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Extended LMI Characterizations for Stability and Performance of Linear Systems

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Abstract

Over the past ten years, extensive research has been devoted to extended LMI characterizations for stability and performance of linear systems. These characterizations constitute a valuable tool for reducing conservatism in hard problems like multi-objective control and robust stability and performance analysis. The present paper proposes a general, projection lemma based methodology for deriving such extended LMIs and hereby provides a straightforward and unified proof for all known literature results as well as some currently missing extended LMIs.

Key words: Linear matrix inequalities, Stability, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm

1. Introduction

Starting from the work of Lyapunov [21, 27], linear matrix inequalities (LMIs) have gained increasing interest in systems and control theory. As surveyed in [4, 25], stability and many performance specifications for linear systems translate into LMIs involving a so-called Lyapunov matrix, in this paper referred to as *standard LMIs*. While these LMI formulations allow a semi-definite programming reformulation for many linear analysis and control problems, others remain hard to solve. For instance, no LMI reformulation has yet been proposed for the design of controllers subject to multiple design specifications, whereas the verification of robust stability/performance for uncertain systems with state-space matrices belonging to a convex polytopic set is known to be *NP*-hard in general.

To tackle these problems, convex but conservative formulations have been derived by constraining the Lyapunov matrices involved to be equal. This approach is referred to as the Lyapunov shaping paradigm in multi-objective control [24] and quadratic stability/performance in robust analysis [3, 20]. In search for less conservative solutions, de Oliveira et al. [7, 9] derived so-called *extended LMI* characterizations for stability and performance of discrete-time

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systems. The benefit of extended LMIs stems from the fact that, in contrast to the standard LMIs, they do not involve products of the Lyapunov matrix and the system's state-space matrices. This separation is obtained by the introduction of an additional matrix variable that takes over the multiplication with the state-space matrices. The results of [7, 9] initiated active research on extended LMI formulations for stability and performance, yielding continuous-time counterparts [1, 6, 14, 26, 30] for the discrete-time results [7, 9], as well as more general extended LMIs [11, 22]. In addition, extended LMIs have been shown to be a valuable tool for reducing conservatism in applications besides multi-objective control and robust analysis, such as robust static state and output feedback control [9], and robust and multi-objective filtering [13, 17, 28, 29].

As its main contribution, the present paper proposes a general, projection lemma based methodology to derive extended LMI characterizations for stability and performance of linear systems. This methodology is able to reproduce all known extended LMIs and completes some currently missing results. As a by-product, the methodology provides a straightforward and unified proof for all these extended LMIs. Currently, the extended LMI formulations are proved in various ways, clouding their mutual relations.

As further elaborated in Section 2, the methodology comprises four types of extended LMIs, where Table 1 classifies known literature results according to these types. A cross $\times$ indicates extended LMIs that are to the best of our knowledge not yet presented in literature, while a slash $/$ indicates that the corresponding type of extension is impossible to derive, as will become clear in Section 3. The 'type I' extended LMIs are the most general, featuring the highest dimensional slack matrix variable. Whereas there is no general ordering between extensions II and III, extensions IV feature the lowest dimensional slack matrix.

The proposed methodology generalizes the approach of [11], which only derives the type I extended LMIs. Extending the stability result of [22] to $\mathcal{H}_2$ and quadratic performance constitutes an alternative approach for deriving the corresponding type I and III extensions, motivating the indication of this reference for these results. Table 1 will be further discussed throughout Sections 3 to 5, together with a discussion on related, yet not equivalent, results.

The content of this paper is laid out as follows: Section 2 details the general methodology for deriving extended LMIs, whereas the subsequent sections apply it to stability (Section 3), $\mathcal{H}_2$ performance (Section 4) and quadratic performance (Section 5) of linear systems. To illustrate the benefit of extended LMIs, Section 6 applies them to a robust analysis and synthesis problem, while Section 7 concludes the paper.

The notation used in this paper is standard. $\mathbf{R}$ and $\mathbf{C}$ denote the set of real and complex numbers, respectively, whereas $\mathbf{S}_n$ indicates the set symmetric $n \times n$ matrices. For a matrix X , X^T and X^H indicate its transpose and complex conjugate transpose, respectively. The $n \times n$ identity matrix is denoted I_n , where the subscript is omitted if the dimension is clear from the context. For a matrix $X \in \mathbf{S}_n$, $X \prec (\preceq) 0$ indicates that X is negative (semi-)definite. Symbol $\otimes$ denotes the matrix Kronecker product.

		extension I	extension II	extension III	extension IV
stability	discr.	[10, 11, 22]	[2, 7]	/	/
	cont.	[11, 18, 22]	[6, 14]	/	/
$\mathcal{H}_2$ perf.	discr.	[11, 22]	$\times$	[22]	[8, 9]
	cont.	[11, 22]	$\times$	[22]	[14]
quadr. perf.	discr.	[11, 22]	$\times$	[22]	[8, 9]
	cont.	[11, 22]	$\times$	[22]	[30]

Table 1: Overview of extended LMI characterizations for stability and $\mathcal{H}_2$ and quadratic performance of linear systems.

2. General Methodology

The proposed methodology to derive extended LMI formulations for linear system stability and performance relies on the projection lemma [15]:

Projection Lemma: *Given a symmetric matrix $Z \in \mathbf{S}_m$ and two matrices U and V of column dimension m ; there exists an unstructured matrix X that satisfies*

$$U^T X V + V^T X^T U + Z \prec 0, \quad (1)$$

if and only if the following projection inequalities with respect to X are satisfied:

$$N_U^T Z N_U \prec 0, \quad (2a)$$

$$N_V^T Z N_V \prec 0, \quad (2b)$$

where N_U and N_V are arbitrary matrices whose columns form a basis of the nullspaces of U and V , respectively.

