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Robust High-Order Repetitive Control: Optimal Performance Trade-offs [★]

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Abstract

High-order repetitive control has previously been introduced to either improve the robustness for period-time uncertainty or reduce the sensitivity for non-periodic inputs of standard repetitive control schemes. This paper presents a systematic, semidefinite programming based approach to compute high-order repetitive controllers that yield an optimal trade-off between these two performance criteria. The methodology is numerically illustrated through trade-off curves for various controller orders and levels of period-time uncertainty. Moreover, existing high-order repetitive control approaches are shown to correspond to specific points on these curves.

Key words: Repetitive control; Robustness; Semidefinite programming; KYP lemma

1 Introduction

Including a repetitive controller [3,6,8,18] in the feedback loop constitutes a common way to improve the attenuation/tracking of periodic disturbance/reference inputs. Repetitive controllers apply the internal model principle [4], which states that, if a disturbance/reference signal can be regarded as the output of an autonomous system, including this system in a stable feedback loop guarantees asymptotically perfect rejection/tracking. Figure 1 shows the more frequently used generator of periodic signals with period T_0 [s]. Including it in the controller results in an infinite loop gain and hence, a zero closed-loop sensitivity at all multiples of $\omega_0 = 2\pi/T_0$ [rad/s]. Consequently, the closed-loop system features perfect *nominal periodic performance*: any periodic input is perfectly rejected/tracked asymptotically, provided that its period is exactly T_0 .

Most repetitive controller designs in literature suffer from two disadvantages. First, the periodic performance of the closed-loop system is very sensitive to uncertainty

Fig. 1. Generator of periodic signals with period T_0 .

on the period of the input. Hence, while yielding perfect nominal periodic performance, the *robust periodic performance* of the closed-loop system is in general not satisfactory. As a result, the implementation of these repetitive controllers requires the period T_0 to be constant or accurately measurable ($\pm 0.1\%$), which may be jeopardized in practice by clock error drift, jitter, measurement noise, Second, the repetitive controller negatively affects the closed-loop performance for non-periodic inputs, i.e., the *non-periodic performance*: due to the Bode Sensitivity Integral [5] pushing the sensitivity down to zero at the multiples of ω_0 is paid for by an increased sensitivity at intermediate frequencies.

To deal with these disadvantages, so-called high-order repetitive control has been proposed. Inoue [7] and Chang et al. [1] design high-order repetitive controllers to improve the non-periodic performance under the constraint of perfect nominal periodic performance, while the design of Steinbuch [14] improves the robust periodic performance under the same constraint. A unified framework able to reproduce the results of both [1] and [14] is proposed in [15].

This paper proposes a novel design approach for single-input single-output (SISO) high-order repetitive con-

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trollers, that allows a systematic and quantitative treatment of combined non-periodic and period-uncertain inputs. In contrast with the current design approaches [1,7,14,15], perfect nominal periodic performance of the closed-loop system is no longer the starting-point of the repetitive control design. Instead, the repetitive controller is designed to yield an optimal trade-off between two performance indices that quantify its effect on the closed-loop non-periodic and robust periodic performance. The optimal repetitive controller design is reformulated as a semidefinite program (SDP), guaranteeing an efficient and reliable computation of the global optimum and allowing a straightforward computation of trade-off curves between the performance indices. These trade-off curves aid the control engineer in answering questions like: “What is the best improvement in robust periodic performance a repetitive controller can yield, regardless of the corresponding non-periodic performance degradation?”, or “How much robust periodic performance has to be given up in order to reduce the non-periodic performance degradation to an acceptable level?”. In addition, the proposed design approach is able to reproduce and outperform the results of [1,14], as illustrated in Sec. 5.

The paper is organized as follows: Section 2 details the control setup and analyzes the closed-loop system without and with repetitive controller. High-order repetitive control is introduced in Sec. 3, while Sec. 4 develops the semidefinite programming framework. Section 5 illustrates the potential of the proposed design approach and compares it with the current high-order repetitive controller design approaches [1,14]. Section 6 summarizes the conclusions.

