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DEPARTEMENT TOEGEPASTE ECONOMISCHE WETENSCHAPPEN

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INTERNATIONAL PORTFOLIO CHOICE AND ASSET PRICING WITH CONSTRAINTS ON BORROWING AND SHORTSELLING

by
P. SERCU

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International Portfolio Choice and Asset Pricing with Constraints on Borrowing and Shortselling

by

Piet Sercu*

Abstract This paper reviews and re-interprets static mean-variance demand for assets under shortselling constraints at the individual level, and then asks the question under what conditions there still exist portfolios or "funds" that are universal, that is, common across all investors regardless of their country (or type of real unit consumed).

Key words: mean-variance asset demand, portfolio theory, shortselling constraints, international finance

In the standard static, one-country, homogenous-expectations CAPM with a risk-free real asset, shortselling of risky assets cannot be optimal, as everybody holds the market portfolio. Other CAPMs, however, lack this property. For example, in Black (1972)'s version, risk-free borrowing is restricted or a (real) risk-free asset may even be entirely absent, and some or all investors hold positions in two funds that are are both fully loaded on all risky assets. In neither of these funds, and a fortiori not in combinations where one of these funds is held short, can negative investments be ruled out a priori (see e.g. Fama, 1976). Likewise, the international model of Solnik (1973) and Sercu (1980) may require systematic shortselling of foreign risk-free assets. In reality, shortselling of foreign bonds is not an option that would naturally occur to non-institutional investors.[1] One issue we address in this paper is the implications of shortselling constraints—on risk-free borrowing in home or foreign currency, or even on stocks—for (i) two-fund separation, (ii) numeraire-independence of the log-utility portfolio. We also list conditions under which a modified CAPM still holds.

In Section 1 we review asset demand for one particular reference country. The international ramifications are discussed in Section 2. Section 3 discusses the implications for the international CAPM (INCAPM). Section 4 concludes.

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*KU Leuven Graduate School of Business Studies, Naamsestraat 69, 3000 Leuven, Belgium; piet.sercu@econ.kuleuven.ac.be

[1]A no-foreign-borrowing constraint has also been invoked as an explanation of home bias, another phenomenon that is in contradiction with Sercu's (1980) asset demand.

1. The demand equations with shortsale restrictions

We consider an individual i who chooses mean-variance-efficient weights $x_{i,j}$ for securities j = 1,..., n+1 subject to non-negativity constraints. One way of writing the portfolio part of the static Merton (1971) objective function with constraints $x_{i,j} \geq 0$ is

$$\max_{\mathbf{X}_i} \; [r + \mathbf{X}_i^{\mathsf{T}} (\mathbf{m}-r\mathbf{u})] - \frac{\eta_i}{2} \; \mathbf{X}_i^{\mathsf{T}} \mathbf{V} \mathbf{X}_i + \lambda_{i,n+1} (1 - \mathbf{X}_i^{\mathsf{T}} \mathbf{u}) - \sum_{j=1}^{n} \lambda_{i,j} x_{i,j} , \tag{1.1}$$

where r is the risk-free rate

$\mathbf{X}_i$ is the n×1 vector of asset weights $x_{i,i}$ chosen by individual i for risky assets 1,...,n

m is the n×1 vector of expected returns for risky assets 1,...,n

u is the n×1 vector with all elements equal to unity

$\eta_i = -W_i \dfrac{\partial^2 J_i/\partial W_i^2}{\partial J_i/\partial W_i}$ is relative risk aversion (where $J_i(W_i)$ is the investor's derived utility of wealth W_i).

V is the nxn covariance matrix of the returns on the risky assets

$\lambda_{i,n+1}$ is the Kuhn-Tucker multiplier for the constraint $x_{i,n+1} = 1 - \mathbf{X}_i^{\mathsf{T}} \mathbf{u} \geq 0$ on the risk-free asset (asset n+1)

$\lambda_{i,j}$ is the Kuhn-Tucker multiplier for the constraint $x_{i,j} \geq 0$ for risky asset j = 1,...,n

$\Lambda_i = [\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,n}]^{\mathsf{T}}$.

Lemma 1.1. *Total demand can be written as*

$$\begin{bmatrix} \mathbf{X}_i \\ \cdots \\ x_{i,n+1} \end{bmatrix} = (1 - \frac{1}{\eta_i}) \begin{bmatrix} \mathbf{0} \\ \cdots \\ 1 \end{bmatrix} + \frac{1}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1} (\mathbf{m}-r\mathbf{u}) \\ \cdots \\ 1-\mathbf{u}^{\mathsf{T}}\mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u}) \end{bmatrix} - \frac{\lambda_{i,n+1}}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1} \mathbf{u} \\ \cdots \\ -\mathbf{u}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{u} \end{bmatrix}$$

$$- \sum_{j=1}^{n} \frac{\lambda_{i,j}}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1}\mathbf{J} \\ \cdots \\ -\mathbf{u}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{J} \end{bmatrix} \tag{1.2}$$

*where **J** is a nx1 vector with a 1 in the j-th position and zeroes elsewhere*

Proof. Let $\Lambda_i = [\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,n}]^{\mathsf{T}}$. The Euler conditions for the risky assets are

$$(\mathbf{m}-r\mathbf{u}) - \eta_i \mathbf{V} \mathbf{X}_i - \lambda_{i,n+1} \mathbf{u} - \Lambda_i = \mathbf{0} , \tag{1.3}$$

where the multipliers satisfy the Kuhn-Tucker conditions,

- for asset n+1: $\lambda_{i,n+1} (1 - \mathbf{X}_i^{\mathsf{T}} \mathbf{u}) = 0,$ (1.4)

- for risky assets j=1,...,n: $\lambda_{i,j} x_{i,j} = 0.$ (1.5)

From (1.3) and the constraint $x_{i,n+1} = 1 - \mathbf{u}^\top \mathbf{X}_i$, we can write asset demand as

$$\mathbf{X}_i = \frac{1}{\eta_i}\, \mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u}) - \frac{\lambda_{i,n+1}}{\eta_i}\, \mathbf{V}^{-1}\mathbf{u} - \frac{1}{\eta_i}\, \mathbf{V}^{-1}\Lambda_i\,, \tag{1.6}$$

and[2]

$$x_{i,n+1} = 1 - \frac{1}{\eta_i}\, \mathbf{u}^\top \mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u}) + \frac{\lambda_{i,n+1}}{\eta_i}\, \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + \frac{1}{\eta_i}\, \mathbf{u}^\top\mathbf{V}^{-1}\Lambda_i \tag{1.7}$$

$$= \frac{1}{\eta_i}\left[1 - \mathbf{u}^\top\mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u})\right] + (1 - \frac{1}{\eta_i}) + \frac{\lambda_{i,n+1}}{\eta_i}\, \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + \frac{1}{\eta_i}\, \mathbf{u}^\top\mathbf{V}^{-1}\Lambda_i. \tag{1.8}$$

Rearranging, this leads to (1.2). **QED**

The first two terms on the right-hand side of (1.2) are regular portfolios with weights summing to unity, while the n+1 funds associated with the non-negativity constraints are zero-investment portfolios, or debt-equity swap portfolios. As a log-utility investor has $\eta_i = 1$, the second portfolio can be interpreted as the log-utility portfolio (LUP) in the absence of constraints.[3]

Corollary to Lemma 1.1: *To satisfy all investors regardless of relative risk aversion, one may need up to n+1 "funds" in the sense of Merton (1971).*

Proof: Equations (1.2) has, in total, n+3 portfolios, but as there are only n+1 assets the maximum number of funds needed is obviously n+1. **QED**

Thus, two-fund separation disappears as soon as the non-negativity constraints are binding. Figure 1 interprets this in the familiar mean/standard-deviation plane. Low-risk-aversion investors chose portfolios in the line segment r-t, which are combinations of the risk-free asset (r) and the tangency portfolio (t). In Black's no-borrowing model without shortselling constraints, all higher-risk portfolios (like a, b, c) are generated as convex combinations of, for instance, the tangency portfolio t (long) and the minimum-variance portfolio m (short). In the presence of shortselling constraints, however, more and more assets leave the portfolios as we move upwards on the efficient frontier. For instance, portfolios in the segment b-c may contain just two assets, and c then is the portfolio where the weight of one of these two becomes zero. In

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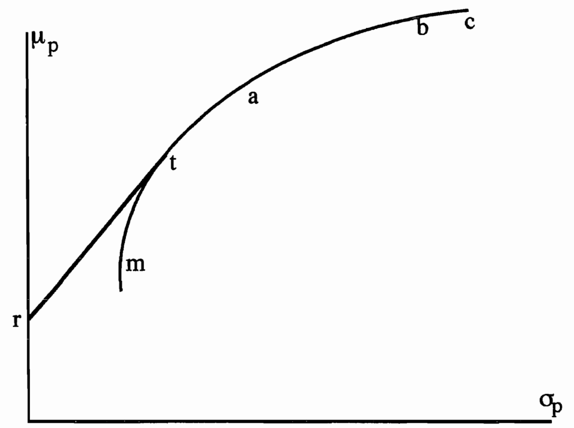
[2]To get the last line, add and subtract $1/\eta_i$ and rearrange.

