Equivalence Checking of Static Affine Programs using Widening to Handle Recurrences

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Equivalence Checking Example

```
A[0] = in[0];  
for (i = 1; i < N; ++i)  
    A[i] = f(in[i]) + g(A[i-1]);  
out = A[N-1];
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Are the two programs on the left equivalent?

⇒ Same **output** when given same **input**

Assumptions

- no pointers
- no recursion
- functions called are pure
- static control flow
- quasi-affine loop bounds
- quasi-affine conditions
- quasi-affine index expressions

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Main challenges:

- Recurrences
- Transformations exploiting commutativity

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Context

Multi-media applications for embedded systems

- Manipulation of large multi-dimensional arrays
- Relatively simple control flow, mainly loops
- Strict timing constraints

Transformation are applied

- mainly loop transformations and data transformations
- to improve time and energy efficiency
- (semi)automatically or manually

Question:

Is the transformed program (input-output) equivalent to the original?

⇒ equivalence checking should be fully automated

Outline

- 1 Motivation
- 2 Related Work
- 3 Our Approach
 - Dependence Graphs
 - Propagation
 - Commutative Operations
 - Associative Operations
 - Recurrences
- 4 Implementation
- 5 Experiments
 - CLooG
 - Results
- 6 Conclusion

Related Work

Many approaches have been proposed for equivalence checking, e.g.,

- Translation validation
 - ⇒ requires hints from the compiler
 - ⇒ not available for manual transformations
- General theorem provers
 - ⇒ require user interaction, e.g., to set up induction hypothesis
 - ⇒ *not* fully automated

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Other approaches for static affine programs

- Barthou et al. (2002)
 - ⇒ no support for commutative operations
 - Shashidhar et al. (2005)
 - ⇒ limited support for commutative operations
 - ⇒ broken support for recurrences
- ⇒ Both rely on transitive closure operation for handling recurrences
- ⇒ only uniform recurrences

Basic Proof Strategy

Prove output arrays are equal given that input arrays are equal

- ① Compute weakest precondition for equal output
 - ⇒ propagate correspondences between values computed in both programs
- ② Check which part is covered by equal input
- ③ Propagate results back to output
 - ⇒ which elements of the output arrays are equal
 - ⇒ recurrence handling
 - ⇒ commutativity handling

All functions are assumed to be pure and are treated as black-boxes

⇒ results of function calls equal iff all arguments are equal

Equivalence checking performed on *dependence graph* abstraction

Dependence Graphs

Dependence graph contains the following information:
for each value computed in the program, on which values does it depend?

Each computed value is represented by

- function call/operation used to compute value
- iteration vector of the enclosing loops

⇒ complete abstraction of where/how values are stored in memory

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Dependence graph

- vertices: “computations”
 - ▶ a (static) function call or operation
 - ▶ iteration domain D : set of iterations of enclosing loops
- edges: dependences
 - ▶ argument position p
 - ▶ dependence relation M : maps use to definition

Dependences are obtained through exact dataflow analysis

Dependence Graphs: Trivial Example

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for (i = 0; i <= N; ++i)
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- Input computations

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- Similarly if u_2 performs “copy” operation (id)

Commutative Operations

- Correspondence between computations u_1 and u_2 that perform the same (non-commutative) operation
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    } else {  
        B[i] = g(A[i-1]);  
        C[i] = f(in[i]);  
    }  
    A[i] = B[i] + C[i];  
}  
out = A[N-1];
```

$$R_1^{\text{lost}} = \{(i, i) \mid i = 2\alpha + 1\}$$

Equivalence Checking Example

```
A[0] = in[0];  
for (i = 1; i < N; ++i)  
    A[i] = f(in[i]) + g(A[i-1]);  
out = A[N-1];
```

```
A[0] = in[0];  
for (i = 1; i < N; ++i) {  
    if (i%2 == 0) {  
        B[i] = f(in[i]);  
        C[i] = g(A[i-1]);  
    } else {  
        B[i] = g(A[i-1]);  
        C[i] = f(in[i]);  
    }  
    A[i] = B[i] + C[i];  
}  
out = A[N-1];
```

$$R_1^{\text{lost}} = \{(i, i) \mid i = 2\alpha + 1\}$$

$$R_2^{\text{lost}} = \{(i, i) \mid i = 2\alpha\}$$

$$R^{\text{lost}} = R_1^{\text{lost}} \cap R_2^{\text{lost}} = \emptyset$$

Associative Operations

Associative operations are “flattened” at the level of dependence graphs, prior to equivalence checking