The notation used in this paper is standard. $\mathcal{S}_n$ denotes the set of symmetric $n \times n$ matrices, while for a matrix $X \in \mathcal{S}_n$, $X \preceq 0$ indicates that X is negative semi-definite. I_n corresponds to the $n \times n$ identity matrix, while $0_{n,m}$ indicates the zero-matrix of dimension $n \times m$. To alleviate notation, the frequency response function (FRF) of a discrete-time system $H(z)$ is denoted by $H(\omega)$ instead of $H(e^{j\omega T_s})$.

2 Add-on Repetitive Control System

The repetitive controller is considered as an add-on device, implying that it is added to the loop of an existing feedback control system. This is the more common repetitive control scheme in literature [3]. Section 2.1 details the corresponding control setup, while Sec. 2.2 analyzes the closed-loop system without and with repetitive controller. Section 2.3 introduces two performance indices to quantify the effect of the repetitive controller on the closed-loop non-periodic and robust periodic performance.

Fig. 2. Control setup of an add-on repetitive control system.

2.1 Control Setup

Figure 2 shows the control setup of an add-on repetitive control system. The controller is designed in discrete time, where the sampling period is denoted by T_s [s] and the index k refers to the discrete time instants kT_s . The plant is a SISO linear time-invariant system with transfer function $P(z)$. The signals $r(k)$, $d(k)$ and $e(k)$ correspond to the reference input, the disturbance input and the tracking error, respectively. In closed loop, $e(k)$ is given by

$$e(k) = S(q) \underbrace{(r(k) - d(k))}_{w(k)}, \quad (1)$$

where q denotes to the one-sample advance operator and $S(q)$ corresponds to the closed-loop sensitivity function. The signal $w(k)$ is referred to as the *input* of the closed-loop system and features both a periodic and a non-periodic part. The periodic part has a nominal period T_0 where the corresponding fundamental frequency is denoted by ω_0 . The sample frequency is chosen such that T_0 corresponds to an integer number N of samples: $T_0 = NT_s$ ¹. Uncertainty on T_0 is modeled as a relative uncertainty on ω_0 , bounded by Δ :

$$\omega_{0,\delta} = \omega_0 (1 + \delta), \quad |\delta| \leq \Delta. \quad (2)$$

The index l indicates the harmonics of the periodic part of the input:

$$l \in \{0, 1, \dots, \Lambda\}, \quad (3)$$

where Λ corresponds to the highest harmonic to be suppressed.

2.2 Closed-Loop System Without and With $K_{RC}(z)$

The system without $K_{RC}(z)$ is referred to as the original feedback system and its closed-loop sensitivity function $S_o(z)$ and complementary sensitivity function $T_o(z)$ are given by

$$S_o(z) = \frac{1}{1 + K_o(z)P(z)}, \quad (4)$$

$$T_o(z) = \frac{K_o(z)P(z)}{1 + K_o(z)P(z)}. \quad (5)$$

¹ If N is not integer, it is rounded to the nearest integer N_{int} and the effect of the rounding error is accounted for as uncertainty on $T_0 = N_{\text{int}}T_s$.

The feedback controller $K_o(z)$ is assumed to be designed properly, which implies [13] (i) a stable closed-loop system; (ii) high-gain feedback at low frequencies; (iii) sufficient roll-off of $|T_o(\omega)|$ at high frequencies; and (iv) a small modulus margin $\|S_o(z)\|_\infty$.

Property (ii) is referred to as *good non-periodic performance*, since it yields small values of $|S_o(\omega)|$ at low frequencies, resulting in a small tracking error $e(k)$ for *any* low-frequency input $w(k)$. The bandwidth of the original closed-loop system, ω_{BW} , is defined as the frequency where $|S_o(\omega)|$ first crosses -3dB from below. Hence, $K_o(z)$ yields good non-periodic performance up to ω_{BW} .

