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Generalized Repetitive Control: Relaxing the Period-Delay Based Structure

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Abstract

Most current repetitive controllers apply the internal model principle by embedding one or multiple period delays into the control scheme. While being simple and often effective, this structure involves a number of disadvantages : (i) in many applications, large memory buffers are needed ; (ii) information about the spectrum of the periodic input cannot be fully exploited and (iii) accounting for period-time uncertainty results in controllers that are overly robust for lower harmonic frequencies. In order to overcome these disadvantages, the present paper proposes a novel repetitive controller structure that sacrifices the simplicity of the period-delay based structure for the benefit of more design freedom. The resulting generalized repetitive controller encompasses the period-delay based structure as a special case and is designed based on semi-definite programming. Numerical results reveal that a generalized repetitive controller yields significantly better performance with the same amount of memory or, conversely, requires substantially less memory to achieve equivalent performance.

1 Introduction

Including a repetitive controller in the feedback loop constitutes a common way to improve the attenuation/tracking of periodic disturbance/reference inputs [4, 8, 12, 22]. Repetitive controllers rely upon the internal model principle [6], which states that, if a disturbance/reference signal can be regarded as the output of an autonomous system, including this system in a stable feedback loop guarantees asymptotically perfect rejection/tracking. Most current repetitive controllers, in this paper referred to as *typical* repetitive controllers, rely on the periodic signal generator shown in Figure 1(a), which can autonomously generate any signal with period $T_p = NT_s$ [s], where N is integer and T_s [s] corresponds to the sample period. This system has an infinite gain at all multiples of $\omega_p = 2\pi/T_p$ [rad/s] and hence, embedding it in the controller guarantees perfect asymptotic rejection/tracking of any input with period T_p . As illustrated in Figure 1(b), typical repetitive controllers generally connect a low-pass filter, denoted $Q(z)$, in series with the delay z^{-N} for guaranteeing robust closed-loop stability [3, 4, 8, 19]. In so-called high-order typical repetitive controllers the delay z^{-N} is extended to a M 'th order polynomial of delays :

$$W(z) = \sum_{m=1}^M W_m z^{-mN} , \quad (1)$$

which gives the designer more freedom to improve the controller's performance through a proper choice of the parameters W_m [1, 11, 15, 18, 19].

While the typical repetitive controller structure is simple and can be adapted easily for varying periods by adjusting N [5, 20], it also involves some drawbacks : Since in many applications

Figure 1: Typical repetitive controller structure (b) that relies on periodic signal generator (a) with period $T_p = NT_s$.

N is large, the implementation of a typical repetitive controller generally requires a large memory buffer. Furthermore, a typical repetitive controller cannot properly account for information about the periodic input spectrum, since the frequency response function (FRF) of z^{-N} is periodic with $\omega_p = 2\pi N/T_s$ and hereby forces an equal treatment of all harmonic components. In addition, the FRF periodicity renders it hard to robustify a typical repetitive controller for uncertainty on T_p [15, 18], since such uncertainty implies a linearly growing uncertainty interval on higher harmonic frequencies.

This paper proposes a novel type of single-input single-output (SISO) repetitive controllers, called *generalized* repetitive controllers, that sacrifices the simple structure of typical repetitive controllers for the benefit of more design freedom to overcome the aforementioned disadvantages. The additional design freedom is created by replacing the series connection $W(z)Q(z)$ in a typical repetitive controller by an unstructured finite impulse response (FIR) filter $X(z)$. In typical repetitive control $W(z)Q(z)$ generally corresponds to a FIR filter with a particular structure, since $Q(z)$ is commonly designed as a linear-phase FIR filter [3, 8, 19]. In a generalized repetitive controller no structure is a priori enforced on $X(z)$. In addition, all coefficients of $X(z)$ are used to optimize the controller's performance, whereas in a typical repetitive controller design performance is only optimized through the M parameters W_m .