[3]For the purpose of international portfolio choice, the merit in writing the first two portfolios this way, rather than as

$$\frac{1}{\eta_i}\, \mathbf{X}^\top\mathbf{V}^{-1}(\mu - r\mathbf{u}) \begin{bmatrix} \dfrac{\mathbf{V}^{-1}(\mu - r\mathbf{u})}{\mathbf{X}^\top\mathbf{V}^{-1}(\mu - r\mathbf{u})} \\ \dots \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\eta_i}\, \mathbf{X}^\top\mathbf{V}^{-1}(\mu - r\mathbf{u})\right) \begin{bmatrix} 0 \\ \dots \\ 1 \end{bmatrix}$$

is that the log-utility portfolio portfolio is numeraire-independent.

that point, the portfolio contains just one asset (whose λ is zero) but the λ for the other asset is also zero.



Key to Figure 1. Low-risk-aversion investors chose portfolios in the line segment r-t, i.e. combinations of the risk-free asset (r) and the tangency portfolio (t). In Black's no-borrowing model without shortselling constraints, all higher-risk portfolios (like a, b, c) are generated as convex combinations of, for instance, the tangency portfolio t and the shorted minimum-variance portfolio m. In the presence of shortselling constraints, more and more assets leave the higher-risk portfolios. For instance, portfolios in the segment b-c may contain just two assets, and c then is the portfolio where the weight of one of these is zero.

1.1. Partitioning the solution in the absence of constraints

Throughout this paper, we denote the solution in the absence of constraints by Greek characters:

$$\begin{bmatrix} \Xi_i \\ \cdots \\ \xi_{i,n+1} \end{bmatrix} = (1 - \frac{1}{\eta_i}) \begin{bmatrix} \mathbf{0} \\ \cdots \\ 1 \end{bmatrix} + \frac{1}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1}\,(\mathbf{m}-\mathbf{r}\mathbf{u}) \\ \cdots \\ 1-\mathbf{u}^{\mathsf{T}}\mathbf{V}^{-1}(\mathbf{m}-\mathbf{r}\mathbf{u}) \end{bmatrix} . \tag{1.9}$$

For later reference, we study the demand for particular partitionings of the risky-asset menu into a set "1" and a set "2". For instance, in Sercu (1980) the first set contains the stocks, and the second set the foreign-currency risk-free assets. For current purposes, a useful partitioning may be the sets for which the no-shortselling constraints are binding and not binding, respectively. At this stage we just consider a general partitioning of the matrices,

$$\Xi_i = \begin{bmatrix} \Xi_{1i} \\ \cdots \\ \Xi_{2i} \end{bmatrix} \;;\; \mathbf{m}-\mathbf{r}\mathbf{u} = \begin{bmatrix} \mathbf{m}_1-\mathbf{r}\mathbf{u}_1 \\ \cdots \\ \mathbf{m}_2-\mathbf{r}\mathbf{u}_2 \end{bmatrix} \;;\; \mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & : & \mathbf{C}_{12}^{\mathsf{T}} \\ \cdots & : & \cdots \\ \mathbf{C}_{12} & : & \mathbf{V}_2 \end{bmatrix} . \tag{1.10}$$

We also define

$\Delta_{12}^{\top} = \mathbf{C}_{12}\,\mathbf{V}_2^{-1}$, the $n_1 \times n_2$ matrix of coefficients $\Delta_{j,k}$ in the n_1 multivariate regressions of the return from each asset $j(1) = 1,\ldots,n_1$ on all returns from assets k in set 2:

$$\frac{dV_{j(1)}}{V_{j(1)}} = \alpha_j + \sum_{k(2)=1}^{n_2} \Delta_{j(1),k(2)} \frac{dV_{k(2)}}{V_{k(2)}} + \varepsilon_{j(1)|2} \tag{1.11}$$

$\mathbf{V}_{1|2} = \mathbf{V}_1 - \Delta_{12}\,\mathbf{V}_2\,\Delta_{12}^{\top}$, the matrix of covariances between the errors of all these regressions

Lemma 1.2. *Demand can be decomposed as follows:*

(i) Demand for the assets in set 1 can be interpreted as the demand for these assets hedged, in the Johnson (1960) - Stein (1961) sense, against risks from assets in set 2 by selling forward $\Delta_{j(1),k(2)}$ units of each asset in set 2:

$$\Xi_{1i} = \frac{1}{\eta_i}\,\mathbf{V}_{1|2}^{-1}\,[(\mathbf{m}_1 - r\mathbf{u}_1) - \Delta_{12}^{\top}(\mathbf{m}_2 - r\mathbf{u}_2)]. \tag{1.12}$$

(ii) Demand for assets in set 2 consists of (a) the demand for these assets as if set 1 had not existed, plus (b) the demand for these assets as hedges of the assets in set 1:

$$\Xi_{2i} = \frac{1}{\eta_i}\,\mathbf{V}_2^{-1}(\mathbf{m}_2 - r\mathbf{u}_2) - \Delta_{12}\,\Xi_{1i}\,. \tag{1.13}$$

Proof: One way to write the partitioned inverse is

$$\mathbf{V}^{-1} = \begin{bmatrix} \mathbf{V}_{1|2}^{-1} & : -\mathbf{V}_{1|2}^{-1}\,\Delta_{12}^{\top} \\ \ldots & : \ldots \\ -\Delta_{12}\,\mathbf{V}_{1|2}^{-1} & : \mathbf{V}_2^{-1} + \Delta_{12}\,\mathbf{V}_{1|2}^{-1}\,\Delta_{12}^{\top} \end{bmatrix}, \tag{1.14}$$

where $\Delta_{12}^{\top}$ and $\mathbf{V}_{1|2}$ are defined below (1.10). Equation (1.14) implies that the demand for risky assets can be rewritten as

$$\begin{bmatrix} \Xi_{1i} \\ \ldots \\ \Xi_{2i} \end{bmatrix} = \frac{1}{\eta_i} \begin{bmatrix} \mathbf{V}_{1|2}^{-1}\,[(\mathbf{m}_1 - r\mathbf{u}_1) - \Delta_{12}^{\top}(\mathbf{m}_2 - r\mathbf{u}_2)] \\ \ldots \\ \mathbf{V}_2^{-1}(\mathbf{m}_2 - r\mathbf{u}_2) - \Delta_{12}\,\mathbf{V}_{1|2}^{-1}[(\mathbf{m}_1 - r\mathbf{u}_1) - \Delta_{12}^{\top}(\mathbf{m}_2 - r\mathbf{u}_2)] \end{bmatrix}. \tag{1.15}$$

Sercu (1980) interprets this as demand for hedged assets 1. Hedging of an asset j(1) from set 1 using assets k(2) from set 2 is defined as in Johnson (1960) and Stein (1961): find the set of positions $\Delta_{j(1),k(2)}$ in forward sales contracts on assets $k(2)=1,\ldots,n_2$ that minimizes the variance of the hedged asset, or, equivalently, makes the hedged return uncorrelated with the returns on each of the hedge assets k(2). The familiar solution is to sell forward amounts $\Delta_{j(1),k(2)}$ that are given by the regression coefficients in (1.11).[4] Thus, $(\mathbf{m}_1-r\mathbf{u}_1) - \Delta_{12}^{\mathsf{T}}(\mathbf{m}_2-r\mathbf{u}_2)$ is the vector of excess returns on each of the assets in set 1 after hedging them using assets in set 2, and $\mathbf{V}_{1|2}$ is the variance-covariance matrix of these hedged returns. This provides the interpretation of (1.12) as the demand for assets 1 hedged.

Using (1.12), the second set of equations in (1.15) simplifies to (1.13). In that expression, $(1/\eta_i)\,\mathbf{V}_2^{-1}(\mathbf{m}_2-r\mathbf{u}_2)$ is the demand that would have been the solution if set 1 had not existed or if the returns on assets 1 had been uncorrelated with the returns on set 2; and the positions $-(1/\eta_i)\,\Delta_{12}\,\Xi_{1i}$ serve to orthogonalize (or hedge) each of the returns from assets 1 on each of the returns from assets 2. **QED**.