In most applications, the projection lemma is used to eliminate X from LMI (1) by replacing it by the set of LMIs (2). As extended LMI characterizations benefit from introducing an additional matrix variable, the proposed methodology applies the projection lemma in the “opposite” direction, that is, (2) is replaced by (1), where X is the additional matrix variable.

As elaborated in the subsequent sections, the standard LMI conditions for stability and performance of linear systems can be traced to the form (2a), where only Z depends on the Lyapunov matrix P , while N_U , and hence, U , depends on the system’s state-spaces matrices. The standard LMI condition (2a) is equivalent to the extended LMI (1) provided that matrix V is constructed such that (2b) does not introduce conservatism. The methodology proposed here distinguishes four design strategies for V satisfying this *equivalency requirement*:

- I Set $V = I$, since in this way the nullspace of V equals $\{0\}$ and hence, (2b) vanishes.¹
- II Design V such that (2b) is equivalent to $P \succ 0$, since the system's stability (which is also included in the performance specifications) requires the Lyapunov matrix P to be positive definite.
- III Design V such that (2b) is equivalent to a trivial matrix inequality, as for instance $I \succ 0$.
- IV Combine II and III, that is: design V such that (2b) is equivalent to the combination of $P \succ 0$ and a trivial matrix inequality.

In addition to the equivalency requirement, the design of V must satisfy a *structural requirement*: V must neither depend on the Lyapunov matrix, nor on the system's state-space matrices. This way, the extended LMI (1) benefits from the following properties:

- In contrast to the standard LMI (2a), the extended LMI (1) does not involve products of the Lyapunov matrix, contained in Z , and the state-space matrices, contained in U .
- The extended LMI (1) does involve products of the slack matrix X and the state-space matrices, but no quadratic terms in the state-space matrices appear, as V is independent of them.

The subsequent sections elaborate the four design strategies for V , and hence, the four types of extended LMI characterizations for stability and performance of linear systems. According to the design of V , the extended LMIs are labeled I, II, III and IV. As the column dimension of X is determined by the row dimension of V , which equals m minus the column dimension of N_V (assuming V and N_V of full rank), extension I features the highest dimensional slack matrix X , whereas design strategy IV gives rise to the smallest dimension of X . As will become clear in Sections 3 to 5, extensions II, III and IV are derived from extension I by substituting particular equality constraints in X .

Throughout this paper, matrices $A \in \mathbf{R}_{n \times n}$, $B \in \mathbf{R}_{n \times m_w}$, $C \in \mathbf{R}_{m_y \times n}$ and $D \in \mathbf{R}_{m_y \times m_w}$ correspond to a minimal (i.e. controllable and observable) state-space model of the system $G(\lambda)$:

$$G(\lambda) : \begin{cases} \delta x &= Ax + Bw \\ y &= Cx + Dw \end{cases}, \quad (3)$$

where in discrete time, δ equals the one-sample-advance operator q and $\lambda = z$, while for continuous time: $\delta x = \frac{dx}{dt}$ and $\lambda = s$.

¹Applying this design strategy, the equivalence between (1) and (2a) is referred to as the elimination lemma [11].

3. Stability

Both in discrete and continuous time, stability of $G(\lambda)$ requires the eigenvalues of A to lie in a stability region of the form

$$\Lambda(\Phi) = \left\{ \lambda \in \mathbf{C} \mid \begin{bmatrix} 1 \\ \lambda \end{bmatrix}^H \Phi \begin{bmatrix} 1 \\ \lambda \end{bmatrix} < 0 \right\}, \quad (4)$$

with given $\Phi \in \mathbf{S}_2$. The interior of the unit disk and the open left half plane are obtained by setting Φ equal to Φ_d and Φ_c , respectively, where:

$$\Phi_d = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (5)$$

Section 3.1 resumes the standard LMI condition for linear stability, while the corresponding type I and II extended LMIs are presented in Sections 3.2 and 3.3, respectively. For stability, V cannot be designed according to design strategies III and IV since the matrix Z depends homogeneously on the Lyapunov matrix P (see Eq. (7b), below). To conclude, Section 3.4 discusses related results.

3.1. Standard LMI

Both continuous-time and discrete-time stability are covered by the following LMI condition [4, 5, 22]:

Standard LMI: *All eigenvalues of A lie in $\Lambda(\Phi)$ given by (4) if and only if there exists a matrix $P \in \mathbf{S}_n$ that satisfies the following set of LMIs:*

$$P \succ 0, \quad (6a)$$

$$\begin{bmatrix} I \\ A \end{bmatrix}^T (\Phi \otimes P) \begin{bmatrix} I \\ A \end{bmatrix} \prec 0. \quad (6b)$$

The LMI (6b) is of the general form (2a), where

$$U = \begin{bmatrix} A & -I \end{bmatrix}, \quad (7a)$$

$$Z = \Phi \otimes P. \quad (7b)$$

3.2. Extension I

According to design strategy I, V is set equal to I_{2n} . Hereby, the methodology of Section 2 reverts to approach [11], while the resulting extended LMI is also presented in [10, 18, 22]:

Extension I: *The LMI (6b) is equivalent to*

$$(\Phi \otimes P) + \begin{bmatrix} A^T \\ -I \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} + \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} \begin{bmatrix} A & -I \end{bmatrix} \prec 0, \quad (8)$$

where $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \in \mathbf{R}_{n \times 2n}$ is an additional matrix variable.