The typical repetitive controller designs of [1, 11, 15, 18, 19] can all be extended to generalized repetitive controllers. This paper, however, adopts the design methodology [15] as this approach is able to reproduce and outperform the controllers presented in [1, 18]. According to this design methodology, the generalized repetitive controller is designed to yield an optimal tradeoff between two performance indices that quantify the repetitive controller's effect on the closed-loop performance with respect to non-periodic and periodic inputs, hereby explicitly accounting for period-time uncertainty. The numerical results of Section 4 illustrate that, thanks to its structure being better suited to account for the input spectrum and period-time uncertainty, a generalized repetitive controller yields significantly better performance than a typical repetitive controller designed according to the same specifications and requiring the same amount of memory.

To the best of our knowledge, the approach of [9, 10] is currently the sole repetitive controller design that allows accounting for the spectrum of the periodic input. The controller structure adopted in [9, 10] differs from the generalized repetitive controller proposed here, and their design is based on pole-placement. Hereby, the trade-off between closed-loop periodic and non-periodic performance is disregarded, and period-time uncertainty cannot be accounted for.

This paper is organized as follows : First, Section 2 provides general background on repetitive control, which is independent of the particular controller structure. It derives the design specifications for a repetitive controller and quantifies its effect on the closed-loop performance in terms of two performance indices. Subsequently, Section 3 presents the generalized repetitive controller structure and elaborates on its design, where the differences with a typical repetitive controller design are highlighted. Section 4 numerically illustrates the advantage of generalized repetitive controllers over typical repetitive controllers, while Section 5 concludes the paper. The notation used in this paper is standard. To alleviate notation, the FRF of a discrete-time

Figure 2: Control setup of an add-on repetitive control system, where repetitive controller $K_{RC}(z)$ is added to the original feedback system governed by $K_o(z)$.

system $H(z)$ is denoted by $H(\omega)$ instead of $H(e^{j\omega T_s})$.

2 Add-on Repetitive Control System

As is common in literature [2, 4, 5, 18, 19, 22], the repetitive controller is considered as an add-on device. This implies that a stabilizing feedback controller has been designed prior to the repetitive controller, yielding good overall performance. In a second step, the repetitive controller is added to this so-called original feedback system in order to improve the tracking/attenuation of the periodic inputs. This paper focusses on the repetitive controller design, while the original feedback system is considered given. Subsection 2.1 details the corresponding control setup, while Subsection 2.2 analyzes the effect of the repetitive controller on the closed-loop system in order to derive its design specifications. Subsection 2.3 defines two performance indices that quantify the repetitive controller's effect on the closed-loop performance with respect to non-periodic and periodic inputs, respectively.

2.1 Control setup

Figure 2 shows the control setup of an add-on repetitive control system, where the repetitive controller, $K_{RC}(z)$, is added to the original feedback system governed by $K_o(z)$. The controller design is considered in discrete time, where T_s [s] denotes the sample period and index k refers to the discrete time instants kT_s . The plant is a SISO linear time-invariant system with transfer function $P(z)$, while signals $r(k)$, $d(k)$ and $e(k)$ correspond to reference input, disturbance input and tracking error, respectively. In closed loop, $e(k)$ is given by

$$e(k) = S(q) \underbrace{(r(k) - d(k))}_{v(k)}, \quad (2)$$

where q denotes the one-sample-advance operator and $S(q)$ corresponds to the closed-loop sensitivity. Signal $v(k)$ is referred to as the input of the closed-loop system and features both a periodic and a non-periodic part. The periodic part has nominal period T_p [s], where the corresponding fundamental frequency is denoted by ω_p [rad/s]. Uncertainty on T_p is modeled as relative uncertainty on ω_p , bounded by Δ :

$$\omega_{p,\delta} = \omega_p (1 + \delta), \quad |\delta| \leq \Delta. \quad (3)$$

Index l indicates the harmonics of the periodic part of $v(k)$, where $\mathcal{L} \subset \mathbb{N}$ corresponds to the set of harmonics to be suppressed. To each harmonic $l \in \mathcal{L}$ a positive weight V_l is attributed, quantifying its relative importance in $v(k)$. Hence, V_l generally corresponds to the amplitude of the corresponding Fourier coefficient, or a (rough) estimate thereof.

