A convergent finite element method with adaptive $\sqrt{3}$ refinement

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Abstract

We develop an adaptive finite element method (AFEM) using piecewise linears on a sequence of triangulations obtained by adaptive $\sqrt{3}$ refinement. The motivation to consider $\sqrt{3}$ refinement stems from the fact that it is a slower topological refinement than the usual red or red-green refinement, and that it alternates the orientation of the refined triangles, such that certain features or singularities that are not aligned with the initial triangulation might be detected more quickly. On the other hand, the use of $\sqrt{3}$ refinement introduces the additional difficulty that the corresponding finite element spaces are nonnested. This makes the setting nonconforming. First we derive a BPX-type preconditioner for piecewise linears on the adaptively refined triangulations and we show that it gives rise to uniformly bounded condition numbers, so that we can solve the linear systems arising from the AFEM in an efficient way. Then we introduce the AFEM of Morin, Nochetto, and Siebert adapted to our special case for solving the Poisson equation. We prove that this adaptive strategy converges to the solution within any prescribed error tolerance in a finite number of steps. Finally we present some numerical experiments that show the optimality of both the BPX preconditioner and the AFEM.

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Abstract

We develop an adaptive finite element method (AFEM) using piecewise linears on a sequence of triangulations obtained by adaptive $\sqrt{3}$ refinement. The motivation to consider $\sqrt{3}$ refinement stems from the fact that it is a slower topological refinement than the usual red or red-green refinement, and that it alternates the orientation of the refined triangles, such that certain features or singularities that are not aligned with the initial triangulation might be detected more quickly. On the other hand, the use of $\sqrt{3}$ refinement introduces the additional difficulty that the corresponding finite element spaces are nonnested. This makes the setting nonconforming. First we derive a BPX-type preconditioner for piecewise linears on the adaptively refined triangulations and we show that it gives rise to uniformly bounded condition numbers, so that we can solve the linear systems arising from the AFEM in an efficient way. Then we introduce the AFEM of Morin, Nochetto, and Siebert [21] adapted to our special case for solving the Poisson equation. We prove that this adaptive strategy converges to the solution within any prescribed error tolerance in a finite number of steps. Finally we present some numerical experiments that show the optimality of both the BPX preconditioner and the AFEM.

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1 Introduction

The use of adaptive procedures for the numerical solution of partial differential equations has the potential advantage of a significant reduction of the computational cost when compared to non-adaptive methods. Starting from a given triangulation T_j an adaptive finite element method (AFEM) typically consists of loops of the form

$$\text{SOLVE} \quad \rightarrow \quad \text{ESTIMATE} \quad \rightarrow \quad \text{MARK} \quad \rightarrow \quad \text{REFINE}$$

to produce the next triangulation T_{j+1}. This loop invokes the solution of the finite element discretized problem (SOLVE), the a posteriori error estimation of the global error (ESTIMATE) by determining easily computable local error quantities, the marking (MARK) of certain triangles $\tau \in T_j$ that correspond to large local error quantities computed in ESTIMATE, and the refinement (REFINE) of the marked triangles.

In this paper we develop an AFEM by considering adaptively refined conforming triangulations obtained by $\sqrt{3}$-refinement [17, 19]. This type of refinement introduces the difficulty that the corresponding finite element spaces of continuous piecewise linear functions on these triangulations are nonnested. On the other hand, $\sqrt{3}$-refinement is a slower topological refinement than, for instance, the usual dyadic split operation (red-refinement). This implies that we can have more levels of refinement if a prescribed target complexity of the finite element space must not be exceeded. Moreover it reduces the overhead that is created when a triangle slightly fails the stopping criterion for the adaptive refinement, but the result of the refinement falls significantly below the threshold. Furthermore $\sqrt{3}$-refinement alternates the orientation of the refined triangles. This property potentially might have the advantage of reducing the amount of work in an adaptive strategy, since certain features or singularities that are not aligned with the initial triangulation might be detected more quickly.

The development and implementation of optimal solvers for the procedure SOLVE has been the subject of intensive research in the past, see, e.g., [11, 18, 23] and references therein. On the other hand, optimal methods on adaptively refined triangulations have only been established for a sequence of nested conforming finite element spaces. Efficient preconditioners for nonconforming finite element discretizations have been investigated in, e.g., [4, 7, 22, 24, 25], but, as far as we know, optimality could only be established for quasi-uniform triangulations. Recently an optimal BPX-type preconditioner was derived for piecewise linears on a sequence of triangulations obtained by regular $\sqrt{3}$ refinement [20]. We will elaborate on the results from [20] to derive a similar BPX-type preconditioner on a sequence of triangulations obtained by adaptive $\sqrt{3}$ refinement. This optimality result is accomplished through an extension of the framework developed in [11].

Reliable a posteriori error estimators have been developed through the years, and this field of research has reached some level of maturity by now, see, e.g., the monograph [29] and references therein. On the other hand, relying on appropriate error reduction properties, it was only in 1996 when the first AFEM that was proven to converge was constructed by Dörfler [15]. Later work by Morin, Nochetto and Siebert [21] extended the results by Dörfler, and by adding a coarsening step to the method from [21] Binev, Dahmen and DeVore [3] could prove that the resulting AFEM was quasi-optimal in the following sense: if for some $s > 0$ the solution can be approximated to accuracy $\mathcal{O}(n^{-s})$ in the energy norm by continuous piecewise linear functions on a triangulation consisting of n triangles, then the adaptive procedure constructs an approximation of the same type with the same asymptotic accuracy in only $\mathcal{O}(n)$ operations. One of the main ingredients in the convergence analysis from [21] is the Galerkin orthogonality of the sequence of discrete solutions arising from the procedure SOLVE due to the nestedness of the sequence of finite element spaces. This Galerkin orthogonality does not hold anymore when working with nonnested spaces and one has to rely on an appropriate substitute quasi-orthogonality property that reflects the error that is introduced by the nonnestedness of the subsequent finite element spaces (see, e.g., [8], where a rigorous convergence analysis has been carried out for an AFEM discretized by the nonconforming lowest order Crouzeix–Raviart elements). To prove convergence of the AFEM based on adaptive $\sqrt{3}$ refinement we show that the error introduced by the nonnestedness eventually becomes arbitrarily small.

In order to develop the theory we shall restrict ourselves to the most simple elliptic equation: the Poisson problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and $\partial\Omega$ is its boundary. The variational formulation of (1.1) is given by:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{1.2}$$

Here $a(\cdot,\cdot)$ is the symmetric, continuous, and coercive bilinear form induced by (1.1), given by $a(\cdot,\cdot) \equiv (\nabla\cdot, \nabla\cdot)$, and $(\cdot,\cdot)$ denotes the $L_2(\Omega)$ scalar product. We will use the notation

$$\Delta_0 \to \Delta_1 \to \cdots \Delta_j \to \cdots \to \Delta_J \tag{1.3}$$

to denote a sequence of uniformly refined triangulations of Ω and

$$T_0 \to T_1 \to \cdots \to T_j \to \cdots \to T_J \tag{1.4}$$

to denote a sequence of adaptively refined triangulations of Ω. We discretize (1.2) by the space S_j of continuous piecewise linear polynomials with respect to a conforming triangulation Δ_j or T_j of Ω. It will always be clear from the context whether S_j is defined on Δ_j or on T_j.

One can obtain a sequence (1.3) using uniform $\sqrt{3}$ refinement as follows. We start from an initial conforming triangulation Δ_0 of Ω. For every triangle in Δ_0 we compute a new vertex inside this triangle and we connect this new vertex with the three surrounding old vertices of the triangle. Flipping the original edges then yields the new triangulation Δ_1 which, in the uniform setting, is a 30 degree rotated triangulation, see Figure 1. When we apply the $\sqrt{3}$ refinement twice we get a triadic refinement, i.e., each original triangle is split into 9 new triangles. The choice of the new interior vertex is not arbitrary. The implemented $\sqrt{3}$ topology refinement scheme should yield a sequence of triangulations $\{\Delta_j\}$ such that the subsequence $\{\Delta_{2j}\}$ is given by regular triadic refinement of the initial triangulation Δ_0. This is obvious if in the $\Delta_{2j} \to \Delta_{2j+1}$ refinement step the inserted vertices are the triangle barycenters while in the $\Delta_{2j+1} \to \Delta_{2(j+1)}$ refinement step the points $(2P_- + P_+)/3$, $(P_- + 2P_+)/3$ on an edge with vertices P_-, P_+ from Δ_{2j} are inserted. For this to work in agreement with the geometric idea of $\sqrt{3}$ refinement, one has to require that the union of any two triangles with a common edge in Δ_0 is a convex quadrilateral (a sufficient condition for this to hold is that the maximal interior angle of any triangle in Δ_0 does not exceed $\pi/2$). Then, $\{\Delta_j\}$ is automatically regular and semi-uniform, with stepsize parameters $h_j \sim 3^{-j/2}$. The boundary treatment is implemented by extending the triangulation in the following way. To every boundary edge of Δ_{2j}, we will attach a virtual exterior triangle, and then follow the above refinement rules. To obtain Δ_{2j+1}, we only keep the triangles inside Ω and the triangles that intersect the boundary of Ω. To obtain $\Delta_{2(j+1)}$ only triangles inside Ω are kept.

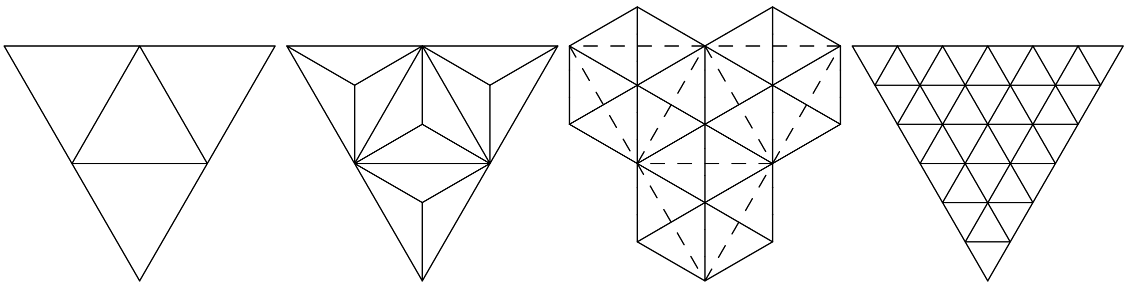


Figure 1: $\sqrt{3}$-refinement of the triangulation. One new vertex per triangle is computed, this vertex is connected with the vertices of the triangle, and the edges between old vertices are flipped. Applying the $\sqrt{3}$-refinement twice yields a triadic refinement.

No special additional rules are needed to obtain a sequence (1.4) with adaptive $\sqrt{3}$ refinement. A triangle is subdivided by inserting a vertex into a triangle and connecting this new vertex to the old vertices of the triangle. If a neighboring triangle is already refined we flip the edge between the two triangles. This process is illustrated in Figure 2. A triangle can only be further refined if it has been generated by an edge flipping operation. Note that all triangles that are generated during the adaptive $\sqrt{3}$-refinement form a proper subset of the uniform refinement hierarchy from Figure 1. The worst triangles are those generated by an 1-to-3 split. Edge flipping then mostly re-improves the shape.

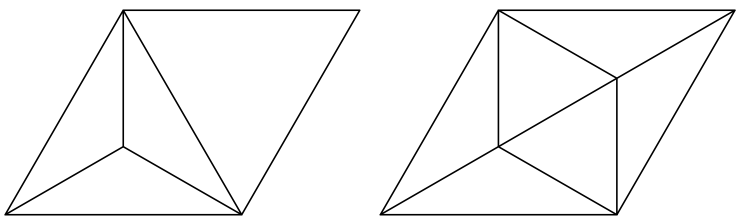


Figure 2: Adaptive $\sqrt{3}$-refinement.

The remaining part of this paper is organized as follows. Section 2 is devoted to preliminaries concerning multilevel preconditioning of elliptic partial differential equations. We derive a general theory concerning multilevel splittings and preconditioning related to nonnested finite element spaces in Section 3. In Sections 4 and 5 we apply this general theory to the specific setting of piecewise linears on triangulations generated by $\sqrt{3}$ refinement. The main result is Theorem 5.3 which gives us an optimal BPX-type preconditioner for problem (1.2). In Section 6 we introduce the AFEM from [21] and we prove that this method also converges when working on triangulations generated by adaptive $\sqrt{3}$ refinement. We conclude the paper with some numerical experiments in Section 7.

2 Preliminaries on multilevel preconditioning

Let p_{2k} be a polynomial of degree $2k$ on $\mathbb{R}^2$. Denote by $p_{2k}(D)$ the differential operator where each variable is replaced by the corresponding partial derivative. We are interested in solving boundary value problems of the form

$$p_{2k}(D)u = f \quad \text{on} \quad \Omega, \qquad Bu = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where B is a suitable trace operator expressing the boundary conditions, and $\Omega \subset \mathbb{R}^2$ has a local Lipschitz continuous boundary $\partial\Omega$. The weak formulation of (2.1) is given by

$$a(u, v) = (f, v) \quad \text{for all } v \in H^k_B(\Omega), \tag{2.2}$$

where $a(\cdot, \cdot)$ is the bilinear form induced by (2.1), and $H^k_B(\Omega)$ is a suitable subspace of the Sobolev space $H^k(\Omega)$ depending on the boundary conditions in terms of the operator B. Here we adopt the standard convention of writing $H^k(\Omega) = W^k_2(\Omega)$ where the Sobolev space $W^k_p(\Omega)$ is defined by the norm $\|\cdot\|^p_{W^k_p(\Omega)} := \|\cdot\|^p_{L_p(\Omega)} + \sum_{|\alpha|=k} \|D^\alpha \cdot\|^p_{L_p(\Omega)}$ (see, e.g., [1]), and p may range between 1 and infinity with the usual modification for $p = \infty$.