When $K_{RC}(z)$ is added to the loop, the closed-loop sensitivity changes from $S_o(z)$ to $S(z)$:

$$S(z) = S_o(z) M_S(z), \quad (6)$$

where $M_S(z)$ is given by

$$M_S(z) = \frac{1}{1 + K_{RC}(z)T_o(z)}. \quad (7)$$

$M_S(z)$ is called the *modifying sensitivity function* [15] and represents the effect of $K_{RC}(z)$ on the closed-loop sensitivity.

The purpose of $K_{RC}(z)$ is to improve the closed-loop periodic performance, while keeping the inevitable non-periodic performance degradation within acceptable limits. Additionally, $K_{RC}(z)$ must not compromise the modulus margin and high-frequency roll-off of the original feedback system. These design specifications for $K_{RC}(z)$ are not conflicting provided that all harmonics $l = 0, \dots, \Lambda$ lie well below ω_{BW} : at harmonics near or above ω_{BW} better performance simply cannot be achieved without compromising the modulus margin or high-frequency roll-off of the original feedback system. This nonrestrictive and classical assumption is made throughout the remainder of this paper.

2.3 Performance Indices of $K_{RC}(z)$

The influence of $K_{RC}(z)$ on the closed-loop nominal periodic performance is quantified by the *nominal periodic performance index* γ_p , defined as the smallest relative reduction of $|S(l\omega_0)|$ over all harmonics l :

$$\gamma_p \equiv \max_{l \leq \Lambda} \left\{ \frac{|S(l\omega_0)|}{|S_o(l\omega_0)|} \right\}, \quad (8a)$$

$$= \max_{l \leq \Lambda} \{|M_S(l\omega_0)|\}. \quad (8b)$$

In the presence of uncertainty on the period of the input, the worst-case value of (8b) is taken over all potential

values $\omega_{0,\delta}$. Hence, the *robust periodic performance index* $\gamma_{p,\Delta}$ is given by

$$\gamma_{p,\Delta} \equiv \max_{|\delta| \leq \Delta} \left\{ \max_{l \leq \Lambda} \{|M_S(l\omega_{0,\delta})|\} \right\}, \quad (9a)$$

$$= \max_{l \leq \Lambda} \left\{ \max_{\omega \in \Omega_l} \{|M_S(\omega)|\} \right\}, \quad (9b)$$

where

$$\Omega_l = [l\omega_0(1 - \Delta), l\omega_0(1 + \Delta)]. \quad (10)$$

The effect of $K_{RC}(z)$ on the closed-loop non-periodic performance is quantified by the *non-periodic performance index* γ_{np} , defined as the highest relative amplification of $|S(\omega)|$ below ω_{BW} :

$$\gamma_{np} \equiv \max_{\omega \leq \omega_{BW}} \left\{ \frac{|S(\omega)|}{|S_o(\omega)|} \right\}, \quad (11a)$$

$$= \max_{\omega \leq \omega_{BW}} \{|M_S(\omega)|\}, \quad (11b)$$

$$= \|M_S(z)\|_\infty. \quad (11c)$$

The equivalence between (11b) and (11c) holds since $|M_S(\omega)|$ reaches its maximum in the frequency range $0 \leq \omega \leq \omega_0$, which is further discussed in Sec. 3.2.

The goal of a repetitive controller is to improve the robust periodic performance: $\gamma_{p,\Delta} < 1$, at the expense of worse non-periodic performance: $\gamma_{np} > 1$.

3 High-Order Repetitive Control

This section introduces high-order repetitive control. Section 3.1 details the general structure of a high-order repetitive controller. In Sec. 3.2 some of the design freedom is fixed to preserve the stability, modulus margin and high-frequency roll-off of the original feedback system while Sec. 3.3 derives the performance indices γ_{np} and $\gamma_{p,\Delta}$ as a function of the remaining design parameters.

3.1 Structure of a High-Order Repetitive Controller

Fig. 3. Structure of a repetitive controller of order M .