2.2 Design specifications for $K_{RC}(z)$

The original feedback system comprises plant $P(z)$ and controller $K_o(z)$, while its sensitivity and complementary sensitivity function are denoted by $S_o(z)$ and $T_o(z)$, respectively. The repetitive controller design relies on the assumption that $K_o(z)$ is designed properly, which implies [17] : (i) a stable closed-loop system ; (ii) high-gain feedback at low frequencies ; (iii) sufficient roll-off of $|T_o(\omega)|$ at high frequencies ; and (iv) a large modulus margin $\|S_o(z)\|_\infty^{-1}$.

The combination of properties (i), (iii) and (iv) guarantees robust stability of the original closed-loop system. Property (ii) is referred to as good *non-periodic performance*, since it yields small $|S_o(\omega)|$ at low frequencies, resulting in a small tracking error $e(k)$ for *any* low-frequency input $v(k)$. Defining the bandwidth ω_{BW} of the original closed-loop system as the frequency where $|S_o(\omega)|$ first crosses-3dB from below implies that $K_o(z)$ yields good non-periodic performance up to ω_{BW} .

In a second step, repetitive controller $K_{RC}(z)$ is added to the loop, without altering $K_o(z)$, and hereby the closed-loop sensitivity changes from $S_o(z)$ to $S(z)$:

$$S(z) = S_o(z) M_S(z) , \quad (4)$$

where $M_S(z)$ is called the *modifying sensitivity function* [19], given by

$$M_S(z) = \frac{1}{1 + K_{RC}(z)T_o(z)} . \quad (5)$$

The repetitive controller design faces two major specifications :

- 1. Robust stability :** Adding $K_{RC}(z)$ must not compromise the robust stability of the original feedback system, and this design specification invokes two requirements : **(1.a)** $K_{RC}(z)$ must yield a nominally stable closed-loop system and **(1.b)** the modulus margin and high-frequency roll-off of the original feedback system should be preserved.
- 2. Performance :** The design of $K_{RC}(z)$ should address two conflicting performance specifications : The main purpose of $K_{RC}(z)$ is to improve the attenuation/tracking of the periodic part of input $v(k)$, that is, the closed-loop *periodic performance*, as much as possible. However, improved periodic performance implies degraded non-periodic performance, since pushing the closed-loop sensitivity down at multiples of ω_p is paid for by an increased sensitivity at intermediate frequencies, as dictated by the Bode Sensitivity Integral [7]. This inevitable non-periodic performance degradation should be kept within acceptable limits.

These design specifications for $K_{RC}(z)$ are not conflicting provided that all harmonics $l \in \mathcal{L}$ lie well below ω_{BW} : at harmonics near or above ω_{BW} better performance simply cannot be achieved without compromising the modulus margin or the high-frequency roll-off of the original feedback system. This assumption is made throughout the remainder of this paper.

2.3 Performance indices of $K_{RC}(z)$

This section defines two performance indices that quantify, respectively, the periodic performance improvement and the non-periodic performance degradation caused by $K_{RC}(z)$. These definitions are adopted from [15], with a minor modification of the periodic performance index to account for the weights V_l of the input harmonics. Since $M_S(z)$ represents the effect of the repetitive controller on the closed-loop sensitivity, both performance indices are expressed in terms of this transfer function.

Figure 3: Structure of a typical (a) and a generalized (b) repetitive controller.

