, n_4, n_7, \ldots, n_{4k}, n_{4k+3}, \ldots\}$ and let $B' = \{n_0, n_1, n_2, n_5, n_6, n_9, \ldots, n_{4k+2}, n_{4k+5}, \ldots\}$. Then, $A_0 = B_0 \cap B'_0$. Hence, we have $\mu(B_0) \geq 0.5$ and $\mu(B'_0) \geq 0.5$. Furthermore,

$$[\,0, n_0[\, \cup (B_0 \,\Delta\, B'_0) \ \cup \ A_0 \ = \ \mathbb{N}.$$

Conclude that the symmetric difference $B_0 \,\Delta\, B'_0$ has a measure equal to 0.5. Hence, at least one of the sets B_0 or B'_0 has a measure strictly larger than 0.5. Select the subset B or B' of A for which the corresponding set B_0 or B'_0 has the highest measure. The selected subset of A belongs to S.

Case 3. $A \in S$. Similar to Case 1, we put $B = A - \{n_0\}$. Conclude that $B \subseteq A$ and that $B \notin S$. □

Finally, we indicate how this result extends to purely finitely additive measures on $\mathbb{R}$. Consider such a measure μ. Since, μ is not countably additive, there exists a countable sequence of pairwise disjoint sets A_1, $A_2, \ldots$, $A_n, \ldots$ (subsets of $\mathbb{R}$) such that

$$\mu(A_1) + \mu(A_2) + \cdots + \mu(A_n) + \cdots \ < \ \mu(A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots).$$

Define a measure μ' on $\mathbb{N}$ by $\mu'(C) = \mu(\cup_{j \in C} A_j)$. This measure has a purely finitely additive component.

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