We assume that the partial differential operator is elliptic, which can be expressed by the equivalence

$$a(v, v) \sim \|v\|^2_{H^k(\Omega)}, \quad v \in H^k_B(\Omega). \tag{2.3}$$

We always mean by $a \sim b$ that $a \lesssim b$ and $a \gtrsim b$ hold, where $a \lesssim b$ means that a can be bounded by a constant multiple of b uniformly in any parameters on which a, b may depend, and $a \gtrsim b$ means $b \lesssim a$. We will also refer to $\|\cdot\|^2_{H^k(\Omega)}$ as the energy norm induced by the bilinear form $a(\cdot, \cdot)$. The ellipticity condition (2.3) implies, by the Theorem of Lax–Milgram (see, e.g., [30]), the existence of a unique solution u to the problem (2.2) for all $f \in \big(H^k_B(\Omega)\big)'$. Here the prime denotes that $\big(H^k_B(\Omega)\big)'$ is the dual function space of $H^k_B(\Omega)$.

Denote by V a (finite or infinite dimensional) subspace of $H^k_B(\Omega)$, and let $\mathcal{A}$ denote the positive definite self-adjoint operator on V that is uniquely defined by

$$a(u, v) = (\mathcal{A}u, v)_{V' \times V}, \quad v \in V. \tag{2.4}$$

The operator $\mathcal{A}$ maps the function space V into the topological dual function space V', and $(\cdot, \cdot)_{V' \times V}$ is the dual form. We have to solve the linear operator equation

$$\mathcal{A}u = b \tag{2.5}$$

for some $u \in V$, where $b \in V'$ is defined by $(b, v)_{V' \times V} = (f, v)_{V' \times V}$, $v \in V$. The conjugate gradient method is a very efficient solver for large linear systems arising from problems such as (2.5). However, because of stability reasons, it is necessary that these systems have been suitably preconditioned. The extreme eigenvalues of some linear self-adjoint positive definite operator $\mathcal{Q}$ can be characterized by

$$\begin{aligned} \lambda_{\min}(\mathcal{Q}) &= \min_{v \in V, v \neq 0} \frac{(\mathcal{Q}v, v)_{V' \times V}}{(v, v)_{V' \times V}}, \\ \lambda_{\max}(\mathcal{Q}) &= \max_{v \in V, v \neq 0} \frac{(\mathcal{Q}v, v)_{V' \times V}}{(v, v)_{V' \times V}}, \end{aligned}$$

and the spectral condition number of $\mathcal{Q}$ is given by

$$\kappa(\mathcal{Q}) = \frac{\lambda_{\max}(\mathcal{Q})}{\lambda_{\min}(\mathcal{Q})}. \tag{2.6}$$

It is known (see, e.g., [11, 18]) that if for some constants $0 < \gamma, \Gamma < \infty$ and some positive definite self-adjoint linear invertible operator $\mathcal{C}$

$$\gamma(\mathcal{C}^{-1}u, u)_{V' \times V} \leq a(u, u) \leq \Gamma(\mathcal{C}^{-1}u, u)_{V' \times V} \text{ for all } u \in V, \tag{2.7}$$

then the spectral condition number $\kappa(\mathcal{C}^{1/2}\mathcal{A}\mathcal{C}^{1/2})$ is bounded by Γ/γ.

Let us represent the operator $\mathcal{A}$ by the stiffness matrix

$$A_\Phi := (a(\varphi_i, \varphi_{i'}))_{i,i' \in I}$$

with respect to some typical nodal basis $\Phi := \{\varphi_i \mid i \in I\}$ of V. Then it is known that the spectral condition number $\kappa(A_\Phi)$ of A_Φ grows at least like $(\#I)$, see, e.g., [28, Chap. 5]. In order to precondition the system

$$A_\Phi c = b_\Phi, \quad (b_\Phi)_i := (f, \varphi_i), \ i \in I, \tag{2.8}$$

one can perform a change of basis. Let $\Psi := \{\psi_i \mid i \in I\}$ be another basis of V, and L be the transfer matrix between the two bases. Then we can transform the stiffness matrix with respect to the nodal basis into the stiffness matrix with respect to Ψ by

$$A_\Psi = L^T A_\Phi L.$$

Now, suppose that the equivalence

$$\gamma \sum_{i \in I} |d_i|^2 \leq \left\| \sum_{i \in I} d_i \psi_i \right\|^2_{H^k(\Omega)} \leq \Gamma \sum_{i \in I} |d_i|^2 \tag{2.9}$$

holds, then by (2.3) and (2.7) we find that

$$\kappa(A_\Psi) \lesssim \frac{\Gamma}{\gamma}.$$

Suppose that the quotient Γ/γ is small compared to the bound $(\#I)$, then the matrix $C := LL^T$ is a candidate for a preconditioner. Ideally γ and Γ are constants, such that $\kappa(A_\Psi) = \mathcal{O}(1)$.

3 A general estimate for multilevel splittings related to nonnested spaces

We are particularly interested in well-conditioned bases when the space V from (2.4) belongs to a sequence of nonnested finite element subspaces

$$V_0 \to V_1 \to V_2 \to \cdots$$

with respect to an increasing sequence of triangulations (1.4) obtained by some adaptive or non-adaptive refinement procedure. We assume that the spaces V_j are comprised of piecewise polynomials of degree $d > k - 1$, and that they allow for local reproduction of polynomials of total degree $\leq d$. Since the V_j are not necessarily subspaces of $H^k(\Omega)$ we have to be careful and we define for $u_j \in V_j$

$$\|u_j\|^2_{H^k(\Omega)} := \|u_j\|^2_{L_2(\Omega)} + \sum_{\tau \in T_j} \|u_j\|^2_{H^k(\tau)}.$$

In order to construct a multilevel splitting for the finest space V_J one needs a suitable operator to project functions from V_j onto V_J. Therefore we introduce linear prolongation operators

$$P_j \ : \ V_{j-1} \to V_j, \quad j = 1, \ldots, J, \quad P_0 := 0, \tag{3.1}$$

and their iterates

$$\widetilde{P}_j := P_J P_{J-1} \cdots P_{j+1} \ : \ V_j \to V_J, \quad j = 0, \ldots, J-1, \qquad \widetilde{P}_J := \mathrm{Id}, \quad \widetilde{P}_{-1} := 0. \tag{3.2}$$

We also need some kind of restriction operator. We define

$$Q_j : L_2(\Omega) \to V_j, \quad j = 0, \ldots, J, \quad Q_{-1} := 0, \tag{3.3}$$

as the L_2-orthogonal projection onto V_j, i.e., $(Q_j f, u_j) = (f, u_j)$ for all $u_j \in V_j$, with $f \in L_2(\Omega)$.

To carry out our analysis we introduce the difference operator

$$(\Delta_h^r f)(x) := \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} (-1)^{r-j} f(x + jh), \qquad x \in \mathbb{R}^2, \tag{3.4}$$

and define the r-th order L_p-modulus of smoothness of $f \in L_p(\Omega)$ (see, e.g., [12])

$$\omega_r(f, t, \Omega)_p := \sup_{|h| \le t} \|\Delta_h^r f\|_{L_p(\Omega(rh))}, \tag{3.5}$$

where $|h|$ is the Euclidean length of vector h and $\Omega(rh) := \{x \in \Omega : x + jh \in \Omega, \ j = 0, \ldots, r\}$. If $s, p, q > 0$, we say that f is in the Besov space $B_q^s(L_p(\Omega))$ whenever the following is finite:

$$|f|_{B_q^s(L_p(\Omega))} \sim \begin{cases} \left(\sum_{j=0}^{\infty} \left[\rho^{js} \omega_r(f, \rho^{-j}, \Omega)_p\right]^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j \ge 0} \rho^{js} \omega_r(f, \rho^{-j}, \Omega)_p \ , & q = \infty. \end{cases} \tag{3.6}$$

where the choice of $r > s$ is arbitrary and $\rho > 1$. See [12] for more details concerning Besov spaces. It is well known that on a domain Ω with Lipschitz boundary the equivalences

$$W_2^s(\Omega) \cong B_2^s(L_2(\Omega)) \quad \text{and} \quad W_p^k(\Omega) \cong B_p^k(L_p(\Omega)), \quad k \in \mathbb{N}, \tag{3.7}$$

hold [14].

Similar to the work of Dahmen and Kunoth [11] we will formulate below in Theorem 3.1 a general estimate of condition numbers in terms of the following quantity

$$\nu_J := \max\{1, \ \nu_{J,j} \mid j = 0, \ldots, J\} \tag{3.8}$$

with

$$\nu_{J,j} := \sup_{g \in V_J} \frac{\|(Q_j - P_j Q_{j-1}) g\|_{L_2(\Omega)}}{\omega_{k+1}(g, \rho^{-j})_2}. \tag{3.9}$$

As opposed to the work in [11] we are working here with a sequence of nonnested spaces which introduces additional difficulties. The aim of Theorem 3.1 is to provide a flexible framework of conditions that yield good estimates for condition numbers of various types of nonconforming finite elements. When all conditions of Theorem 3.1 are satisfied it remains to estimate the quantity ν_J, e.g., for a sequence of uniformly refined triangulations (1.3) ν_J can be bounded by Jackson-type estimates (see, e.g., [20]).

Let $\mathcal{A}_J$ denote the operator defined by (2.4) for $V = V_J$ and let $\mathcal{C}_J^{-1}$ be the self-adjoint positive definite operator on V_J defined by

$$(\mathcal{C}_J^{-1} u_J, v_J) = \sum_{j=0}^{J} \rho^{2jk} \left((\widetilde{Q}_j - \widetilde{Q}_{j-1}) u_J, (\widetilde{Q}_j - \widetilde{Q}_{j-1}) v_J\right), \quad \forall v_J \in V_J, \tag{3.10}$$

where $\widetilde{Q}_j$ are the L_2-orthogonal projectors onto the spaces $\widetilde{V}_j := \mathrm{span}\{\widetilde{P}_j v_j \mid v_j \in V_j\}$, i.e.,

$$(\widetilde{Q}_j f, \widetilde{v}_j) = (f, \widetilde{v}_j), \qquad \forall \widetilde{v}_j \in \widetilde{V}_j, \quad f \in L_2(\Omega).$$

Note that

$$\widetilde{V}_0 \subset \widetilde{V}_1 \subset \widetilde{V}_2 \subset \cdots \subset V_J. \tag{3.11}$$

We now formulate the following general estimate.

Theorem 3.1. *Suppose that the iterated prolongation operators $\widetilde{P}_j : V_j \to V_J$ satisfy*

$$\left\| \widetilde{P}_j v_j \right\|_{L_2} \sim \|v_j\|_{L_2} \text{ for all } v_j \in V_j, \tag{3.12}$$

and suppose that the Bernstein estimates

$$\left\| \widetilde{P}_j v_j \right\|_{H^{k\pm\epsilon}(\Omega)} \lesssim \rho^{(k\pm\epsilon)j} \|v_j\|_{L_2(\Omega)} \tag{3.13}$$

hold for some $\epsilon > 0$. Then one has for $\mathcal{C}_J$ defined by (3.10)

$$\kappa(\mathcal{C}_J^{1/2} \mathcal{A}_J \mathcal{C}_J^{1/2}) = \mathcal{O}((\nu_J)^2). \tag{3.14}$$

We will postpone the proof of Theorem 3.1 to the end of this section. The following theorem is a direct consequence of Theorem 3.1 and gives us a simple criterion to obtain uniformly bounded condition numbers.

Theorem 3.2. *Suppose that the hypotheses of Theorem 3.1 are satisfied. Moreover, assume that the prolongation operators $P_j : V_{j-1} \to V_j$ satisfy the Jackson-type estimate*

$$\|Q_j g - P_j Q_{j-1} g\|_{L_2} \lesssim \rho^{-mj} |g|_{H^m(\Omega)} \tag{3.15}$$

for arbitrary $g \in H^m(\Omega)$ and for some integer $m > k$. Then

$$\kappa(\mathcal{C}_J^{1/2} \mathcal{A}_J \mathcal{C}_J^{1/2}) = \mathcal{O}(1).$$

Proof. From the triangle inequality, the L_2-boundedness of the operators Q_j and P_j, and the Jackson estimate (3.15), we get

$$\begin{aligned}
\|(Q_j - P_j Q_{j-1})g\|_{L_2(\Omega)} &\leq \|Q_j g - Q_j h\|_{L_2(\Omega)} + \|Q_j h - P_j Q_{j-1} g\|_{L_2(\Omega)} \\
&\lesssim \|g - h\|_{L_2(\Omega)} + \|Q_j h - P_j Q_{j-1} h\|_{L_2(\Omega)} + \|P_j Q_{j-1} h - P_j Q_{j-1} g\|_{L_2(\Omega)} \\
&\lesssim \|g - h\|_{L_2(\Omega)} + \rho^{-mj} |h|_{H^m(\Omega)}
\end{aligned}$$

with $h \in H^m(\Omega)$ arbitrary. The characterization of moduli of smoothness by K-functionals (see, e.g., [10] for the result and some references) yields

$$\|(Q_j - P_j Q_{j-1})g\|_{L_2(\Omega)} \lesssim \inf_{h \in H^m(\Omega)} \left(\|g - h\|_{L_2(\Omega)} + \rho^{-mj} |h|_{H^m} \right) \lesssim \omega_m(g, \rho^{-j}, \Omega)_2.$$

The definition of ν_J, and the fact that $\omega_m(\cdot, \rho^{-j}, \Omega)_p \leq c\omega_{k+1}(\cdot, \rho^{-j}, \Omega)_p$ holds for some constant c depending on $m > k$, proves our claim. ☐

The proof of Theorem 3.1 makes use of techniques from the theory of function spaces (cf., e.g., [13, 23]). The idea is to introduce a discrete norm associated with the increasing sequence of approximating subspaces (3.11). In view of (2.9) the objective is to relate this discrete norm to the Sobolev norm $\|\cdot\|_{H^k(\Omega)}$.

Theorem 3.3. *Suppose that the Bernstein estimates (3.13) hold for some $\epsilon > 0$. Then the discrete norm*

$$|||u_J|||_{k,J} := \inf_{v_j \in V_j:\ u_J = \sum_{j=0}^{J} \widetilde{P}_j v_j} \left(\sum_{j=0}^{J} \left(\rho^{kj} \left\| v_j \right\|_{L_2} \right)^2 \right)^{1/2} \tag{3.16}$$

associated with the sequence (3.11) satisfies the two-sided inequality

$$\frac{\gamma}{\nu_J} |||u_J|||_{k,J} \leq \|u_J\|_{H^k(\Omega))} \leq \Gamma |||u_J|||_{k,J},$$

where the constants $0 \leq \gamma, \Gamma \leq \infty$ are independent of $u_J \in V_J$, $J \in \mathbb{N}_0$.