The above interpretation is just one of the many possible ones. For instance, demand for assets in group 2 can also be written in the style of (1.12), and demand for asset in group 1 can be written in the style of (1.13). The current interpretation is, however, quite convenient for our purpose.

1.2. Conditions under which none of the constraints are binding

The constraints do not affect the unrestricted solution provided that $\xi_{i,j} \geq 0$ for all j = 1,..., n+1. We first discuss the no-borrowing constraint and then the no-shortselling constraints, each time assuming that the other constraints are not binding.

First consider the risk-free asset. From (1.7), if the no-shortselling constraints are not binding, the no-borrowing constraint is not binding either when the individual's risk aversion is sufficiently high:

$$\xi_{i,n+1} \gtreqless 0 \Leftrightarrow \eta_i \gtreqless \mathbf{u}^{\mathsf{T}}\,\mathbf{V}^{-1}\,(\mathbf{m}-r\mathbf{u})\,. \qquad (1.16)$$

Next consider the risky assets. To identify the condition under which the weight $\xi_{i,j}$ of a risky asset is non-negative we consider (without loss of generality) risky asset 1. In terms of the partitioned demand (1.15), we let set 1 contain just asset 1. It follows that $\xi_{j,1}$ is given by the return/variance ratio of the first asset hedged against all other asset risks:

$$\xi_{i,1} = \frac{1}{\eta_i}\,\frac{(m_1-r) - \sum_{k=2}^{n}\Delta_{1,k}\,(m_k-r)}{v_{1|\text{all } k\neq 1}}\,. \qquad (1.17)$$

Therefore the condition under which the Kuhn-Tucker constraint for any risky asset j is not binding, given that the no-borrowing constraint does not bind either, is

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[4]In the absence of a direct forward market, one shorts, per unit of value invested in stock j(1), $\delta_{j(1),k(2)}$ units of each asset k(2) and invests the proceeds in the risk-free asset.

$$(m_j - r) \geq \sum_{\text{all } j \neq k}^{n} \Delta_{j,k} (m_k - r). \tag{1.18}$$

Equation (1.18) may be useful for portfolio allocation models that rely on estimated inputs. If the prior is that all assets should be held long, a violation of (1.18) suggests underestimation of the expected return relative to the other expected returns and exposures.

Equations (1.16) and (1.18) provide conditions under which each constraint is non-binding provided that the other constraints are not binding either. In the next section we discuss the counterparts to (1.16) and (1.18) when some of the other constraints *are* binding.

1.3. The general constrained solution

We partition the set of risky assets into a subset "1" for which $\lambda_{i,j}$ is non-zero (implying $x_{i,j}$ = 0), and a set "2" for which $\lambda_{i,j}$ is zero (implying $x_{i,j} \geq 0$). Obviously, the portfolio must be efficient with reference to set 2.

We can infer the value of a positive multiplier $\lambda_{i,n+1}$ from the property that, when the no-borrowing restriction holds, total holdings for asset n+1 is zero. Total demand for that asset is given by (1.7). If we use the properties that (i) for all assets in set 1 the weights are zero and (ii) for assets in set 2 all λs are zero except possibly $\lambda_{i,n+1}$, then (1.7) can be written as

$$1 - \frac{1}{\eta_i} \mathbf{u}_2^T \mathbf{V}_2^{-1} (\mathbf{m}_2 - r\mathbf{u}_2) + \frac{\lambda_{i,n+1}}{\eta_i} \mathbf{u}_2^T \mathbf{V}_2^{-1} \mathbf{u}_2 = 0 .$$

This immediately leads to a result obtained by Tepla (1997),

$$\lambda_{i,n+1} = \text{Max}\left(\frac{\mathbf{u}_2^T \mathbf{V}_2^{-1} (\mathbf{m}_2 - r\mathbf{u}_2) - \eta_i}{\mathbf{u}_2^T \mathbf{V}_2^{-1} \mathbf{u}_2}, 0 \right) \tag{1.19}$$

$$> 0 \text{ if } \eta_i < \mathbf{u}_2^T \mathbf{V}_2^{-1} (\mathbf{m}_2 - r\mathbf{u}_2) .$$

The term $\mathbf{u}_2^T \mathbf{V}_2^{-1} (\mathbf{m}_2 - r\mathbf{u}_2)$ can be interpreted as the total weight of all risky investments in the LUP from set 2, or alternatively as the expected excess return on the minimum-variance swap portfolio.

Now consider situations where only the no-borrowing constraint is binding, that is, where set 2 contains all assets. Then the investor shorts the minimum-variance portfolio of risky assets. The question we address under what conditions an investor with relative risk aversion η_i is not constrained by a possible no-shorting rule for risky assets. If $\lambda_{i,n+1} > 0$ but $\lambda_{i,j} = 0$ for all $j \leq n$, then (1.2) can be written as

$$\begin{bmatrix} \mathbf{X}_i \\ \cdots \\ x_{i,n+1} \end{bmatrix} = (1 - \frac{1}{\eta_i}) \begin{bmatrix} \mathbf{0} \\ \cdots \\ 1 \end{bmatrix} + \frac{1}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1} (\mathbf{m}-(r+\lambda_{i,n+1})\mathbf{u}) \\ \cdots \\ 1-\mathbf{u}^T\mathbf{V}^{-1}(\mathbf{m}-(r+\lambda_{i,n+1})\mathbf{u}) \end{bmatrix}, \tag{1.20}$$

with the interpretation that $\lambda_{i,n+1}$ is the lowest bid-offer spread that, when added to the risk-free rate, would make borrowing unattractive to investor i.[5] If (1.20) is the solution, then for each risky asset j condition (1.18) still holds except that r is increased by the critical spread $\lambda_{i,n+1}$:

$$m_j-(r+\lambda_{i,n+1}) \geq \sum_{\text{all } j\neq k}^{n} \Delta_{j,k} [m_k-(r+\lambda_{i,n+1})] ,$$

or

$$(m_j-r) - \sum_{\text{all } j\neq k}^{n} \Delta_{j,k} (m_k-r) \geq (1 - \sum_{\text{all } j\neq k}^{n} \Delta_{j,k}) \lambda_{i,n+1} . \tag{1.21}$$

Comparing to (1.18), we see that (1.21) is stricter than (1.18) for assets that are less sensitive to returns on other assets (or more precisely, for assets with $\sum_{j\neq k} \Delta_{1j,k} < 1$)[6], and for investors with low risk-aversions (that is, high $\lambda_{i,n+1}$). Conversely, (1.21) is easier to meet than (1.18) in the case of assets highly related to others, that is, assets for which the right hand side of (1.21) is negative. Thus, assets that have negative weights in the unconstrained solution may be held positively in the constrained one.

For sufficiently unusual values of η, then one or more $\lambda_{i,j}$s will be positive. To characterize the suppressed assets 1, we write (1.2) as an efficient portfolio selected from set 2:

$$\begin{bmatrix} \mathbf{X}_{1i} \\ \cdots \\ \mathbf{X}_{2i} \\ \cdots \\ x_{i,n+1} \end{bmatrix} = \frac{1}{\eta_i} \begin{bmatrix} \mathbf{0}_1 \\ \cdots \\ \mathbf{V}_2^{-1} [\mathbf{m}_2-(r+\lambda_{i,n-1})\mathbf{u}_2] \\ \cdots \\ 1-\mathbf{u}_2^T\mathbf{V}_2^{-1}[\mathbf{m}_2-(r+\lambda_{i,n-1})\mathbf{u}_2] \end{bmatrix} + (1 - \frac{1}{\eta_i}) \begin{bmatrix} \mathbf{0}_1 \\ \cdots \\ \mathbf{0}_2 \\ \cdots \\ 1 \end{bmatrix} . \tag{1.22}$$

Comparing (1.22) to (1.15), we see that, holding constant the distribution parameters, the constrained solution differs from the unconstrained one in four respects. First, if $\lambda_{i,n+1}$ is positive, we again have to use a modified risk-free rate that includes the investor-specific critical spread $\lambda_{i,n+1}$ needed to repress negative holdings. Second, and most obviously, the unconstrained demand for asset set 1 is suppressed. Third, there is no more demand for asset set 2 as hedging devices for assets 1. Lastly, in the LUP the suppressed demand (demand for asset set 1 and the "hedging" demand for asset set 2) shows up as increased demand for the risk-free asset.

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[5]Note, in passing, that this equation does not imply two-fund separation because the required spread $\lambda_{i,n+1}$ is investor-specific. The same comment holds for Tepla's way of correcting the risky-asset returns as $\mu + \Lambda_i$.