Substituting $\Phi = \Phi_d$ in (8) yields the type I extended LMI for discrete-time stability, while $\Phi = \Phi_c$ yields the continuous-time stability result.

3.3. Extension II

Strategy II designs V such that $N_V^T Z N_V \prec 0$ is equivalent to $P \succ 0$. This design depends on the matrix Φ and is elaborated below for discrete-time and continuous-time stability.

Substituting $\Phi = \Phi_d$ in (7b) yields $Z = \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix}$, and strategy II designs V as follows:

$$N_V = \begin{bmatrix} I \\ 0 \end{bmatrix} \rightarrow V = \begin{bmatrix} 0 & I \end{bmatrix}. \quad (9)$$

This gives rise to the extended LMI condition presented in [2, 7]:

Extension II (discrete time): For discrete-time stability ($\Phi = \Phi_d$), the set of LMIs (6) is equivalent to

$$\begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} + \begin{bmatrix} A^T \\ -I \end{bmatrix} \begin{bmatrix} 0 & X \end{bmatrix} + \begin{bmatrix} 0 \\ X^T \end{bmatrix} \begin{bmatrix} A & -I \end{bmatrix} \prec 0, \quad (10)$$

where $X \in \mathbf{R}_{n \times n}$ is an additional matrix variable.

Hence, for discrete-time stability, enforcing $X_1 = 0$ in extension I (8) does not introduce conservatism.

For continuous-time stability: $Z = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$, and designing V according to strategy II yields:

$$N_V = \begin{bmatrix} \alpha I \\ -I \end{bmatrix} \rightarrow V = \begin{bmatrix} I & \alpha I \end{bmatrix}, \quad (11)$$

with arbitrary scalar $\alpha > 0$. This design yields the extended LMI proposed in [6] which can also be derived as a limit case of the results presented in [14]:

Extension II (continuous time): For continuous-time stability ($\Phi = \Phi_c$), the set of LMIs (6) is equivalent to

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \begin{bmatrix} A^T \\ -I \end{bmatrix} \begin{bmatrix} X & \alpha X \end{bmatrix} + \begin{bmatrix} X^T \\ \alpha X^T \end{bmatrix} \begin{bmatrix} A & -I \end{bmatrix} \prec 0 \quad (12)$$

for arbitrary scalar $\alpha > 0$, where $X \in \mathbf{R}_{n \times n}$ is an additional matrix variable.

Hence, for continuous-time stability, enforcing $X_2 = \alpha X_1$ in extension I (8), with arbitrary $\alpha > 0$, does not introduce conservatism.

3.4. Related results

Similar extended LMI characterizations can be derived by formulating the standard LMI (6b) in terms of A^T instead of A . In addition, the methodology also applies to more general stability regions, involving $\Phi \in \mathbf{S}_{2l}$ where $l > 1$:

$$\Lambda(\Phi) = \left\{ \lambda \in \mathbf{C} \mid \begin{bmatrix} I_l \\ \lambda I_l \end{bmatrix}^H \Phi \begin{bmatrix} I_l \\ \lambda I_l \end{bmatrix} \prec 0 \right\}. \quad (13)$$

To allow for $l > 1$, the standard LMI (6b) should be replaced by

$$\begin{bmatrix} I_{nl} \\ I_l \otimes A \end{bmatrix}^T (\Phi \otimes P) \begin{bmatrix} I_{nl} \\ I_l \otimes A \end{bmatrix} \prec 0. \quad (14)$$

The derivation of the corresponding type I extended LMI proceeds similarly to Section 3.2, whereas design strategy II for V depends on the particular choice of Φ and is only feasible if Φ has at least one negative eigenvalue.

Apkarian et al. [1] propose an extended LMI characterization for continuous-time stability, similar to (12). Their result also features an $n \times n$ slack matrix variable, but the obtained LMI is of dimension $\mathbf{S}_{3n}$ instead of $\mathbf{S}_{2n}$. This extended LMI can be derived by applying the methodology of Section 2 after augmenting the set of LMIs (6) to (Schur complement):

$$\begin{bmatrix} A^T P + P A - P & P \\ P & -P \end{bmatrix} = \begin{bmatrix} A & I \\ I & 0 \\ 0 & I \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & P & 0 \\ P & -P & 0 \\ 0 & 0 & -P \end{bmatrix}}_Z \underbrace{\begin{bmatrix} A & I \\ I & 0 \\ 0 & I \end{bmatrix}}_{N_U} \prec 0. \quad (15)$$

By setting $V = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$, $N_V^T Z N_V \prec 0$ is equivalent to $P \succ 0$, and hence, Hurwitz stability of A is equivalent to the existence of matrices $P \in \mathbf{S}_n$ and $X \in \mathbf{R}_{n \times n}$ for which

$$\begin{bmatrix} 0 & P & 0 \\ P & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} -I \\ A^T \\ I \end{bmatrix} [X \ 0 \ 0] + \begin{bmatrix} X^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -I & A & I \end{bmatrix} \prec 0. \quad (16)$$

4. $\mathcal{H}_2$ Performance

To unify the continuous-time and discrete-time results in this section, system $G(\lambda)$ is assumed strictly proper², yielding $D = 0$ in (3).