Proof. By definition of $\widetilde{P}_j$ and Q_j we have that

$$u_J = \sum_{j=0}^{J} \left(\widetilde{P}_j Q_j - \widetilde{P}_{j-1} Q_{j-1} \right) u_J = \sum_{j=0}^{J} \widetilde{P}_j \left(Q_j - P_j Q_{j-1} \right) u_J.$$

From this decomposition we infer

$$\begin{aligned} |||u_J|||_{k,J}^2 &\lesssim \sum_{j=0}^{J} \left(\rho^{kj} \left\| \left(Q_j - P_j Q_{j-1} \right) u_J \right\|_{L_2} \right)^2 \\ &\lesssim \sum_{j=0}^{J} \left(\rho^{kj} \omega_{k+1}(u_J, \rho^{-j})_2 \ \nu_J \right)^2, \end{aligned}$$

which implies

$$\frac{1}{\nu_J} |||u_J|||_{k,J} \lesssim \|u_J\|_{H^k(\Omega))}.$$

To prove the other bound we let $\sum_{j=0}^{J} \widetilde{P}_j v_j$ be an arbitrary decomposition of u_J. We use the strengthened Cauchy–Schwarz inequalities to derive that

$$\begin{aligned} \|u_J\|_{H^k(\Omega)}^2 &= \left\| \sum_{j=0}^{J} \widetilde{P}_j v_j \right\|_{H^k(\Omega)}^2 \\ &= \left(\sum_{j=0}^{J} \widetilde{P}_j v_j, \sum_{\ell=0}^{J} \widetilde{P}_\ell v_\ell \right)_{H^k(\Omega)} \\ &\lesssim \sum_{j=0}^{J} \sum_{\ell=j}^{J} \left\| \widetilde{P}_j v_j \right\|_{H^{k+\epsilon}(\Omega)} \left\| \widetilde{P}_\ell v_\ell \right\|_{H^{k-\epsilon}(\Omega)} \end{aligned}$$

The Bernstein estimates (3.13) yield

$$\begin{aligned} \|u_J\|_{H^k(\Omega)}^2 &\lesssim \sum_{j=0}^{J} \sum_{\ell=j}^{J} \rho^{\epsilon j} \rho^{-\epsilon \ell} \left(\rho^{kj} \left\| v_j \right\|_{L_2} \right) \left(\rho^{k\ell} \left\| v_\ell \right\|_{L_2} \right) \\ &\lesssim \sum_{j=0}^{J} \rho^{2kj} \left\| v_j \right\|_{L_2}^2. \end{aligned}$$

Since the decomposition $\sum_{j=0}^{J} \widetilde{P}_j v_j$ was arbitrary we have proven the theorem. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. First we note that

$$(\mathcal{C}_J^{-1} u_J, u_J) = \inf_{\widetilde{v}_j \in \widetilde{V}_j : u_J = \sum_{j=0}^{J} \widetilde{v}_j} \sum_{j=0}^{J} \rho^{2kj} \, \|\widetilde{v}_j\|^2_{L_2(\Omega)} \tag{3.17}$$

where the operator $\mathcal{C}_J$ is defined by (3.10). If we combine Theorem 3.3 with (3.12) and (2.3), we have proven that

$$\frac{1}{(\nu_J)^2} (\mathcal{C}_J^{-1} u_J, u_J) \lesssim a(u_J, u_J) \lesssim (\mathcal{C}_J^{-1} u_J, u_J), \tag{3.18}$$

and the theorem follows from (2.7). □

4 Piecewise linears on $\sqrt{3}$ refined triangulations

In this section we recall some results that were established in [20]. Suppose that we are given a sequence (1.3) of conforming triangulations where Δ_j is obtained from Δ_{j-1} by applying uniform $\sqrt{3}$ refinement. We recall that S_j is defined as the space of continuous piecewise linear polynomials on Δ_j. Any spline function in S_j is completely determined by its function values at the vertices of Δ_j. Because Δ_j is obtained from Δ_{j-1} by $\sqrt{3}$ refinement, the subsequent spaces are nonnested, i.e., $S_{j-1} \not\subset S_j$. Therefore we will first introduce suitable prolongation operators that project a function from S_{j-1} onto S_j.

The prolongation operator that we describe here averages the vertex values of the triangle containing a newly inserted point. We will denote this prolongation operator by $I_j : S_{j-1} \to S_j$, and its iterates by $\widetilde{I}_j := I_J I_{J-1} \cdots I_{j+1} : S_j \to S_J$, similar to (3.1) and (3.2). Given a function $s_{j-1} \in S_{j-1}$, we associate with each vertex in Δ_{j-1} the corresponding function value of s_{j-1}. The prolongation operator I_j assigns to each new vertex in the triangulation Δ_j a function value which is the average of the three function values associated with the three surrounding vertices in the coarser triangulation Δ_{j-1}. The function values of the old vertices from Δ_{j-1} are copied to the corresponding vertices in Δ_j. Figure 3 depicts the prolongation process schematically and shows some pictures of the basis functions $\widetilde{I}_j \varphi_{j,P}$, where $\varphi_{j,P}$ is the hat function in S_j centered around vertex P.

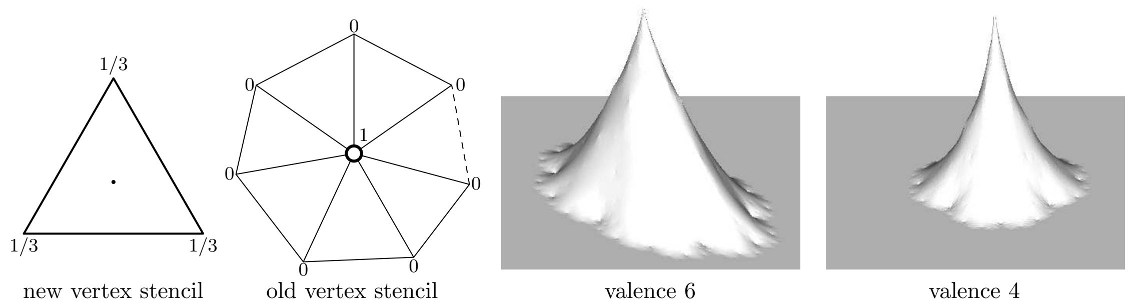
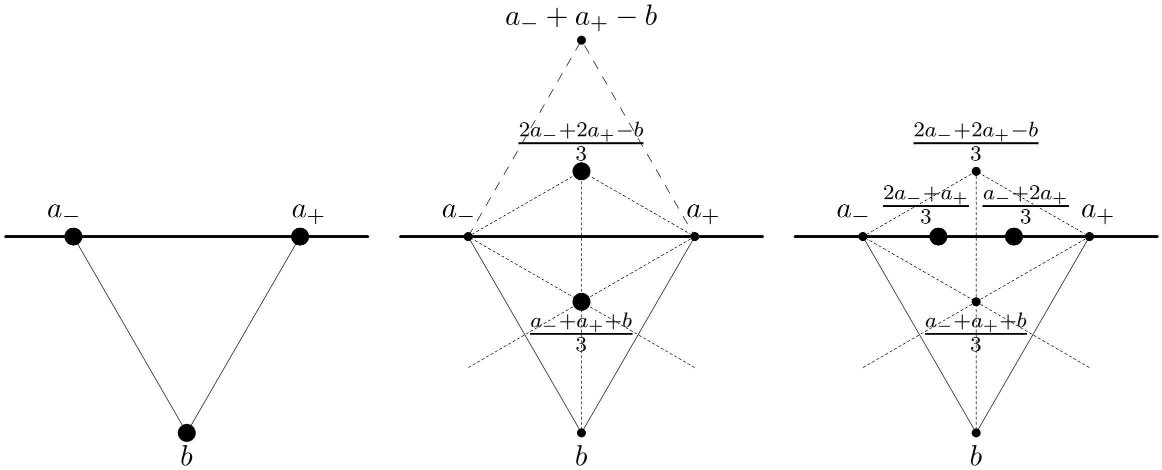


Figure 3: The values at the newly introduced vertices are computed by averaging the values at the old vertices of the triangle (left). The values at the old vertices are inherited, hence the scheme is interpolating (second from left). Examples of basis functions centered around vertices of valence 6 respectively 4 are shown on the right.

To get consistency at the boundary, for a boundary edge carrying homogeneous Dirichlet boundary conditions, we use odd extension to obtain the missing vertex value for the virtual triangles, which leads automatically to zero boundary values. More generally, we use extension by preserving linear functions for each of the virtual triangles, which is the same as odd extension used for homogeneous Dirichlet conditions but is also appropriate for all other situations. To be precise, if a_-, a_+ are the values at the vertices of the boundary edge, and b the value at the remaining vertex of the boundary triangle, then at the vertex of the virtual exterior triangle we take the value $a_- + a_+ - b$, see Figure 4. Figure 5 shows a basis function near the boundary that satisfies homogeneous Dirichlet boundary conditions.



(a) (b) (c)

Figure 4: (a) Function values near the boundary. (b) Function values after one step of $\sqrt{3}$ refinement. We introduce a virtual triangle to compute the function value outside the domain. (c) The function values that are computed on the boundary after the second step of $\sqrt{3}$ refinement preserve linear functions on the boundary.

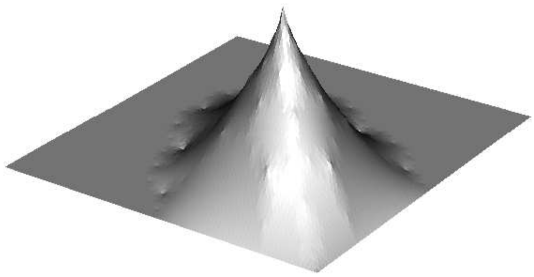


Figure 5: Basis function $\widetilde{I}_j\varphi_{j,P}$ that satisfies homogeneous Dirichlet boundary conditions.

Theorem 4.1 and Corollary 4.2 are the main results of [20].

Theorem 4.1 ([20]). *Let Δ_0 be an arbitrary conforming triangulation of the domain Ω and suppose that the sequence* (1.3) *is obtained by regular $\sqrt{3}$-refinement. Let S_j be the space of continuous piecewise linear polynomials on Δ_j, let $I_j : S_{j-1} \to S_j$ be the prolongation operator defined by Figure 3, and let $\widetilde{I}_j : S_j \to S_J$ be its iterate. Then*

i) for any $s_j \in S_j$

$$\left\|\widetilde{I}_j s_j\right\|_{L_2(\Omega)} \sim \|s_j\|_{L_2(\Omega)}$$

ii) the Jackson estimate

$$\|I_j Q_{j-1} g - Q_j g\|_{L_2(\Omega)} \lesssim \sqrt{3}^{-2j} |g|_{H^2(\Omega)}$$

holds for arbitrary $g \in H^2(\Omega)$.

iii) the Bernstein inequality

$$\left\| \widetilde{I}_j v_j \right\|_{H^s(\Omega)} \lesssim \sqrt{3}^{js} \|v_j\|_{L_2(\Omega)}, \qquad v_j \in S_j, \tag{4.1}$$

holds in the range $0 < s < s_0$ *with* $s_0 := 1 + \log_3(9/5)/2 > 1$.

Corollary 4.2. *The BPX preconditioner given by*

$$\mathcal{B}_J := \sum_{j=0}^{J} \sum_{P \in \Delta_j \cap \operatorname{int} \Omega} (\,\cdot\,, \widetilde{I}_j \varphi_{j,P}) \widetilde{I}_j \varphi_{j,P}$$

yields a uniformly bounded condition number for the variational formulation (1.2) *of the Poisson problem* (1.1).

Proof. From Theorems 3.1, 3.2, and 4.1, we find that

$$\kappa(\mathcal{C}_J^{1/2} \mathcal{A}_J \mathcal{C}_J^{1/2}) = \mathcal{O}(1),$$

with $\mathcal{A}_J$ the operator on S_J defined by (2.4) for the Poisson problem, and $\mathcal{C}_J$ the operator defined by (3.10) for $V_j = S_j$ and $P_j = I_j$. Standard techniques (see, e.g., [18, 20]) show that the operator $\mathcal{C}_J$ is spectrally equivalent to $\mathcal{B}_J$ so that

$$\kappa(\mathcal{B}_J^{1/2} \mathcal{A}_J \mathcal{B}_J^{1/2}) = \mathcal{O}(1)$$

holds. ☐

The above results remain valid when $J \to \infty$. In fact the limit function $\lim_{J\to\infty} \widetilde{I}_j v_j$ exists in $H^s(\Omega)$ for all $0 < s < s_0$. The computation of $\lim_{J\to\infty} \widetilde{I}_j v_j$ generates the whole sequence

$$\{\ I_{j+1} v_j\ ,\ I_{j+2} I_{j+1} v_j\ ,\ \ldots\ ,\ I_k I_{k-1} \cdots I_{j+1} v_j\ ,\ \ldots\ \}. \tag{4.2}$$

The following lemma shows that this sequence is Cauchy with respect to the norm in $H^s(\Omega)$. We will need it in the next section to prove Bernstein estimates (3.13) for the spaces S_j when they are defined on adaptively refined triangulations (1.4) instead of uniformly refined triangulations (1.3). The proof of this lemma is based on the proof of Theorem 4.1 iii).

Lemma 4.3. *Let* Δ_0 *be an arbitrary triangulation of the polygonal domain* Ω *and suppose that* (1.3) *is a sequence of triangulations where* Δ_j *is obtained from* Δ_{j-1} *by applying uniform* $\sqrt{3}$ *refinement. Let* S_j *be the space of continuous piecewise linear polynomials on* Δ_j, *let* $I_j : S_{j-1} \to S_j$ *be the prolongation operator defined by Figure 3, and let* $\widetilde{I}_j : S_j \to S_J$ *be its iterate. Furthermore we define* $I_j^k : S_j \to S_k$ *by* $I_j^k := I_k I_{k-1} \cdots I_{j+1}$. *Then for arbitrary* $u_j \in S_j$ *and for arbitrary* $\epsilon > 0$ *there exists an* $N > 0$ *such that*

$$\left\| \widetilde{I}_j u_j - I_j^k u_j \right\|_{H^s(\Omega)} < \epsilon \quad \text{as} \quad J \geq k > N$$

for all s *in the range* $0 < s < s_0$ *with* $s_0 := 1 + \log_3(9/5)/2 > 1$.

