

Closed-Form Approximations for Constant Continuous Annuities*

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Abstract

For a series of cash flows, its stochastically discounted or compounded value is often a key quantity of interest in finance and actuarial science. Unfortunately, even for most realistic rate of return models, it may be too difficult to obtain analytic expressions for the risk measures involving this discounted sum. Some recent research has demonstrated that in the case where the return process follows a Brownian motion, the so-called comonotonic approximations usually provide excellent and robust estimates of risk measures associated with discounted sums of cash flows involving log-normal returns.

In this paper, we derive analytic approximations for risk measures in case one considers the continuous counterpart of a discounted sum of log-normal returns. Although one may consider the discrete sums as providing a more realistic situation than its continuous counterpart, considering in this case, the continuous setting leads to more tractable explicit formulas and may therefore provide further insight necessary to expand the theory and to exploit new ideas for later developments.

Moreover, the closed-form approximations we derive in this continuous set-up can then be compared more effectively with some exact results, thereby facilitating a discussion about the accuracy of the approximations. Indeed, in the discrete setting, one always must compare approximations with results from simulation procedures which always give rise to room of debate. Our numerical comparisons reveal that the comonotonic ‘maximal variance’ lower bound approximation provides an excellent fit for several risk measures associated with integrals involving log-normal returns. Similar results as we derive here for continuous annuities can also be obtained in case of continuously compounding which therefore opens a roadmap for deriving closed-form approximations for the prices of Asian options. Future research will also focus on optimal portfolio selection problems.

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1 Introduction

The stochastically discounted or compounded value of a series of cash flows is a random variable (r.v.) of importance in finance and actuarial science. Such a discounted or compounded sum is most often introduced as a r.v. S given by

$$S = \sum_{i=1}^n \alpha_i e^{Z_i}. \quad (1)$$

Here, the α_i are non-negative real numbers and (Z_1, Z_2, \dots, Z_n) is a random vector.

The accumulated value at time n of a series of future deterministic saving amounts α_i can be written in the form (1), where Z_i denotes the random accumulation factor over the period $[i, n]$. Also the present value of a series of future deterministic payments α_i can be written in the form (1), where now Z_i denotes the random discount factor over the period $[0, i]$. The valuation of Asian or basket options, the setting of provisions and required capitals in an insurance context boils down to the evaluation of risk measures related to the distribution function of a random variable S as defined in (1). We define a risk measure to be a mapping from the set of random variables, usually representing the risks at hand, to the set of real number \mathbb{R} . Risk measures are a helpful tool for decision-making since they reduce the information available about the random variable X into one single number $\rho[X]$.

Common risk measures in actuarial science are premium principles, see for instance Goovaerts, De Vijlder & Haezendonck (1984), or also Chapter 5 in Kaas, Goovaerts, Dhaene & Denuit (2001). Other risk measures are used for determining provisions and capital requirements of an insurer, in order to avoid insolvency. Then risk measures are based on the upper tails of distribution functions. Such measures of risk are considered in Artzner, Delbaen, Eber & Heath (1999), Wirch & Hardy (2000), Panjer (2002), Dhaene, Goovaerts & Kaas (2003), Dhaene, J., Vanduffel, S., Tang, Q., Goovaerts, M.J., Kaas, R. & Vyncke, D. (2004) and Tsanakas & Desli (2003), among others.

In this paper, we consider the p -quantile risk measure, often called the ‘VaR’ (Value-at-Risk) at level p in the financial and actuarial literature. For any p in $(0, 1)$, the p -quantile risk measure for a random variable X , which will be denoted by $Q_p[X]$, is defined to be

$$Q_p[X] = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in (0, 1), \quad (2)$$

where $F_X(x) = \Pr[X \leq x]$. Note that expression (2) can also be used to define $Q_0[X]$ and $Q_1[X]$. For the latter quantile, we take the convention $\inf \emptyset = +\infty$. We then find that $Q_0(X) = -\infty$. For a bounded random variable X , we have that $Q_1[X] = \max(X)$. The quantile function $Q_p[X]$ is a non-decreasing and left-continuous function of p . Finally, we also note that for all $x \in \mathbb{R}$ and $p \in [0, 1]$ that

$$Q_p[X] \leq x \Leftrightarrow p \leq F_X(x). \quad (3)$$

Note that the equivalence relation (3) holds with equalities if F_X is continuous at this particular value of x .

Another very popular risk measure, which is also being considered in this paper, is the Conditional Tail Expectation for which at level p , we denote it by $\text{CTE}_p[X]$. It is defined as

$$\text{CTE}_p[X] = E[X \mid X > Q_p[X]], \quad p \in (0, 1). \quad (4)$$

Loosely speaking, the conditional tail expectation at level p is equal to the mean of the top $(1-p)\%$ losses. It can also be interpreted as the VaR at level p augmented by the average exceeding of the claims X over that quantile, given that such exceeding occurs. Conditional Tail Expectations have been considered in Panjer (2002) and Landsman & Valdez (2003).

We also define the stop-loss premium with retention d of the random variable X to be $E[(X-d)_+]$, with the notation $(x-d)_+ = \max(x-d, 0)$. By using partial integration, we obtain

$$E[(X-d)_+] = \int_d^\infty (1-F_X(x)) dx, \quad -\infty < d < +\infty, \quad (5)$$

from which we see that the stop-loss premium with retention d can be considered as the weight of an upper tail of the c.d.f. (cumulative distribution function) of X : it is the surface between the c.d.f. F_X of X and the constant function 1, from d on. Also useful is the observation that $E[(X-d)_+]$ is a decreasing continuous function of d , with derivative $F_X(d) - 1$ at d , which vanishes when d reaches infinity.

Yet even for the most known stochastic return model i.e. when (Z_1, Z_2, \dots, Z_n) is a multivariate normal distributed random vector, it is difficult to obtain analytic expressions for most of the the risk measures involving these discounted or compounded sum (1). This is because the dependency structure of the terms involved in the sum is too cumbersome to work.