The $\mathcal{H}_2$ norm of a linear system $G(s)$ equals

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \text{Tr} \left\{ \int_{-\infty}^{\infty} G(\omega) G(\omega)^H d\omega \right\}}, \quad (17)$$

where $G(\omega)$ indicates the frequency response function (FRF) of $G(\lambda)$. For discrete time, the integrator bounds equal $-\pi$ and π . This system norm can be computed from the controllability or the observability gramian, and verifying $\|G(\lambda)\|_2 < \nu$ using LMIs involves the corresponding Lyapunov inequality.

²While a continuous-time system $G(s)$ must be strictly proper to have a finite $\mathcal{H}_2$ norm, for a discrete-time system $G(z)$ strict properness is not necessary. For a discrete-time system with $D \neq 0$, the LMI (18b) should

be replaced by $\begin{bmatrix} P & 0 & C^T \\ 0 & \nu I & D^T \\ C & D & Q \end{bmatrix} \succ 0$.

Section 4.1 resumes the standard LMI condition for $\mathcal{H}_2$ performance, while the corresponding extended LMIs are derived in Sections 4.2 to 4.5. A discussion on related results concludes the section (Section 4.6).

4.1. Standard LMI

Relying on the controllability gramian, $\|G(\lambda)\|_2 < \nu$ translates into the following set of LMIs [4]:

Standard LMI: *For strictly proper system $G(\lambda)$ given by (3), $\|G(\lambda)\|_2 < \nu$ holds if and only if there exist matrices $P \in \mathbf{S}_n$ and $Q \in \mathbf{S}_{m_y}$ that satisfy the following set of LMIs:*

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T (\Phi \otimes P) \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\nu I \end{bmatrix} \prec 0, \quad (18a)$$

$$\begin{bmatrix} P & C^T \\ C & Q \end{bmatrix} \succ 0, \quad (18b)$$

$$\text{Tr}\{Q\} < \nu. \quad (18c)$$

For discrete-time system $G(z)$, $\Phi = \Phi_d$ is substituted in (18a), while $\Phi = \Phi_c$ yields the continuous-time result.

To put LMI (18a) in the general methodology, it is reformulated as

$$\begin{bmatrix} I & 0 \\ A & B \\ 0 & I \end{bmatrix}^T \begin{bmatrix} (\Phi \otimes P) & 0 \\ 0 & -\nu I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \end{bmatrix} \prec 0, \quad (19)$$

which is of the form (2a) with

$$U = \begin{bmatrix} A & -I & B \end{bmatrix}, \quad (20a)$$

$$Z = \begin{bmatrix} \Phi_{11}P & \Phi_{12}P & 0 \\ \Phi_{12}P & \Phi_{22}P & 0 \\ 0 & 0 & -\nu I \end{bmatrix}. \quad (20b)$$

4.2. Extension I

Design strategy I yields $V = I_{2n+m_w}$. Hereby, the methodology of Section 2 reverts to approach [11], while the resulting extended LMI can also be derived by extending the stability results of [22] to $\mathcal{H}_2$ performance:

Extension I: *The LMI (18a) is equivalent to*

$$\begin{bmatrix} (\Phi \otimes P) & 0 \\ 0 & -\nu I \end{bmatrix} + \begin{bmatrix} A^T \\ -I \\ B^T \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} + \begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \begin{bmatrix} A & -I & B \end{bmatrix} \prec 0, \quad (21)$$

where $X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \in \mathbf{R}_{n \times (2n+m_w)}$ is an additional matrix variable.

4.3. Extension II

Strategy II designs V such that $N_V^T Z N_V \prec 0$ is equivalent to $P \succ 0$. This design depends on the matrix Φ and hence differs for discrete and continuous time. The design of V is facilitated by the observation that the Z -matrix for stability (7b) is part of the Z -matrix for $\mathcal{H}_2$ performance (20b).

For discrete time, designing V according to strategy II yields:

$$N_V = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \rightarrow V = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (22)$$

which results in the following novel extended LMI condition:

Extension II (discrete time): *For discrete-time system $G(z)$ ($\Phi = \Phi_d$), enforcing $X_1 = 0$ in LMI (21) does not introduce conservatism.*

For continuous time, V is designed as follows:

$$N_V = \begin{bmatrix} \alpha I \\ -I \\ 0 \end{bmatrix} \rightarrow V = \begin{bmatrix} I & \alpha I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (23)$$

where $\alpha > 0$ is an arbitrary scalar. This gives rise to the following novel extended LMI condition:

Extension II (continuous time): *For continuous-time system $G(s)$ ($\Phi = \Phi_c$), enforcing $X_2 = \alpha X_1$ in LMI (21), with arbitrary scalar $\alpha > 0$, does not introduce conservatism.*

4.4. Extension III

Considering the structure of Z (20b), $-\nu I \prec 0$ is the only matrix inequality available for design strategy III and it is obtained by setting

$$N_V = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \rightarrow V = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}. \quad (24)$$

This gives rise to the following extended LMI characterization, which can alternatively be derived according to [22]:

Extension III: *Enforcing $X_3 = 0$ in LMI (21) does not introduce conservatism.*

4.5. Extension IV

Design strategy IV combines the second and third design for V . For discrete time, this yields

$$N_V = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \rightarrow V = \begin{bmatrix} 0 & I & 0 \end{bmatrix}, \quad (25)$$

giving rise to the extended LMI characterization proposed in [8, 9]:

