The use of a Stochastic Loss Given Default in a Credit Default Economic Capital framework.

J. Dhaene[†] M.J. Goovaerts[†] S. Vanduffel[§*]
R. Koch[‡] R. Olieslagers[‡] O. Romijn[‡†]

[†] KULeuven and University of Amsterdam

[‡] Fortis Central Risk Management

[§] KULeuven

March 9, 2005

Abstract

Despite the success of advanced credit portfolio models, many financial institutions still continue using a variance-covariance approach to portfolio modelling. When setting up such a framework, the parameters must be quantified and a certain number of assumptions has to be made. Assessing the level of the parameters is beyond the scope of this paper since they should ultimately pertain to peculiar features of the actual dataset. The different assumptions however should at least be mutually consistent, and a model with an inconsistent set of parameters is clearly unacceptable. We found that the concept of a stochastic loss given default in conjunction with default correlations can give rise to an inconsistent set of axioms. We propose two consistent methodologies that do not add (too much) complexity to the

*Corresponding author. His adress is Department of Applied Economics, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium. emailadress: steven.vanduffel@econ.kuleuven.ac.be

†External consultant at Fortis

initial approach. These two models are also extremal in the sense that the first alternative will provide a lower bound for the variance of the portfolio loss whilst the second (comonotonic) alternative will provide an upperbound.
Keywords: Default correlation, Loss correlation, Comonotonicity, Variance-Covariance, Economic Capital.

1 Introduction

Advanced credit portfolio models such as J.P. Morgan's CreditMetrics (1997), Credit Suisse Financial Products' CreditRisk+ (1997), PortfolioManager of KMV (Kealhofer, 1995) and McKinsey & Company's CreditPortfolioView (Wilson 1997a,b) are nowadays widespread among numerous banks allowing them to assess the credit default risk of their diverse loan portfolios. They rely on these results to put important capital buffers aside to protect against this risk.

The implementation of such models is often for the banks their first step towards developing what is now called an enterprise risk framework, which can support consistent risk and reward management on an enterprise-wide basis by integrating all risk components. Indeed, the capital consumption of the different business units within a financial conglomerate may seriously affect various centrally-taken investment decisions, which in turn may impact the performance measurement of the different business units with possible repercussions on individual bonuses.

Despite the commercial success of the above mentioned models, many other financial institutions are, according to Deloitte & Touch's global Risk Management Survey (2004), yet to set up such an integrated framework. Amongst them, some prefer sticking with a variance-covariance approach as portfolio model for the sake of transparency and actual applicability. In contrast to the bottom-up credit risk model that computes directly the distribution function of the random portfolio loss, the variance-covariance approach focuses on the computation of the mean and the variance of this random variable. Next, these two moments are linked to the capital through a calibration on a known two-parameter distribution such as, for example, the Beta distribution.

The parameters used in the variance-covariance framework are: the probability of default, a deterministic exposure at default, a stochastic loss given default (LGD) modelled by its mean, variance and the default correlation between the obligors. These parameters can also be found in the quantitative groundings of the New Basel Accord (2004). Before setting up that variance-covariance framework, we must quantify and establish a certain number of assumptions. Assessing the level of the parameters is beyond the scope of this paper since they should ultimately pertain to the actual dataset features. However, the different assumptions should at least be mutually consistent: indeed, a model that produces an inconsistent set of parameters is clearly unacceptable. When introducing the variance-covariance framework, Fortis, a financial conglomerate based in Belgium, considered in a first stage the variance of LGD to be equal to zero. In a second stage, a more sophisticated approach would be followed by assuming a stochastic LGD. We found that this concept in conjunction with the use of default correlations can give rise to an inconsistent set of assumptions and should therefore be considered with caution.

In this paper we propose two consistent variance-covariance frameworks. Both methods maintain the concept of a stochastic loss given default but improve some other (correlation) assumptions. Whilst the first alternative relies on an independency assumption between the different LGD's, we assume in the second alternative that the different LGD's are comonotonic, meaning that they are all monotonic functions of a common random variable. We observe that these two alternatives do not add (too much) complexity to the initial approach. These two models are also extremal in the sense that the first alternative will provide a lower bound for the variance of the portfolio loss whilst the second (comonotonic) alternative will provide an upper bound.

The structure of the paper is as follows: after introducing some general results (section 2) we will describe a typical set of assumptions that is used when setting up a variance-covariance framework for portfolio modelling (section 3) and we will prove its inconsistency (section 4). In sections 5 and 6 we describe the two consistent alternatives and demonstrate their boundary properties. We conclude in section 7 with some final remarks.

