ON AN INEQUALITY OF OPREA FOR LAGRANGIAN SUBMANIFOLDS

FRANKI DILLEN AND JOHAN FASTENAKELS

ABSTRACT. We show that a Lagrangian submanifold of a complex space form attaining equality in the inequality obtained by Oprea in [8], must be totally geodesic.

1. INTRODUCTION

In recent years there has been some progress in the field of inequalities relating intrinsic and extrinsic curvatures. T. Oprea improved in [7] an inequality of B.-Y. Chen (see for instance [6]). Since then several authors have studied the equality case of this improved inequality, see for instance [1], [2] and [3].

In another article, [8], T. Oprea introduced another Riemannian invariant of a submanifold and related this to the mean curvature with an inequality. In particular for a Lagrangian submanifold of a complex space form, again this inequality can be improved. In this article we consider Lagrangian submanifolds which attain equality in this new inequality. We investigate what this property implies for the second fundamental form and then prove that only totally geodesic submanifolds can attain the equality identically.

2. PRELIMINARIES

Let (M, g) be a Riemannian manifold of dimension n , p a point of M and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space $T_p M$ at p . The scalar curvature τ at p is defined by $\tau(p) = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_j, e_i)$ where R is the Riemann Christoffel curvature tensor of M and the Ricci curvature is defined by $\text{Ric}(X) = \sum_{j=1}^n R(X, e_j, e_j, X)$.

Now following [8] we have the following Riemannian invariant on M

$$(1) \quad \delta_n : M \rightarrow \mathbb{R} : p \mapsto \delta_n(p) = \tau(p) - \frac{1}{n-1} \min_{X \in T_p M, \|X\|=1} \text{Ric}(X).$$

We denote by $\widetilde{M}^{2n}(4c)$ a complex space form of constant holomorphic sectional curvature $4c$. Hence $\widetilde{M}^{2n}(4c)$ is the complex projective space $\mathbb{C}P^{2n}$, the complex hyperbolic space $\mathbb{C}H^{2n}$ or $\mathbb{C}^{2n}$ if respectively $c > 0$, $c < 0$ or $c = 0$.

Recall that for $\widetilde{M}^{2n}(4c)$ the Riemann curvature tensor $\widetilde{R}$ is given by

2000 *Mathematics Subject Classification.* 53B25.

Key words and phrases. Curvature inequalities, Lagrangian submanifolds.

The second author is Research assistant of the Fund for Scientific Research - Flanders (Belgium) (FWO). This research was supported by Research Grant G.0432.07 of the Research Foundation-Flanders (FWO).

$$(2) \quad \tilde{R}(X, Y)Z = c(g(X, Z)Y - g(Y, Z)X + g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ)$$

for all vector fields X, Y and Z on $\tilde{M}$, where J is the complex structure.

Consider an n -dimensional submanifold M of $\tilde{M}^{2n}(4c)$. We say that M is a Lagrangian submanifold if $J(T_p M) = T_p^\perp M$ for every $p \in M$.

We have the formulas of Gauss and Weingarten which state that for every X and Y tangent to M and for every N normal to M there holds that

$$(3) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(4) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

with $\tilde{\nabla}$, respectively ∇ , the Levi Civita connection of $\tilde{M}$, respectively M . Here h is a symmetric $(1, 2)$ -tensorfield, taking values in the normal bundle, called the second fundamental form of the submanifold, A_N is a symmetric $(1, 1)$ -tensorfield, called the shape operator associated to N and $\nabla^\perp$ is a connection in the normal bundle.

With the previous notations, the equations of Gauss and Codazzi are given by

$$(5) \quad \begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle A_{h(Y, Z)}X, W \rangle \langle A_{h(X, Z)}Y, W \rangle \\ &\quad + c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle), \end{aligned}$$

and

$$(6) \quad (\bar{\nabla}h)(X, Y, Z) = (\bar{\nabla}h)(Y, X, Z),$$

for every tangent vector fields X, Y, Z and W where

$$(\bar{\nabla}h)(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X, Z).$$

Moreover we know that for Lagrangian submanifolds the cubic form $C(X, Y, Z) = \langle h(X, Y), JZ \rangle$ is totally symmetric.

From [8] we have the following theorem.