Extension IV (discrete-time) *For discrete-time system $G(z)$ ($\Phi = \Phi_d$), enforcing both $X_3 = 0$ and $X_1 = 0$ in LMI (21) does not introduce conservatism.*

For continuous time, combining the second and third design for V yields

$$N_V = \begin{bmatrix} \alpha I & 0 \\ -I & 0 \\ 0 & I \end{bmatrix} \rightarrow V = \begin{bmatrix} I & \alpha I & 0 \end{bmatrix}, \quad (26)$$

with arbitrary $\alpha > 0$. This results in the extended LMI characterization presented in [14]:

Extension IV (continuous-time) *For continuous-time system $G(s)$ ($\Phi = \Phi_c$), enforcing both $X_3 = 0$ and $X_2 = \alpha X_1$ in LMI (21), with arbitrary scalar $\alpha > 0$, does not introduce conservatism.*

4.6. Related results

Similar extended LMI characterizations for linear $\mathcal{H}_2$ performance can be derived by starting from the observability gramian based standard LMI formulation instead of (18) which relates to the controllability gramian. In addition, the general methodology also applies if both standard LMI formulations are formulated in terms of P^{-1} instead of P .

Apkarian et al. [1] derive an extended LMI characterization for continuous-time $\mathcal{H}_2$ performance, similar to extension IV. Their result also features an $n \times n$ slack matrix variable but the obtained LMI is of dimension $\mathbf{S}_{3n+m_w}$ instead of $\mathbf{S}_{2n+m_w}$. Similarly to their stability result (see Eqs. (15,16)), the $\mathcal{H}_2$ result of [1] can be derived by applying the methodology of Section 2 after augmenting (18a) to (Schur complement):

$$\begin{bmatrix} A^T P + P A - P & P B & P \\ B^T P & -\nu I & 0 \\ P & 0 & -P \end{bmatrix} = \begin{bmatrix} A & B & I \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & P & 0 & 0 \\ P & -P & 0 & 0 \\ 0 & 0 & -\nu I & 0 \\ 0 & 0 & 0 & -P \end{bmatrix}}_Z \underbrace{\begin{bmatrix} A & B & I \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}_{N_U} \prec 0. \quad (27)$$

As the standard LMI formulation for the $\mathcal{L}_2$ - $\mathcal{L}_\infty$ induced system norm (or “energy-to-peak” norm) is very similar to the one for the $\mathcal{H}_2$ norm, the corresponding extended LMIs are obtained by similar application of the general methodology. In [16] the corresponding discrete-time extension IV is presented, while its continuous-time counterpart, being based on [1], features an unnecessarily large LMI.

5. Quadratic Performance

Quadratic performance comprises performance measures determined by a frequency-domain inequality (FDI) of the form

$$\begin{bmatrix} I \\ G(\omega) \end{bmatrix}^H M \begin{bmatrix} I \\ G(\omega) \end{bmatrix} \prec 0, \quad \forall \omega \in \mathbf{R} \setminus \Omega, \quad (28)$$

where Ω is the set of poles of $G(\lambda)$ on the boundary of the corresponding stability region. The matrix M

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}, \quad (29)$$

corresponds to a given symmetric matrix³, where this section makes the classical assumption $M_{22} \succeq 0$ [25]. As special cases, $\|G(\lambda)\|_\infty < \gamma$ corresponds to (28) with

$$M = \begin{bmatrix} -\gamma I & 0 \\ 0 & \gamma^{-1} I \end{bmatrix}, \quad (30)$$

while positive-realness of $G(\lambda)$ requires (28) with

$$M = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}. \quad (31)$$

Section 5.1 resumes the standard LMI condition for quadratic performance, while the corresponding extended LMIs are derived in Sections 5.2 to 5.5. A discussion on related results concludes the section (Section 5.6).

5.1. Standard LMI

Application of the Kalman-Yakubovich-Popov (KYP) lemma [23] translates FDI (28) into the following set of LMIs:

Standard LMI: *System $G(\lambda)$ given by (3) is stable and satisfies FDI (28) if and only if there exists a matrix $P \in \mathbf{S}_n$ that satisfies the following set of LMIs:*

$$P \succ 0, \quad (32a)$$

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T (\Phi \otimes P) \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T M \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}}_{R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}} \prec 0. \quad (32b)$$

³By extension, Hermitian matrices M are allowed, requiring the Lyapunov matrix to be Hermitian as well.

In continuous time: $\Phi = \Phi_c$, while $\Phi = \Phi_d$ for discrete time. To put LMI (32b) in the general framework, it is reformulated as

$$\begin{bmatrix} I & 0 \\ A & B \\ 0 & I \end{bmatrix}^T \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} (\Phi \otimes P) & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \end{bmatrix} \prec 0, \quad (33)$$

which is of the form (2a) with

$$U = \begin{bmatrix} A & -I & B \end{bmatrix}, \quad (34a)$$

$$Z = \begin{bmatrix} \Phi_{11}P + R_{11} & \Phi_{12}P & R_{12} \\ \Phi_{12}P & \Phi_{22}P & 0 \\ R_{12}^T & 0 & R_{22} \end{bmatrix}. \quad (34b)$$

5.2. Extension I

Design strategy I yields $V = I_{2n+m_w}$. Hereby, the methodology reverts to approach [11], while the resulting extended LMI can also be derived by extending the stability results of [22] to quadratic performance:

Extension I: *The LMI (32b) is equivalent to*

$$Z + \begin{bmatrix} A^T \\ -I \\ B^T \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} + \begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \begin{bmatrix} A & -I & B \end{bmatrix} \prec 0 \quad (35)$$

with Z given by (34b), where $X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \in \mathbf{R}_{n \times (2n+m_w)}$ is an additional matrix variable.

