Component Reuse in Iterative Solvers for the Solution of Fuzzy Partial Differential Equations

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September 21, 2009

- **PDEs** of elliptic type with **fuzzy** parameters.
- Our goal is to determine fuzzy characteristics of the PDE solution.
- Problem can be rewritten as large set of related **global optimization** problems.
- We consider **component reuse** in function and gradient evaluations to accelerate optimization.

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Fuzzy Numbers

What are they for?

Increasingly popular method to model uncertainty in engineering applications. Stochastic methods are criticized in case of little knowledge on distribution function. Fuzzy numbers are useful for *early design*, *worst-case analysis* (e.g. for structures) and *sensitivity analysis*.

What are they?

Special kind of fuzzy set. Fuzzy number $\tilde{x}$ is represented by membership function $\mu_{\tilde{x}}: \Omega \rightarrow [0, 1]$, where the value $\mu_{\tilde{x}}(x)$ represents the degree of membership of x .

A classical set either contains an element or not, which means its membership function is a function $\mu_{\tilde{x}}: \Omega \rightarrow \{0, 1\}$.

Operations on Fuzzy Numbers

- The fuzzy extension $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ of a function $y = f(x_1, x_2, \dots, x_n)$ is given by **Zadeh's extension principle**:

$$\mu_{\tilde{y}}(y) = \sup_{y=f(x_1, x_2, \dots, x_n)} \min \{ \mu_{\tilde{x}_1}(x_1), \mu_{\tilde{x}_2}(x_2), \dots, \mu_{\tilde{x}_n}(x_n) \}$$

- Using the **α -cut approach**, this can be reformulated:

$$[\tilde{y}]_{\alpha} = f([\tilde{x}_1]_{\alpha}, [\tilde{x}_2]_{\alpha}, \dots, [\tilde{x}_n]_{\alpha}) := \{ y : y = f(\mathbf{x}), \mathbf{x} \in [\tilde{\mathbf{x}}]_{\alpha} \}$$

with $[\tilde{y}]_{\alpha} = \{ y : \mu_{\tilde{y}}(y) \geq \alpha \}$. This is **Interval Analysis**.

Example

$\cos(\tilde{x})$, where $\tilde{x} = \text{triag}(-1, 1, 2)$

Optimization

The α -cut approach is equivalent to max- and minimization on the search space $[\tilde{\mathbf{x}}]_\alpha$, for every α -cut we take. For f continuous:

$$[\tilde{y}]_\alpha = [f(\tilde{\mathbf{x}})]_\alpha = [\min_{\mathbf{x}}(f(\mathbf{x})), \max_{\mathbf{x}}(f(\mathbf{x}))] \text{ s.t. } \mathbf{x} \in [\tilde{\mathbf{x}}]_\alpha$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 1$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$\alpha = 0.9$

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

Optimization with $G\alpha D$

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$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 0.8$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 0.7$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 0.6$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$\alpha = 0.5$

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$\alpha = 0.4$$

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 0.3$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 0.2$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

$$\alpha = 0.1$$

Optimization with $G\alpha D$

A dedicated global optimization algorithm:

Gradual α -level Decreasing ($G\alpha D$) Algorithm

Optimization on growing hyperrectangular search space from α -level 1 to α -level 0.

$\alpha = 0$

$$y = \log(x_1^3 + 3.2x_1^2 + 3.08x_1 + x_1x_2 + 0.4x_2 - 3x_1x_2^2 - 0.2x_2^2 - x_2^3 + 1)$$

$$\tilde{x}_1 = \text{triag}(-1, 0, 1)$$

$$\tilde{x}_2 = \text{triag}(-1, 0, 1)$$

PDE with fuzzy parameters

PDE of elliptic type:

$$-\nabla \cdot (k \nabla u) + cu = f \text{ in } \Omega \subset \mathbb{R}^2 \text{ s.t. } \alpha u + \beta \frac{\partial u}{\partial \vec{n}} = \gamma \text{ on } \partial\Omega,$$