[6]$\sum_{j\neq k} \Delta_{1j,k}$ can ve interpreted as the "beta" w.r.t. an equally-weighted portfolio of all other assets.

We now demonstrate a property for the risk-return characteristics of assets with binding constraints on shortselling. Suppose we offer the investors the asset menu "2" plus an asset j(1) from set 1. In the absence of a shortselling constraint on that extra asset, the demand for it would be

$$\xi_{i,j(1)} = \frac{1}{\eta_i} \frac{m_{j(1)}-(r+\lambda_{i,n+1}) - \Delta_{j(1),2}^{\mathsf{T}}(\mathbf{m}_2-(r+\lambda_{i,n+1})\mathbf{u}_2)}{v_{1|2}} , \qquad (1.23)$$

where $\Delta_{j(1),2}$ contains the regression coefficients of the return from asset j (a member of set 1) on all returns from assets in set 2. This demand is negative (and, therefore, violates the short-sale constraint) if

$$m_{j(1)}-(r+ \lambda_{i,n+1}) < \Delta_{j(1),2}^{\mathsf{T}}(\mathbf{m}_2-(r+\lambda_{i,n+1})\mathbf{u}_2) . \qquad (1.24)$$

Thus, (1.24) holds for any asset in set 1. (Note, in passing, that if there is a no-borrowing constraint, this property depends on the investor's relative risk aversion through $\lambda_{i,n+1}$.) This result also allows us to identify the no-shortselling multiplier $\lambda_{i,j(1)}$ in the general solution (1.2). Specifically, the multiplier can be interpreted as a lowest extra dividend that, when added to the return $m_{j(1)}$, would have eliminated the incentive to go short. From (1.23), the superdividend $\lambda_{i,j(1)}$ is the one that satisfies

$$0 = (m_{j(1)}+\lambda_{i,j(1)}) - (r+\lambda_{i,n+1}) - \Delta_{j(1),2}^{\mathsf{T}}(\mathbf{m}_2-(r+\lambda_{i,n+1})\mathbf{u}_2) ,$$

that is,

$$\lambda_{i,j(1)} = [(m_{j(1)} - r) - \sum_{k(2)=1}^{n_2} \Delta_{j(1),k(2)}(m_k-r)] - [1 - \sum_{k(2)=1}^{n_2} \Delta_{j(1),k]} \lambda_{i,n+1} . \qquad (1.25)$$

We summarize our review into the following proposition:

Proposition 1.1. *When risk-free borrowing and shortselling of risky assets are prohibited, mean-variance efficient portfolios can be written as*

$$\begin{bmatrix} \mathbf{X}_{1i} \\ \cdots \\ \mathbf{X}_{2i} \\ \cdots \\ x_{i,n+1} \end{bmatrix} = \frac{1}{\eta_i} \begin{bmatrix} \mathbf{0}_1 \\ \cdots \\ \mathbf{V}_2^{-1}(\mathbf{m}_2 - r\mathbf{u}_2) \\ \cdots \\ 1-\mathbf{u}_2^T\mathbf{V}_2^{-1}(\mathbf{m}_2 - r\mathbf{u}_2) \end{bmatrix} + (1 - \frac{1}{\eta_i}) \begin{bmatrix} \mathbf{0}_1 \\ \cdots \\ \mathbf{0}_2 \\ \cdots \\ 1 \end{bmatrix}$$

$$+ \mathrm{Min}\left(0, \frac{1 - \mathbf{u}_2^T\mathbf{V}_2^{-1}(\mathbf{m}_2 - r\mathbf{u}_2)/\eta_i}{\mathbf{u}_2^T\mathbf{V}_2^{-1}\mathbf{u}_2}\right) \begin{bmatrix} \mathbf{0}_1 \\ \cdots \\ -\mathbf{V}_2^{-1}\mathbf{u}_2 \\ \cdots \\ \mathbf{u}_2^T\mathbf{V}_2^{-1}\mathbf{u}_2 \end{bmatrix},$$

where set 2 is such that

- for all j(1) not in the set :

$$m_{j(1)} - (r + \lambda_{i,n-1}) < \sum_{k(2)=1}^{n_2} \Delta_{j(1),k}[m_k - (r + \lambda_{i,n-1})] ; \qquad (1.26)$$

- for all j(2) in the set,

$$m_{j(2)} - (r + \lambda_{i,n-1}) \geq \sum_{\substack{k(2)=1 \\ k(2)\neq j(2)}}^{n_2} \Delta_{j(2),k}[m_k - (r + \lambda_{i,n-1})] ; \qquad (1.27)$$

and where

$$\lambda_{i,n+1} = \mathrm{Max}\left(\frac{\mathbf{u}_2^T\mathbf{V}_2^{-1}(\mathbf{m}_2 - r\mathbf{u}_2) - \eta_i}{\mathbf{u}_2^T\mathbf{V}_2^{-1}\mathbf{u}_2}, 0\right). \qquad (1.28)$$

1.4. The CAPM

For completeness, we review the familiar implications for the CAPM. In general, the demand from investor i is given by

$$\begin{bmatrix} \mathbf{X}_i \\ \cdots \\ x_{i,n+1} \end{bmatrix} = (1 - \frac{1}{\eta_i}) \begin{bmatrix} \mathbf{0} \\ \cdots \\ 1 \end{bmatrix} + \frac{1}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1}[(\mathbf{m}+\Lambda_i) - r\mathbf{u}] \\ \cdots \\ 1 - \mathbf{u}^T\mathbf{V}^{-1}[(\mathbf{m}+\Lambda_i) - r\mathbf{u}] \end{bmatrix} - \frac{\lambda_{i,n+1}}{\eta_i} \begin{bmatrix} \mathbf{V}^{-1}\mathbf{u} \\ \cdots \\ -\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} \end{bmatrix}. \qquad (1.29)$$

To identify the market portfolio, we multiply both sides by the agent's invested wealth W_i; we sum across investors i; and we divided by aggregate wealth $W_m = \sum_{i=1}^{n} W_i$:

$$\begin{bmatrix} \mathbf{X}_m \\ \cdots \\ x_{m,n+1} \end{bmatrix} = \sum_{i=1}^{n} \frac{W_i}{W_m} \begin{bmatrix} \mathbf{X}_i \\ \cdots \\ x_{i,n+1} \end{bmatrix}$$

$$= (1-\frac{1}{\eta_m})\begin{bmatrix}\mathbf{0}\\ \cdots\\ 1\end{bmatrix} + \frac{1}{\eta_m}\begin{bmatrix}\mathbf{V}^{-1}[(\mathbf{m}+\Lambda_m)-r\mathbf{u}]\\ \cdots\\ 1-\mathbf{u}^T\mathbf{V}^{-1}[(\mathbf{m}+\Lambda_m)-r\mathbf{u}]\end{bmatrix} - \kappa_{m,n+1}\begin{bmatrix}\mathbf{V}^{-1}\mathbf{u}\\ \cdots\\ -\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}\end{bmatrix}, \tag{1.30}$$

where $\eta_m = \left[\sum_{i=1}^{n}\frac{W_i}{W_m}\eta_i^{-1}\right]^{-1}$, $\lambda_{m,j} = \sum_{i=1}^{n}\frac{W_i}{W_m}\lambda_{i,j}$, and $\kappa_{m,n+1} = \sum_{i=1}^{n}\frac{W_i}{W_m}\frac{\lambda_{i,n+1}}{\eta_i}$.

The market portfolio portfolio is efficient with respect to the set of risky assets if it is a combination of the funds with weights proportional to $\mathbf{V}^{-1}\,\mathbf{m}$ and $\mathbf{V}^{-1}\mathbf{u}$. This holds only in a limited number of cases (Fama, 1976; Ross, 1977):

- First, there is no restriction on borrowing, or any such restriction does not bind for any individual investor—that is, all η_i meet (1.16).
  In this case, $\kappa_{m,n+1} = 0$. In addition, all investors hold the same tangency portfolio, which must be the market portfolio of risky assets and therefore has positive loadings on all assets, implying $\Lambda_m = \mathbf{0}$.
- The second case is when $\kappa_{m,n+1} > 0$ and $\Lambda_m = \mathbf{0}$. This, in turn, is possible when (i) there are no shortselling restrictions, or (ii) the shortselling restrictions do not bind any individual investor.
  Case (ii) would hold if all η_i are equal across investors—then everybody holds the same portfolio, which must be the market portfolio of risky assets, again implying $\Lambda_m = \mathbf{0}$—or when the variability of the ηs is so small that everybody still selects positive weights for all risky assets. Then zero-beta assets or portfolios earn an expected return equal to $r+\kappa_{m,n+1}$.
- Lastly, there is the very unlikely case where $\Lambda_m = \kappa_m\mathbf{u}$, that is, many individuals face shortselling constraints but the wealth-weighted averages $\lambda_{m,j}$ happen to be identical across all risky assets j.