Figure 3 shows the structure of a repetitive controller of order M [14]. Taking into account that $e^{-sT_0} = z^{-N}$ reveals the similarity between the first-order structure ($M = 1$) and the periodic signal generator of Fig. 1. A high-order repetitive controller ($M > 1$) generalizes a

first-order controller in that z^{-N} is replaced by a polynomial in z^{-N} :

$$W(z) = \sum_{m=1}^M W_m z^{-mN}, \quad (12)$$

giving rise to the following expressions for $K_{RC}(z)$ and $M_S(z)$:

$$K_{RC}(z) = \frac{W(z)Q(z)L(z)}{1 - W(z)Q(z)}, \quad (13)$$

$$M_S(z) = \frac{1 - W(z)Q(z)}{1 - W(z)Q(z)[1 - L(z)T_1(z)]}. \quad (14)$$

Designing the high-order repetitive controller corresponds to designing $W(z)$ and the filters $Q(z)$ and $L(z)$. $Q(z)$ and $L(z)$ are designed first (Sec. 3.2) to preserve the stability, modulus margin and high-frequency roll-off of the original feedback system, irrespective of $W(z)$. Subsequently, $W(z)$ is designed such that the corresponding repetitive controller realizes an optimal trade-off between the performance indices $\gamma_{p,\Delta}$ and γ_{np} (Sec. 4).

High-order repetitive controllers involve high memory requirements for in most applications N is large and the implementation of a M th order controller requires the storage of MN data samples. If this is unacceptable, a so-called multi-rate implementation of the controller is advised in order to reduce the required memory, see e.g. [2] for more details.

3.2 Design of $Q(z)$ and $L(z)$

The filters $Q(z)$ and $L(z)$ are usually designed as follows [6,7,14]:

- $Q(z)$ is a low-pass linear-phase FIR filter with unity dc-gain and cut-off frequency ω_{co} .
- $L(z)$ is set equal to $T_o^{-1}(z)$. If $T_o(z)$ is non-minimum phase, the “zero phase error tracking” inverse [17] is used to obtain a stable $L(z)$. In fact, it suffices that $L(\omega) = T_o^{-1}(\omega)$ in the pass-band of the filter $Q(z)$.

Substituting $L(z) = T_o^{-1}(z)$ in (14) yields a modifying sensitivity function

$$M_S(z) = 1 - W(z)Q(z) \quad (15)$$

that does no longer depend on the original control system, that is, on $S_o(z)$, $T_o(z)$ nor $K_o(z)$.

To illustrate the effect of $Q(z)$ on $M_S(z)$, Fig. 4 compares, for a typical repetitive control system, the FRF of

Fig. 4. Comparison between $M_S(\omega)$ and $\widehat{M}_S(\omega)$ for a typical repetitive control system. At low frequencies the curves coincide.

$M_S(z)$ with the FRF of $\widehat{M}_S(z)$, defined as the modifying sensitivity function for $Q(z) \equiv 1$:

$$\widehat{M}_S(z) = 1 - W(z). \quad (16)$$

The low-pass filter $Q(z)$ turns off $K_{RC}(z)$ outside its pass-band, since $Q(\omega) \approx 0$ yields $M_S(\omega) \approx 1$. Hence, if ω_{co} lies sufficiently below ω_{BW} , neither the modulus margin nor the high-frequency roll-off are altered by the repetitive controller, regardless of $W(z)$.

The remainder of the paper relies on two nonrestrictive assumptions, made in all current high-order repetitive control design approaches [7,1,14,15]. First, all harmonics $l = 0, \dots, \Lambda$ are assumed to lie in the pass-band of $Q(z)$. This can be guaranteed by a proper design of $Q(z)$ provided that all harmonics $l = 0, \dots, \Lambda$ lie well below ω_{BW} . Second, in its pass-band $Q(z)$ is assumed to equal its dc-gain: $Q(\omega) = 1$ for all $\omega < \omega_{co}$. These two assumptions imply that

$$\forall \omega \leq \Lambda \omega_0 (1 + \Delta) : M_S(\omega) = \widehat{M}_S(\omega). \quad (17)$$

The implications of these assumptions for the computation of $\gamma_{p,\Delta}$ and γ_{np} are discussed in the following section.