To quantify the effect of $K_{RC}(z)$ on the closed-loop periodic performance in the presence of uncertainty Δ on ω_p , the *periodic performance index* $\gamma_{p,\Delta}$ is defined as follows :

$$\gamma_{p,\Delta} \equiv \max_{l \in \mathcal{L}} \left\{ V_l \max_{\omega \in \Omega_l} \{|M_S(\omega)|\} \right\} , \quad (6)$$

where

$$\Omega_l = [l\omega_p(1 - \Delta) , l\omega_p(1 + \Delta)] . \quad (7)$$

At each harmonic, the worst-case reduction of $|S(l\omega_{p,\delta})|$ with respect to $|S_o(l\omega_{p,\delta})|$ is taken over all possible $\omega_{p,\delta}$ (3), and $\gamma_{p,\Delta}$ is set equal to the weighted maximum of these values for all harmonics $l \in \mathcal{L}$, where the V_l values are used as weights.

The effect of $K_{RC}(z)$ on the closed-loop non-periodic performance is quantified by the *non-periodic performance index* γ_{np} , defined as the highest amplification of $|S(\omega)|$ with respect to $|S_o(\omega)|$, below ω_{BW} :

$$\gamma_{np} \equiv \max_{\omega \leq \omega_{BW}} \{|M_S(\omega)|\} = \|M_S(z)\|_\infty . \quad (8)$$

The second equivalence holds for all practically relevant repetitive controller designs, since they improve periodic performance, $\gamma_{p,\Delta} < 1$, at the expense of degraded non-periodic performance, $\gamma_{np} > 1$. On the other hand, the robust stability requirement imposes $|M_S(\omega)| \approx 1$ for all $\omega > \omega_{BW}$ (see Section 3.1), whereby (8) follows.

3 Generalized Repetitive Controller Design

Figure 3(a) shows the structure of a typical repetitive controller, $K_{tRC}(z)$, which embeds the system shown in Figure 1(b). The M 'th order polynomial of delays z^{-N} is denoted $W(z)$, Equation (1), while

$$N = \text{int} (T_p/T_s) , \quad (9)$$

and the effect of a potential rounding error is modeled as uncertainty on $T_p \equiv NT_s$. In typical repetitive control, $W(z)$ and filters $Q(z)$ and $L(z)$ each address one of the designs specifications listed in Subsection 2.2 [1, 11, 15, 18, 19] : $L(z)$ guarantees nominal closed-loop stability **(1.a)**, $Q(z)$ ensures preservation of the original feedback system's modulus margin and high-frequency roll-off **(1.b)**, and the performance of $K_{tRC}(z)$ is optimized through the design of $W(z)$ **(2)**.

Figure 4: Practical, causal, implementation of a typical (a) and generalized (b) repetitive controller with non-causal filters $L(z)$ and $Q(z)$.

This paper proposes the novel, generalized repetitive controller structure, shown in Figure 3(b). The series connection of $W(z)$ and $Q(z)$ in a typical repetitive controller is replaced by FIR filter $X(z)$ of length $K + 1$ with design variables X_k :

$$X(z) = \sum_{k=\tau}^K X_k z^{-k} , \quad (10)$$

where τ denotes the relative degree of $T_o(z)$. Typical repetitive controllers constitute a subclass of generalized repetitive controllers since a FIR filter design of $Q(z)$, as is common practice in typical repetitive control [3, 8, 19], turns $W(z)Q(z)$ into a FIR filter with a particular structure. In addition, a typical repetitive controller uses only M parameters in this FIR filter to optimize its performance, whereas a generalized repetitive controller optimizes its performance through all $(K - \tau + 1) \gg M$ filter coefficients X_k . Although specification **(1.a)** invokes an additional constraint in this design problem, the increased design freedom generally results in better performance (see Section 4).

Below, it is elaborated how the design specifications of Subsection 2.2 are addressed in typical and generalized repetitive control, with emphasis on the latter controller structure.

3.1 Robust stability

The generalized repetitive controller structure of Figure 3(b) gives rise to the following expression for $M_S(z)$:

$$M_S(z) = \frac{1 - X(z)}{1 - X(z) [1 - L(z)T_o(z)]} , \quad (11)$$

while replacing $X(z)$ by $W(z)Q(z)$ yields the expression for a typical repetitive controller.