This finishes our review of shortselling or borrowing restrictions in a one-country setting. We now address the implications of the existence of many "countries", defined as subsets of investors whose real numeraires are identical with the subset but differ stochastically across subsets.

2. International asset demand with shortsale restrictions

Within the same setting as in Section 1, we now consider other ("foreign") investors whose real numeraires differ stochastically from the reference-country's. In each of these numeraires, there still is a risk-free asset available (which, however, is risky in terms of each of the other numeraires because the real exchange rate is stochastic). There may be many foreign countries; we arbitrarily select one.

2.1. Asset demand by a foreign investor

The foreign investor's problem is similar to the one in (1.1), and its solution similar to the one in (1.2), except that all variables now have an asterisk indicating the f-th foreign country. Thus,

$$\begin{bmatrix} \mathbf{x}_i^* \\ \cdots \\ x_{i,n+1}^* \end{bmatrix} = \left(1 - \frac{1}{\eta_i^*}\right)\begin{bmatrix} \mathbf{0} \\ \cdots \\ 1 \end{bmatrix} + \frac{1}{\eta_i^*}\begin{bmatrix} \mathbf{V}^{*-1}(\mathbf{m}^* - r^*\mathbf{u}) \\ \cdots \\ 1 - \mathbf{u}^T\mathbf{V}^{*-1}(\mathbf{m}^* - r^*\mathbf{u}) \end{bmatrix} - \frac{\lambda_{i,n+1}^*}{\eta_i^*}\begin{bmatrix} \mathbf{V}^{*-1}\mathbf{u} \\ \cdots \\ -\mathbf{u}^T\mathbf{V}^{*-1}\mathbf{u} \end{bmatrix}$$

$$- \sum_{j=1}^{n} \frac{\lambda_{ij}^*}{\eta_i^*}\begin{bmatrix} \mathbf{V}^{*-1}\mathbf{J} \\ \cdots \\ -\mathbf{u}^T\mathbf{V}^{*-1}\mathbf{J} \end{bmatrix} \tag{2.1}$$

In the sections below, we address the issue to what extent the above funds are numeraire-independent.

2.2. The unconstrained log-utility portfolio (LUP) is numeraire invariant

To link domestic and country-f demand, we need the mean and (co-)variances of the f-th real exchange rate change. This information is implicit in **V** and (**m**–r**u**): in the reference-country opportunity set, one of the risky assets is the country-f risk-free asset, the random part of whose return is the exchange-rate change. For ease of notation, and without loss of generality, we assume that, in the home country's list of risky asset, the country-f risk-free asset is the first risky asset. Likewise, in the country-f menu of risky asset, the home-country risk-free asset is the first risky asset. Thus,

- $x_{i,n+1}$ and $x_{i,1}^*$ both refer to the reference-currency risk-free asset
- $x_{i,n+1}^*$ and $x_{i,1}$ both refer to the foreign risk-free asset
- $x_{i,j}^*$ and $x_{i,j}$, $1 < j \leq n$, both refer to the same stock or third-country risk-free asset.

Define

$$\mathbf{G}_{n\times n} = \begin{bmatrix} -1 & : & -\mathbf{u}_{n-1}^T \\ \cdots & : & \cdots \\ \mathbf{0}_{n-1} & : & \mathbf{I}_{n-1} \end{bmatrix} = \mathbf{G}^{-1}, \tag{2.2}$$

where $\mathbf{u}_{n-1}$ and $\mathbf{0}_{n-1}$ are unit and zero column vectors, respectively, with dimension n–1, and $\mathbf{I}_{n-1}$ is the identity matrix with dimension n–1. Using Ito's Lemma, the vector of foreign-currency excess returns ($\mathbf{m}^*$–$r^*\mathbf{u}$) and the variance-covariance matrix $\mathbf{V}^*$ in terms of currency f are linked to the home-currency counterparts by

$$(\mathbf{m}^* - r^*\mathbf{u}) = \mathbf{G}^T\,[\mathbf{m} - r\mathbf{u} + \mathbf{V}\,\mathbf{G}^T\mathbf{u}], \tag{2.3}$$

$$\mathbf{V}^* = \mathbf{G}^T\,\mathbf{V}\,\mathbf{G} \;\Leftrightarrow\; \mathbf{V}^{*-1} = \mathbf{G}\,\mathbf{V}^{-1}\,\mathbf{G}^T. \tag{2.4}$$

We are now able to parsimonously prove a familiar result (Sercu, 1980; Stulz, 1981; Adler and Prasad, 19XX):

Proposition 2.1: *The unconstrained log-utility portfolio (LUP) is numeraire-independent.*

Proof: Denote the risky-asset weights in the domestic and foreign unconstrained log-utility portfolios (LUPs) by $\mathbf{Y}$ and $\mathbf{Y}^*$, respectively. From (2.3) and (2.4), then,

$$\mathbf{Y}^* \equiv \mathbf{V}^{*-1}(\mathbf{m}^*-r^*\mathbf{u}) = [\mathbf{G}^{-1}\mathbf{V}^{-1}(\mathbf{G}^T)^{-1}]\,\mathbf{G}^T\,[\mathbf{m} - r\mathbf{u} + \mathbf{V}\,\mathbf{G}^T\mathbf{u}]$$

$$= \mathbf{G}^{-1}\,[\mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u}) + \mathbf{G}^T\,\mathbf{u}]\,.$$

We solve this for $\mathbf{Y} \equiv \mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u})$ and find

$$\mathbf{Y} = \mathbf{G}\,\mathbf{Y}^* - \mathbf{G}^T\,\mathbf{u} = \begin{bmatrix} -1 & : & -\mathbf{u}_{n-1}^T \\ \dots & : & \dots \\ \mathbf{0}_{n-1} & : & \mathbf{I}_{n-1} \end{bmatrix} \begin{bmatrix} y_1^* \\ \dots \\ \mathbf{Y}_2^* \end{bmatrix} + \begin{bmatrix} 1 \\ \dots \\ \mathbf{0} \end{bmatrix}, \tag{2.5}$$

that is,

$$\begin{bmatrix} y_1 \\ \dots \\ \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} -y_1^* - \mathbf{u}_{n-1}^T\mathbf{Y}^* + 1 \\ \dots \\ \mathbf{Y}^*_2 \end{bmatrix}. \tag{2.6}$$

The first element on the right-hand side is $1-\mathbf{u}^T\mathbf{Y}^* \equiv y^*_{1,n+1}$, the weight of the foreign risk-free asset in the foreign LUP. Thus, both LUPs have the same weight for that asset. (2.6) further says that this also holds for assets j=2,..., n. Lastly, since in each portfolio the asset weights sum to unity, it must be that also $y_{i,n+1} = y^*_{1,1}$. **QED**.

The intuition is as familiar as the result itself (Sercu, 1980, or even Hakansson,1969): for a log-utility investor, the unconstrained objective function is numeraire independent. To see this, consider a log-utility investor from the reference country, country 0, and denote her wealth, measured in her home currency, by W_0. In a foreign currency, that same agent's wealth is worth W_0^*, where $W_0^*\,S = W_0$. Thus, unconstrained maximization of $E(\ln W_0)$ can equivalently be written as Max $[E(\ln W_0^*\,S) + E(\ln S)]$, which has the same solution as Max $E(\ln W_0^*)$ because $\partial E(\ln S)/\partial x_{ij} = 0$ (price takership).