Proof. Suppose that $J = 2M$. We rely on the norm equivalence

$$\|u_{2M}\|_{H^s} \sim \inf_{w_{2j} \in S_{2j} : u_{2M} = \sum_{j=0}^{M} w_{2j}} \left(\sum_{j=0}^{M} 3^{2js} \|w_{2j}\|_{L_2}^2 \right)^{1/2}, \tag{4.3}$$

which holds for all $u_{2M} \in S_{2M}$, $0 < s < 3/2$. The proof of (4.3) is standard (see, e.g., [23]) and is based on the nestedness of the sequence $S_0 \subset S_2 \subset S_4 \subset \cdots$, which follows by construction. A simple consequence is that if we denote $v_{2j} = I_{2m}^{2j} u_{2m}$ and $w_{2j} = v_{2j} - v_{2j-2}$, then

$$\left\| \widetilde{I}_{2m} u_{2m} - I_{2m}^{2k} u_{2m} \right\|_{H^s}^2 \lesssim \sum_{j=k+1}^{M} 3^{2js} \left\| w_{2j} \right\|_{L_2}^2 . \tag{4.4}$$

The main observation is that the L_2-norms of w_{2j} decay at a geometric rate

$$\| w_{2j} \|_{L_2}^2 \lesssim 3^{-2(j-m)s_0} \| u_{2m} \|_{L_2}^2 , \qquad j > m, \tag{4.5}$$

where $3/2 > s_0 := 2 - (\log_3 5)/2 = 1 + \log_3(9/5)/2 > 1$. The proof of (4.5) is technical but elementary and can be found in [20].

Substitution of (4.5) into (4.4) yields

$$\begin{aligned} \left\| \widetilde{I}_{2m} u_{2m} - I_{2m}^{2k} u_{2m} \right\|_{H^s}^2 &\lesssim 3^{2ms_0} \| u_{2m} \|_{L_2(\Omega)}^2 \sum_{j=k+1}^{M} 3^{-2j(s_0 - s)} \\ &\lesssim 3^{2ms_0} \| u_{2m} \|_{L_2(\Omega)}^2 3^{-2k(s_0 - s)} \end{aligned}$$

which goes to zero as k increases for all $0 < s < s_0$.

The case of odd m reduces to the previous case by using the L_2-boundedness of the prolongations: $\|v_{2m}\|_{L_2} = \|I_{2m} v_{2m-1}\|_{L_2} \sim \|v_{2m-1}\|_{L_2}$. For odd $J = 2M + 1$, one can use the H^s-boundedness $(1 < s < 3/2)$ of the finite element interpolation projector $\mathcal{I}_{2M+1}$ mapping onto S_{2M+1} together with the obvious fact that $\mathcal{I}_{2M+1} v_{2(M+1)} = \mathcal{I}_{2M+1} I_{2(M+1)} v_{2M+1} = v_{2M+1}$. This proves the lemma in full generality. □

5 Adaptive $\sqrt{3}$ refinement for piecewise linears

In this section we establish an optimal BPX-type preconditioner for piecewise linears on triangulations obtained from adaptive $\sqrt{3}$ refinement. When one uses the standard dyadic split of triangles in an adaptive way, problems occur where different refinement levels of the triangulation meet. The non-conformity of the triangulation can be fixed with the so-called red-green refinement [29, 11]. There are several reasons why $\sqrt{3}$-subdivision seems better suited for adaptive refinement than red-green refinement. The slower topological refinement reduces the overhead that is created when a coarse triangle slightly fails the stopping criterion for the adaptive refinement but the result of the refinement falls significantly below the threshold. Furthermore the localization, i.e., the extend to which the side-effects of a local refinement step spread out over the triangulation, is better than for dyadic refinement and no temporary triangles (green refinement) are necessary to preserve the conformity of the triangulation. Furthermore all triangles that are generated during the adaptive $\sqrt{3}$ refinement form a proper subset of the uniform refinement hierarchy. This property implies that the shape of the triangles does never degenerate. We refer to [17] for a detailed discussion on adaptive $\sqrt{3}$ refinement.

To implement the adaptive refinement one can use a simple recursive procedure ([17]). First we assign a generation index to each triangle in the triangulation. Initially we set $T_0 = \Delta_0$ and all triangles in T_0 get a generation index equal to zero. If a triangle with even generation index is split into three by inserting a new vertex at its center, the generation index increases by 1 (giving an odd index to the new triangles). Splitting a triangle with odd generation index requires to find its neighbor, perform an edge flip, and assign even indices to the resulting triangles. For an already adaptively refined triangulation T_j, further splits are performed by the following recursive procedure:

```
split(τ)
  if (τ.index is even) then
    compute midpoint P
    split τ(A,B,C) into τ[1](P,A,B), τ[2](P,B,C), τ[3](P,C,A)
    for i = 1,2,3 do
      τ[i].index = τ.index + 1
      if (τ[i].neighbor[1].index == τ[i].index) then
        swap(τ[i], τ[i].neighbour[1])
  else
    if (τ.neighbor[1].index == τ.index - 2)
      split(τ.neighbor[1])
  split(τ.neighbor[1])
```

By this approach one automatically preserves the conformity of the triangulation and a mild balancing of the mesh is achieved: it is not possible that triangles with a generation index difference greater than one are neighbors. The ordering of the vertices in the 1-to-3 split is chosen such that `neighbor[1]` always points to the correct neighboring triangle (outside the parent triangle τ). The edge flipping procedure is implemented as

```
swap(τ1, τ2)
  change τ1(A,B,C), τ2(B,A,D) into τ1(C,A,D), τ2(D,B,C)
  τ1.index++
  τ2.index++
```

Notice that edge flipping at the boundary is not possible since the opposite triangle-neighbor is missing. The application of a second $\sqrt{3}$-step has the overall effect of a triadic split where each original triangle is replaced by 9 new ones. Therefore, every second $\sqrt{3}$-step we split each boundary edge into three new edges and connect the new vertices to the corresponding interior ones such that the new boundary triangles form a proper subset of the uniform 1-to-9 split of the parent boundary triangle. Alternatively one can use the idea of the virtual triangles that we introduced in the introduction and in the previous section. Both approaches essentially give the same result.

To prove optimality of a BPX-preconditioner we first need to satisfy the conditions (3.12) and (3.13) of Theorem 3.1. Let $L(\tau) := \min\{j \mid \tau \in T_j\}$ denote the level of τ. We will assume that the following refinement condition holds:

(R) Only triangles $\tau \in T_j$ with $L(\tau) = j$ are marked for refinement in order to construct T_{j+1}.

Of course, because of the built-in balancing, triangles τ with $L(\tau) < j$ might be refined as well by the recursive refinement procedure. Let X_j denote the set of vertices in T_j. Then we define S_j as the space spanned by the piecewise linear 'hat' functions defined by

$$\varphi_{j,P}(Q) = \sqrt{3}^{L_{j,P}} \delta_{PQ}, \quad P, Q \in X_j, \tag{5.1}$$

where $L_{j,P} := \min\{L(\tau) \mid \tau \in T_j, P \in \tau\}$. Obviously

$$\|\varphi_{j,P}\|_{L_2(\Omega)} \sim 1, \quad P \in X_j, \ j > 0. \tag{5.2}$$

Following standard techniques one easily derives that

$$\left\| \sum_P c_P \varphi_{j,P} \right\|^2_{L_2(\Omega)} \sim \sum_P |c_P|^2. \tag{5.3}$$

The following two theorems establish conditions (3.12) and (3.13) of Theorem 3.1.

Theorem 5.1. *Let $s_j \in S_j$. Then*

$$\left\| \widetilde{I}_j s_j \right\|_{L_2(\Omega)} \sim \|s_j\|_{L_2(\Omega)},$$

where the constants of equivalence are independent of j and J.

Proof. First we prove that $\|\widetilde{I}_j \varphi_{j,P}\|_{L_2(\Omega)} \sim 1$. Because $\widetilde{I}_j \varphi_{j,P} \in S_J$ we get from (5.3)

$$\left\| \widetilde{I}_j \varphi_{j,P} \right\|_{L_2(\Omega)}^2 \sim \sum_{Q \in T_J} |\sqrt{3}^{-L_{J,Q}} \widetilde{I}_j \varphi_{j,P}(Q)|^2$$

and because $\widetilde{I}_j \varphi_{j,P}(Q) \leq \varphi_{j,P}(Q) \leq \sqrt{3}^{L_{j,P}}$ we get

$$\left\| \widetilde{I}_j \varphi_{j,P} \right\|_{L_2(\Omega)}^2 \lesssim 3^{L_{j,P}} \sum_{Q \in T_J \cap \operatorname{supp} \widetilde{I}_j \varphi_{j,P}} 3^{-L_{J,Q}} \lesssim 3^{L_{j,P}} \|1\|_{L_2(\operatorname{supp} \varphi_{j,P})}^2 \lesssim 1.$$

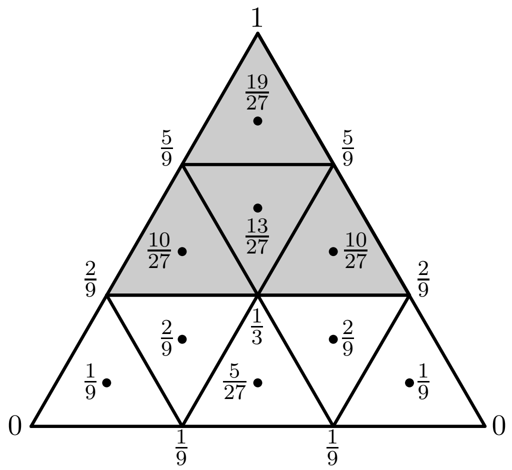


Figure 6: Scheme of $I_3 I_2 I_1 \varphi_{0,P}$.

On the other hand, let τ be such that $P \in \tau$ and $L(\tau) = L_{j,P}$. Obviously $\|\widetilde{I}_j \varphi_{j,P}\|_{L_2(\Omega)} \geq \|\widetilde{I}_j \varphi_{j,P}\|_{L_2(\tau)}$. Let us consider the action of the operator $I_{j+3} I_{j+2} I_{j+1}$ on the function $\varphi_{j,P}$ restricted to τ. Only a bounded number of different cases are possible. Figure 6 depicts the most common case which corresponds to uniform refinement of τ. In this case we easily derive that $\widetilde{I}_j \varphi_{j,P}(Q) \geq \frac{2}{9} \sqrt{3}^{L_{j,P}}$ for all $Q \in T_J$ that lie in the grey region R indicated on Figure 6. For all other cases one trivially derives the similar bound $\widetilde{I}_j \varphi_{j,P}(Q) \gtrsim \sqrt{3}^{L_{j,P}}$ for all vertices $Q \in T_J$ that lie in the same indicated grey region R. We find that

$$\left\| \widetilde{I}_j \varphi_{j,P} \right\|_{L_2(\Omega)} \gtrsim 3^{L_{j,P}} \sum_{Q \in T_J \cap R} |\sqrt{3}^{-L_{J,Q}}|^2 \gtrsim 3^{L_{j,P}} \|1\|_{L_2(R)}^2 \gtrsim 1.$$

Suppose that $s_j = \sum_{P \in T_j} c_P \varphi_{j,P}$, then

$$\|s_j\|_{L_2(\Omega)}^2 \sim \sum_{P \in T_j} |c_P|^2 \sim \sum_{P \in T_j} |c_P|^2 \left\| \widetilde{I}_j \varphi_{j,P} \right\|_{L_2(\Omega)}^2 \sim \left\| \sum_{P \in T_j} c_P \widetilde{I}_j \varphi_{j,P} \right\|_{L_2(\Omega)}^2 = \left\| \widetilde{I}_j s_j \right\|_{L_2(\Omega)}^2,$$

which holds because $\left\| \widetilde{I}_j \varphi_{j,P} \right\|_{L_2(\Omega)} \sim 1$ and because $\widetilde{I}_j \varphi_{j,P}$ is locally supported. Figure 7 depicts such a basis function $\widetilde{I}_j \varphi_{j,P}$. □

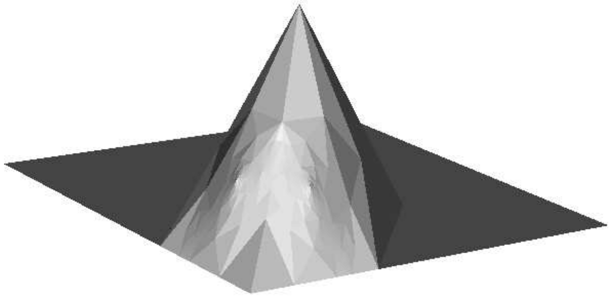


Figure 7: Basis function $\widetilde{I}_j\varphi_{j,P}$ on an adaptively refined triangulation.

Theorem 5.2. *The Bernstein inequality*

$$\left\|\widetilde{I}_j v_j\right\|_{H^s(\Omega)} \lesssim \sqrt{3}^{js} \|v_j\|_{L_2(\Omega)}, \qquad v_j \in S_j, \quad 0 \le j \le J < \infty \tag{5.4}$$

holds in the range $0 < s < s_0$ *with* $s_0 := 1 + \log_3(9/5)/2 > 1$.

Proof. First we prove that $\|\widetilde{I}_0 v_0\|_{H^s(\Omega)} \lesssim \|v_0\|_{L_2(\Omega)}$. Because the adaptive $\sqrt{3}$ refinement only generates triangles that are a proper subset of the uniform refinement hierarchy, we find that on any triangle $\tau \in T_J$ the function $\widetilde{I}_0 v_0$ corresponds with some function in the sequence (4.2) (for $j = 0$) associated with the uniform $\sqrt{3}$ refinement hierarchy. For all those functions in the sequence (4.2) we know that the Bernstein estimate (4.1) holds, and, by Lemma 4.3, this sequence (4.2) is Cauchy with respect to the norm in $H^s(\Omega)$. Therefore we conclude that $\|\widetilde{I}_0 v_0\|_{H^s(\Omega)} \lesssim \|v_0\|_{L_2(\Omega)}$.