Theorem 1. *Let M be a Lagrangian submanifold in a complex space form $\tilde{M}^{2n}(4c)$ with $n \geq 3$. Then we have*

$$(7) \quad \delta_n \leq \frac{(n+1)(n-2)}{8}c + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}\|H\|^2.$$

Hereby $H = \sum_{i=1}^n h(e_i, e_i)$ denotes the mean curvature vector.

3. THE EQUALITY CASE

We first characterize Lagrangian submanifolds for which the equality is attained in terms of their second fundamental form.

Theorem 2. *Let M be a Lagrangian submanifold in a complex space form $\widetilde{M}^{2n}(4c)$ with $n \geq 3$. Then we have*

$$(8) \quad \delta_n(p) = \frac{(n+1)(n-2)}{8}c + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}\|H(p)\|^2,$$

if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM such that

$$(9) \quad \begin{aligned} h(e_1, e_1) &= 2bJe_2, & h(e_2, e_2) &= 9bJe_2, & h(e_1, e_j) &= 0, \\ h(e_1, e_2) &= 2bJe_1, & h(e_2, e_j) &= 3bJe_j, & h(e_j, e_k) &= \delta_{jk}3bJe_2 \end{aligned}$$

with $j, k \in \{3, \dots, n\}$ and b a function on M .

Proof. Carefully reading the proof of inequality (7) in [8] we see that $h_{ij}^r = 0$ if i, j and r are all different and that there exist functions $a_1, \dots, a_n$ on M such that

$$\begin{aligned} h_{11}^1 &= 2a_1, \\ h_{22}^1 &= h_{jj}^1 = a_1, \end{aligned}$$

and

$$\begin{aligned} h_{11}^j &= 2a_j, \\ h_{kk}^j &= 3a_j, \\ h_{jj}^j &= 9a_j, \end{aligned}$$

with $j, k \in \{3, \dots, n\}$ and $j \neq k$. Here we used the notation $h_{ij}^r = \langle h(e_i, e_j), Je_r \rangle$.

Moreover we see from [8] that a_1 must be zero in order to have equality. Now choosing our orthonormal basis in such a way that the mean curvature vector H is parallel with Je_2 and using that the cubic form is totally symmetric, we get the result. $\square$

Using the equations of Gauss (5) and Codazzi (6) we will prove the following theorem.

Theorem 3. *A Lagrangian submanifold M in a complex space form $\widetilde{M}^{2n}(4c)$ with $n \geq 3$ satisfies equality (8) identically if and only if M is totally geodesic.*

Proof. Suppose M is not totally geodesic and consider a point p where h doesn't vanish. From Theorem 2 we know that there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM such that (9) holds at p . Note that our assumption implies that $b \neq 0$. Since Je_2 is a multiple of the mean curvature vector, we can extend e_2 to a differentiable unit vector field on an open neighborhood O of p such that Je_2 is a multiple of the mean curvature vector field.

At every point of O , the shape operator A_{Je_2} then has three distinct eigenvalues: $2b$, $9b$ and $3b$ of multiplicity respectively 1, 1 and $n-2$. So we can extend the orthonormal basis of T_pM to an orthonormal frame $\{e_1, \dots, e_n\}$ on O such that (9) holds at every point of O for some differentiable function b .

Now take $k, \ell \in \{3, \dots, n\}$ with $k \neq \ell$. Using the notation $\nabla_{e_i}e_j = \sum_{r=1}^n \omega_j^r(e_i)e_r$, we find from the equation of Codazzi by comparing $(\bar{\nabla}h)(e_2, e_2, e_k)$ and $(\bar{\nabla}h)(e_k, e_2, e_2)$ that

the 1-forms ω_j^r must satisfy

$$(10) \quad \omega_k^1(e_2) = 5\omega_2^1(e_k),$$

$$(11) \quad -b\omega_k^2(e_2) = 3e_k(b),$$

$$(12) \quad e_2(b) = b\omega_2^k(e_k),$$

$$(13) \quad \omega_2^\ell(e_k) = 0.$$

In an analogous way we have

$$(14) \quad 3\omega_2^1(e_2) = \omega_k^1(e_k), \quad b\omega_k^1(e_1) = -2b\omega_k^2(e_2) = 2e_k(b),$$

$$(15) \quad e_2(b) = -b\omega_k^2(e_k), \quad -3\omega_k^2(e_1) = \omega_k^1(e_2) = -5\omega_1^2(e_k),$$