Unsurprisingly, in the literature, a variety of approximation techniques have been suggested and in a series of papers so-called comonotonic approximations for the c.d.f. and risk measures related to the random variable (r.v.) S have been proposed. We refer to Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b) for an extensive overview on the theory of comonotonicity and its applications.

The accuracy of the comonotonic upper bound and lower bound approximations has been demonstrated, amongst other results, by Huang, H., Milevsky, M. & Wang, J. (2004) and Vanduffel, Hoedemakers & Dhaene (2004).

The discrete case (sums of random variables) has a continuous counterpart (integrals of stochastic processes) and in this paper, we focus on some explicit results in the case where the stochastic process under consideration is a geometric Brownian motion which is a continuous equivalent of the Gaussian setting (1).

Hence, in line with Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002b), we here consider the continuous equivalent of (1) which is the continuous temporary annuity S_t defined by

$$S_t = \int_0^t \alpha(\tau) e^{-\delta\tau - \sigma B(\tau)} d\tau, \quad (6)$$

where $\{B(\tau), \tau \geq 0\}$ represents a standard Brownian motion, i.e. the process has independent and stationary increments, $B(0) = 0$ and for any $\tau \geq 0$, the random variable $B(\tau)$ is Normally distributed with mean 0 and variance τ . Furthermore, the drift δ and the volatility σ are positive real numbers. Finally, the payments are described by $\alpha(\tau)$ which is a non-negative and continuous function of τ .

Recall also that the convex ordering, denoted by \leq_{cx} , reflects the common preferences of all risk-averse decision takers when choosing between risks with equal mean. This holds in both the classical utility theory from von Neuman & Morgenstern as in Yaari's dual utility theory, see Dhaene, J., Vanduffel, S., Tang, Q., Goovaerts, M.J., Kaas, R. & Vyncke, D. (2004) for more details.

In this paper we show that, in case of a constant annuity the comonotonic upper bound approximation gives rise to closed-form results for the quantiles, conditional tail expectation and stop-loss premiums. We also demonstrate that, for some specific choices of the conditioning random variable Λ , explicit results for these risk measures can be obtained in case one uses the comonotonic lower bound approximation.

Note that we agree that this is of course a rather theoretical exercise because in reality, one almost always deal with discrete sums and not with continuous integrals. However, we observe that several research is done usually based on a continuous setting for the particular problem of interest. This is because this often leads to more tractable formulas and may therefore provide initial insights and as such pave the way for developing the results in real-life discrete settings. In case of constant perpetuities, for instance, it is known from Merton (1975) that in a continuous setting the cumulative distribution function of S_∞ can be calculated very easily since one can prove that S_∞^{-1} is indeed a Gamma distributed random variable with parameters $\frac{2\delta}{\sigma^2}$ and $\frac{\sigma^2}{2}$. In this paper, we say that the random variable X is Gamma distributed with parameters α and β when its probability density function (p.d.f.) is expressed as

$$f_X(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x > 0,$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma(\cdot)$ denotes the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du \quad (\alpha > 0).$$

Its reciprocal $Y = 1/X$ is said to be reciprocal Gamma distributed whose p.d.f. we can write as

$$f_Y(y; \alpha, \beta) = f_X(1/y; \alpha, \beta)/y^2, \quad y > 0.$$

It is straightforward to prove that the quantiles and conditional tail expectations of Y are given by

$$Q_p[Y] = \frac{1}{F_X^{-1}(1-p; \alpha, \beta)}, \quad p \in (0, 1) \tag{7}$$

and

$$\text{CTE}_p[Y] = \frac{F_X(F_X^{-1}(1-p; \alpha, \beta); \alpha-1, \beta)}{(1-p)(\alpha-1)\beta}, \quad p \in (0, 1), \tag{8}$$

where $F_X(\cdot; \alpha, \beta)$ is the cumulative d.f. of the Gamma distribution with parameters α and β . Since the Gamma distribution is readily available in many statistical software packages, these risk measures can easily be determined.

With this insight it has been proposed by Huang, Milevsky and Wang (2004) to use the reciprocal Gamma distribution function as a suitable choice for approximating the risk measures of finite annuities. Therefore, a first reason to mention the explicit results is that it might be useful for future research. As a second reason, we can compare our approximations with available explicit results. In the discrete setting we always need to compare with simulated results which always give rise to room of debate. In the continuous setting, however, there exist explicit closed-form results

for some interesting actuarial quantities and hence, these can be compared with the results obtained by using the comonotonic approximations. Although the results we mention here are only true in the discounting case, similar results can also be obtained in case of continuously compounding. Obviously, this creates the framework for deriving closed-form approximations for the prices of Asian options. Future research will also focus on optimal portfolio selection problems.

For the remainder of this paper, it has been organized as follows. In Section 2, as the analogue of the continuous case developed in this paper, we give the general results and briefly describe the 'maximal variance' lower bound for the discrete case of the sum in (1). The main focus of this paper is developing the upper and lower bound approximations of several risk measures for the sum in (6) so that in Section 3, we derive explicit closed-form formulas for these risk measures related to continuous annuities. In Section 4, we compare these upper and lower bound approximations with explicit results in the case of constant continuous perpetuities. Section 5 provides some concluding remarks.

2 Comonotonic approximations - the discrete case

2.1 General results

Let the random variable S_n be given by (1), where the α_i are non-negative real numbers and the random vector (Z_1, Z_2, \dots, Z_n) has a multivariate Normal distribution. Consider the conditioning random variable Λ , given by

$$\Lambda = \sum_{i=1}^n \gamma_i Z_i. \quad (9)$$

and also the random variables S^l and S^c defined by

$$S^l = E[S|\Lambda] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i}\Phi^{-1}(U)} \quad (10)$$

and

$$S^c = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \sigma_{Z_i}\Phi^{-1}(U)}, \quad (11)$$

respectively. Here U is a Uniform(0, 1) r.v., Φ is the c.d.f. of the $N(0, 1)$ distribution and r_i is the correlation between Z_i and Λ .