5.3. Extension II

According to strategy II, V is designed such that $N_V^T Z N_V \prec 0$ is equivalent to $P \succ 0$. The resulting V matrices are similar to the ones derived for $\mathcal{H}_2$ performance (Section 4.3).

For discrete time, design strategy II yields:

$$N_V = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \rightarrow V = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (36)$$

whereby (2b) corresponds to

$$N_V^T Z N_V = R_{11} - P = C^T M_{22} C - P \prec 0. \quad (37)$$

Since $M_{22} \succeq 0$, this matrix inequality is more strict than $P \succ 0$. However, no conservatism is introduced as this stricter LMI is also implied by (32), which, for discrete time, is equivalent to:

$$\begin{bmatrix} -P & PA & PB \\ A^T P & R_{11} - P & R_{12} \\ B^T P & R_{12}^T & R_{22} \end{bmatrix} \prec 0. \quad (38)$$

Design (36) results in the following novel extended LMI condition:

Extension II (discrete time): *For discrete-time system $G(z)$ ($\Phi = \Phi_d$), enforcing $X_1 = 0$ in LMI (35) renders it equivalent to the set of LMIs (32).*

For continuous time, V is designed as follows

$$N_V = \begin{bmatrix} \alpha I \\ -I \\ 0 \end{bmatrix} \rightarrow V = \begin{bmatrix} I & \alpha I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (39)$$

which yields

$$N_V^T Z N_V = \alpha^2 R_{11} - 2\alpha P = \alpha^2 C^T M_{22} C - 2\alpha P \prec 0. \quad (40)$$

Hence, in order not to introduce conservatism with respect to $P \succ 0$, scalar $\alpha > 0$ must be considered as a variable in the extended LMI. This contrasts the corresponding result for stability (11) and $\mathcal{H}_2$ performance (23), where α could be fixed to an arbitrary positive value without introducing conservatism.

Extension II (continuous time): *For continuous-time system $G(s)$ ($\Phi = \Phi_c$), enforcing $X_2 = \alpha X_1$ in LMI (35), where $\alpha > 0$ is an additional scalar variable, renders it equivalent to the set of LMIs (32).*

5.4. Extension III

Both for discrete and continuous time, the set of LMIs (32) implies $R_{22} \prec 0$. This matrix inequality is equivalent to $N_V^T Z N_V \prec 0$ for

$$N_V = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \rightarrow V = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}. \quad (41)$$

This gives rise to the following extended LMI characterization, which can also be derived according to [22]:

Extension III: *Enforcing $X_3 = 0$ in LMI (35) does not introduce conservatism.*

5.5. Extension IV

Design strategy IV combines the second and third design for V . For discrete time, this yields

$$N_V = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \rightarrow V = \begin{bmatrix} 0 & I & 0 \end{bmatrix}, \quad (42)$$

and

$$N_V^T Z N_V = \begin{bmatrix} R_{11} - P & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \prec 0. \quad (43)$$

Although this LMI is more strict than the combination of $P \succ 0$ and $R_{22} \prec 0$, no conservatism is introduced since (43) is implied by the standard LMI (32), as clarified by (38). Design (42) gives rise to the following extended LMI, which is proposed in [8, 9] for the $\mathcal{H}_\infty$ norm:

Extension IV (discrete time): For discrete-time system $G(z)$ ($\Phi = \Phi_d$), enforcing both $X_1 = 0$ and $X_3 = 0$ in LMI (35) renders it equivalent to the set of LMIs (32).

For continuous time, design strategy IV yields

$$N_V = \begin{bmatrix} \alpha I & 0 \\ -I & 0 \\ 0 & I \end{bmatrix} \rightarrow V = \begin{bmatrix} I & \alpha I & 0 \end{bmatrix}, \quad (44)$$

whereby

$$N_V^T Z N_V = \begin{bmatrix} \alpha^2 R_{11} - 2\alpha P & \alpha R_{12} \\ \alpha R_{12}^T & R_{22} \end{bmatrix} \prec 0, \quad (45)$$

This is equivalent to the combination of $R_{22} \prec 0$, and $P \succ 0.5\alpha(R_{11} - R_{12}R_{22}^{-1}R_{12}^T) \succeq 0$. Hence, no conservatism is introduced provided that $\alpha > 0$ is a variable in the extended LMI:

Extension IV (continuous time): For continuous-time system $G(s)$ ($\Phi = \Phi_c$), enforcing both $X_3 = 0$ and $X_2 = \alpha X_1$ in LMI (35), where $\alpha > 0$ is an additional scalar variable, renders it equivalent to the set of LMIs (32).

This result is presented in [30] for the $\mathcal{H}_\infty$ norm.

5.6. Related results

For the continuous-time case, similar extended LMI formulations can be derived by introducing scalar α in an alternative way. Since $P \succ 0$, LMI (32b) is equivalent to the existence of a positive scalar variable α such that

$$\begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} + \alpha \begin{bmatrix} A^T P A & A^T P B \\ B^T P A & B^T P B \end{bmatrix} + R \prec 0. \quad (46)$$

Reformulating this LMI as

$$\begin{bmatrix} I & 0 \\ I + \alpha A & \alpha B \\ 0 & I \end{bmatrix}^T \underbrace{\begin{bmatrix} \alpha R_{11} - P & 0 & \alpha R_{12} \\ 0 & P & 0 \\ \alpha R_{12}^T & 0 & \alpha R_{22} \end{bmatrix}}_Z \underbrace{\begin{bmatrix} I & 0 \\ I + \alpha A & \alpha B \\ 0 & I \end{bmatrix}}_{N_U} \prec 0 \quad (47)$$

allows applying the methodology of Section 2 similarly to the discrete-time case. The result proposed in [19, 26] is equivalent to the corresponding extension IV.