We now prove (5.4) in full generality. Let τ be an arbitrary triangle in T_j. We define

$$\tau^{(1)} := \{\tau_i \in T_j \mid \tau \cap \tau_i \neq \emptyset\}. \tag{5.5}$$

We say that $\tau \cap \tau_i \neq \emptyset$ when τ and τ_i share an edge or a vertex. Similarly we define

$$\tau^{(2)} := \{\tau_i \in T_j \mid \tau^{(1)} \cap \tau_i \neq \emptyset\}.$$

The spline function v_j is completely determined on $\tau^{(2)}$ by its function values at the vertices of $\tau^{(2)}$. Furthermore, because the prolongation operators I_j act locally, one can check that the function $\widetilde{I}_j v_j$ restricted to the triangle τ only depends on the function values of v_j at the vertices of $\tau^{(2)}$. This is related to the concept of "dependency region" or "invariant neighborhood" in subdivision theory [26]. The triangles in $\tau^{(2)}$ have a generation index difference of at most 2, which implies that all triangles in $\tau^{(2)}$ have a diameter $\sim |\tau|$ with $|\tau|$ the diameter of τ (obviously $|\tau| \gtrsim \sqrt{3}^{-j}$). We now apply a similar reasoning as we used for the case $j = 0$ to derive that $\|\widetilde{I}_j v_j\|_{H^s(\tau)} \lesssim \sqrt{3}^{js}\|v_j\|_{L_2(\tau^{(2)})}$. Summing over all triangles $\tau \in T_j$ yields the result. $\square$

To establish the optimality of the preconditioner $\mathcal{C}_J$ defined by (3.10) it remains to bound the expression ν_J from (3.8). For the remainder of the paper we will assume that $T_0 \equiv \Delta_0$ is a uniform triangulation of the domain Ω. We first show that

$$\|(Q_j - I_j Q_{j-1})g\|_{L_2(\Omega)} \lesssim \|Q_j g - g\|_{L_2(\Omega)} + \|Q_{j-1} g - g\|_{L_2(\Omega)}. \tag{5.6}$$

The geometric $\sqrt{3}$ refinement process always inserts a vertex at the barycenter of the triangle. This implies that there exist extensions $I_j^{\text{ext}} : S_{j-1} + S_j \to S_j$ of the prolongation operators I_j such that

$$I_j^{\text{ext}}\big|_{S_{j-1}} = I_j, \quad I_j^{\text{ext}}\big|_{V_j} = \text{Id}, \quad \left\|I_j^{\text{ext}}(v_{j-1} + v_j)\right\|_{L_2} \lesssim \|v_{j-1} + v_j\|_{L_2},$$

for all $v_{j-1} \in S_{j-1}$, $v_j \in S_j$. Indeed, define $I_j^{\text{ext}}(v_{j-1}+v_j)$ as the unique spline in S_j that interpolates the function values of $v_{j-1}+v_j$ at the vertices of the triangulation Δ_j. It is obvious that $I_j^{\text{ext}} s_j = s_j$ holds, and $I_j^{\text{ext}} s_{j-1} = I_j s_{j-1}$ follows from the fact that the interior vertices are always inserted at the barycenter of the triangle. From the observation $\|I_j^{\text{ext}}(v_{j-1}+v_j)\|_{L_\infty(\tau)} = \|v_{j-1}+v_j\|_{L_\infty(\tau)}$ one easily infers $\|I_j^{\text{ext}}(v_{j-1}+v_j)\|_{L_2(\tau)} \sim \|v_{j-1}+v_j\|_{L_2(\tau)}$ by using the fact that all norms on a finite dimensional space are equivalent. Summing over all triangles $\tau \in T_{j-1}$ gives $\|I_j^{\text{ext}}(v_{j-1}+v_j)\|_{L_2(\Omega)} \sim \|v_{j-1}+v_j\|_{L_2(\Omega)}$. We find that

$$\begin{aligned} \|I_j Q_{j-1} g - g\|_{L_2(\Omega)} &\leq \|I_j Q_{j-1} g - Q_j g\|_{L_2(\Omega)} + \|Q_j g - g\|_{L_2(\Omega)} \\ &\leq \left\|I_j^{\text{ext}}(Q_{j-1} g - Q_j g)\right\|_{L_2(\Omega)} + \|Q_j g - g\|_{L_2(\Omega)} \\ &\lesssim \|Q_{j-1} g - Q_j g\|_{L_2(\Omega)} + \|Q_j g - g\|_{L_2(\Omega)}, \end{aligned}$$

and from the triangle inequality we find (5.6).

If we can show that

$$\|Q_j g - g\|_{L_2(\Omega)} \lesssim \omega_2(g, \sqrt{3}^{-j}, \Omega)_2 \tag{5.7}$$

for arbitrary $g \in S_J$, we conclude from (3.8) and (3.9) that $\nu_J = \mathcal{O}(1)$. The techniques needed to prove (5.7) are exactly the same as the ones used in [11]. We present the proof here for completion.

For any triangle $\tau \in T_j$ with vertices A,B,C, the restrictions of $\varphi_{j,P}$, $P \in \{A, B, C\}$, to τ are linearly independent over τ. Therefore there exists a unique collection of linear polynomials $\zeta_A^\tau, \zeta_B^\tau, \zeta_C^\tau$ such that

$$\int_\tau \varphi_{j,P'}(x) \zeta_{P''}^\tau(x) dx = \delta_{P'P''}, \quad P', P'' \in \{A, B, C\}.$$

Now we define for every vertex $P \in X_j$, and $\tau \in T_j$ such that $P \in \tau$

$$\eta_{j,P}(x) = \begin{cases} \frac{1}{N_P} \zeta_P^\tau(x) & , \quad x \in \tau, \\ 0 & , \quad x \notin \operatorname{supp} \varphi_{j,P} \end{cases},$$

where N_P is the number of triangles in T_j contained in the support of $\varphi_{j,P}$. Then one easily confirms that $(\varphi_{j,P}, \eta_{j,Q}) = \delta_{PQ}$ for all vertices $P, Q \in X_j$ so that

$$\widehat{Q} f := \sum_{P \in X_j} (f, \eta_{j,P}) \varphi_{j,P} \tag{5.8}$$

defines a projector onto S_j which reproduces Π_1, the space of all polynomials of degree at most one on Ω. Some simple computations yield

$$\|\eta_{j,P}\|_{L_2(\Omega)} \sim 1, \quad P \in X_j,$$

and, with $\tau^{(1)}$ as in (5.5),

$$\left\|\widehat{Q}_j f\right\|_{L_2(\tau)} = \left\|\sum_{P \in X_j} (f, \eta_{j,P}) \varphi_{j,P}\right\|_{L_2(\tau)} \lesssim \|f\|_{L_2(\tau^{(1)})}. \tag{5.9}$$

Let

$$T_j^* := \{\tau \in T_j \mid L(\tau) < j \text{ and } \tau^{(1)} \cap \tau' = \emptyset \text{ for all } \tau' \in T_j \text{ with } L(\tau') = j\}. \tag{5.10}$$

Because we only allow triangles τ with $L(\tau) = j$ to be marked for refinement one can check that all triangles in T_j^* will also belong to T_J for any $J \geq j$. Now let $g \in S_J$. Due to the local support of the dual basis functions $\eta_{j,P}$ and since $\widehat{Q}_j$ is a projector, we have

$$\left\|\widehat{Q}_j g - g\right\|_{L_2(\tau)} = 0, \quad \tau \in T_j^*. \tag{5.11}$$

For $\tau \in T_j \setminus T_j^*$, we obtain from (5.9)

$$\left\|\widehat{Q}_j g - g\right\|_{L_2(\tau)} \le \left\|\widehat{Q}_j (g-\pi)\right\|_{L_2(\tau)} + \|g-\pi\|_{L_2(\tau)} \lesssim \|g-\pi\|_{L_2(\tau^{(1)})},$$

with $\pi \in \Pi_1$. Now we employ Whitney's theorem which states that

$$\inf_{\pi\in\Pi_1} \|f-\pi\|_{L_2(\sigma)} \lesssim \omega_2(f, |\sigma|, \sigma)_2$$

with σ some simplex in $\mathbb{R}^2$ and $|\sigma|$ its diameter. A proof of Whitney's theorem for multivariate functions can be found in [27]. Because the adaptive $\sqrt{3}$ refinement algorithm allows at most a generation index difference of one between neighboring triangles one easily infers that

$$|\tau^{(1)}| \sim \sqrt{3}^{-j},$$

with $|\tau^{(1)}|$ the diameter of $\tau^{(1)}$. Since the choice of $\pi \in \Pi_1$ was arbitrary we find that

$$\left\|\widehat{Q}_j g - g\right\|_{L_2(\tau)} \lesssim \omega_2(g, \sqrt{3}^{-j}, \tau^{(1)})_2$$

which yields

$$\left\|\widehat{Q}_j g - g\right\|_{L_2(\Omega)} \lesssim \omega_2(g, \sqrt{3}^{-j}, \Omega)_2.$$

Since one trivially has

$$\|Q_j g - g\|_{L_2(\Omega)} \le \left\|\widehat{Q}_j g - g\right\|_{L_2(\Omega)}$$

we conclude that

$$\nu_J = \mathcal{O}(1). \tag{5.12}$$

Theorem 3.2, Theorem 5.1, Theorem 5.2 and (5.12) yield the following result.

Theorem 5.3 (Optimal BPX preconditioner). *Suppose that T_0 is a uniform triangulation of Ω and let $T_0 \to T_1 \to \cdots \to T_J$ be a sequence of triangulations obtained by adaptive $\sqrt{3}$ refinement such that condition (R) is satisfied. Then the BPX preconditioner given by*

$$\mathcal{B}_J := \sum_{j=0}^{J} \sum_{P\in T_j\cap \operatorname{int}\Omega} (\,\cdot\,, \widetilde{I}_j \varphi_{j,P}) \widetilde{I}_j \varphi_{j,P}$$

yields a uniformly bounded condition number for the variational formulation (1.2) *of the Poisson problem* (1.1).

6 A convergent adaptive finite element method

Adaptive procedures for the numerical solution of partial differential equations are by now standard tools in science and engineering, because they have the potential advantage of a significant reduction of the computational cost when compared to non-adaptive methods. Only in the last decade [15, 21] such adaptive methods for solving Poisson's equation, discretized by nested finite element spaces, have been shown to converge. In this section we modify the adaptive method of Morin, Nochetto, and Siebert [21] by using adaptive $\sqrt{3}$ refinement to construct the subsequent finite element spaces. The main result of this section is Theorem 6.5 which proves convergence of the AFEM within any prescribed error tolerance in a finite number of steps.

We first recall the residual-type a posteriori error estimators from [29] for the variational formulation (1.2) of the Poisson problem (1.1). A posteriori error estimators are a key ingredient of adaptivity.

They are easily computable local quantities, depending on the computed approximate solution and data, that provide information about the quality of the current approximate solution. The ultimate goal is to construct a sequence of triangulations that would eventually equidistribute the approximation errors, such that the computational effort is minimized.

To carry out the subsequent analysis we introduce the natural interpolation operators $\mathcal{I}_j : C^0(\overline{\Omega}) \to S_j$ onto the spaces S_j, i.e., $\mathcal{I}_j f(P) = f(P)$ for all vertices $P \in T_j$. Let E_j denote the set of non-boundary edges in T_j. Using Galerkin orthogonality and Green's formula we derive that

$$\begin{aligned} a(u-u_j, v) &= a(u-u_j, v-v_j) \\ &= \sum_{\tau \in T_j} \int_\tau f(v-v_j) - \sum_{\tau \in T_j} \int_\tau \nabla u_j \cdot \nabla (v-v_j) \\ &= \sum_{\tau \in T_j} \int_\tau f(v-v_j) + \sum_{e \in E_j} \int_e [\nabla u_j]_e \cdot n_e (v-v_j), \end{aligned} \tag{6.1}$$

where $[\nabla u_j]_e \cdot n_e$ represents the jump of flux across edge e which is independent of the orientation of the unit normal n_e. Cauchy–Schwarz yields

$$\int_\tau f(v-v_j) \le \|f\|_{L_2(\tau)} \|v-v_j\|_{L_2(\tau)}.$$

If we set $v_j = \mathcal{I}_j v$, then, from the Bramble–Hilbert lemma [5], we find

$$\|v - \mathcal{I}_j v\|_{L_2(\tau)} \lesssim |\tau| \, |v|_{H^1(\tau)}$$

and, from Cauchy–Schwarz again,

$$\left| \sum_{\tau \in T_j} \int_\tau f(v-v_j) \right|^2 \lesssim \left(\sum_{\tau \in T_j} \|f\|^2_{L_2(\tau)} |\tau|^2 \right) \left(\sum_{\tau \in T_j} |v|^2_{H^1(\tau)} \right) \lesssim \left(\sum_{\tau \in T_j} \|f\|^2_{L_2(\tau)} |\tau|^2 \right) |v|^2_{H^1(\Omega)}.$$

Now we bound the second term from (6.1) in a similar way. Cauchy–Schwarz gives

$$\int_e [\nabla u_j]_e \cdot n_e (v-v_j) \le \|[\nabla u_j]_e \cdot n_e\|_{L_2(e)} \|v - \mathcal{I}_j v\|_{L_2(e)}$$

and from the Trace Theorem we find

$$\|v - \mathcal{I}_j v\|_{L_2(e)} \lesssim |e|^{1/2} \, |v|_{H^1(\tau_e)}$$

with τ_e the union of $\tau_1, \tau_2 \in T_j$ and $e = \tau_1 \cap \tau_2$. We get

$$\left| \sum_{e \in E_j} \int_e [\nabla u_j]_e \cdot n_e (v-v_j) \right|^2 \lesssim \left(\sum_{e \in E_j} \|[\nabla u_j]_e \cdot n_e\|^2_{L_2(e)} |e| \right) |v|^2_{H^1(\Omega)}.$$

By the ellipticity (2.3) and by substituting $v = u - u_j$ we find the a posteriori error bound

$$|u-u_j|^2_{H^1(\Omega)} \lesssim \left(\sum_{\tau \in T_j} \|f\|^2_{L_2(\tau)} |\tau|^2 \right) + \left(\sum_{e \in E_j} \|[\nabla u_j]_e \cdot n_e\|^2_{L_2(e)} |e| \right) \tag{6.2}$$

Hence, if we set

$$\eta_e^2 := \| \, |\tau_e| \; f\|^2_{L_2(\tau_e)} + \left\| \; |e|^{1/2} [\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)} \tag{6.3}$$

as local error indicator associated with the edge e, and if we define a global error indicator by

$$\eta_j^2 := \sum_{e \in E_j} \eta_e^2,$$

then (6.2) implies

$$|u - u_j|_{H^1(\Omega)}^2 \lesssim \eta_j^2. \tag{6.4}$$

We now give the marking strategy for error reduction due to Dörfler [15]. It ensures that sufficiently many sides e are chosen such that their contributions η_e constitute a fixed portion of the global error estimation η_j.