As demonstrated in Kaas, Dhaene & Goovaerts (2000), it follows that for the r.v.'s S , S^l and S^c , the following convex order relations hold:

$$S^l \leq_{cx} S \leq_{cx} S^c.$$

For example, in the case of lognormal distribution, we have the expressions for the risk measures:

$$Q_p[S^c] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \sigma_{Z_i}\Phi^{-1}(p)}, \quad (12)$$

$$\text{CTE}_p[S^c] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2 \frac{\Phi(\sigma_{Z_i} - \Phi^{-1}(p))}{1-p}}, \quad p \in (0, 1). \quad (13)$$

Now provided all coefficients r_i are positive, we also find, still in the lognormal case, that for $p \in (0, 1)$:

$$Q_p [S^l] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i}\Phi^{-1}(p)}, \quad (14)$$

$$\text{CTE}_p [S^l] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \frac{\Phi(r_i\sigma_{Z_i} - \Phi^{-1}(p))}{1-p}.. \quad (15)$$

Notice that the expected values of the random variables S , S^c and S^l are all equal:

$$E(S) = E(S^l) = E(S^c) = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2}, \quad (16)$$

whereas their variances are given by

$$\text{Var}(S) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j e^{E[Z_i] + E[Z_j] + \frac{1}{2}(\sigma_{Z_i}^2 + \sigma_{Z_j}^2)} (e^{\text{cov}(Z_i, Z_j)} - 1), \quad (17)$$

$$\text{Var}(S^l) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j e^{E[Z_i] + E[Z_j] + \frac{1}{2}(\sigma_{Z_i}^2 + \sigma_{Z_j}^2)} (e^{r_i r_j \sigma_{Z_i} \sigma_{Z_j}} - 1) \quad (18)$$

and

$$\text{Var}(S^c) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j e^{E[Z_i] + E[Z_j] + \frac{1}{2}(\sigma_{Z_i}^2 + \sigma_{Z_j}^2)} (e^{\sigma_{Z_i} \sigma_{Z_j}} - 1), \quad (19)$$

respectively.

2.2 The ‘maximal variance’ lower bound approach

Comparing variances is meaningful when comparing stop-loss premiums of convex ordered random variables, see, e.g. Kaas, van Heerwaarden & Goovaerts (1994, p. 68). The following relation links variances and stop-loss premiums:

$$\frac{1}{2} \text{Var}[X] = \int_{-\infty}^{\infty} (E[(X-t)_+] - (E[X]-t)_+) dt. \quad (20)$$

To prove this relation, write

$$\int_{-\infty}^{\infty} (E[(X-t)_+] - (E[X]-t)_+) dt = \int_{-\infty}^{E[X]} E[(t-X)_+] dt + \int_{E[X]}^{\infty} E[(X-t)_+] dt.$$

Interchanging the order of the integrations and using integration by parts, one finds

$$\int_{-\infty}^{E[X]} E[(t-X)_+] dt = \int_{-\infty}^{E[X]} \int_{-\infty}^t F_X(x) dx dt = \frac{1}{2} \int_{-\infty}^{E[X]} (x - E[X])^2 dF_X(x).$$

Similarly,

$$\int_{E[X]}^{\infty} E[(X-t)_+] dt = \frac{1}{2} \int_{E[X]}^{\infty} (x - E[X])^2 dF_X(x).$$

This proves (20). We deduce from this that if $X \leq_{cx} Y$,

$$\int_{-\infty}^{\infty} (E[(Y-t)_+] - E[(X-t)_+]) dt = \frac{1}{2} \{Var[Y] - Var[X]\} \quad (21)$$

Thus, if $X \leq_{cx} Y$, their stop-loss distance, i.e. the integrated absolute difference of their respective stop-loss premiums, equals half the variance difference between these two random variables. As the integrand in (21) is non-negative, we find that if $X \leq_{cx} Y$ whilst $Var[X] = Var[Y]$, then this means that X and Y must have equal stop-loss premiums and hence the same d.f. We also find that $\frac{1}{2}\{Var[Y] - Var[X]\}$ can be interpreted as a measure for the "average error" one makes when approximating the stop-loss premiums of Y by those of the less convex X . This indicates that if we want to replace S by the less convex S^l , the best approximations will be the ones where the variance of S^l is 'as close as possible' to the variance of S . In other words, we should try to choose the coefficients γ_i of the conditioning variable Λ defined in (9) such that the variance of S^l is maximized.

Vanduffel, Hoedemakers & Dhaene (2004) proved that the first order approximation of the variance of S^l will be maximized for the following choice of the parameters γ_i :

$$\gamma_i = \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2}, \quad i = 1, \dots, n. \quad (22)$$

Indeed, from (19) we find that

$$\begin{aligned} Var(S^l) &\approx \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j e^{E[Z_i] + E[Z_j] + \frac{1}{2}(\sigma_{Z_i}^2 + \sigma_{Z_j}^2)} (r_i r_j \sigma_{Z_i} \sigma_{Z_j}) \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j e^{E[Z_i] + E[Z_j] + \frac{1}{2}(\sigma_{Z_i}^2 + \sigma_{Z_j}^2)} \left(\frac{Cov[Z_i, \Lambda] Cov[Z_j, \Lambda]}{Var(\Lambda)} \right) \\ &= \frac{(Cov(\sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} Z_i, \Lambda))^2}{Var(\Lambda)} \\ &= (Corr(\sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} Z_i, \Lambda))^2 Var(\sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} Z_i). \end{aligned} \quad (23)$$

Hence, the first order approximation of $Var(S^l)$ is maximized when Λ is given by

$$\Lambda = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} Z_i. \quad (24)$$

Note that also in case the α_i are not all positive, the choice (22) will optimize the first order approximation of the variance of S^l . In the remainder of this chapter and also further in the second part of this work, we will always assume that the conditioning r.v. Λ is given by (24). Notice that this optimal choice for Λ is slightly different from the choice that was made for this r.v. in Dhaene, Denuit, Kaas, Goovaerts & Vyncke (2002b).