Dependence relations are adapted accordingly

Recurrences

```
buf[0] = f1(f2(in[0]));  
for (k = 1; k < N; k++)  
    buf[k] = f1(f2(buf[k-1]));  
out = buf[N-1];
```

```
tmp = f2(in[0]);  
for (k = 1; k < N; k++)  
    tmp = f2(f1(tmp));  
out = f1(tmp);
```

Recurrences

Recurrences

out

Recurrences

$\text{out}^{\bullet} \rightarrow \text{id}_{f1}^{\bullet}$

Recurrences

Recurrences

Recurrences

Recurrences

Recurrences

Recurrences

Recurrences

Recurrences

Recurrences

Apply widening: affine hull

Recurrences

Recurrences

Recurrences

Recurrences

Apply induction

Recurrences

Recurrences

Recurrences

Recurrences

Recurrences—Widening

Sequence of $R^{\text{want}} = \{(N-1, N-1)\}$ and $R^{\text{want}} = \{(N-2, N-2)\}$
replaced by $R^{\text{want}} = \{(i, i) \mid 1 \leq i < N\}$

Recurrences—Widening

Sequence of $R^{\text{want}} = \{(N-1, N-1)\}$ and $R^{\text{want}} = \{(N-2, N-2)\}$
replaced by $R^{\text{want}} = \{(i, i) \mid 1 \leq i < N\}$

⇒ application of **widening**

⇒ turns ascending chain into eventually stationary ascending chain

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$$

$$\text{IN} \quad \text{IN} \quad \text{IN} \quad \exists N : \forall n \geq N : B_{n+1} = B_n$$

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$$

Recurrences—Widening

Sequence of $R^{\text{want}} = \{(N-1, N-1)\}$ and $R^{\text{want}} = \{(N-2, N-2)\}$
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$$\text{IN} \quad \text{IN} \quad \text{IN} \quad \exists N : \forall n \geq N : B_{n+1} = B_n$$

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$$

Here: affine hull (smallest affine subspace containing A_i)

$$\{(N-1; N-1)\} \subseteq \{(N-1; N-1), (N-2; N-2)\}$$

$$\text{IN} \quad \text{IN} \\ \{(N-1; N-1)\} \subseteq \{(i; i)\}$$

Recurrences and Tabling

Widening may be overestimate

⇒ important to keep track of unproved correspondence R^{lost}

Further occurrence of same pair of computations

⇒ further widening, or

⇒ apply induction hypothesis

⇒ after backpropagation of R^{lost} , check hypotheses do not intersect R^{lost}

⇒ perform “narrowing” if needed

Recurrences and Tabling

Widening may be overestimate

⇒ important to keep track of unproved correspondence R^{lost}

Other occurrence of same pair of computations

⇒ widening, or

⇒ apply induction hypothesis

⇒ after backpropagation of R^{lost} , check hypotheses do not intersect R^{lost}

⇒ perform “narrowing” if needed

or

⇒ reuse tabled results (R^{lost})

Implementation Details

All operations on sets/relations of integers performed by `isl`

- intersection
- union
- application of a map (relation) to a set, the domain of a relation or the range of a relation
- inverse of a relation (or application of an inverse map)
- integer affine hull (widening)
- subset testing
- equality testing
- emptiness testing

Implementation Details

All operations on sets/relations of integers performed by `isl`

- intersection
- union
- application of a map (relation) to a set, the domain of a relation or the range of a relation
- inverse of a relation (or application of an inverse map)
- integer affine hull (widening)
- subset testing
- equality testing
- emptiness testing