Marking Strategy E

`Given a parameter` $0 < \theta < 1$:

1. `Select in` $\widehat{E}_j$ `some sides such that`

$$\left(\sum_{e \in \widehat{E}_j} \eta_e^2 \right)^{1/2} \geq \theta \eta_j.$$

2. `Let` $\widehat{T}_j$ `be the set of elements with one side in` $\widehat{E}_j$ `and mark all these elements.`

Let us define the norm $\|\cdot\|_{\mathcal{A}}$ induced by the bilinear form $a(\cdot,\cdot)$, i.e. $\|u\|_{\mathcal{A}} := a(u,u)$. Then, by (2.3), $\|\cdot\|_{\mathcal{A}} \sim \|\cdot\|_{H^1(\Omega)}$.

Let f_τ denote the integral mean of f over triangle τ, i.e.,

$$f_\tau := \mathrm{vol}(\tau)^{-1} \int_\tau f(x)dx.$$

The weighted $L^2(\tau)$ norm of the difference $f - f_\tau$ is called the oscillation of f over D and written

$$\mathrm{osc}(f,\tau) := |\tau| \, \|f - f_\tau\|_{L_2(\tau)}.$$

It was shown in [21] that the following quantity, called data oscillation, plays a fundamentol role

$$\mathrm{osc}(f, T_j) := \left(\sum_{\tau \in T_j} |\tau|^2 \, \|f - f_\tau\|_{L_2(\tau)}^2 \right)^{1/2}. \tag{6.5}$$

The following lemma is based on Lemma 4.2 in [21].

Lemma 6.1. *Let T_{j+1} be a triangulation obtained from T_j by refining at least every element according to Strategy E. Then, for all $e \in \widehat{E}_j$, we have*

$$\eta_e^2 \lesssim \|u_{j+1} - u_j\|_{\mathcal{A}(\tau_e)}^2 + \| \, |\tau_e| \, (f - f_{\tau_e})\|_{L_2(\tau_e)}^2.$$

Here $\tau_e = \tau_1 \cup \tau_2$ with $\tau_1, \tau_2 \in T_j$ such that $e = \tau_1 \cap \tau_2$, and $f_{\tau_e} := \begin{cases} f_{\tau_1} & on \quad \tau_1 \\ f_{\tau_2} & on \quad \tau_2 \end{cases}$.

Proof. We construct an auxiliary function $\phi \in S_{j+1}$ that satisfies $\phi|_{\partial\tau_e} = 0$, together with the conditions

$$\int_{\tau_e} f_{\tau_e}\phi + \int_e [\nabla u_j]_e \cdot n_e \; \phi = \| \; |\tau_e| \; f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\| |e|^{1/2}[\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)}, \tag{6.6}$$

$$\|\nabla\phi\|^2_{L_2(\tau_e)} \lesssim \| \; |\tau_e| \; f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\| |e|^{1/2}[\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)}. \tag{6.7}$$

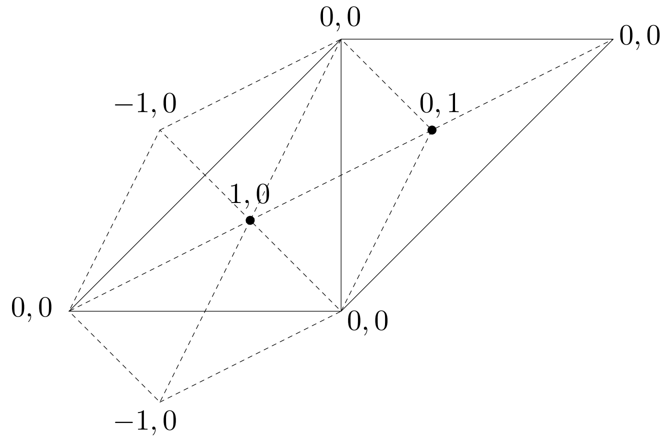


Figure 8: Example of a refined patch $\tau_e = \tau_1 \cup \tau_2$. We associate basis functions φ_1 and φ_2 with the two marked dots. At each vertex we give the function values of φ_1 resp. φ_2. Note that both φ_1 and φ_2 vanish on the boundary of τ_e.

Consider $\phi = \alpha_1\varphi_1 + \alpha_2\varphi_2$, where $\varphi_i \in S_{j+1}$, $i = 1, 2$, have function value 1 at the vertex inserted in triangle τ_i when going from T_j to T_{j+1} and $\varphi_i|_{\partial\tau_e} = 0$. Figure 8 shows in detail how such functions φ_i can be created. By construction the functions φ_i, $i = 1, 2$, have the following properties:

$$\varphi_i|_{\partial\tau_e} = 0, \quad \|\nabla\varphi_i\|_{L_2(\tau_e)} \sim 1, \quad \int_{\tau_i} \varphi_i \sim |\tau_i|^2, \quad \int_e \varphi_i \sim |e|. \tag{6.8}$$

In order to satisfy (6.6) we choose

$$\begin{cases} \alpha_1 = \frac{1}{2}\frac{\||\tau_e|f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\||e|^{1/2}[\nabla u_j]_e\cdot n_e\right\|^2_{L_2(e)}}{\sum_{i=1}^2 \int_{\tau_i} f_{\tau_i}\varphi_1 + \int_e [\nabla u_j]_e\cdot n_e\varphi_1} \\ \alpha_2 = \frac{1}{2}\frac{\||\tau_e|f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\||e|^{1/2}[\nabla u_j]_e\cdot n_e\right\|^2_{L_2(e)}}{\sum_{i=1}^2 \int_{\tau_i} f_{\tau_i}\varphi_2 + \int_e [\nabla u_j]_e\cdot n_e\varphi_2} \end{cases}.$$

From the properties (6.8), the equivalence $|\tau_1|, |\tau_2| \sim |\tau_e|$, and the fact that f_{τ_e} resp. $[\nabla u_j]_e \cdot n_e$ is constant on τ_1, τ_2 resp. e, we infer

$$\begin{aligned} |\alpha_1|, |\alpha_2| \; &\lesssim \; \frac{\sum_{i=1}^2 |\tau_i|^4 |f_{\tau_i}|^2 + |e|^2 |[\nabla u_j]_e \cdot n_e|^2}{\sum_{i=1}^2 |f_{\tau_i}||\tau_i|^2 + |[\nabla u_j]_e \cdot n_e| \; |e|} \\ &\lesssim \; \frac{\sum_{i=1}^2 |\tau_i|^4 |f_{\tau_i}|^2 + |e|^2 |[\nabla u_j]_e \cdot n_e|^2}{\max\{|f_{\tau_1}||\tau_1|^2, |f_{\tau_2}||\tau_2|^2, |[\nabla u_j]_e \cdot n_e| \; |e|\}} \\ &\lesssim \; \sum_{i=1}^2 |\tau_i|^2 |f_{\tau_i}| + |e||[\nabla u_j]_e \cdot n_e| \\ &\lesssim \; \left(\| \; |\tau_e| \; f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\| |e|^{1/2}[\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)} \right)^{1/2} \end{aligned}$$

Since $\|\nabla\phi\|_{L_2(\tau_e)} \leq \sum_{i=1}^2 |\alpha_i| \|\nabla\varphi_i\|_{L_2(\tau_e)}$ we find (6.7).

Because $\phi \in S_{j+1}$ and because $\phi|_{\partial\tau_e} = 0$, Green's formula yields

$$\begin{aligned} a(u_{j+1} - u_j, \phi)_{\tau_e} &= \int_{\tau_e} f\phi + \int_e [\nabla u_j]_e \cdot n_e \phi \\ &= \int_{\tau_e} f_j\phi + \int_e [\nabla u_j]_e \cdot n_e \phi + \int_{\tau_e} (f - f_j)\phi. \end{aligned}$$

From (6.6) and the Poincaré inequality $\|\phi\|_{L_2(\tau_e)} \lesssim |\tau_e| \|\nabla\phi\|_{L_2(\tau_e)}$ we obtain

$$\begin{aligned} &|\tau_e|^2 \|f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\| |e|^{1/2} [\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)} \\ &= a(u_{j+1} - u_j, \phi)_{\tau_e} - \int_{\tau_e} (f - f_j)\phi \\ &\leq \|u_{j+1} - u_j\|_{\mathcal{A}(\tau_e)} \|\phi\|_{\mathcal{A}(\tau_e)} + \|f - f_{\tau_e}\|_{L_2(\tau_e)} \|\phi\|_{L_2(\tau_e)} \\ &\lesssim \left(\|u_{j+1} - u_j\|_{\mathcal{A}(\tau_e)} + \||\tau_e|(f - f_{\tau_e})\|_{L_2(\tau_e)} \right) \|\nabla\phi\|_{L_2(\tau_e)}. \end{aligned}$$

From (6.7) we thus obtain

$$\| \, |\tau_e| \; f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\| |e|^{1/2} [\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)} \lesssim \|u_{j+1} - u_j\|^2_{\mathcal{A}(\tau_e)} + \| \, |\tau_e| \; (f - f_{\tau_e})\|^2_{L_2(\tau_e)}.$$

The triangle inequality and (6.3) give

$$\begin{aligned} \eta_e^2 &= \| \, |\tau_e| \; f\|^2_{L_2(\tau_e)} + \left\| \; |e|^{1/2} [\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)} \\ &\lesssim \| \, |\tau_e| \; f_{\tau_e}\|^2_{L_2(\tau_e)} + \left\| \; |e|^{1/2} [\nabla u_j]_e \cdot n_e \right\|^2_{L_2(e)} + \| \, |\tau_e| \; (f - f_{\tau_e})\|^2_{L_2(\tau_e)} \\ &\lesssim \|u_{j+1} - u_j\|^2_{\mathcal{A}(\tau_e)} + \| \, |\tau_e| \; (f - f_{\tau_e})\|^2_{L_2(\tau_e)}, \end{aligned}$$

which proves the lemma. □

We can now prove the following error reduction property.

Theorem 6.2 (Error reduction property). *There holds*

$$\|u - u_{j+1}\|^2_{\mathcal{A}} \leq \alpha \|u - u_j\|^2_{\mathcal{A}} + C \inf_{v_{j+1} \in S_{j+1}} \|u_j - v_{j+1}\|^2_{\mathcal{A}} + \operatorname{osc}(f, T_j)^2 \qquad (6.9)$$

for some $0 < \alpha < 1$.

Proof. A straightforward computation gives

$$\|u_{j+1} - u_j\|^2_{\mathcal{A}} + \|u - u_{j+1}\|^2_{\mathcal{A}} - \|u - u_j\|^2_{\mathcal{A}} = -2a(u - u_{j+1}, u_{j+1} - u_j).$$

From Lemma 6.1 we get

$$\begin{aligned} \|u - u_j\|^2_{\mathcal{A}} &\lesssim \eta_j^2 \lesssim \|u_{j+1} - u_j\|^2_{\mathcal{A}} + \operatorname{osc}(f, T_j)^2 \\ &\lesssim \|u - u_j\|^2_{\mathcal{A}} - \|u - u_{j+1}\|^2_{\mathcal{A}} - 2a(u - u_{j+1}, u_{j+1} - u_j) + \operatorname{osc}(f, T_j)^2. \end{aligned}$$

By Galerkin orthogonality we have that $a(u - u_{j+1}, u_{j+1} - u_j) = a(u - u_{j+1}, v_{j+1} - u_j)$ for any $v_{j+1} \in S_{j+1}$ and Cauchy–Schwarz yields

$$\|u - u_j\|^2_{\mathcal{A}} \lesssim \|u - u_j\|^2_{\mathcal{A}} - \|u - u_{j+1}\|^2_{\mathcal{A}} + 2\|u - u_{j+1}\|_{\mathcal{A}} \|u_j - v_{j+1}\|_{\mathcal{A}} + \operatorname{osc}(f, T_j)^2.$$

Hence, there exists a constant $0 < \kappa < 1/2$ such that

$$2\kappa \|u - u_j\|^2_{\mathcal{A}} \leq \|u - u_j\|^2_{\mathcal{A}} - \|u - u_{j+1}\|^2_{\mathcal{A}} + 2\|u - u_{j+1}\|_{\mathcal{A}} \|u_j - v_{j+1}\|_{\mathcal{A}} + \operatorname{osc}(f, T_j)^2.$$

Now we use Young's inequality ($ab \le a^2/(2\varepsilon) + (\varepsilon b^2)/2$) to derive that

$$2\|u - u_{j+1}\|_{\mathcal{A}}\|u_j - v_{j+1}\|_{\mathcal{A}} \le \kappa\|u - u_{j+1}\|_{\mathcal{A}}^2 + \frac{4}{4\kappa}\|u_j - v_{j+1}\|_{\mathcal{A}}^2,$$

which implies

$$\|u - u_{j+1}\|_{\mathcal{A}}^2 \le \frac{1-2\kappa}{1-\kappa}\,\|u - u_j\|_{\mathcal{A}}^2 + \frac{1}{\kappa(1-\kappa)}\|u_j - v_{j+1}\|_{\mathcal{A}}^2 + \operatorname{osc}(f, T_j)^2.$$

Since v_{j+1} was chosen arbitrarily we have proven the theorem. □

Theorem 6.2 shows the importance of controlling the quantity $\operatorname{osc}(f, T_j)^2$. The following marking strategy guarantees a data oscillation decrease by some factor $\widehat{\alpha} < 1$.