One can easily prove that the first order approximation for $Var(S^l)$ with Λ given by (24) is equal to the first order approximation of $Var(S)$. This observation gives an additional indication that this particular choice for Λ will provide a good fit. We emphasize that the conditioning r.v. Λ defined in (24) does not necessarily maximize

the variance of S^l , but has to be understood as an approximation for the optimal Λ . Theoretically, one could use numerical procedures to determine the optimal Λ , but this would outweigh one of the main features of the convex bounds, namely that the quantiles and conditional tail expectations (and also other actuarial quantities such as stop-loss premiums) can easily be determined analytically. Having a ready-to-use approximation that can be implemented easily is important from a practical point of view.

3 Closed-form comonotonic approximations for the continuous case

3.1 General results

Let $Y(\tau) = \delta\tau + \sigma B(\tau)$ and $X(\tau) = \exp\{-Y(\tau)\}$. Analogous to the discrete setting discussed in Kaas, Dhaene & Goovaerts (2000), it can be shown that $S_t^l \leq_{cx} S_t \leq_{cx} S_t^c$, where the random variable S_t^c and S_t^l are defined by

$$S_t^c = \int_0^t F_{\alpha(\tau)X(\tau)}^{-1}(U) d\tau = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma \sqrt{\tau} \Phi^{-1}(U)} d\tau \quad (25)$$

and

$$S_t^l = E[S_t | \Lambda] = \int_0^t \alpha(\tau) e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(V)} d\tau, \quad (26)$$

where U is a Uniform(0,1) random variable, the conditioning variable Λ follows a Normal distribution and $V = \Phi\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right)$ is standard uniformly distributed. Furthermore, $r(\tau)$ is defined by

$$r(\tau) = \frac{\text{cov}[Y(\tau), \Lambda]}{\sigma_\Lambda \sigma \sqrt{\tau}}. \quad (27)$$

Since $B(\tau)$ is a Brownian motion process, it follows that the conditional random variable $Y(\tau) | \Lambda = \lambda$ is Normally distributed with mean

$$E[Y(\tau) | \Lambda = \lambda] = \delta\tau + r(\tau)\sigma\sqrt{\tau}\frac{\lambda}{\sigma_\Lambda}$$

and variance

$$\text{Var}[Y(\tau) | \Lambda = \lambda] = \sigma^2\tau(1 - r^2(\tau)).$$

We also define the quantity δ^* :

$$\delta^* = \left(\delta - \frac{1}{2}\sigma^2\right). \quad (28)$$

Throughout the remainder of this paper, we assume that $\delta^* > 0$.

Since $\alpha(\tau)$ is assumed to be non-negative, S_t^c will be an integral of comonotonic random variables. Hence, the quantiles, conditional tail expectations and stop-loss

premiums of S_t^c follow from

$$Q_p[S_t^c] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma \sqrt{\tau} \Phi^{-1}(p)} d\tau, \quad (29)$$

$$\text{CTE}_p[S_t^c] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \frac{\Phi[\sigma\sqrt{\tau} - \Phi^{-1}(p)]}{1-p} d\tau, \quad (30)$$

$$E[S_t^c - d]_+ = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \Phi[\sigma\sqrt{\tau} - \Phi^{-1}(p)] d\tau - d(1-p), \quad (31)$$

with $0 < p < 1$ and $d > 0$ determined as the unique root of $Q_p[S_t^c] = d$. The expressions (29), (30) and (31) are the continuous counterparts of the formulas derived in Vanduffel, Hoedemakers & Dhaene (2004).

Likewise, we find that S_t^l will be an integral of comonotonic random variables in case the function $f(\tau) = \text{cov}[Y(\tau), \Lambda]$ remains non-negative. Hence, we find the following continuous analogues of similar expressions in Vanduffel, Hoedemakers & Dhaene (2004):

$$Q_p[S_t^l] = \int_0^t \alpha(\tau) e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(p)} d\tau, \quad (32)$$

$$\text{CTE}_p[S_t^l] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \frac{\Phi[r(\tau)\sigma\sqrt{\tau} - \Phi^{-1}(p)]}{1-p} d\tau, \quad (33)$$

$$E[S_t^c - d]_+ = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \Phi[r(\tau)\sigma\sqrt{\tau} - \Phi^{-1}(p)] d\tau - d(1-p), \quad (34)$$

with again $0 < p < 1$ and $d > 0$ uniquely determined by $Q_p[S_t^l] = d$. The formula's (28)-(34) can also be found in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002b).

3.2 Upper bound approach

In the remainder of the paper we will assume that $\alpha(\tau) = 1$. Hence we have that S_t^c is an integral of comonotonic random variables. The quantiles of S_t^c follow from (29):

$$Q_p[S_t^c] = \int_0^t e^{-\delta\tau + \sigma \sqrt{\tau} \Phi^{-1}(p)} d\tau, \quad (0 < p < 1). \quad (35)$$

By substituting $y = \sqrt{\tau}$ and realizing that the resulting integral can be rewritten in terms of the standard-normal c.d.f. we find the following analytical expression:

$$\begin{aligned} Q_p[S_t^c] &= \frac{1}{\delta} \\ &\quad - \frac{1}{\delta} e^{-\delta t + \sigma \sqrt{t} \Phi^{-1}(p)} \\ &\quad + \frac{1}{\delta} \sqrt{2\pi} a e^{\frac{a^2}{2}} (\Phi[\sqrt{2\delta t} - a] - \Phi[-a]), \end{aligned} \quad (36)$$

with

$$a = \frac{\sigma \Phi^{-1}(p)}{\sqrt{2\delta}}, \quad (0 < p < 1). \quad (37)$$

From (30) we have that the conditional tail expectations are given by

$$\text{CTE}_p[S_t^c] = \int_0^t e^{-\delta\tau + \sigma^2\tau/2} \frac{\Phi[\sigma\sqrt{\tau} - \Phi^{-1}(p)]}{1-p} d\tau, \quad (0 < p < 1). \quad (38)$$