Other features

- explicit support for existentially quantified variables
- uses PIP for solving (P)ILP problems
- all computations in exact integer arithmetic using GMP
- built-in incremental LP solver
- released under LGPL license

Experiments—CLooG

Most experiments performed on output of CLooG

- CLooG generates code for scanning polyhedra
- Different backends (PolyLib/isl) may generate different code
- Experiment: prove different outputs are equivalent

Experiments—CLoog

Most experiments performed on output of CLoog

- CLoog generates code for scanning polyhedra
- Different backends (PolyLib/isl) may generate different code
- Experiment: prove different outputs are equivalent

S1: $\{(i,j) \mid 1 \leq i \leq n \leq m \wedge j = i\}$

S2: $\{(i,j) \mid 1 \leq i \leq n \leq m \wedge i \leq j \leq n\}$

S3: $\{(i,j) \mid 1 \leq i \leq m \wedge j = n \leq m\}$

```
for (i=1; i<=m; i++) {  
  if (i <= n) {  
    S1(i,i);  
  }  
  for (j=i; j<=n; j++) {  
    S2(i);  
  }  
  S3(i,n);  
}
```


CLooG Experiment Example

```
if (N >= 1) {  
  for (i=1;i<=N;i++) {  
    if (i <= 2*M) {  
      S1(i);  
    }  
    if (i>=max(M,2*M+1)) {  
      S1(i);  
    }  
  }  
  for (i=N+1;i<=2*N;i++) {  
    S2(i);  
  }  
}
```

```
if (N >= 1) {  
  for (i=1;i<=N;i++) {  
    if (i >= M) {  
      S1(i);  
    }  
    if (i<=min(2*M,M-1)) {  
      S1(i);  
    }  
  }  
  for (i=N+1;i<=2*N;i++) {  
    S2(i);  
  }  
}
```

CLooG Experiment Example

```
token = 0;
if (N >= 1) {
    for (i=1;i<=N;i++) {
        if (i <= 2*M) {
            token = f1(token, i);
        }
        if (i>=max(M,2*M+1)) {
            token = f1(token, i);
        }
    }
    for (i=N+1;i<=2*N;i++) {
        token = f2(token, i);
    }
}
out = token;
```

```
token = 0;
if (N >= 1) {
    for (i=1;i<=N;i++) {
        if (i >= M) {
            token = f1(token, i);
        }
        if (i<=min(2*M,M-1)) {
            token = f1(token, i);
        }
    }
    for (i=N+1;i<=2*N;i++) {
        token = f2(token, i);
    }
}
out = token;
```

Experimental Results

Program pair	cases	lines	loops	time	widenings
Motivating example	1	47	2	0.03s	1
USVD	1	652	98	0.63s	0
CLooG test 1	1	62	4	0.06s	4
CLooG test 2	1	291	34	0.59s	51
CLooG test 3	1	1634	108	19.85s	116
CLooG test suite	99	10804	1384	44.41s	1269

Conclusions

Novel approach for equivalence checking of static affine programs

- uses widening (affine hull) instead of transitive closure for handling recurrence
- supports non-uniform recurrences
- supports associative and commutative operations with fixed number of arguments
- publicly available prototype implementation
(<http://www.kotnet.org/~skimo/loop/isa-0.08.tar.bz2>)

Future work

- associative operations with a variable number of arguments, e.g., different encodings of $\sum_{0 \leq i < N} a[i]$
- general reductions
- data-dependent and non-affine conditions and index expressions

Partial Loop Unrolling

```
out = 0;
if (n >= 0) {
    for (i = 1; i <= n; ++i)
        out = out + i;
}
```

```
out = 0;
if (n >= 0) {
    for (i = 1; i <= n/2; ++i) {
        out = out + (2*i-1);
        out = out + 2*i;
    }
    if (n % 2 == 1)
        out = out + n;
}
```