To obtain a nominally stable closed-loop system, a generalized repetitive controller design adopts the common design of $L(z)$ in typical repetitive control [4, 8, 12, 22]. Accordingly, $L(z)$ is set equal to $T_o^{-1}(z)$, which gives rise to a plant-independent expression for the modifying sensitivity function :

$$M_S(z) = 1 - X(z) , \quad (12)$$

and the expression for a typical repetitive controller is obtained by replacing $X(z)$ by $W(z)Q(z)$. Non-causality of $L(z)$ caused by relative degree $\tau > 0$ of $T_o(z)$ is accounted for as illustrated in Figure 4, where the generalized repetitive controller implementation of Figure 4(b) is causal since τ samples delay are enforced in $X(z)$ (10). If $T_o(z)$ is non-minimum phase, the “zero phase error tracking” inverse [21] is used, as is common practice in typical repetitive control [4, 22].

To preserve the modulus margin and high-frequency roll-off of the original feedback system, the repetitive controller must be turned off from bandwidth ω_{BW} . In a typical repetitive controller design this is commonly achieved by designing $Q(z)$ as a low-pass zero-phase FIR filter [3, 8, 19]. However, a zero-phase FIR filter $Q(z)$ comprises $n_Q/2$ non-causal impulse response samples, where $n_Q + 1$ corresponds to the FIR length of $Q(z)$ and n_Q is assumed even. Hence, $z^{-n_Q/2}Q(z)$ is causal while $Q(z)$ is non-causal. As illustrated in Figure 4, the non-causality of $Q(z)$ can be absorbed in the period delays z^{-N} of $W(z)$, as long as $N - \tau - n_Q/2 \geq 0$. The length and cut-off frequency of $Q(z)$ are chosen such that such all harmonics $l \in \mathcal{L}$ lie in the pass-band of $Q(z)$, while $Q(\omega) \approx 0$ for all $\omega > \omega_{BW}$ [3, 8, 19]. $Q(z)$ can comply with both requirements provided that all harmonics $l \in \mathcal{L}$ lie well below ω_{BW} , as is assumed.

In the generalized repetitive controller design, $Q(z)$ is absorbed in $X(z)$ and hence, design parameters X_k must guarantee that $K_{gRC}(z)$ is turned off from ω_{BW} . The low-pass characteristic of $Q(z)$ is mimicked by the following constraint on $X(z)$:

$$|X(\omega)| \leq \epsilon, \quad \forall \omega \geq \omega_{BW}, \quad (13)$$

where ϵ corresponds to a small positive number, e.g. 10^{-3} .

3.2 Optimal performance

Dictated by the Bode Sensitivity Integral, improved closed-loop periodic performance comes at the expense of degraded non-periodic performance. Therefore, performance of the repetitive controller is translated into a trade-off between performance indices $\gamma_{p,\Delta}$ and γ_{np} .

In a typical repetitive controller, only the M design parameters W_m are used to optimize performance, and in [15] these parameters are designed according to the following optimization problem :

$$\underset{W_m, \gamma_{p,\Delta}, \gamma_{np}}{\text{minimize}} \quad \gamma_{p,\Delta} + \alpha \gamma_{np} \quad (14a)$$

$$\text{subject to} \quad \|1 - W(z)Q(z)\|_\infty \leq \gamma_{np} \quad (14b)$$

$$V_l |1 - W(\omega)Q(\omega)| \leq \gamma_{p,\Delta}, \quad \forall \omega \in \Omega_l, \quad \forall l \in \mathcal{L} \quad (14c)$$

α is a user-specified optimization weight, quantifying the relative importance of $\gamma_{p,\Delta}$ and γ_{np} . Optimization problem (14) is not numerically tractable, since its constraints require evaluation on infinitely many frequencies. Therefore, (14) is transformed into an equivalent semi-definite program (SDP) by application of the (generalized) Kalman-Yakubovich-Popov (KYP) lemma [13, 16]. In [15], the resulting SDP is further simplified such that it can be solved with a general-purpose interior point solver, like SDPT3 [23], at a computational cost that is independent of N : $\mathcal{O}(M^6)$.