The relevance of this result is the simplicity of the structure of the world market portfolio and the corresponding InCAPM (Solnik, 1974; Sercu, 1980). In particular, as all investors' demand for stocks is through the LUP, the world stockmarket portfolio is the corresponding part of the LUP. Interpreting set 1 in equation (1.15) as the stocks, we immediately obtain Sercu (1981)'s generalisation of the Solnik (1974) CAPM for hedged stocks:

$$(\mathbf{m}_1-r\mathbf{u}_1) - \Delta_{12}{}^{T}(\mathbf{m}_2-r\mathbf{u}_2) = \eta_w \mathbf{V}_{1|2}\, \Xi_{1w}\,, \tag{2.6a}$$

with η_w being the wealth-weighted harmonic mean η. Alternatively, one can aggregate the full demand equation; denoting η_k as the wealth-weighted harmonic mean η for country k, k=1,...,M, W_k as the country's aggregate wealth, and W_w as world aggragate wealth

$$\Xi_w = \frac{1}{\eta_w}\mathbf{V}^{-1}(\mathbf{m}-r\mathbf{u}) + \sum_{k=1}^{M}\frac{W_k(1-\eta_k{}^{-1})}{W_w}\,\mathbf{k},$$

where **k** has all elements equal to zero except the position that corresponds to the country-k riskless asset. Solving for the expected returns we find

$$(\mathbf{m}-r\mathbf{u}) = \eta_w\, \mathbf{V}\, \Xi_w + \eta_w \sum_{k=1}^{M}\frac{W_k(1-\eta_k{}^{-1})}{W_w}\,\mathbf{Vk}, \tag{2.6b}$$

where $\mathbf{V}\,\Xi_w$ is the vector of covariances of all assets' returns with the world market portfolio and **Vk** is the vector of covariances of all assets with the country-k exchange rate.

2.3. Conditions for numeraire semi-invariance of the combined swap funds.

We next address the question whether the additional sources of demand are numeraire independent too. Obviously, when considered in itself, each of the swap portfolios $[\mathbf{V}^{-1}\mathbf{J}, -\mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{J}]$ in (1.2) is numeraire-dependent because it swaps risky assets for the investor's own risk-free asset, the latter of which is country-specific. Still, there are conditions under which all swap portfolios, taken together, are the same for two investors from different countries that have the same risk aversion η_i. We say that a portfolio is numeraire semi-invariant if investors that are from different countries but have the same risk aversion η_i still hold the same portfolio.[7]

Define the spread differential on the home-country risk-free asset—asset 1 abroad, asset n+1 at home—as $\lambda^*_{i,1} - \lambda_{i,n+1}$, and the spread differential on risky assets $j \geq 2$ as $\lambda^*_{i,j} - \lambda_{i,j}$.

Proposition 2.2. *When at least one constraint is binding, asset demand is numeraire semi-invariant if all spread differentials are equal across assets:*

- *for the foreign risk-free asset (asset 1 at home, asset n+1 abroad)*

$$\lambda^*_{i,n+1} - \lambda_{i,1} = \lambda^*_{i,1} - \lambda_{i,n+1}\,; \tag{2.7}$$

[7]This is not the same as verifying the numeraire-independence of a constrained LUP, because the lambdas depend on η_i. Thus, the constrained portfolio of an investor with $\eta_i \neq 1$ will generally differ from the constrained LUP by more than merely a premultiplication by $1/\eta_i$ and an addition of $(1-1/\eta_i)$ risk-free investments.

- *for all other assets j =2,....n*

$$\lambda^*_{i,j} - \lambda_{i,j} = \lambda^*_{i,1} - \lambda_{i,n+1} \,. \tag{2.8}$$

Proof. We can write the total demand associated with the non-negativity constraints, given in (1.2), as

$$-\frac{\lambda_{i,n+1}}{\eta_i}\begin{bmatrix}\mathbf{V}^{-1}\mathbf{u}\\ \cdots \\ -\mathbf{u}^{\mathsf T}\mathbf{V}^{-1}\mathbf{u}\end{bmatrix} - \sum_{j=1}^{n}\frac{\lambda_{i,j}}{\eta_i}\begin{bmatrix}\mathbf{V}^{-1}\mathbf{J}\\ \cdots \\ -\mathbf{u}^{\mathsf T}\mathbf{V}^{-1}\mathbf{J}\end{bmatrix} = -\frac{1}{\eta_i}\begin{bmatrix}\mathbf{V}^{-1}(\Lambda_i-\lambda_{i,n+1}\mathbf{u})\\ \cdots \\ -\mathbf{u}^{\mathsf T}\mathbf{V}^{-1}(\Lambda_i-\lambda_{i,n+1}\mathbf{u})\end{bmatrix}.$$

We identify the condition on the multipliers under which this demand becomes numeraire independent. Let

$$\begin{bmatrix}\mathbf{Z}\\ \cdots \\ z_{i,n+1}\end{bmatrix} \equiv \begin{bmatrix}\mathbf{V}^{-1}(\Lambda_i-\lambda_{i,n+1}\mathbf{u})\\ \cdots \\ -\mathbf{u}^{\mathsf T}\mathbf{V}^{-1}(\Lambda_i-\lambda_{i,n+1}\mathbf{u})\end{bmatrix}, \tag{2.9}$$

and similarly for the foreign counterpart. Again using $\mathbf{V}^{*-1} = \mathbf{G}\,\mathbf{V}^{-1}\,\mathbf{G}^{\mathsf T}$, we obtain

$$[\mathbf{G}\,\mathbf{V}^{-1}\,\mathbf{G}^{\mathsf T}]\,(\Lambda^*-\lambda^*_{i,n+1}\mathbf{u}) = \mathbf{Z}^* \,. \tag{2.10}$$

Below, we have pre-multiplied both sides of (2.10) by $\mathbf{G}^{-1}$, and on the right-hand side we have used $\mathbf{G}^{-1} = \mathbf{G}$. As before, this G-premultiplication reflects the rearrangement that goes with the redefinition of the risky assets:

$$\begin{aligned}\mathbf{V}^{-1}\,\mathbf{G}^{\mathsf T}\,(\Lambda^*-\lambda^*_{i,n+1}\mathbf{u}) &= \mathbf{G}\,\mathbf{Z}^*\\ &= \begin{bmatrix}z^*_{i,n+1}\\ \cdots \\ \mathbf{Z}^*_2\end{bmatrix}.\end{aligned} \tag{2.11}$$

Comparing with (2.9), to obtain numeraire independence—that is: $\mathbf{Z}^*_2 = \mathbf{Z}_2$; $z^*_{i,n+1} = z_1$; and, by implication, $z^*_1 = z_{i,n+1}$—we need

$$\Lambda-\lambda_{i,n+1}\mathbf{u} = \mathbf{G}^{\mathsf T}\,(\Lambda^*-\lambda^*_{i,n+1}\mathbf{u}) \tag{2.12}$$

$$\begin{bmatrix}\lambda_{i,1}-\lambda_{i,n+1}\\ \cdots \\ \Lambda_2-\lambda_{i,n+1}\mathbf{u}\end{bmatrix} = \begin{bmatrix}-1 & : & \mathbf{0}_{n-1}{}^{\mathsf T}\\ \cdots & : & \cdots \\ -\mathbf{u}_{n-1} & : & \mathbf{I}_{n-1}\end{bmatrix}(\Lambda^*-\lambda^*_{i,n+1}\mathbf{u})$$

$$= \begin{bmatrix} \lambda^*_{i,n+1}-\lambda^*_{i,1} \\ \dots \\ \Lambda^*_2-\lambda^*_{i,1}\mathbf{u} \end{bmatrix} . \qquad (2.13)$$

Rearranging, we obtain (2.7)-(2.8). **QED.**

The proposition is cast in technical terms. Economically, there may be less behind conditions (2.7)-(2.8) than meets the eye:

i) For assets $j \geq 2$ that have positive lambdas, numeraire-invariance of the total swap investments holds trivially: for these assets—say, set 1—the total swap portfolio positions $1/\eta_i\ \mathbf{Z_1}$ just wipe out the corresponding investments $1/\eta_i\ \mathbf{Y_1}$ via the LUP, and we already know that $1/\eta_i\ \mathbf{Y_1}$ is numeraire semi-invariant.[8] Thus, the proposition is interesting only for assets with a zero lambda in at least one country. We consider two cases (that will turn out to be mutually exclusive):

ii) If, for at least one asset $j \geq 2$, $\lambda_{i,j}$ and $\lambda^*_{i,j}$ are both zero, then, from (2.8) and (2.7) all lambdas for the risk-free assets must be equal to each other if the portfolio is semi-invariant. From (1.19) and (1.25), this is possible only under symmetry conditions (identical countries) that are quite unlikely to hold in practice.

   Also, if for one asset the spread differential is zero, then semi-invariance requires that all spread differentials be zero, too—that is, either both lambdas are zero, or both are non-zero (and identical across countries, case i) above). Thus, case ii) is incompatible with the last one:

iii) If, for one or more $j \geq 2$, one lambda is zero while the other is not, then (i) all spread differentials must be non-zero, and (ii) invariance is possible only when, for each and every asset, the zero lambda happens to correspond with a zero investment $x_{i,j}$. This is conceivable but unlikely to be true in practice, to say the least.

We conclude that semi-invariance holds only under exceptional circumstances. Thus, like two-fund separation in the one-country model, numeraire independence in the international model is quite sensitive to shorting restrictions.