Alternative extended LMI characterizations for the quadratic performance of linear systems (both discrete and continuous) are obtained by formulating LMI (32b) as

$$\begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} \Phi \otimes P & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix} \prec 0 \quad (48)$$

and applying the methodology of Section 2 to this formulation. The corresponding extensions I are presented in [11].

If $M_{22} \succ 0$, the quadratic term $\begin{bmatrix} C & D \end{bmatrix}^T M_{22} \begin{bmatrix} C & D \end{bmatrix}$ in LMI (32b) can be elaborated using the Schur complement, prior to the application of the proposed methodology. In addition, the methodology also applies to this LMI, yet formulated in terms of P^{-1} instead of P .

Zhou et al. [31] present an extended LMI characterization for positive realness of discrete-time systems, similar to extension IV. It also features an $n \times n$ slack matrix variable, but the extended LMI is of dimension $\mathbf{S}_{3n+m_w}$ instead of $\mathbf{S}_{2n+m_w}$. This result can also be derived by applying the methodology of Section 2 to the elaborated form (38) of the standard LMI (32), which equals:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ -0.5I & A & B \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & P \\ 0 & R_{11} - P & R_{12} & 0 \\ 0 & R_{12}^T & R_{22} & 0 \\ P & 0 & 0 & 0 \end{bmatrix}}_Z \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ -0.5I & A & B \end{bmatrix}}_{N_U} \prec 0. \quad (49)$$

6. Application Example of Extended LMIs

As indicated in Section 1, the point of extended LMIs is their capability to reduce conservatism in some hard problems in linear systems and control theory. This section illustrates this benefit for a discrete-time robust performance analysis problem (Section 6.1), while in a second step, the robust analysis is extended to a robust state feedback control problem (Section 6.2). Although extension I yields the least conservative analysis result, extension IV is shown more appropriate for controller synthesis.

6.1. Robust analysis

The analysis problem deals with uncertain linear discrete-time systems $G_\psi(z)$, of which the state-space matrices, grouped in $\mathcal{L}_\psi$

$$\mathcal{L}_\psi = \begin{bmatrix} A_\psi & B_\psi \\ C_\psi & D_\psi \end{bmatrix}, \quad (50)$$

depend on an uncertain parameter ψ , but are known to be contained in a convex bounded polyhedron Ψ :

$$\Psi = \left\{ \mathcal{L}_\psi \mid \mathcal{L}_\psi = \sum_{i=1}^m \psi_i \mathcal{L}_i, \psi_i \geq 0, \sum_{i=1}^m \psi_i = 1 \right\}. \quad (51)$$

The robust $\mathcal{H}_\infty$ norm, indicated by γ^* , corresponds to the worst-case $\mathcal{H}_\infty$ norm over all systems $G_\psi(z)$ and hence, γ^* corresponds to the optimal value of the following program:

$$\underset{\gamma}{\text{minimize}} \quad \gamma \quad (52a)$$

$$\text{subject to} \quad \|G_\psi(z)\|_\infty < \gamma, \quad \forall \mathcal{L}_\psi \in \Psi. \quad (52b)$$

Applying the standard LMI formulation to constraint (52b) requires a parameter dependent Lyapunov matrix P_ψ . While the search for parameter dependent P_ψ results in a non-convex program, a convex but conservative approximation of (52) is obtained by constraining the Lyapunov matrix to be constant over the entire polytope Ψ . This approach is known as quadratic performance analysis and translates into the following optimization problem:

$$\underset{P, \gamma_0}{\text{minimize}} \quad \gamma_0 \quad (53a)$$

$$\text{subject to} \quad \begin{bmatrix} -P & A_i P & B_i & 0 \\ P A_i^T & -P & 0 & P C_i^T \\ B_i^T & 0 & -\gamma_0 I & D_i^T \\ 0 & C_i P & D_i & -\gamma_0 I \end{bmatrix} \prec 0 \quad \forall i = 1, \dots, m. \quad (53b)$$

Due to conservatism, the optimal value γ_0^* is an upper bound on γ^* : $\gamma_0^* \geq \gamma^*$.

Applying the extended LMI formulations to constraint (52b) involves a parameter dependent Lyapunov matrix P_ψ and parameter dependent slack matrix X_ψ . While this results in a non-convex program, a convex but conservative approximation of (52) is obtained by considering the slack matrix X parameter independent, while linear parameter dependency is allowed for P_ψ :

$$P_\psi = \sum_{i=1}^m \psi_i P_i. \quad (54)$$

Applying this approach to extension I yields⁴

$$\begin{array}{ll} \underset{P_i, X, \gamma_I}{\text{minimize}} & \gamma_I \end{array} \quad (55a)$$

$$\text{subject to} \quad P_i \succ 0, \quad \forall i = 1, \dots, m \quad (55b)$$

$$\begin{aligned} & \begin{bmatrix} -P_i & 0 & B_i & 0 \\ 0 & P_i & 0 & 0 \\ B_i^T & 0 & -\gamma_I I & D_i^T \\ 0 & 0 & D_i & -\gamma_I I \end{bmatrix} + \begin{bmatrix} A_i \\ -I \\ 0 \\ C_i \end{bmatrix} X + \\ & X^T \begin{bmatrix} A_i^T & -I & 0 & C_i^T \end{bmatrix} \prec 0, \quad \forall i = 1, \dots, m. \end{aligned} \quad (55c)$$

