Optimal H_2-design and the smoothed spectral abscissa

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1 Introduction

The stabilization of systems with control laws having a smaller amount of free variables than the system's dimension is a difficult problem, and optimizing the stability robustness is even more challenging. In particular, given a plant matrix A, a control matrix B and an observer matrix C that are all dependent on some parameter vector K, we are interested in finding values for K such that

1. the system matrix A is stable,
2. the H_2-norm of the system's transfer function $\mathbf{T}$

$$\mathbf{T}(s;K) = C(K)(sI - A(K))^{-1}B(K),$$

is minimal.

It is assumed that the amount of free parameters is much smaller than the system's dimension, i.e. $\dim(K) \ll \dim(A)$, and that (A,B) is controllable and (C,A) is observable.

Concisely put, we want to solve the minimization problem

$$\min_K \|\mathbf{T}(s;K)\|_{H_2}$$

over the set of stabilizing K. The H_2-norm, defined as

$$\|\mathbf{T}(s;K)\|_{H_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(\mathbf{T}(j\omega;K)^*\mathbf{T}(j\omega;K)\right)\mathrm{d}\omega,$$

is a smooth function and thus relatively easy to optimize. For the H_2-norm to exist however, the system matrix A must be stable, i.e. all its eigenvalues must reside in the left half complex plane. This is equivalent to demanding that the real part of the rightmost eigenvalue of A, the so-called spectral abscissa $\alpha(A)$, be negative. Objective functions derived from $\alpha(A)$ unfortunately are in general not smooth, so one typically ends up with a nonsmooth optimization problem for the stabilization.

Recently, several nonsmooth optimization approaches have been devised to overcome this difficulty. These methods then result in a two stage approach for the H_2-minimization problem, with a first step that stabilizes the system, and a second step that subsequently robustifies the system.

We present a framework that, making use of the smoothed spectral abscissa [1], uses only smooth optimization techniques to stabilize a system. If desired, the method can also be adapted to optimize the H_2-norm, which is performed in a way simultaneously with the stabilization, thus without having to worry about the (nonsmooth) first step of the traditional two-stage approach.

2 Stabilization and optimal H_2-design

The method is based on the smoothed spectral abscissa. This stability measure, denoted by $\tilde{\alpha}_\varepsilon$, is defined as the shift σ for which it holds that the function f

$$f(A,B,C,\sigma) := \|C(sI-(A-\sigma I))^{-1}B\|_{H_2}^2. \qquad (1)$$

is equal to ε^{-1}. The function f is nothing else than the H_2-norm of a *shifted* transfer function. The larger the shift σ, the easier this function is to optimize. We therefore do a series of optimizations with decreasing σ, where the new, smaller σ is computed as a smoothed spectral abscissa. The algorithm looks as follows.

Algorithm 1 Optimal H_2 algorithm

Require: $E > 1$, $\varepsilon > 0$, initial K^*, $\sigma = +\infty$
1: **while** $\sigma > 0$ **do**
2: $\quad\sigma \leftarrow \tilde{\alpha}_\varepsilon(A(K^*))$
3: $\quad$Solve
$$f^* \leftarrow \min_K f(A,B,C,\max\{\sigma,0\})$$
$\quad$with K^* as initial guess for K.
4: $\quad$Update K^* with the minimizer of step 1.
5: $\quad$Decrease ε: $\varepsilon \leftarrow \varepsilon/E$
6: **end while**
7: **return** f^* and corresponding minimizer K^*

By putting simply σ instead of $\max\{\sigma,0\}$ on line 3, the algorithm reduces to a pure stabilization method.

References

[1] J. Vanbiervliet, B. Vandereycken, W. Michiels, S. Vandewalle, M. Diehl, "The smoothed spectral abscissa for robust stability optimization", to appear in SIAM Journal on Optimization.