2.4. Conditions for weak numeraire invariance of demand

We define portfolios to be weakly numeraire-independent if the domestic and foreign portfolios both have positive (but not necessarily identical) weights for the same set of assets. Since we are primarily interested in situations where a CAPM holds (which requires zero lambdas for all risky assets), we focus on situations where *all* assets are held positively.

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[8](The link between this simple argument and the conditions on the spread differential can be established using (1.25) and the lemmas in the next section, but this formal proof adds little insight.

2.4.1. Conditions under which both countries hold all risky assets j≥2.

To find the conditions for weak invariance for all risky assets, we first derive an interim result:

Lemma 2.1: *For any risky asset j other than the country-1 risk-free asset (asset 1),*

i) in home and foreign currency, the asset's multivariate exposures $\Delta_{j,k}$ to the n–1 other assets are related as

- *(foreign-country bond:)* $$\Delta_{j,1} = 1 - \sum_{\text{all } k \neq j} \Delta^*_{j,k}\,, \tag{2.14}$$
- *(home country bond:)* $$\Delta^*_{j,1} = 1 - \sum_{\text{all } k \neq j} \Delta_{j,k}\,, \tag{2.15}$$
- *(other assets:)* $$\Delta^*_{j,k} = \Delta_{j,k}\,. \tag{2.16}$$

ii) the variance of the hedged return is numeraire-independent.

iii) the hedged expected return is numeraire-independent..

Proof: Without loss of generality, consider asset risky asset n. (As of now, position j=1 remains reserved for the country-1 risk-free asset.) Then we can partition **G** and **V** into (i) a set, denoted by $\bar{n}$, that includes all assets except n; and(ii) a set that contains just asset n:

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{\bar{n}} & : & \mathbf{-1} \\ \dots & : & \dots \\ \mathbf{0}^{\mathsf{T}} & : & 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_{\bar{n}} & : & C_{n,\bar{n}} \\ \dots & : & \dots \\ \mathbf{C}_{n,\bar{n}}{}^{\mathsf{T}} & : & v_n \end{bmatrix},$$

where $\mathbf{G}_{\bar{n}}$ has dimension (n–1)× (n–1) but is otherwise similar to **G**,

$\mathbf{1} = [1, \mathbf{0}]^{\mathsf{T}}$.

Then

$$\mathbf{V}^* = \mathbf{G}^{\mathsf{T}}\,\mathbf{V}\,\mathbf{G} = \begin{bmatrix} G_{\bar{n}}{}^{\mathsf{T}}\,V_{\bar{n}}\,G_{\bar{n}} & : & G_{\bar{n}}{}^{\mathsf{T}}\,(C_{n,\bar{n}} - V_{\bar{n}}\,\mathbf{1}) \\ \dots & : & \dots \\ (C_{n,\bar{n}} - V_{\bar{n}}\,\mathbf{1})^{\mathsf{T}}\,G_{\bar{n}} & : & \mathbf{1}^{\mathsf{T}}V_{\bar{n}}\mathbf{1} - C_{n,\bar{n}}{}^{\mathsf{T}}\mathbf{1} - \mathbf{1}^{\mathsf{T}}C_{n,\bar{n}} + V_n \end{bmatrix}.$$

Proof of part i): From the first n-1 rows of **V***, we can rewrite $\Delta^*_{n,\bar{n}}{}^{\mathsf{T}} \equiv \mathbf{C}^*_{n,\bar{n}}{}^{\mathsf{T}}\,\mathbf{V}^*_{\bar{n}}$ as

$$\begin{aligned} \Delta^*_{n,\bar{n}}{}^{\mathsf{T}} &= [G_{\bar{n}}{}^{\mathsf{T}}\,(C_{n,\bar{n}} - V_{\bar{n}}\;\mathbf{1})]^{\mathsf{T}}\,[G_{\bar{n}}{}^{\mathsf{T}}\,V_{\bar{n}}\;\;G_{\bar{n}}]^{-1} \\ &= (C_{n,\bar{n}}{}^{\mathsf{T}}\,V_{\bar{n}}{}^{-1} + \mathbf{1}^{\mathsf{T}})\,G_{\bar{n}}{}^{\mathsf{T}} \\ &= (\Delta_{\bar{n}}{}^{\mathsf{T}} + \mathbf{1}^{\mathsf{T}})\,G_{\bar{n}}{}^{\mathsf{T}} \\ &= [(1 - \textstyle\sum_{k \neq n} \Delta_{n,k}\,, \Delta_{n,2}\,, \Delta_{n,3}\,, \dots, \Delta_{n,n-1}]. \end{aligned} \tag{2.17}$$

This proves (2.15) and (2.16). Property (2.14) then follows because the choice of the base currency is arbitrary. **QED.**

Proof of part ii): The conditional variance of asset n is

$$v^*_{n|\bar{n}} \equiv v^*_n - \Delta^{*}_{n,\bar{n}}{}^T V^{*}_{\bar{n}} \Delta^*_{n,\bar{n}} = [\mathbf{1}^T V_{\bar{n}} \mathbf{1} - C_{n,\bar{n}}{}^T \mathbf{1} - \mathbf{1}^T C_{n,\bar{n}} + V_{\bar{n}}]$$

$$- [(\Delta_{\bar{n}}{}^T + \mathbf{1}^T)\, G_{\bar{n}}{}^T]\ [G_{\bar{n}}{}^T V_{\bar{n}}\ G_{\bar{n}}]\ [G_{\bar{n}} (\Delta_{\bar{n}} + \mathbf{1})]$$

$$= [\mathbf{1}^T V_{\bar{n}} \mathbf{1} - C_{n,\bar{n}}{}^T \mathbf{1} - \mathbf{1}^T C_{n,\bar{n}} + V_{\bar{n}}] - (\Delta_{\bar{n}}{}^T + \mathbf{1}^T)\, V_{\bar{n}} (\Delta_{\bar{n}} + \mathbf{1})$$

$$= v_n - \Delta_{n,\bar{n}}{}^T V_{\bar{n}}{}^{-1} \Delta_{n,\bar{n}}$$

$$= v_{n|\bar{n}}\ , \textbf{QED} \tag{2.18}$$

Proof of part iii): From (1.17) and Proposition 2.1 it follows that for any risky asset n,

$$\frac{(m_n-r) - \sum_{k\neq n}\Delta_{n,k}(m_k-r)}{v_{n|\bar{n}}} = \frac{(m^*_n-r^*) - \sum_{k\neq n}\Delta^*_{n,k}(m^*_k-r)}{v^*_{n|\bar{n}}}\ . \tag{2.19}$$

Since the conditional variances for assets $j \geq 2$ are equal across currencies, we immediately obtain

$$(m_n-r) - \sum_{k\neq n} \Delta_{j,k}(m_k-r) = (m^*_n-r^*) - \sum_{k\neq n} \Delta^*_{j,k}(m^*_k-r),\ \textbf{QED}. \tag{2.20}$$

Note, in passing, that parts ii) and iii) of the Lemma hold for assets $j \geq 2$ only, since it assumes that asset 1 (the exchange rate asset) is one of the regressors. For asset 1, parts ii) and iii) no longer hold:

Lemma 2.2: *For the foreign risk-free asset,*

i) the conditional variance is given by $v_{1|\text{all } k\neq 1} = \mathbf{u}^T \mathbf{V}^{*-1} \mathbf{u}$;

ii) in general, $v_{1|\text{all } k\neq 1} \neq v^*_{1|\text{all } k\neq 1}$;

iii) the excess hedged return on the foreign risk-free asset is not generally equal to foreign-currency excess return on the hedged domestic risk-free asset.

Proof of Part i): From (2.4), $\mathbf{u}^T \mathbf{V}^{*-1}\mathbf{u} = \mathbf{u}^T \mathbf{G}\, \mathbf{V}^{-1} \mathbf{G}^T \mathbf{u}$. As $\mathbf{u}^T\mathbf{G} = -\mathbf{1}$, $\mathbf{u}^T \mathbf{G}\, \mathbf{V}^{-1} \mathbf{G}^T \mathbf{u}$ is the first diagonal element of $\mathbf{V}^{-1}$, which, from (1.14), is the variance of asset 1 conditional on all other assets' returns, $v_{1|\text{all } k\neq 1}$.