Discretization results in a linear system:

$$A(\mathbf{k})\mathbf{u} = \mathbf{b}$$

If we assume k is a fuzzy field $\tilde{k}$, We can fuzzify the system:

$$A(\tilde{\mathbf{k}})\tilde{\mathbf{u}} = \mathbf{b}$$

Problem Formulation

Optimization Problem

$$\begin{aligned} \forall i \in \mathcal{I}, \forall \alpha_j \in \mathcal{A}: \max_{\mathbf{k}} u_i \text{ and } \min_{\mathbf{k}} u_i \\ \text{s.t. } A(\mathbf{k})\mathbf{u} = \mathbf{b} \text{ and } \mathbf{k} \in [\tilde{\mathbf{k}}]_{\alpha_j} \end{aligned}$$

The many **function and gradient evaluations** in optimization differ from each other through a possibly **small change in parameters**. Therefore we consider **component reuse** to accelerate re-analysis.

Reuse in matrix assembly phase

Assume the fuzzy field $\tilde{\mathbf{k}}$ can be written as finite sum of basis functions: $\tilde{\mathbf{k}} = \sum_{l=1}^n \tilde{k}_l \varphi_l$, then:

$$A(\tilde{\mathbf{k}}) = \sum_{l=1}^n \tilde{k}_l K_l + M$$

Construction of a new $A(\mathbf{k})$ in the optimization algorithm becomes very cheap once the K_l and M are known.

Reuse of earlier solution as initial guess

Assume $A(\mathbf{k})\mathbf{u} = \mathbf{b}$ has been solved iteratively in some iteration point $\mathbf{k}^{(i)}$ until relative residual error is below some threshold:

$$\frac{\|\mathbf{b} - A^{(i)}\mathbf{u}^{(i)}\|}{\|\mathbf{b}\|} \leq \epsilon_{\text{tol}}$$

then we can prove:

$$\frac{\|\mathbf{b} - A^{(i+1)}\mathbf{u}^{(i)}\|}{\|\mathbf{b}\|} \leq \epsilon_{\text{tol}} + \sum_{l=1}^n |\Delta k_l^{(i+1)}| \|K_l\| \|(A^{(i)})^{-1}\| (1 + \epsilon_{\text{tol}})$$

The smaller the step, the better the earlier solution is as initial guess.

Reuse intergrid operators in algebraic multigrid

Basic two-grid to solve $A\mathbf{u} = \mathbf{b}$

- 1 Choose initial guess $\mathbf{u}_{\text{init}}$
- 2 relax ν_1 times on $A_1\mathbf{u}_1 = \mathbf{b}_1$
- 3 $\mathbf{r}_2 \leftarrow l_1^2(\mathbf{b}_1 - A_1\mathbf{u}_1)$
- 4 solve $A_2\mathbf{e}_2 = \mathbf{r}_2$
- 5 $\mathbf{u}_1 \leftarrow \mathbf{u}_1 + l_2^1\mathbf{e}_2$
- 6 relax ν_2 times on $A_1\mathbf{u}_1 = \mathbf{b}_1$

Reuse intergrid operators in algebraic multigrid

- Discrete operator A_k for grid Ω_k is constructed like:
 $A_{k+1} = I_k^{k+1} A_k I_{k+1}^k$, with $A_1 = A$.
- The intergrid transfer operators I_k^{k+1} and I_{k+1}^k are constructed during **costly** setup phase in algebraic multigrid.

Therefore we construct $I_k^{k+1}(\mathbf{k}_{\text{ref}})$ and $I_{k+1}^k(\mathbf{k}_{\text{ref}})$ in some reference point $\mathbf{k}_{\text{ref}}$ and reuse them in other iteration points to construct $A_{k+1} = I_k^{k+1}(\mathbf{k}_{\text{ref}}) A_k I_{k+1}^k(\mathbf{k}_{\text{ref}})$, with $A_1 = A(\mathbf{k}^{(i)})$.

- 2 uncertain parameters: $\tilde{k}_{CPU}$ and $\tilde{k}_{board}$
- grid of 100×100
- about 1000 function evaluations
- AMG: Ruge-Stüben

intergrid operator reuse	initial guess reuse	multigrid setup time (s)	solution time (s)	total time (s)
x	x	2.89	23.25	26.14
x		2.89	51.43	54.32
	x	86.24	22.27	108.51
		86.71	49.05	135.76

(Add about 30 min. extra in case of no matrix assembly reuse.)

Thank You!