3.3 Performance Indices as a Function of $\widehat{M}_S(z)$

Assumption (17) allows the replacement of $M_S(\omega)$ by $\widehat{M}_S(\omega)$ in the definition (9) of $\gamma_{p,\Delta}$:

$$\gamma_{p,\Delta} = \max_{l \leq \Lambda} \left\{ \max_{\omega \in \Omega_l} \left\{ |\widehat{M}_S(\omega)| \right\} \right\}. \quad (18)$$

Due to the periodicity of $\widehat{M}_S(\omega)$, see e.g. Fig. 4, each uncertainty interval Ω_l can be replaced by a corresponding interval $\tilde{\Omega}_l$ around the origin:

$$\tilde{\Omega}_l = [-l\omega_0\Delta, l\omega_0\Delta]. \quad (19)$$

Since for all $l < \Lambda$, $\tilde{\Omega}_l \subset \tilde{\Omega}_\Lambda$, it follows that

$$\gamma_{p,\Delta} = \max_{\omega \in \tilde{\Omega}_\Lambda} \left\{ |\widehat{M}_S(\omega)| \right\}. \quad (20)$$

For $\Delta = 0$, this result reduces to

$$\gamma_p = |\widehat{M}_S(0)|. \quad (21)$$

Since (i) $\|M_S(z)\|_\infty$ is determined by $|M_S(\omega)|$ for $0 \leq \omega \leq \omega_0$, as illustrated in Fig. 4, and (ii) by assumption (17) $|M_S(\omega)| = |\widehat{M}_S(\omega)|$ at these frequencies, Eq. (11c) reduces to

$$\gamma_{np} = \|\widehat{M}_S(z)\|_\infty. \quad (22)$$

4 Optimal Robust Repetitive Control Design

The design variables W_m (12) are computed such that the corresponding repetitive controller realizes an optimal trade-off between the periodic performance improvement $\gamma_{p,\Delta}$ and the inevitable non-periodic performance degradation γ_{np} . The repetitive controller design amounts to solving the following optimization problem in $W_1, \dots, W_M$, γ_{np} and $\gamma_{p,\Delta}$:

$$\text{minimize } \gamma_{p,\Delta} + \alpha \gamma_{np} \quad (23a)$$

$$\text{subject to } |\widehat{M}_S(\omega)| \leq \gamma_{np}, \quad \forall \omega \in \mathbb{R} \quad (23b)$$

$$|\widehat{M}_S(\omega)| \leq \gamma_{p,\Delta}, \quad \forall \omega \in \tilde{\Omega}_\Lambda \quad (23c)$$

where the weight α quantifies the relative importance of $\gamma_{p,\Delta}$ and γ_{np} . This optimization problem is semi-infinite since the constraints require evaluation for infinitely many values of ω . To render this problem numerically tractable, the semi-infinite constraints are transformed into linear matrix inequalities (LMIs) by application of the KYP lemma [11] and the generalized KYP lemma [9,12]. These transformations are detailed below. The global optimum of the resulting semidefinite program (SDP) is computed using SDPT3 [16], a dedicated solver for conic programming problems, called via the YALMIP interface [10]. The required CPU time is typically less than one CPU second.

The *KYP lemma* states that constraint (23b) is equivalent to the matrix inequality

$$\begin{bmatrix} A^T P_{np} A - P_{np} & A^T P_{np} B & C^T \\ B^T P_{np} A & B^T P_{np} B - \gamma_{np} & D^T \\ C & D & -\gamma_{np} \end{bmatrix} \preceq 0, \quad (24)$$

where $P_{np} \in \mathcal{S}_M$ is a slack matrix variable. The matrices A , B , C and D correspond to a state-space realization

of $\widehat{M}_S(z^N)$ ²:

$$\widehat{M}_S(z^N) = C(z^N I - A)^{-1} B + D. \quad (25)$$

The matrix inequality (24) only corresponds to an LMI if all entries are linear in the optimization variables. For this reason, the control canonical state-space form is used:

$$\begin{aligned} A &= \begin{bmatrix} 0_{M-1,1} & I_{M-1} \\ 0 & 0_{1,M-1} \end{bmatrix} & B &= \begin{bmatrix} 0_{M-1,1} \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} -W_M & -W_{M-1} & \dots & -W_1 \end{bmatrix} & D &= \begin{bmatrix} 1 \end{bmatrix}. \end{aligned}$$