Using the same substitution $y = \sqrt{\tau}$ as in the case of the quantiles, we find after some long computations the following closed-form expression for the conditional tail expectations:

$$\begin{aligned} \text{CTE}_p[S_t^c] &= \frac{1}{\delta^*} \\ &\quad - \frac{1}{\delta^*} e^{-\delta^*t} \frac{\Phi[\sigma\sqrt{t} - \Phi^{-1}(p)]}{1-p} \\ &\quad + \frac{\sigma}{\delta^*} e^{\frac{a^2 - (\Phi^{-1}(p))^2}{2}} \sqrt{\frac{1}{2\delta}} \frac{\Phi[\sqrt{2\delta t} - a] - \Phi[-a]}{1-p}, \quad (0 < p < 1). \end{aligned} \quad (39)$$

Finally, we obtain that the stop-loss premiums with retentions $d > 0$ are given by

$$\begin{aligned} E[(S_t^c - d)_+] &= \frac{1}{\delta^*} (1-p) \\ &\quad - \frac{1}{\delta^*} e^{-\delta^*t} \Phi[\sigma\sqrt{t} - \Phi^{-1}(p)] \\ &\quad + \frac{\sigma}{\delta^*} e^{\frac{a^2 - (\Phi^{-1}(p))^2}{2}} \sqrt{\frac{1}{2\delta}} (\Phi[\sqrt{2\delta t} - a] - \Phi[-a]) \\ &\quad - d(1-p), \quad (0 < p < 1), \end{aligned} \quad (40)$$

where p can be obtained by solving $Q_p[S_t^c] = d$. We remark that the expressions (39) and (40) are valid under the condition that $\delta^* > 0$.

3.3 Lower bound approaches

3.3.1 General results

In order to compute the risk measures of S_t^l , Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002b) proposed to use the conditioning random variable $\Lambda = \int_0^t e^{-\delta\tau} B(\tau) d\tau$ because this can be seen as a linear transformation of a kind of first order approximation of S_t . However, in the same way as in the discrete case developed in the previous section, one can prove that the alternative choice $\Lambda = \int_0^t e^{-\delta^*\tau} B(\tau) d\tau$ will maximize the first order Taylor approximation for the variance of S_t^l . The latter choice for Λ is therefore likely to provide better approximations for the risk measures of S_t . We have in this case that Λ is Normally distributed with mean 0 and variance

$$\begin{aligned} \sigma_\Lambda^2 = \text{Var}[\Lambda] &= \int_0^t \int_0^t e^{-\delta^*(\tau+\nu)} \min(\tau, \nu) d\tau d\nu \\ &= \frac{1}{2\delta^{*3}} + \frac{3 + 2\delta^*t - 4e^{\delta^*t}}{2\delta^{*3}e^{2\delta^*t}}. \end{aligned}$$

$r(\tau)$ is given by

$$r(\tau) = \frac{\text{cov}[Y(\tau), \Lambda]}{\sigma_\Lambda \sigma \sqrt{\tau}} = \frac{1}{\sigma_\Lambda \sqrt{\tau}} \left[\frac{1 - e^{-\delta^*\tau}}{\delta^{*2}} - \frac{\tau e^{-\delta^*t}}{\delta^*} \right], \quad \tau \leq t.$$

Since the function $f(\tau) = \text{cov}[Y(\tau), \Lambda]$ is a non-negative function, S_t^l will be an integral of comonotonic random variables. Unfortunately, in case of finite annuities, there seems to be no closed-form solutions for the quantiles, conditional tail expectations and stop-loss premiums of S_t^l , in case one uses any of the two discussed choices for Λ . We now propose two other choices for Λ , so that explicit form approximations for these risk measures can be obtained.

3.3.2 $\Lambda = \int_0^\infty e^{-\delta^*\tau} B(\tau) d\tau$

The quantiles of S_t^l ($0 < p < 1$) follow from

$$Q_p[S_t^l] = \int_0^t e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(p)} d\tau, \quad (41)$$

with

$$r(\tau) = \sqrt{\frac{2}{\delta^*\tau}}(1 - e^{-\delta^*\tau}).$$

Expression (41) can be rewritten as

$$Q_p[S_t^l] = \int_0^t e^{-\delta^*\tau - \frac{1}{2}c^2(1-e^{-\delta^*\tau})^2 + c\Phi^{-1}(p)(1-e^{-\delta^*\tau})} d\tau, \quad (42)$$

with

$$c = \sigma\sqrt{\frac{2}{\delta^*}}.$$

By making the substitution $y = e^{-\delta^*\tau}$ one sees that the integral (42) can be rewritten in terms of the standard-normal c.d.f. Hence, we find the following analytical expression for the quantiles of S_t^l :

$$Q_p[S_t^l] = \frac{1}{c\delta^*} \sqrt{2\pi} e^{\frac{(\Phi^{-1}(p))^2}{2}} (\Phi(k_t) - 1 + p), \quad (43)$$

with

$$k_t = c(1 - e^{-\delta^*t}) - \Phi^{-1}(p).$$

The conditional tail expectations of S_t^l are now given by

$$\text{CTE}_p[S_t^l] = \int_0^t e^{-\delta^*\tau} \frac{\Phi[c(1 - e^{-\delta^*\tau}) - \Phi^{-1}(p)]}{1 - p} d\tau,$$

After some tedious algebra we find that

$$\begin{aligned} \text{CTE}_p[S_t^l] &= \frac{1}{\delta^*(1-p)} \Phi(k_t)(1 - e^{-\delta^*t}) \\ &\quad + \frac{1}{\delta^*(1-p)} \left(\frac{\Phi^{-1}(p)}{c}\right)(1 - p - \Phi(k_t)) \\ &\quad - \frac{1}{\sqrt{2\pi}c\delta^*(1-p)} (e^{-\frac{1}{2}[\Phi^{-1}(p)]^2} - e^{-\frac{1}{2}k_t^2}), \end{aligned}$$

Finally, the stop-loss premiums of S_t^l with retentions $d > 0$ are now given by

$$\begin{aligned}
E[(S_t^l - d)_+] &= \frac{1}{\delta^*} \Phi(k_t) (1 - e^{-\delta^* t}) \\
&\quad + \frac{1}{\delta^*} \left(\frac{\Phi^{-1}(p)}{c} \right) (1 - p - \Phi(k_t)) \\
&\quad - \frac{1}{\sqrt{2\pi c \delta^*}} (e^{-\frac{1}{2}[\Phi^{-1}(p)]^2} - e^{-\frac{1}{2}k_t^2}) \\
&\quad - d(1 - p),
\end{aligned} \tag{44}$$

where p is the unique root of $Q_p[S_t^l] = d$.