For the other extended LMIs, the corresponding result is obtained by substituting particular equality constraints in X (see Sections 5.3 to 5.5), yielding:

$$\gamma_I^* \leq \gamma_{II}^* \leq \gamma_{IV}^* \quad \text{and} \quad \gamma_I^* \leq \gamma_{III}^* \leq \gamma_{IV}^*, \quad (56)$$

while the relation between γ_{II}^* and γ_{III}^* depends on the application. Due to conservatism, $\gamma^* \leq \gamma_I^*$. However, for all extended LMIs, (55) is less conservative than (53) since the latter optimization problem is obtained by enforcing $P_i = P$ for all $i = 1, \dots, m$, and $X = \begin{bmatrix} 0 & P & 0 & 0 \end{bmatrix}$ in (55). Hence, $\gamma_{IV}^* \leq \gamma_0^*$. On count of relations (56), extension I is generally preferred for the robust analysis problem.

6.2. Robust state-feedback control

In the robust state-feedback problem, the uncertain linear discrete-time system $G_\psi(z)$ features an additional control input u :

$$G_\psi(z) : \begin{cases} \delta x &= A_\psi x + B_\psi w + E_\psi u \\ y &= C_\psi x + D_\psi w + F_\psi u \end{cases}, \quad (57)$$

and its state-space matrices, grouped in $\mathcal{L}_\psi$

$$\mathcal{L}_\psi = \begin{bmatrix} A_\psi & B_\psi & E_\psi \\ C_\psi & D_\psi & F_\psi \end{bmatrix}, \quad (58)$$

are known to be contained in a convex bounded polyhedron Ψ (51). The goal is to compute the state-feedback controller

$$u = Kx, \quad (59)$$

that yields the best robust $\mathcal{H}_\infty$ performance of the closed loop system $H_\psi(z)$:

$$H_\psi(z) : \begin{cases} \delta x &= (A_\psi + E_\psi K)x + B_\psi w \\ y &= (C_\psi + F_\psi K)x + D_\psi w \end{cases}. \quad (60)$$

⁴SDP (55) is also presented in [12], where it results from a robust performance analysis using parameter dependent Lyapunov functions and multipliers.

Starting from the standard LMI characterization of the $\mathcal{H}_\infty$ norm, a convex, conservative approximation of this robust control problem can be constructed similarly to (53). The corresponding synthesis problem equals:

$$\underset{K, P, \gamma_0}{\text{minimize}} \quad \gamma_0 \quad (61a)$$

$$\text{subject to} \quad \begin{bmatrix} -P & (A_i + E_i K)P & B_i & 0 \\ P(A_i + E_i K)^T & -P & 0 & P(C_i + F_i K)^T \\ B_i^T & 0 & -\gamma_0 I & D_i^T \\ 0 & (C_i + F_i K)P & D_i & -\gamma_0 I \end{bmatrix} \prec 0, \quad \forall i = 1, \dots, m \quad (61b)$$

and is converted to an SDP by the change of variables: $\tilde{K} = KP$.

Starting from the type IV extended LMI, the synthesis program corresponding to (55) is transformed to an SDP by a similar change of variables: $\tilde{K} = KX_2$, yielding

$$\underset{\tilde{K}, P_i, X_2, \gamma_{IV}}{\text{minimize}} \quad \gamma_{IV} \quad (62a)$$

$$\text{subject to} \quad \begin{bmatrix} -P_i & 0 & B_i & 0 \\ 0 & P_i & 0 & 0 \\ B_i^T & 0 & -\gamma_{IV} I & D_i^T \\ 0 & 0 & D_i & -\gamma_{IV} I \end{bmatrix} + \begin{bmatrix} A_i X_2 + E_i \tilde{K} \\ -X_2 \\ 0 \\ C_i X_2 + F_i \tilde{K} \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (X_2^T A_i^T + \tilde{K}^T E_i^T) & -X_2^T & 0 & (X_2^T C_i^T + \tilde{K}^T F_i^T) \end{bmatrix} \prec 0, \quad \forall i = 1, \dots, m \quad (62b)$$

Although applying the same strategy to the type I extended LMI would yield a less conservative result, the corresponding linearizing change of controller variables $\tilde{K} = KX$ is generally not invertible. The computation of K from $\tilde{K}$ requires X to have full column rank, which is generally not satisfied since $X \in \mathbf{R}_{n \times (2n+m_w+m_y)}$. On the other hand, for extension IV, $X_2 \in \mathbf{R}_{n \times n}$ is of full rank since (62b) implies $X_2 + X_2^T \succ P \succ 0$, and hence, $\tilde{K} = KX_2$ is invertible.

7. Conclusion

This paper proposes a general, projection lemma based methodology for deriving extended LMI characterizations for stability and performance of linear systems. The methodology covers all known literature results and completes some currently missing extended LMIs. It provides a unified proof for all extended LMIs and emphasizes their mutual relations.

The methodology comprises four types of extended LMIs, where the most general result features the highest dimensional slack matrix variable. The other three extensions are obtained by substituting particular equality constraints for this slack matrix.

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