Since all design variables W_m are grouped into C , the variables in LMI (24) are γ_{np} , C and P_{np} .

The *generalized KYP lemma* states that constraint (23c) is equivalent to the following set of LMI's

$$\begin{aligned} &\begin{bmatrix} A^T P_p A - P_p & A^T P_p B & C^T \\ B^T P_p A & B^T P_p B - \gamma_{p,\Delta} & D^T \\ C & D & -\gamma_{p,\Delta} \end{bmatrix} + \\ &\begin{bmatrix} Q_p A + A^T Q_p + \eta Q_p & Q_p B & 0_{M,1} \\ B^T Q_p & 0 & 0 \\ 0_{1,M} & 0 & 0 \end{bmatrix} \preceq 0, \quad (26a) \\ &-Q_p \preceq 0, \quad (26b) \end{aligned}$$

where $\eta = -2 \cos(2\pi \Lambda \Delta)$. The variables in these LMIs are $\gamma_{p,\Delta}$, C and the slack matrix variables $P_p \in \mathcal{S}_M$ and $Q_p \in \mathcal{S}_M$.

5 Numerical Results

This section numerically illustrates the potential of the proposed design approach. The convexity of the obtained SDPs allows a straightforward and efficient computation of the trade-off curves between γ_{np} and $\gamma_{p,\Delta}$. These trade-off curves are discussed in Sec. 5.1, while Sec. 5.2 compares the proposed design approach with the current high-order repetitive control design approaches [1,14].

5.1 Trade-off Curves

Figure 5 shows the trade-off curve between $\gamma_{p,\Delta}$ and γ_{np} , for different orders M of the repetitive controller. The relative uncertainty $\Lambda \Delta$, ranges from 0% (Fig. 5(a)) to 10% (Fig. 5(c)). These trade-off curves are computed

² Due to the periodicity of the FRF of $\widehat{M}_S(z)$ a state-space model of $\widehat{M}_S(z^N)$ instead of $\widehat{M}_S(z)$ suffices.

Fig. 5. Trade-off curves between $\gamma_{p,\Delta}$ and γ_{np} for different orders M of the repetitive controller and different levels of the relative uncertainty $\Lambda\Delta$.

by solving (23) for different values of α : by increasing α they are traced from left to right.

These trade-off curves indicate the performance limits of a repetitive controller of order M . For a given level of robust periodic performance improvement $\gamma_{p,\Delta}$, the curve indicates the minimal level of non-periodic performance degradation γ_{np} that has to be tolerated. Or, vice-versa, for a fixed level of allowable non-periodic performance degradation, the trade-off curve indicates the best robust periodic performance improvement that can be achieved. Since $\widehat{M}_S(z)$ only depends on the design parameters W_m , the trade-off curves are independent of the original feedback system to which the repetitive controller is added. Hence, they apply to *any* SISO control system.

Given the higher number of design variables, a repetitive controller of a given order M yields at least as good a performance as lower order controllers, for all values of $\Lambda\Delta$: the trade-off curve shifts to the left and/or downwards if M increases. However, increasing the order does not guarantee better performance. Figures 5(b,c) reveal points at which the trade-off curves for order M and $M + 1$ touch. In such a point neither $\gamma_{p,\Delta}$ nor γ_{np} improve if the order of the repetitive controller is increased by one. However, increasing the order by two again results in improved performance.