We point out that this specific choice for $\Lambda = \int_0^\infty e^{-\delta^* \tau} B(\tau) d\tau$ cannot be expected to perform very well for finite annuities. However, when t reaches infinity, this choice for Λ leads to the continuous equivalent of the ‘maximal variance’ lower bound approach that we was discussed in section 2. Hence, our specific choice for Λ is likely to yield excellent results when t reaches infinity whilst allowing for finite t an analytical expression for the selected risk measures of S_t^l too.

3.3.3 $\Lambda = B(t)$

The quantiles of S_t^l ($0 < p < 1$) are still given by

$$Q_p[S_t^l] = \int_0^t e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(p)} d\tau,$$

but now with $r(\tau)$ given by

$$r(\tau) = \sqrt{\frac{\tau}{t}}.$$

It proves easily that one obtains the following closed-form expressions for the quantiles of S_t^l :

$$Q_p[S_t^l] = \frac{\sqrt{2\pi t}}{\sigma} e^{\frac{\gamma^2}{2}} (\Phi[\sigma\sqrt{t} - \gamma] - \Phi[-\gamma]), \tag{45}$$

with

$$\gamma = \Phi^{-1}(p) - \frac{\delta^* \sqrt{t}}{\sigma}.$$

The conditional tail expectations of S_t^l follow from

$$\text{CTE}_p[S_t^l] = \int_0^t e^{-\delta^* \tau} \frac{\Phi\left[\frac{\sigma\tau}{\sqrt{t}} - \Phi^{-1}(p)\right]}{1 - p} d\tau. \tag{46}$$

Again, after some computations we find that

$$\begin{aligned}
\text{CTE}_p[S_t^l] &= \frac{1}{\delta^*} - \frac{1}{\delta^*(1-p)} e^{-\delta^* t} (\Phi[\sigma\sqrt{t} - \Phi^{-1}(p)]) \\
&\quad + \frac{1}{\delta^*(1-p)} e^{-\left(\frac{\delta^* \sqrt{t}}{\sigma}\right)(\gamma + \frac{\delta^* \sqrt{t}}{2\sigma})} \times \left\{ \Phi[\sigma\sqrt{t} - \gamma] - \Phi[-\gamma] \right\}.
\end{aligned}$$

The stop-loss premiums of S_t^l with retentions $d > 0$ now follow from

$$E[(S_t^l - d)_+] = \int_0^t e^{-\delta^* \tau} \Phi\left[\frac{\sigma \tau}{\sqrt{t}} - \Phi^{-1}(p)\right] d\tau - d(1 - p),$$

where p is the root of $Q_p[S_t^l] = d$. We find that

$$\begin{aligned} E[(S_t^l - d)_+] &= +\frac{1}{\delta^*}(1 - p) - \frac{1}{\delta^*}e^{-\delta^* t}(\Phi[\sigma\sqrt{t} - \Phi^{-1}(p)]) \\ &\quad + \frac{1}{\delta^*}e^{-(\frac{\delta^* \sqrt{t}}{\sigma})(\gamma + \frac{\delta^* \sqrt{t}}{2\sigma})} \times \left\{ \Phi[\sigma\sqrt{t} - \gamma] - \Phi[-\gamma] \right\} \\ &\quad - d(1 - p). \end{aligned}$$

4 Application on perpetuities

Consider the perpetuity S_∞ defined by

$$S_\infty = \int_0^\infty \exp[-\delta\tau - \sigma B(\tau)] d\tau. \quad (47)$$

For this annuity, the cumulative distribution function of the perpetuity S_∞ , expressed in (47) can be calculated very easily since one can prove that its reciprocal S_∞^{-1} is Gamma distributed with parameters $\frac{2\delta}{\sigma^2}$ and $\frac{\sigma^2}{2}$. This result can be found in Merton (1975), see also Dufresne (1990) and Milevsky (1997) for various proofs of this result. Hence, we can compare the cumulative distribution functions of the lower bound S_∞^l and the upper bound S_∞^c with the exact cumulative distribution function of S_∞ . We propose to use the ‘maximal variance’ lower bound approach that we discussed in Subsection 2.2, since this is likely to provide the best results in case of infinite annuities.

From (7), (36) and (43), with $t \rightarrow \infty$, we find for $0 < p < 1$ the following expressions for the quantiles of S_∞ , S_∞^c and S_∞^l respectively.

$$\begin{aligned} Q_p[S_\infty] &= \frac{1}{F_X^{-1}(1 - p; \frac{2\delta}{\sigma^2}, \frac{\sigma^2}{2})}, \\ Q_p[S_\infty^c] &= \frac{1}{\delta}(1 + a\sqrt{2\pi}e^{\frac{a^2}{2}}\Phi[a]), \\ Q_p[S_\infty^l] &= \frac{1}{c\delta^*}\sqrt{2\pi}e^{\frac{(\Phi^{-1}(p))^2}{2}}(\Phi[c - \Phi^{-1}(p)] - 1 + p), \end{aligned}$$

with F_X the c.d.f. of the Gamma distribution and

$$\begin{aligned} a &= \frac{\sigma\Phi^{-1}(p)}{\sqrt{2\delta}}, \\ c &= \sigma\sqrt{\frac{2}{\delta^*}}. \end{aligned}$$

From (7), (8), (40) and (44), one also obtains closed-form results for the stop-loss premiums for S_∞ , S_∞^c and S_∞^l . This is left as an easy exercise for the interested reader.