The generalized repetitive controller design adopts the same performance specification, while both the design freedom and design specifications of $Q(z)$ and $W(z)$ are combined into one large design problem for $X(z)$:

$$\underset{X_k, \gamma_{p,\Delta}, \gamma_{np}}{\text{minimize}} \quad \gamma_{p,\Delta} + \alpha \gamma_{np} \quad (15a)$$

$$\text{subject to} \quad \|1 - X(z)\|_\infty \leq \gamma_{np} \quad (15b)$$

$$V_l |1 - X(\omega)| \leq \gamma_{p,\Delta}, \quad \forall \omega \in \Omega_l, \quad \forall l \in \mathcal{L} \quad (15c)$$

$$|X(\omega)| \leq \epsilon, \quad \forall \omega \geq \omega_{BW}, \quad (15d)$$

Figure 5: Comparison of the FRF of $M_S(z)$ for the $K_{tRC,89}(z)$ and the $K_{gRC,89}(z)$ that yield $\gamma_{np} = 1.4$. The $\Delta = 2\%$ relative uncertainty on ω_p results in the grey-shaded uncertainty intervals on the various harmonics.

This optimization is also transformed into an SDP by application of the (generalized) KYP lemma, similarly to (14), but the resulting SDP is much harder to solve. K is generally of order MN (see Section 4) and this high number of optimization variables renders the resulting SDP practically insolvable with standard interior point solvers. To overcome this problem, the specialized solver developed by Liu and Vandenberghe [14] is used, which exploits the particular structure of LMIs obtained by the (generalized) KYP lemma. Although this results in a significant reduction in computational time ($\mathcal{O}(K^3)$ instead of $\mathcal{O}(K^6)$), the computational effort is still larger than for a typical design (tens of CPU minutes compared less than a CPU second).

4 Numerical Results

This section illustrates the potential of the generalized repetitive controller on a numerical example. The input period contains $N = 50$ samples, while the first five harmonics are present in $v(k)$ with equal weight : $V_l = 1, \forall l \in \mathcal{L} = \{1, 2, 3, 4, 5\}$. The original feedback system has bandwidth $\omega_{BW} = 8\omega_p$ and the relative degree of $T_o(z)$ equals $\tau = 2$. These two properties of the original feedback system are the only ones affecting the repetitive controller design.

Subsection 4.1 compares generalized and typical repetitive controllers for this simulation example, while Subsection 4.2 analyzes the performance of a generalized repetitive controller as a function of K .

4.1 Generalized versus typical repetitive control

In a M 'th order typical repetitive controller, $W(z)Q(z)$ corresponds to a particularly structured FIR filter of length $K_M + 1$, where

$$K_M = MN + \frac{n_Q}{2}. \quad (16)$$

As elaborated in Subsection 3.1, only the causal part of zero-phase FIR filter $Q(z)$ contribute to K_M . The FIR filter length of $X(z)$ or $W(z)Q(z)$ equals the number of memory locations

Figure 6: Comparison of the impulse response of $M_S(z)$ for the $K_{tRC,89}(z)$ and the $K_{gRC,89}(z)$ that yield $\gamma_{np} = 1.4$.

required for the real-time generalized/typical repetitive controller implementation, and in addition, $M_S(z)$ has a finite impulse response of the same length (12). This subsection compares typical and generalized repetitive controllers that require the same amount of memory (or, yield the same transient response time for $M_S(z)$), while Subsection 4.2 complements these results by comparing typical and generalized repetitive controllers with different K_M and K values.