Every trade-off curve is defined for $\gamma_{p,\Delta}$ values between an uncertainty and order-dependent lower limit and an uncertainty and order-independent upper limit $\gamma_{p,\Delta} = 1$, corresponding to $K_{RC}(z) = 0$. If no uncertainty is present, the lower limit equals $\gamma_p = 0$ for all orders M , whereas for a nonzero uncertainty, $\gamma_{p,\Delta} = 0$ cannot be achieved by a *finite*-order controller since it corresponds to *infinitely* many equality constraints. The latter fact

is clearly observable in Fig. 5(c), while Fig. 5(b) reveals that for small uncertainty levels, nearly zero $\gamma_{p,\Delta}$ values are still achievable.

The presence of 'knees', i.e., abrupt changes in the slope of the trade-off curve, such as the points x and y in Fig. 5(b), renders the choice of a good engineering design rather straightforward: y would generally be preferred over any repetitive controller design with higher $\gamma_{p,\Delta}$, since the cost for decreasing $\gamma_{p,\Delta}$ to 0.18 is only small an increase in γ_{np} . On the other hand, is it not advisable to decrease $\gamma_{p,\Delta}$ below $1.23 \cdot 10^{-2}$ (point x) since this comes at a very large increase in γ_{np} . If no uncertainty (Fig. 5(a)) or larger relative uncertainty levels (Fig. 5(c)) are considered, such clear knees are not present, such that many more viable designs are available.

5.2 Comparison With Literature

The high-order repetitive controllers of Chang et al. [1] and Steinbuch [14] can be reproduced by the design framework proposed in this paper, as explained in Secs. 5.2.1 and 5.2.2, respectively.

5.2.1 Comparison with Chang et al., 1995

To improve the non-periodic performance of a repetitive controller that yields perfect nominal periodic performance, Chang et al. [1] design a high-order controller according to the following optimization problem, where

$$\gamma_p = 0$$

$$\underset{W_m, \gamma_{np}}{\text{minimize}} \quad \gamma_{np} \quad (27a)$$

$$\text{subject to} \quad |\widehat{M}_S(\omega)| \leq \gamma_{np}, \quad \forall \omega \in \mathbb{R} \quad (27b)$$

$$|\widehat{M}_S(0)| \leq \gamma_p \quad (27c)$$

$$0 \leq W_m \leq 1, \quad m : 1, \dots, M \quad (27d)$$

As already pointed out in [15], the inequality constraints (27d) are redundant. As a result, this optimization problem corresponds to the left-most point of the corresponding trade-off curve in Fig. 5(a), computed by solving (23) with a very small α -value. Whereas Chang et al. [1] solve the semi-infinite optimization problem (27) with a stochastic “evolution strategy”, the proposed framework provides a systematic and efficient approach to compute their solutions with guaranteed global optimality.

5.2.2 Comparison with Steinbuch, 2001

To improve the robust periodic performance of the corresponding closed-loop system, under the constraint of perfect nominal periodic performance, Steinbuch [14] enforces the higher-order derivatives of $|\widehat{M}_S(\omega)|$ to equal zero at the multiples of ω_0 . Hence, for a repetitive controller of order M , the design variables W_m are designed to satisfy the following set of M equations, linear in W_m :

$$\frac{d^i |\widehat{M}_S(\omega_0)|}{d\omega^i} = 0, \quad \forall i = 0, \dots, M-1. \quad (28)$$

While this analytical design approach does not involve numerical optimization, the results can be approximated by solving optimization problem (23) with $\alpha = 0$ for a small uncertainty $\Lambda\Delta$. Figure 6 compares, for $M = 3$ the results of [14] with the optimal solution of (23) with $\alpha = 0$ (corresponding to the left-most points of the trade-off curves). Fig. 6(a) and (b) show the results for $\Lambda\Delta = 2\%$ and $\Lambda\Delta = 20\%$, respectively, where the corresponding uncertainty interval $\tilde{\Omega}_\Lambda$ is shaded grey.