To put some numerical values to the results, Table 1 shows the quantiles of S_∞^l , S_∞^c and S_∞ in the case where $\delta = 0.07$ and $\sigma = 0.1$. These results can be compared with the results reported in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002). The

p	$Q_p[S_\infty^l]$	$Q_p[S_\infty]$	$Q_p[S_\infty^c]$
0.95	23.62	23.63	25.90
0.975	26.09	26.13	29.34
0.99	29.37	29.49	34.08
0.995	31.90	32.10	37.86
0.999	38.00	38.49	47.38

Table 1: The table compares some selected exact quantiles of the constant perpetuity with the ‘maximal variance’ lower bound and upper bound approximations ($\delta=0.07$, $\sigma=0.1$).

small differences we observe, can be explained as follows : Firstly, the authors computed the quantiles of S_∞^c and S_∞^l by numerical evaluation of the expressions (29) and (32) with $t \rightarrow \infty$ and $\alpha(\tau) = 1$. Secondly, they used the conditioning variable $\Lambda = \int_0^\infty e^{-\delta\tau} B(\tau) d\tau$ whereas our explicit results rely on the ‘maximal variance’ lower bound approximation, involving $\Lambda = \int_0^\infty e^{-\delta^*\tau} B(\tau) d\tau$ as conditioning random variable.

In Table 2, we show quantiles of S_∞ , S_∞^c and S_∞^l but now for $\delta = 0.07$ and $\sigma = 0.2$. This example is interesting because it proves that for suitable choices of Λ , the c.d.f.’s of S_∞^l and S_∞ do not necessarily cross only once. In this respect it is worthwhile to mention that Vanduffel et al showed in a discrete setting that the cdf’s of S_t^l and S_t^c can only cross in the region where their distribution functions take a value that is contained in the interval $[p^-, p^+]$ for some p^- and $p^+ > 0$ leaving it as an open question whether this crossing point is unique or not. Exactly the same result can be drawn in the continuous setting, but despite the explicit expressions for the quantiles, we are still unable to answer satisfactorily the question concerning the uniqueness of the crossing point.

Finally, Table 3 compares the stop-loss premiums for different retention values d . The same comments as for Table 1 can be made. Here we give the expressions for the stop-loss premiums for the case of the perpetuities:

$$E[(S_\infty^c - d)_+] = \left(\frac{1}{\delta^*} - d\right) (1 - p) + \frac{\sigma}{\delta^*} e^{\frac{\sigma^2 - (\Phi^{-1}(p))^2}{2}} \sqrt{\frac{1}{2\delta}} \Phi[d]$$

and

$$\begin{aligned} E[(S_\infty^l - d)_+] &= \frac{1}{\delta^*} \Phi(c - \Phi^{-1}(p)) \\ &+ \frac{1}{\delta^*} \left(\frac{\Phi^{-1}(p)}{c}\right) (1 - p - \Phi(c - \Phi^{-1}(p))) \\ &- \frac{1}{\sqrt{2\pi c \delta^*}} (e^{-\frac{1}{2}[\Phi^{-1}(p)]^2} - e^{-\frac{1}{2}(c - \Phi^{-1}(p))^2}) \\ &- d(1 - p), \end{aligned} \tag{48}$$

Corresponding figures to Tables 1 through 3 are drawn as Figures 1 through 3, respectively, to help visualize the resulting upper and lower bound approximations to the true values of the quantiles as well as the stop-loss premiums. As visually seen

p	$Q_p[S_\infty^l]$	$Q_p[S_\infty]$	$Q_p[S_\infty^c]$
0.25	11.13	11.07	9.34
0.50	15.74	15.76	14.29
0.75	23.51	23.50	23.11
0.95	46.30	46.14	51.84
0.99	79.64	80.71	100.45
0.995	98.35	101.09	130.77

Table 2: The table compares some selected exact quantiles of the constant perpetuity with the ‘maximal variance’ lower bound and upper bound approximations ($\delta=0.07$, $\sigma=0.2$).

d	$E[S_\infty^l - d]_+$	$E[S_\infty - d]_+$	$E[S_\infty^c - d]_+$
10	5.4430	5.4457	5.5554
15	1.8590	1.8626	2.2690
20	0.4917	0.4961	0.8337
25	0.1229	0.1270	0.3079
30	0.0316	0.0344	0.1192

Table 3: The table compares some selected exact stop-loss premiums of the constant perpetuity with the ‘maximal variance’ lower bound and upper bound approximations ($\delta=0.07$, $\sigma=0.1$).

from these figures, the comonotonic ‘maximal variance’ lower bound approximations do indeed come very close to the true values.

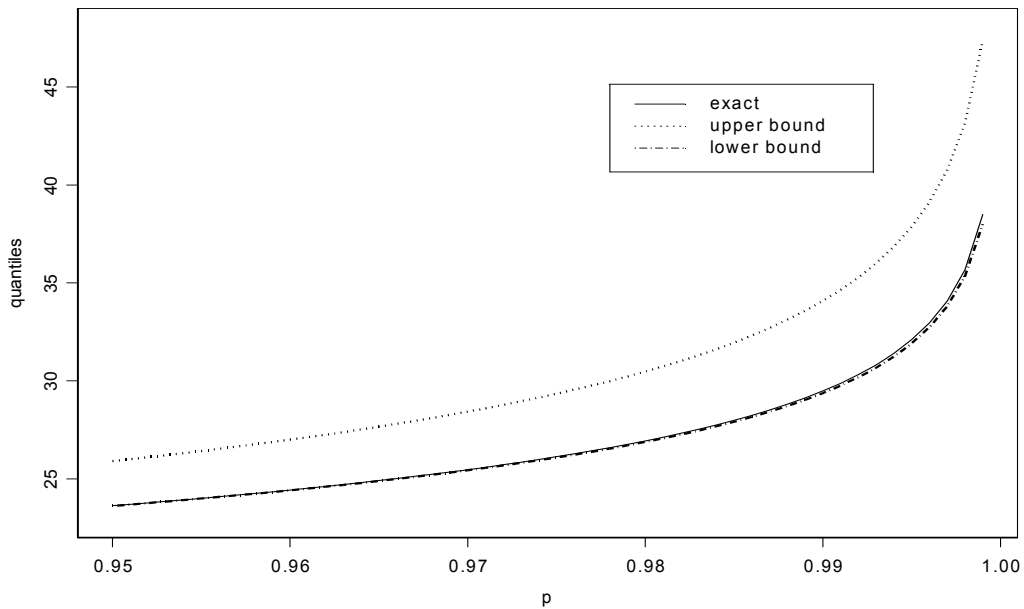


Figure 1: This figure reproduces Table 1 which compares the lower and upper bound

approximations with the exact quantiles ($\delta=0.07, \sigma=0.1$).