Proof of part ii): By symmetry, then, also $v^*_{1|\text{all } k\neq 1} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$. For $v_{1|\text{all } k\neq 1}$ to be equal to $v^*_{1|\text{all } k\neq 1}$, we need the first element in $\mathbf{V}^{*-1}$ to be equal to the first element in $\mathbf{V}^{-1}$. Writing, for ease of notation, $\mathbf{V}^{-1}$ as

$$\mathbf{V}^{-1} = \begin{bmatrix} w_1 & : & \mathbf{W}_{12}{}^T \\ \dots & : & \dots \\ \mathbf{W}_{12} & : & \mathbf{W}_2 \end{bmatrix}, \tag{2.21}$$

we find that

$$\mathbf{V}^{*-1} = \mathbf{G}\mathbf{V}^{-1}\mathbf{G}^{\top} = \begin{bmatrix} w_1 + \mathbf{u}^{\top}\mathbf{W}_{12} + \mathbf{W}_{12}{}^{\top}\mathbf{u} + \mathbf{u}^{\top}\mathbf{W}_2\mathbf{u} & : & -\mathbf{W}_{12}{}^{\top} - \mathbf{u}^{\top}\mathbf{W}_2 \\ \dots & : & \dots \\ -\mathbf{W}_{12} - \mathbf{W}_2\mathbf{u} & : & \mathbf{W}_2 \end{bmatrix}, \quad (2.22)$$

whose first element is not generally equal to w_1.

Proof of part iii) follows immediately from part ii), the first line of (2.6), and (1.17). QED.

We now return to the issue of weak numeraire-independence. Lemma 2.1 allows us to prove

Proposition 2.3: *Investors from different countries hold all assets j≥2 long if, for each risky asset j ≥ 2*

$$(m_j - r) - \sum_{\text{all } j \neq k}^{n} \Delta_{j,k}(m_k - r) \geq \text{Max}\left((1 - \sum_{\text{all } j \neq k}^{n} \Delta_{j,k})\, \lambda_{i,n+1},\ \Delta_{j,1}\, \lambda^*_{i,n+1} \right) \quad (2.22)$$

$$= \text{Max}\left(\Delta^*_{j,1}\, \lambda_{i,n+1},\ \Delta_{j,1}\, \lambda^*_{i,n+1} \right) \quad (2.23)$$

Proof: The conditions for asset j (≥2) to be held long in either country separately have been identified in (1.21):

$$(m_j - r) - \sum_{\text{all } j \neq k}^{n} \Delta_{j,k}(m_k - r) \geq (1 - \sum_{\text{all } j \neq k}^{n} \Delta_{j,k})\, \lambda_{i,n+1}, \quad (2.24)$$

$$(m^*_j - r^*) - \sum_{\text{all } j \neq k}^{n} \Delta^*_{j,k}(m^*_k - r^*) \geq (1 - \sum_{\text{all } j \neq k}^{n} \Delta^*_{j,k})\, \lambda^*_{i,n+1}. \quad (2.25)$$

From Parts i) and iii) of Lemma 2.1, (2.25) can be rewritten as

$$(m_j - r) - \sum_{\text{all } j \neq k}^{n} \Delta_{j,k}(m_k - r) \geq \Delta_{j,1}\, \lambda^*_{i,n+1}, \quad (2.26)$$

Equation (2.22) just combines (2.24) and (2.26). Equation (2.23) again invokes part i) of the Lemma. **QED**

The proposition links the hedged returns to the multivariate currency exposure and the critical bid-ask spreads. Also these spreads themselves can be linked to the exchange rate parameters:

Corollary to Lemma 2.2: *The critical bid-ask spreads are linked to the conditional exchange-rate risks as follows:*

$$\lambda_{i,n+1} = \text{Max}\left(\frac{\mathbf{u}^{\top}\mathbf{V}^{-1}(\mathbf{m} - r\mathbf{u}) - \eta_i}{v^*_{1|\text{all } k \neq 1}}, 0 \right) \quad (2.27)$$

$$\lambda^*_{i,n+1} = \text{Max}\left(\frac{\mathbf{u}^T\mathbf{V}^{*-1}(\mathbf{m}^*-r^*\mathbf{u}) - \eta_i}{v_{1|\text{all } k\neq 1}}, 0\right) \tag{2.28}$$

Proof: This follows immediately from (1.19) and part i) of Lemma 2.2. **QED**.

Thus, currencies that are low-risk from a foreign point of view tend to have higher critical bid-ask spreads. The intuition is that such low-risk currencies are heavily demanded by foreign investors, which lowers the interest rate. The lower interest rate must be offset by a higher bid-ask spread, otherwise the domestic investor would borrow.

2.4.1. Conditions under which both countries are long the same restricted set of risky assets.

The results obtained in the previous section hold as soon as the opportunity set is restricted to set 2:

Corollary to Lemma 2.1*: If the opportunity set is partitioned into sets 1 and 2, and the foreign risk-free asset one is included in set 2, then Lemmas 2.1 and 2.2 remain true relative to set 2 .*

Proof: Throughout, replace $\mathbf{V}$ by $\mathbf{V}_2$ and $\mathbf{m}$ by $\mathbf{m}_2$. **QED**.

3. CAPMs with representative agents under shortsale restrictions

In this section we review cases where the CAPM still holds. We do this in a context of representative-agent models: if all investors in a country have the same η, the CAPM is less likely to be upset by unusually undiversified asset demand from low-η investors. As the consensus is that relative risk-aversion exceeds unity, we first discuss cases where $\eta \geq 1$. We also restrict our discussion to cases where the shortselling constraints are not binding, as any such binding constraint will upset the CAPM.

We start by noting that, when the no-borrowing constraint does not bind, then investors with $\eta>1$ invest positive amounts in their own risk-free asset. If there are no outside risk-free assets or if the positive net supply is too small, this demand can only be met when foreigners lend to the home investor, that is, when the LUP has short positions in that asset. Thus, to rule out binding constraints on shortselling, one needs

Proposition 3.1: *Alternative sufficient conditions for zero lambda's in a representative-consumer model include*

i) relative risk aversion equals unity; or

ii) relative risk aversion exceeds unity and the net outstanding amount of each risk-free asset is at least equal to $W_i\,(1\text{-}1/\eta_i)$, where Wi is the representative investor's invested wealth.

Proof. First, if all investors have log-utility, there is no special role for the local risk-free assets, and the objective function and its solution are numeraire independent, as shown before. As there is no asset with negative outstanding amounts, the representative investor does not want to go short. Thus returns and risks are such that all λs are zero.

Alternatively, assume that $\eta_i > 1$, and that there is a net outstanding amount of the local risk-free asset, M_i, that is no smaller than $W_i\,(1-1/\eta_i)$, where W_i is the wealth of the country's representative investor. The term $W_i\,(1-1/\eta_i)$ represents local demand for the risk-free asset; thus, when $M_i \geq W_i\,(1-1/\eta_i)$, there is a non-zero residual supply of the risky asset that can be taken up by the unconstrained LUP. Thus, again, returns and risks are such that all λs are zero. **QED**

A positive net stock of risk-free assets can exist if there is a lower bound on at least some firms' cashflow.[9] But even if all lending and borrowing is among investors, the net supply of risk-free assets is positive in each currency if there is money. To see this, note that money balances M_i held by investor i can always be decomposed into[10]

$$M_i = \frac{M_i}{1+r} + M_i\,\frac{r}{1+r}\,, \tag{2.29}$$

where the first term on the right hand side is the one-period-bond component of the money stock, and the second term is the present value of the foregone interest. The first term is part of the portfolio problem, the second one part of the consumption problem. If optimal money balances are too large from a portfolio point view, the investor issues risk-free claims so as to reduce the total weight of the risk-free investments, and vice versa. Thus, non-negative LUP weights for the risk-free are possible when there is a sufficiently large money stock.

We now turn to cases where $\eta < 1$. Then the investors would like to short their own risk-free asset, which they cannot do. Thus, they supply zero amounts of their own asset to the other

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[9]If corporate debt is risky, thebasic asset (as priced by static CAPM) is the firm as a whole. The different classes of securities are derivatives of this underlying asset.

An alternative is to define the government as the source of outside bonds. However, in the aggregate, private claims on the government are matched by tax liabilities (including an inflation tax, if debt is monetized). So in the aggregate there is no net government debt outstanding. (There is, of course, no claim that each individual's net position relative to the government is zero and that Ricardian equivalence holds.)

[10]We use a discrete-time notation. In continuous time, $M_i = M_i(1-r\,dt) + M_i\,r\,dt$.

investors (and to the log-utility component in their own demand). Under the weak-invariance conditions outlined in the preceding section, a CAPM still holds.

4. Conclusion

This paper reviews and re-interprets static mean-variance demand for assets under shortselling constraints at the individual level, and then asks the question under what conditions there still exist portfolios or "funds" that are universal, that is, common across all investors regardless of their country (or type of real unit consumed).

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