$$(16) \quad 3\omega_2^k(e_2) = e_k(b), \quad 3e_1(b) = -3b\omega_1^2(e_2) = -b\omega_1^k(e_k),$$

$$(17) \quad \omega_2^\ell(e_2) = 0, \quad \omega_1^\ell(e_k) = 0,$$

$$(18) \quad 5b\omega_2^1(e_1) = 2e_2(b), \quad \omega_1^k(e_k) = 0,$$

$$(19) \quad 9e_1(b) = -5b\omega_1^2(e_2), \quad -\omega_k^2(e_1) = 3\omega_2^1(e_k),$$

$$(20) \quad 3\omega_2^\ell(e_1) = -\omega_1^\ell(e_2), \quad b\omega_k^1(e_1) = 2e_k(b),$$

$$(21) \quad \omega_1^2(e_2) = 0, \quad -3\omega_1^2(e_1) = 2\omega_2^k(e_k),$$

$$(22) \quad e_1(b) = 0, \quad 3\omega_2^k(e_1) = -\omega_1^2(e_k),$$

$$(23) \quad \omega_k^1(e_1) = 0, \quad \omega_2^\ell(e_1) = 0.$$

From (20) and (23) we get that $e_k(b) = 0$ and from (22) and (23) we see that $\omega_1^2(e_k) = 0$. Combining this with (10) we find that $\omega_k^1(e_2) = 0$. Using this together with (17), (18) and (23) we have

$$(24) \quad \omega_1^k = 0.$$

Also combining (15), (18) and (21) we see that $\omega_k^2(e_k) = 0$. From (20) and (24) we find that $\omega_2^\ell(e_1) = 0$. Using this together with (13) and (17) we have

$$(25) \quad \omega_2^k = 0.$$

Now computing the sectional curvature of the plane spanned by e_1 and e_2 with the equation of Gauss we find that

$$(26) \quad R(e_1, e_2, e_2, e_1) = c + 14b^2.$$

On the other hand using the definition of the curvature tensor with (24) and (25) we have that

$$(27) \quad R(e_1, e_2, e_2, e_1) = 0.$$

If $c \geq 0$ we immediately have a contradiction with $b \neq 0$. If $c < 0$ we also look at the sectional curvature of the plane spanned by e_1 and e_3 . From the equation of Gauss we have

$$(28) \quad R(e_1, e_3, e_3, e_1) = c + 6b^2,$$

but from the definition of the curvature tensor with (24) and (25) we have that

$$(29) \quad R(e_1, e_3, e_3, e_1) = 0.$$

So combining (26), (27), (28) and (29) we also get a contradiction for $c < 0$. This proves that $b = 0$ and hence the Lagrangian submanifold is totally geodesic. □

REFERENCES

- [1] J. Bolton, F. Dillen, J. Fastenakels and L. Vrancken, *A best possible inequality for curvature-like tensor fields*, Preprint, 2007.
- [2] J. Bolton, C. Rodriguez Monteleagre and L. Vrancken, *Characterizing warped product Lagrangian immersions in complex projective space*, Proc. Edinb. Math. Soc. **51** (2008), 1–14.
- [3] J. Bolton and L. Vrancken, *Lagrangian submanifolds attaining equality in the improved Chen's inequality*, Bull. Belg. Math. Soc., **14** (2007), 311–315.
- [4] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60** (1993), 568–578.
- [5] B. Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension*, Glasgow Math. J., **41** (1999), 33–41.
- [6] B. Y. Chen, *Riemannian geometry of Lagrangian submanifolds*, Taiwan. J. Math. **5** (2001), 681–723.
- [7] T. Oprea, *Chen's inequality in Lagrangian case*, Colloq. Math., **108** (2007), 163–169.
- [8] T. Oprea, *On a Riemannian invariant of Chen type*, Rocky Mountain J. Math. **38** (2008), 567–581.

KATHOLIEKE UNIVERSITEIT LEUVEN, DEPARTMENT OF MATHEMATICS, CELESTIJNENLAAN 200 B, BOX 2400, BE-3001 LEUVEN, BELGIUM

E-mail address, F. Dillen: `franki.dillen@wis.kuleuven.be`

E-mail address, J. Fastenakels: `johanfastenakels@hotmail.com`