Marking Strategy D

`Given a parameter` $0 < \widehat{\theta} < 1$ `and the subset` $\widehat{T}_j \subset T_j$ `produced by Marking Strategy E:`

1. `Enlarge` $\widehat{T}_j$ `such that`

$$\operatorname{osc}(f, \widehat{T}_j) \ge \widehat{\theta}\operatorname{osc}(f, T_j).$$

2. `Mark all elements in` $\widehat{T}_j$ `for refinement.`

Lemma 6.3 ([21, Lemma 3.8]). *Let T_{j+1} be a triangulation obtained from T_j by refining at least every element in $\widehat{T}_j$ produced by Marking Strategy D, then the following data oscillation reduction occurs*

$$\operatorname{osc}(f, T_{j+1}) \le \widehat{\alpha}\operatorname{osc}(f, T_j), \qquad 0 < \widehat{\alpha} < 1.$$

It remains to bound the term $\inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}}$ in Theorem 6.2. This term reflects the projection error introduced by the nonnestedness of the subsequent finite element spaces.

Lemma 6.4. *There holds*

$$\inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}} \lesssim |T^*_{j+1}|^{\epsilon} \tag{6.10}$$

given that $u \in H^{1+\epsilon}(\Omega)$ for some $0 < \epsilon < 1/2$. Here $|T^*_{j+1}|$ denotes the maximum diameter $|\tau|$ of all triangles $\tau \in T^*_{j+1}$, and $T^*_{j+1} \subset T_{j+1}$ denotes the smallest subset of triangles in T_{j+1} so that for any triangle $\tau \in T_{j+1} \setminus T^*_{j+1}$ we have $\tau \in T_j$, in other words, T^*_{j+1} represents the set of newly created triangles in T_{j+1}.

Proof. Obviously

$$\inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}} = \inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}(T^*_{j+1})}.$$

The standard Jackson estimate for piecewise constant functions yields

$$\inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}(T^*_{j+1})} = \inf_{v_{j+1}\in S_{j+1}} \|\nabla u_j - \nabla v_{j+1}\|_{L_2(T^*_{j+1})} \lesssim \omega_1(\nabla u_j, |T^*_{j+1}|, T^*_{j+1})_2,$$

and we find that

$$\inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}} \lesssim |T^*_{j+1}|^{\epsilon} |u_j|_{H^{1+\epsilon}(\Omega)}.$$

The lemma now follows from the estimate

$$|u_j|_{H^{1+\epsilon}(\Omega)} \lesssim |u|_{H^{1+\epsilon}(\Omega)}. \tag{6.11}$$

Note that $u_j = Q_j^{\mathcal{A}} u_j$, where $Q_j^{\mathcal{A}}$ is the orthogonal projection onto S_j with respect to the $\mathcal{A}$-norm, i.e.,

$$a(Q_j^{\mathcal{A}} u, v_j) = a(u, v_j) \quad \text{for all } v_j \in S_j.$$

In order to establish (6.11) one has to prove that the operator $Q_j^{\mathcal{A}}$ is stable in $H^{1+\epsilon}(\Omega)$. This can be established using techniques from [6] where the H^s stability of the L_2 orthogonal projector onto the space of piecewise linears was established for $0 \le s \le 1$. Establishing the $H^{1+\epsilon}$ stability of $Q_j^{\mathcal{A}}$ is equivalent to establishing the H^{ϵ} stability of the L_2 orthogonal projector onto the space of piecewise constant polynomials. □

We are now in a position to formulate Algorithm C from [21] and to prove its convergence.

Convergent Algorithm C

`Choose parameters` $0 < \theta, \widehat{\theta} < 1$.

1. `Solve the system on` T_0 `for the discrete solution` u_0.
2. `Let` $j = 0$.
3. `Compute the local indicators` η_e.
4. `Construct` $\widehat{T}_j \subset T_j$ `by Marking Strategy E and parameter` θ.
5. `Enlarge` $\widehat{T}_j$ `by Marking Strategy D and parameter` $\widehat{\theta}$.
6. `Refine every element in` $\widehat{T}_j$. `Let` T_{j+1} `be the new mesh.`
7. `Solve the system on` T_{j+1} `for the discrete solution` u_{j+1}.
8. `Let` $j = j + 1$ `and go to Step 3.`

Theorem 6.5 (Convergence). *Let $0 < \theta, \widehat{\theta} < 1$ and $\varepsilon > 0$ arbitrary. Algorithm C finds a solution $u_j \in S_j$ so that*

$$\|u - u_j\|_{\mathcal{A}} < \varepsilon \tag{6.12}$$

in a finite number of steps, given that $u \in H^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$.

Proof. In view of Lemma 6.3 we see that $\operatorname{osc}(f, T_j)$ is reduced by a fixed factor with each iteration step. Therefore, after a fixed number of iterations j_0, we find that $\operatorname{osc}(f, T_j) < \frac{1-\alpha}{3}\varepsilon$ for all $j > j_0$. Similarly, by Lemma 6.4, we can bound the error term $C \inf_{v_{j+1}\in S_{j+1}} \|u_j - v_{j+1}\|_{\mathcal{A}}$ by $\frac{1-\alpha}{3}\varepsilon$, provided that $|T^*_{j+1}|$ is small enough. Hence we have to make sure that the diameter of the largest triangle in T^*_{j+1} is small. There can only be a finite number of T^*_{j+1} for which $|T^*_{j+1}|$ is too large. This is due to the fact that triangles that belong to some T_k^*, $k < j+1$, cannot belong to T_{j+1}^* anymore. Thus, suppose that $\tau \in T_{j+1}^*$ such that $|\tau| = |T_{j+1}^*|$ and $|\tau|$ is "large". Then, or τ is refined after some time and the "smaller" refined triangles belong to some T_k^*, $k > j+1$, or τ is not refined anymore such that, by consequence, there is no projection error on τ since $S_{j+1}|_\tau \subset S_k|_\tau$ for all $k > j+1$.

So far we have shown that, in view of Theorem 6.2, the bound

$$\|u - u_{j+1}\|_{\mathcal{A}}^2 \leq \alpha \|u - u_j\|_{\mathcal{A}}^2 + 2\frac{1-\alpha}{3}\varepsilon$$

holds, after a finite number of iteration steps. Suppose that $\|u - u_j\|_{\mathcal{A}} \geq \varepsilon$, then

$$\|u - u_{j+1}\|_{\mathcal{A}}^2 \leq \frac{\alpha+2}{3} \|u - u_j\|_{\mathcal{A}}^2$$

holds, after a finite number of iteration steps. Since $\frac{\alpha+2}{3} < 1$ we find that eventually (6.12) holds for some finite j. $\square$

7 Numerical experiments

This section starts with some brief comments on the implementation of the BPX preconditioner and Algorithm C. We conclude with two relevant experiments that confirm the theory established in this manuscript.

7.1 Implementation issues

Sequence generation. The BPX preconditioner from Theorem 5.3 is only optimal when the sequence (1.4) satisfies condition (R). Note that the sequence of triangulations obtained from Algorithm C does not necessarily satisfies condition (R). Therefore, each time we solve the linear system on T_j, we need to construct a new sequence

$$T_0 \to \widetilde{T}_1 \to \widetilde{T}_2 \to \cdots \to \widetilde{T}_{j-1} \to T_j \tag{7.1}$$

such that (R) is satisfied. This is straightforward if one stores the original sequence (1.4) obtained from Algorithm C in a tree structure, i.e., for each triangle that is refined we keep references from the parent triangle to its three children, and from each child to its parent triangle. The sequence (7.1) can now be generated as follows. We construct $\widetilde{T}_1$ from T_0 by marking all triangles in T_0 that have children. Then we construct $\widetilde{T}_2$ from $\widetilde{T}_1$ by marking all triangles in $\widetilde{T}_1$ for which the corresponding triangles in the given tree structure have children, and so on.

The BPX preconditioner. We use a conjugate gradient method with the BPX-preconditioner from Theorem 5.3 to solve the linear systems arising from (1.2). More specifically we follow the approach outlined by Griebel in [16] to implement the BPX preconditioner. Roughly speaking the BPX preconditioner is equivalent to a transformation of the stiffness matrix with respect to the nodal basis of S_J into a stiffness matrix with respect to the frame $\{\widetilde{I}_j \varphi_{j,P} \mid P \in T_j \cup \operatorname{int}\Omega, j = 0, \ldots, J\}$. We stop the conjugate gradient iteration if the energy norm (H^1-norm) of the residual is smaller than 10^{-7}. It is well known that for an optimal preconditioner the energy norm of the residual is equivalent to the ℓ_2 norm of the discrete residual of the discretized system, see, e.g., [9].

Integration. In our theortical derivation we have assumed exact integration. In the numerical experiments we used a third order Gaussian quadrature formula on each triangle to compute the integrals.

Implementation of the marking strategies. For both Marking Strategy E and Marking Strategy D we need to collect those locally computed quantities that have the largest absolute value. In order to avoid a complicated sorting algorithm for the ordering of the computed quantities, we use the quasi-optimal algorithm with linear complexity that was proposed in [15].

7.2 Example: Adaptivity driven by a geometric criterion

The first test problem that we consider is as follows:

$$\begin{aligned} -\Delta u + u &= f, \quad \text{on } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{7.2}$$

with $\Omega = [0,1]\times[0,1]$. The right hand side f is constructed so that the true solution is $u = \sin \pi x \sin \pi y$. In this first experiment adaptivity is driven by a geometric criterion. Namely, the triangles which intersect with the quarter circle centered at the origin with radius 0.25 are repeatedly marked for further refinement. This test example is similar to the first test example from [2] where several multilevel preconditioners based on adaptive red refinement or adaptive red-green refinement are compared to each other.

Table 1 shows the results. The first block contains the maximum resolution level J. Then we display the spectral condition number κ of the preconditioned system matrix for the linear system of equations that is solved. Next we show the number of iterations that are needed to reach the stopping criterion (which is when the energy norm of the residual is smaller than 10^{-7}). The starting vector for each iteration is the zero-vector. The next block displays the number of iterations when a nested iteration conjugate gradient method is used, i.e., by means of an outer iteration loop going from a coarse resolution level to the finest resolution level J one computes the solution at each level with the conjugate gradient method and one uses the solution obtained at the previous coarser level as an initial guess. Finally we show the dimension of the spline space S_J that is used to discretize the problem. The dimension corresponds to the number of non-boundary vertices in the underlying triangulation. Figure 9 shows the initial triangulation T_0 up to T_8. Figure 10 shows the final solution at level 16.

Levels	1	2	3	4	5	6	7	8
	9	10	11	12	13	14	15	16
$\kappa(\mathcal{B}_J^{1/2}\mathcal{A}_J\mathcal{B}_J^{1/2})$	3.8732	4.5426	5.3613	6.3116	6.3712	7.7175	7.9293	8.5193
	10.4497	11.3046	13.4034	13.6729	14.0361	14.5026	15.1459	16.9483
Iterations	4	6	9	13	15	17	17	19
	21	22	22	23	24	24	25	26
Nested iteration	4	6	9	11	11	14	13	14
	14	16	14	18	17	16	17	15
Nodes - DOF	6	9	14	24	37	54	74	107
	165	252	376	559	822	1177	1681	2400

Table 1: Iteration history for the first test problem.

7.3 Example: L-shaped domain

The second test problem that we consider is as follows:

$$\begin{aligned} -\Delta u &= f, \quad \text{on } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{7.3}$$

with $\Omega = [-1,1] \times [0,1] \cup [-1,0] \times [-1,0]$ an L-shaped domain. The right-hand side f is chosen so that the exact solution is given by

$$u(r,\theta) = r^{2/3} \sin(\frac{2}{3}\theta) \exp(-10r^2)$$

in polar coordinates. Figure 11 shows the exact solution u and the right-hand side f.