For small uncertainty, $\Lambda\Delta = 2\%$, the results of the two design approaches almost coincide. [14] yields $\gamma_{p,\Delta} = 2 \cdot 10^{-3}$ and $\gamma_{np} = 8$, while the optimal design yields $\gamma_{p,\Delta} = 5.84 \cdot 10^{-4}$ and $\gamma_{np} = 7.97$. If an optimal repetitive controller is designed that yields $\gamma_{p,\Delta} = 2 \cdot 10^{-3}$, the same value of [14], a better non-periodic performance index is achieved: $\gamma_{np} = 6.97$, a reduction by 13% compared to $\gamma_{np} = 8$. However, since both solutions are situated to the left of the knee x in Fig. 5(a), none of them is preferable in practice, as explained in the previous section.

For large uncertainty, $\Lambda\Delta = 20\%$, the presented approach differs significantly from the design [14], which, being sensitivity (derivative) based inherently assumes

small uncertainty levels. The optimal repetitive controller design features both a better robust periodic performance index ($\gamma_{p,\Delta} = 0.37$ compared to $\gamma_{p,\Delta} = 1.62$ for [14]) and a better non-periodic performance index ($\gamma_{np} = 4.83$ compared to $\gamma_{np} = 8$ for [14]). This can be understood from the Bode Sensitivity Integral, which imposes:

$$\int_0^{\omega_0/2} \log(|\widehat{M}_S(\omega)|) d\omega = 0. \quad (29)$$

That is, constraints (28) enforce $|\widehat{M}_S(\omega)| \approx 0$ in the neighborhood of $\omega = 0$, which results in a very large negative contribution of these frequencies in the integral (29). This negative contribution has to be compensated for by high values of $|\widehat{M}_S(\omega)|$ at the intermediate frequencies. The optimal design, on the other hand, does not push $|\widehat{M}_S(\omega)|$ down to zero, yielding a substantially smaller negative contribution in (29), and hence, lower values of $|\widehat{M}_S(\omega)|$ at the intermediate frequencies.

Following the design strategy of [14], the first repetitive controller that yields a robust periodic performance improvement ($\gamma_{p,\Delta} \leq 1$) is of order $M = 18$. This controller features $\gamma_{p,\Delta} = 0.17$ and an unacceptably high $\gamma_{np} = 1360$.

6 Conclusions

This paper proposes an optimal design approach for SISO repetitive controllers that accounts for both non-periodic inputs and period-uncertain inputs that enter the control loop. The high-order repetitive controller is designed to yield an optimal trade-off between its non-periodic performance index γ_{np} and its robust periodic performance index $\gamma_{p,\Delta}$. The computation of this optimal repetitive controller is reformulated as an SDP.

This design approach allows the computation of trade-off curves between $\gamma_{p,\Delta}$ and γ_{np} . These trade-off curves correspond to the limits of performance of a repetitive control design: they indicate the minimal level of non-periodic performance degradation that has to be tolerated if a repetitive controller of a given order is designed to yield a given level of robust periodic performance improvement. These trade-off curves apply to any repetitive controller design since they are independent of the feedback system to which the repetitive controller is added.

The high-order repetitive control design approach by Chang et al. [1] corresponds to the left-most point of the corresponding trade-off curve. The proposed repetitive control design approach outperforms the approach by Steinbuch [14]: the same robust periodic performance index is achieved with significantly less non-periodic performance degradation. This gain in non-periodic perfor-

Fig. 6. Comparison of optimal repetitive control design with $\alpha = 0$ and the repetitive controller designed by Steinbuch (2002) for $M = 3$ and two uncertainty levels $\Lambda\Delta$: (a) $\Lambda\Delta = 2\%$; (b) $\Lambda\Delta = 20\%$. The grey band indicates the corresponding uncertainty interval $\hat{\Omega}_\Lambda$.

mance increases if the uncertainty on the highest harmonic of the input is higher.

In the definition of the robust-periodic performance index $\gamma_{p,\Delta}$, a uniform probability distribution of the fundamental frequency in the uncertainty interval is implicitly assumed. Future work includes accounting for other probability distributions in the definition of $\gamma_{p,\Delta}$.

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