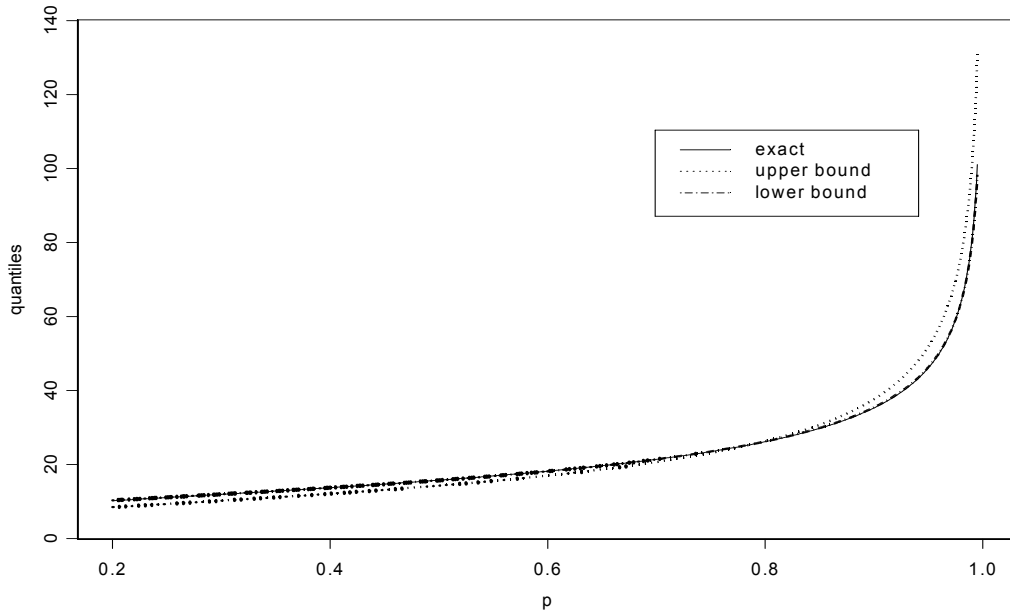


Figure 2: This figure reproduces Table 2 which compares the lower and upper bound approximations with the exact quantiles ($\delta=0.07, \sigma=0.2$).

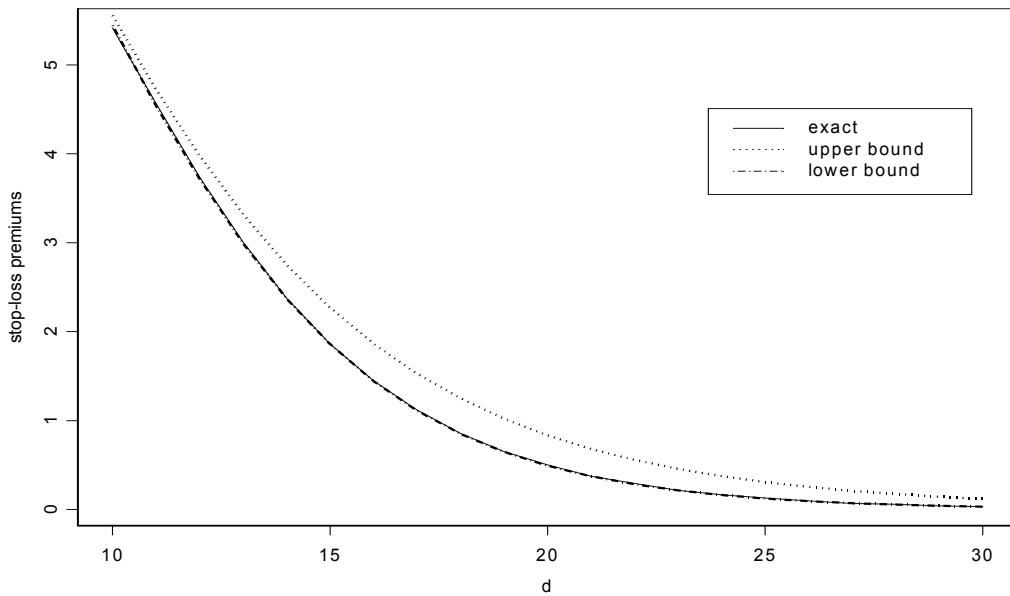


Figure 3: This figure reproduces Table 3 which compares the lower and upper bound approximations with the exact stop loss premiums ($\delta=0.07, \sigma=0.1$).

5 Concluding remarks

The stochastically discounted or compounded value of a series of cash flows is often a key quantity of importance in finance and actuarial science. Yet even for most realistic stochastic return models, it is often difficult to obtain analytic expressions for the risk measures involving these discounted sums. Following the works of Dhaene, et al. (2002a, 2002b), Dhaene, et al. (2004), and Vanduffel, et al. (2004), we show in this paper how to derive explicit comonotonic approximations for risk measures for constant continuous annuities, in the case where discounting is done using a Brownian motion process. We compared these approximations with available explicit results in case of perpetuities. Our numerical comparisons support the conclusions made in Vanduffel, et al. (2004), namely that especially the 'maximal variance' comonotonic lower bound approximation provides an excellent fit for several risk measures associated with integrals or sums that involve lognormal returns. The results we mention here correspond to the discounting case but can be generalised to the compounding case too. The authors are currently studying optimal portfolio selection problems and closed-form approximations for the prices of Asian options.

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