In the typical repetitive controllers, parameters W_m are computed as the solution of (14), while $Q(z)$ is designed as the shortest zero-phase FIR filter such that all harmonics $l \in \mathcal{L}$ lie in its pass-band :

$$1 - 10^{-3} \leq Q(\omega) \leq 1 + 10^{-3}, \quad \forall \omega \leq \omega_p \max_{l \in \mathcal{L}} \{l\}, \quad (17)$$

while $|Q(\omega)| \leq \epsilon = 10^{-3}$ for all $\omega \geq \omega_{BW}$. This way, $n_Q = 78$, and for a first-order typical repetitive controller ($M = 1$), Equation (16) yields $K_M = 89$. These typical repetitive controllers are denoted $K_{tRC,89}(z)$ and they are compared here to generalized repetitive controllers with $K = K_M = 89$, designed according to (15). In constraint (15d), $\epsilon = 10^{-3}$ is used, and the resulting generalized repetitive controllers are indicated by $K_{gRC,89}(z)$.

Figure 5 compares, for $\Delta = 2\%$, the FRFs of $M_S(z)$ corresponding to the $K_{tRC,89}(z)$ and the $K_{gRC,89}(z)$ that yield $\gamma_{np} = 1.4$. While yielding the same level of non-periodic performance degradation, the $K_{gRC,89}(z)$ yields significantly better periodic performance ($\gamma_{p,2\%} = 0.53$) than the $K_{tRC,89}(z)$ ($\gamma_{p,2\%} = 0.72$). Figure 5 reveals the characteristic periodic behavior of $M_S(\omega)$ for a typical repetitive controller, which is imposed by its structure. In addition, this figure illustrates how the additional design freedom of a generalized repetitive controller is applied to obtain better performance. First, in contrast to the $K_{tRC,89}(z)$, the $K_{gRC,89}(z)$ does not waste control effort at harmonics not present in the input, such as $l = 0$ and $l = 6$. In addition, the $K_{gRC,89}(z)$ properly accounts for the linearly growing uncertainty intervals Ω_l around the harmonics, as opposed to the $K_{tRC,89}(z)$. This is revealed by the fact that for the $K_{gRC,89}(z)$ constraint (15c) is active for all harmonics $l \in \mathcal{L}$, that is : in each of the grey-shaded uncertainty intervals there is at least one ω at which $|M_S(\omega)| = \gamma_{p,2\%} = 0.53$. For the $K_{tRC,89}(z)$, on the other hand, constraint (14c) with $\gamma_{p,2\%} = 0.72$ is only active for $l = 5$, the highest harmonic in $\mathcal{L}$, such that the $K_{tRC,89}(z)$ is overly robust at the lower harmonics. Although not illustrated by the example, in a similar way the periodicity of $M_S(\omega)$ impedes a typical repetitive controller to account properly for harmonic dependent weights V_l .

Figure 6 compares the corresponding impulse responses $m_S(k)$ of $M_S(z)$, and as explained

Figure 7: Comparison of the tradeoff-curves between $\gamma_{p,\Delta}$ and γ_{np} for a $K_{tRC,89}(z)$ and a $K_{gRC,89}(z)$. (a) $\Delta = 0\%$, (b) $\Delta = 2\%$; where Δ corresponds to the relative uncertainty on ω_p .

above, for both the $K_{tRC,89}(z)$ and the $K_{gRC,89}(z)$ this impulse response is confined to 90 time instants : from $k = 0$ to $k = 89$. Although the $K_{gRC,89}(z)$ has more freedom in the design of $M_S(z)$, this controller replicates the dominant behavior of $m_S(k)$ for the $K_{tRC,89}(z)$ around $k = 0$ and $k = 50$.