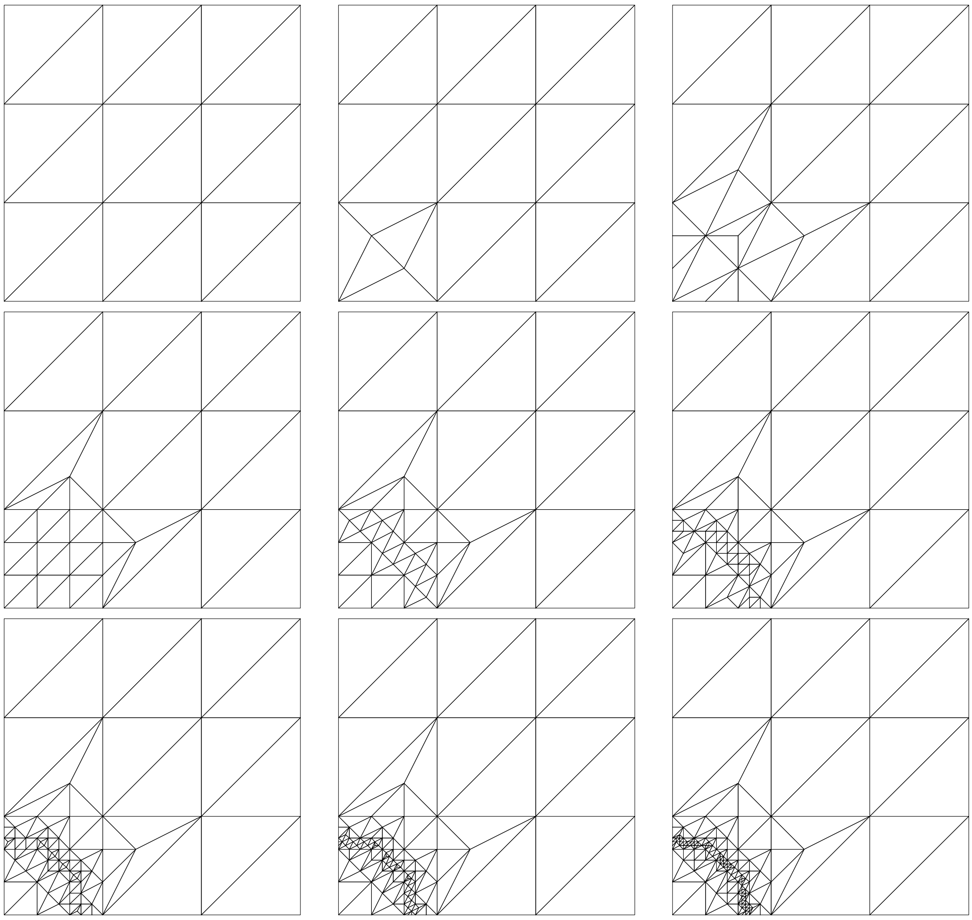
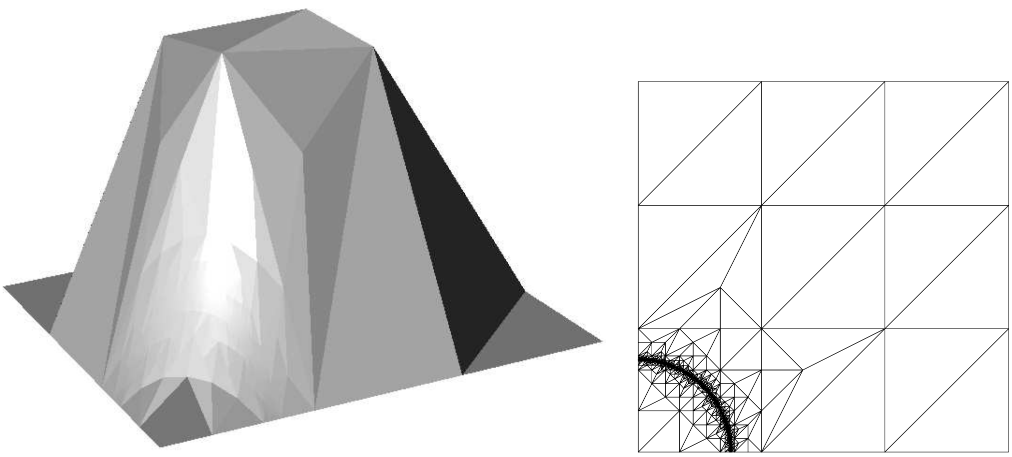


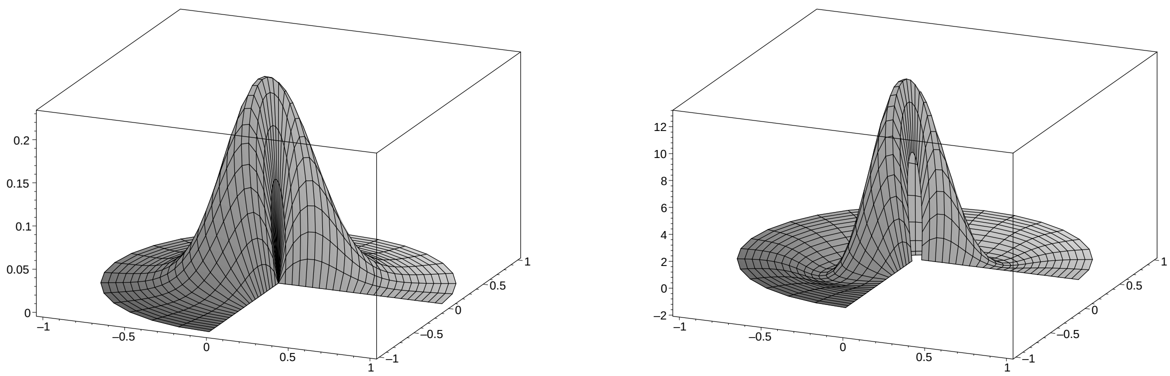
Figure 9: Adaptive $\sqrt{3}$ refinement by a geometric criterion of the initial triangulation for the first test problem.



(a) (b)

Figure 10: The computed solution (a) and final triangulation (b) of the first test problem at level 16.

We use Algorithm C to solve the problem. Figure 12 shows the adaptively refined triangulations $T_0, \ldots, T_{14}$. In Table 2 we give the results concerning the BPX preconditioner of Theorem 5.3. In the first block we show the iteration number j. The next block contains the spectral condition number κ of the BPX-preconditioned system matrix for the linear system of equations that is solved. Then we display the number of iterations when a BPX-preconditioned nested iteration conjugate gradient method is used to solve the linear system. Finally we show the dimension of the spline space S_J that is used to discretize the problem. The dimension corresponds to the number of non-boundary vertices in the underlying triangulation. Figure 13 shows the error reduction and Figure 14 shows the data oscillation reduction. The final solution and final triangulation are shown in Figure 15.



(a) (b)

Figure 11: (a) Exact solution u of the second test problem. (b) Right-hand side f of the second test problem.

Levels	1	2	3	4	5	6	7
	8	9	10	11	12	13	14
	15	16	17	18	19	20	21
	22	23	24	25			
$\kappa(\mathcal{B}_J^{1/2}\mathcal{A}_J\mathcal{B}_J^{1/2})$	3.8493	4.3824	3.9987	4.2215	5.6233	5.4856	6.1213
	6.1042	6.0046	5.8533	5.6274	5.1423	6.7223	6.6252
	6.1374	6.4355	7.4418	6.9306	6.9862	7.3240	7.4959
	7.3850	7.7207	7.7174	7.8435			
Nested iteration	4	6	8	9	10	12	13
	13	13	13	13	10	13	13
	12	12	11	12	11	11	10
	9	11	10	8			
Nodes - DOF	7	10	14	16	20	27	31
	37	51	63	79	99	133	191
	256	344	486	675	904	1228	1841
	2529	3609	5462	7708			

Table 2: Iteration history for the second test problem.

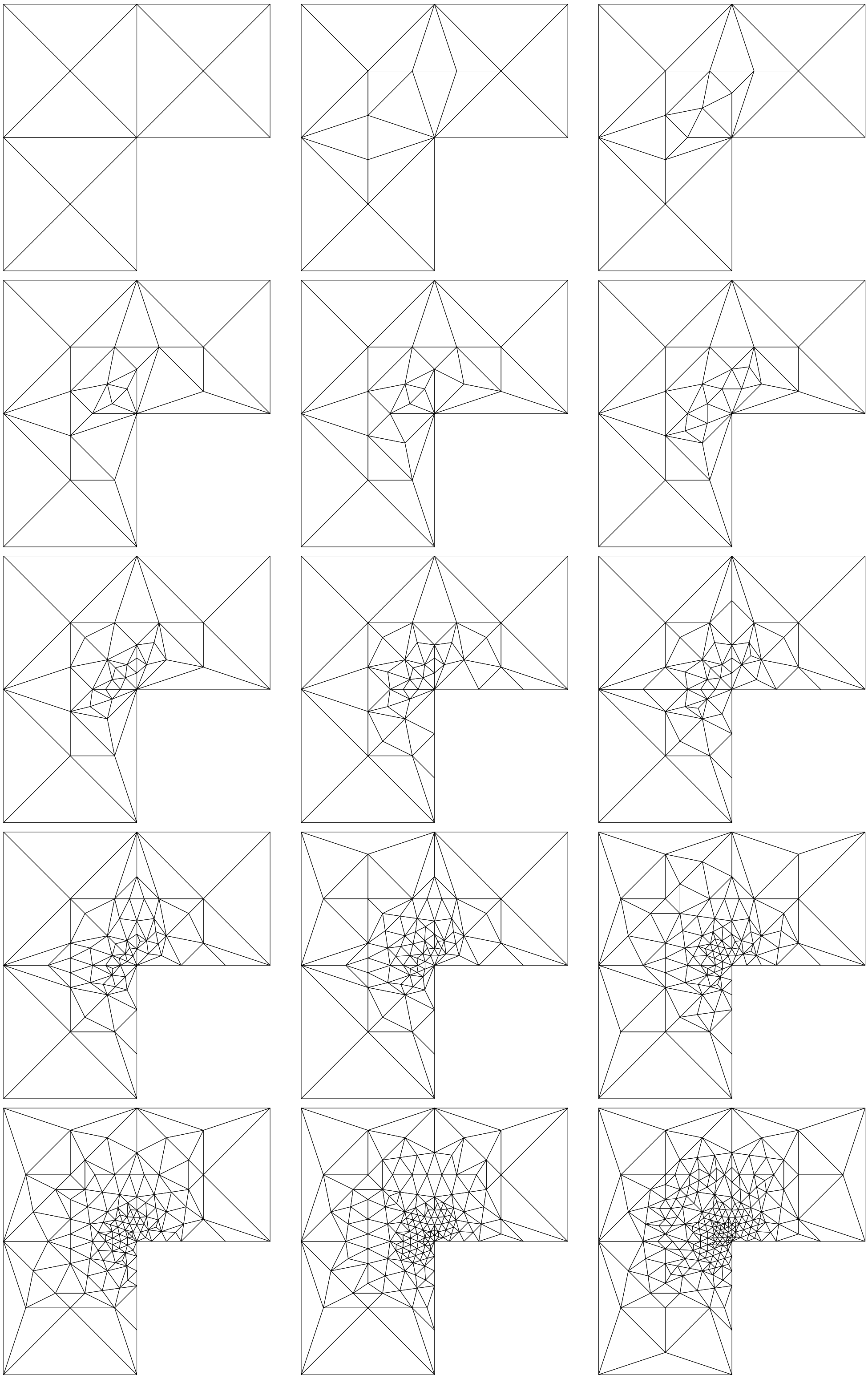


Figure 12: Triangulations $T_0, \ldots, T_{14}$ of the L-shaped domain Ω generated by Algorithm C.

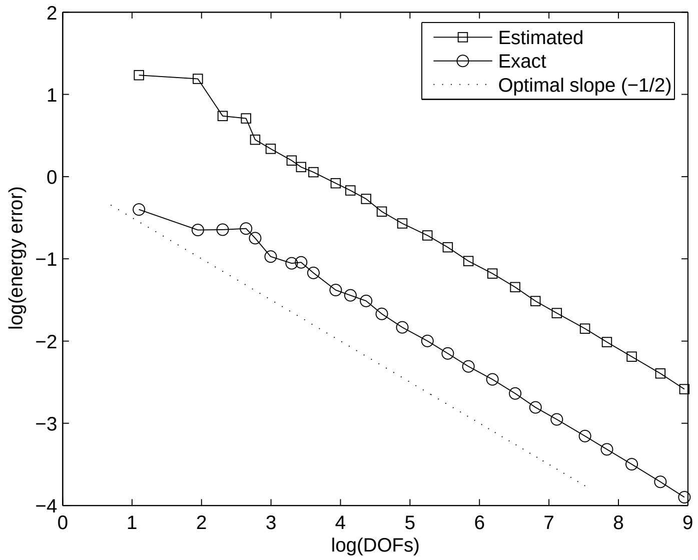


Figure 13: Quasi-optimality of Algorithm C. We show the estimated and true error. The optimal decay is indicated by the line with slope -1/2.

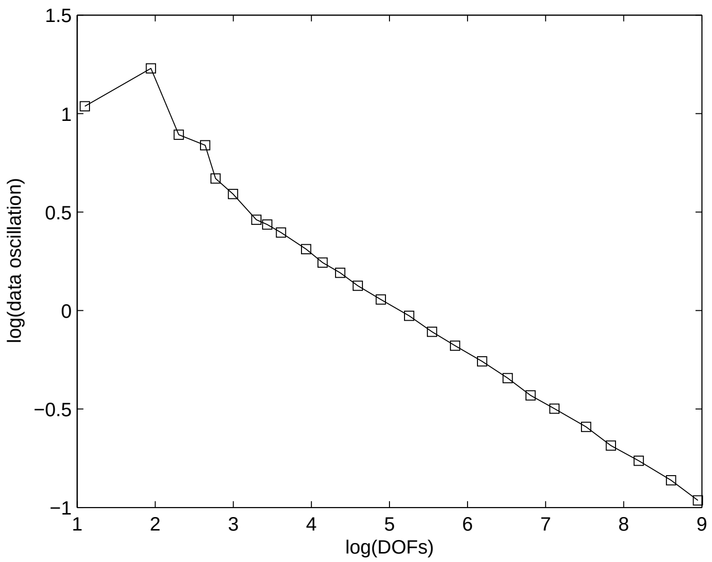
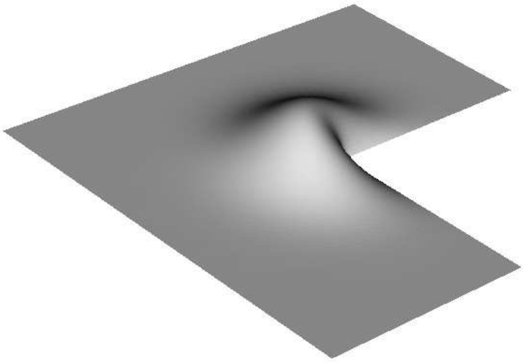
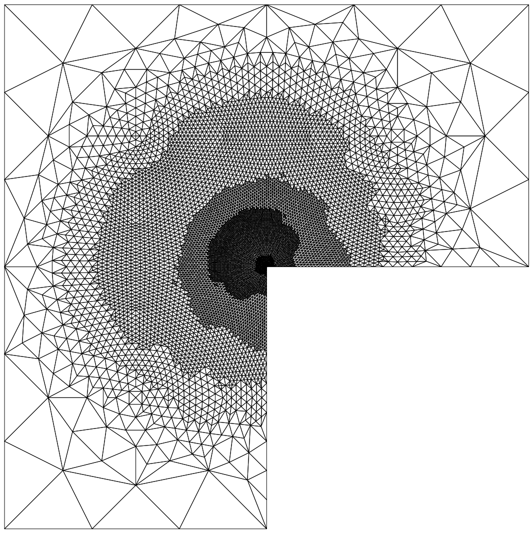


Figure 14: Linear reduction of the data oscillation.



(a)



(b)

Figure 15: The computed solution (a) and final triangulation (b) of the second test problem at level 25.

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