To allow for a more systematic comparison between the performance attainable by a $K_{tRC,89}(z)$ and a $K_{gRC,89}(z)$, Figure 7 shows the corresponding trade-off curves between $\gamma_{p,\Delta}$ and γ_{np} , for $\Delta = 0\%$ (a) and $\Delta = 2\%$ (b). These trade-off curves are computed by solving (14) with $M = 1$, and (15) with $K = 89$, for a range of α values : by increasing α they are traced from left to right. For a given level of robust periodic performance improvement $\gamma_{p,\Delta}$, the curve indicates the minimal level of non-periodic performance degradation γ_{np} that a $K_{tRC,89}(z)$ or a $K_{gRC,89}(z)$ can attain. Vice-versa, for a fixed level of allowable non-periodic performance degradation, the trade-off curve indicates the best robust periodic performance improvement that can be achieved.

If no uncertainty on ω_p is present ($\Delta = 0\%$) the trade-off curves reveal that, for the same level of non-periodic performance degradation γ_{np} , a $K_{gRC,89}(z)$ is able to improve the periodic performance $\gamma_{p,0\%}$ of a $K_{tRC,89}(z)$ with 0.23. In the presence of 2% uncertainty on ω_p , a $K_{gRC,89}(z)$ can gain up to 0.39 periodic performance $\gamma_{p,2\%}$ for a given level of non-periodic

Figure 8: $\gamma_{p,\Delta}$ of a generalized repetitive controller as a function of K , for $\gamma_{np} = 1.4$. Optimal typical repetitive controllers are indicated at their K_M value by a cross. (a) $\Delta = 0$, (b) $\Delta = 2\%$; where Δ corresponds to the relative uncertainty on ω_p .

performance degradation. The advantage of a generalized over a typical repetitive controller gets larger for higher uncertainty on ω_p due to the former's capability to deal more efficiently with this kind of uncertainty.

4.2 Performance as a function of K

To further investigate how the performance of a generalized repetitive controller depends on K , Figure 8 shows $\gamma_{p,\Delta}$ as a function of K for a fixed level of non-periodic performance degradation $\gamma_{np} = 1.4$ and two levels of uncertainty on ω_p . Additionally, the optimal typical repetitive controllers with various orders M are indicated at their K_M value by a cross. The intersections of the trade-off curves in Figure 7 with $\gamma_{np} = 1.4$ are found again in Figure 8 by inspecting the corresponding K value.

Figure 8 is of substantial help to the control engineer who has to select a proper memory buffer length $K + 1$ for the generalized repetitive controller. This choice is facilitated by the fact that the curves feature alternating parts with steep and shallow slopes. From an engineering point of view, K values at the foot of a steep slope are preferred: reducing K any further would

result in a substantial loss of performance, whereas little performance is gained by increasing K . As observed in Figure 8, these preferred K values are separated by approximately $N = 50$ samples.

Figure 8 also facilitates the comparison between typical and generalized repetitive controllers for a fixed level of tolerable non-periodic performance degradation, in this case $\gamma_{np} = 1.4$. The horizontal distance between a typical repetitive controller point and the generalized repetitive controller curve reveals how much memory can be saved without loss of performance, while the vertical distance indicates the periodic performance gained by a generalized repetitive controller using the same amount of memory.

5 Conclusion

This paper proposes the generalized repetitive controller, a novel type of SISO repetitive controllers that generalizes the typical, period-delay based repetitive controller structure. Whereas its structure impedes a typical repetitive controller to account properly for the input spectrum and to handle uncertainty on the period of the input efficiently, the generalized structure provides additional design freedom to overcome these disadvantages. It has been numerically illustrated that a generalized repetitive controller yields significantly better performance with the same amount of memory, or, conversely, requires substantially less memory to achieve equivalent performance.

The generalized repetitive controller design is based on two performance indices, γ_{np} and $\gamma_{p,\Delta}$, that quantify its effect on the closed-loop performance for the non-periodic and the period-uncertain inputs, respectively. The trade-off curves between these performance indices constitute a valuable design tool for control engineers : they provide insight in the fundamental trade-off between γ_{np} and $\gamma_{p,\Delta}$ and aid the engineer to make a proper selection of the memory buffer length for the repetitive controller. In addition, they allow for a clear comparison between the performance of typical and generalized repetitive controllers.

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