2 Description of the problem

Consider a portfolio of n credit risks. Let I_i be defined as the indicator random variable (r.v.) which equals 1 if risk i leads to failure in the next period, and 0 otherwise. The probability that risk i leads to a failure is denoted by q_i:

$$q_i = \Pr\left[I_i = 1\right]. \tag{1}$$

Hence, q_i is the probability of default. Further, let $(EAD)_i$ denote the Exposure-at-Default and $(LGD)_i$ the Loss-Given-Default of risk i.
The Exposure-at-Default is the maximal amount of loss on risk i, given that default occurs. It is assumed to be a deterministic quantity.
The Loss-Given-Default is the percentage of the loss on risk i, given default occurs. In this paper, the Loss-Given-Default is assumed to be a r.v.
The Portfolio Loss during the reference period is then defined as the sum of the losses on the individual credit risks. It is given by:

$$\begin{aligned} L &= \sum_{i=1}^{n} L_i \qquad (2) \\ &= \sum_{i=1}^{n} I_i \ (EAD_i) \ (LGD_i). \qquad (3) \end{aligned}$$

The Credit Economic Capital is determined as a high percentile (e.g. a 99.97%-quantile) of the Portfolio Loss distribution minus its expectation.
The Default Correlation of risk pair (i, j) is denoted by $\rho^D_{i,j}$. It is given by

$$\rho^D_{i,j} = \rho\left(I_i, I_j\right), \tag{4}$$

where $\rho\left(I_i, I_j\right)$ is the Pearson's correlation coefficient for (I_i, I_j):

$$\rho\left(I_i, I_j\right) = \frac{cov\left(I_i, I_j\right)}{\sigma\left(I_i\right)\sigma\left(I_j\right)}. \tag{5}$$

with:

$$\sigma^2(I_i) = q_i\ (1 - q_i). \tag{6}$$

The LGD Correlation of risk pair (i, j) is denoted by $\rho^{LGD}_{i,j}$. It is given by

$$\rho^{LGD}_{i,j} = \rho\left(LGD_i, LGD_j\right). \tag{7}$$

Finally, the Loss Correlation of risk pair (i, j) is denoted by $\rho^L_{i,j}$. It is given by

$$\rho^L_{i,j} = \rho\left(L_i, L_j\right). \tag{8}$$

In the next sections we will discuss how to construct (and how not to construct) a consistent model of correlations $\rho^D_{i,j}, \rho^{LGD}_{i,j}$ and $\rho^L_{i,j}$. Whilst it is of course correct to consider LGD as a r.v. and not as a deterministic quantity, we show that the consequences of such an assumption should be carefully considered. Since the equality between loss and default correlations hold in case the LGD are deterministic, it is tempting to continue assuming that $\rho^L_{i,j} = \rho^D_{i,j}$ holds for all risk pairs (i, j) also when LGD is a r.v. We will show that this assumption, in conjunction with the use of a stochastic LGD, leads to inconsistencies. A number of authors have considered methods of estimating default correlations, including theoretical models of Hull and White (2001) and Zhou (2001). Some estimates from real data are used in Stevenson et al (1995) and Gollinger and Morgan (1993). On the other hand, much less literature seems to exist on the more general concept of loss correlations.

3 Some general results

Throughout this paper we will assume that the vector of Default Bernoulli random variables $(I_i, \cdots, I_n)$ and the vector of Loss Given Default random variables $(LGD_1, \cdots, LGD_n)$ are mutually independent. Hence, in line with most credit default risk models, we will assume

$$\textbf{ASSUMPTION(1)} \ : \ I_i \text{ and } LGD_j \text{ are mutually independent, for any pair } (i,j)\,. \tag{9}$$

We emphasize that the mutual independency of I_i and LGD_i is just a technical assumption. Indeed, only the random variable $LGD_i \mid I_i = 1$ is relevant. So we can choose the distribution function of $LGD_i \mid I_i = 0$ as we want. A convenient choice is to assume that $(LGD_i \mid I_i = 0) \stackrel{d}{=} (LGD_i \mid I_i = 1)$, where $\stackrel{d}{=}$ stands for the equality in distribution. This is indeed a good choice, because it makes the random variables LGD_i and I_i mutually independent which is convenient from a mathematical point of view. The assumption of mutually independency between I_i and LGD_j for $i \neq j$ cannot be considered as a technical assumption.
In the sequel, we will also set all EAD_i equal to 1. Results and conclusions can easily be generalized to Exposures-at-Default with values different from being all equal to one.

From

$$cov\left(L_i, L_j\right) = E\left[cov\left(L_i, L_j\right) \mid I_i, I_j\right] + cov\left[E\left(L_i \mid I_i, I_j\right), E\left(L_j \mid I_i, I_j\right)\right] \quad (10)$$

we find using assumption (1) that

$$\begin{aligned} cov\left(L_i, L_j\right) &= E\left(I_i\ I_j\right)\ cov\left(LGD_i, LGD_j\right) \\ &\quad + E(LGD_i)\ E(LGD_j)\ cov(I_i, I_j) \\ &= \left[cov(I_i, I_j) + q_i\ q_j\right] cov(LGD_i, LGD_j) \\ &\quad + E(LGD_i)\ E(LGD_j)\ cov(I_i, I_j). \end{aligned} \quad (11)$$

Hence,

$$\begin{aligned} \rho_{i,j}^{L}\ \sigma(L_i)\ \sigma(L_j) &= \left[\rho_{i,j}^{D}\ \sigma(I_i)\ \sigma(I_j) + q_i\ q_j\right]\ \rho_{i,j}^{LGD}\ \sigma(LGD_i)\ \sigma(LGD_j) \\ &\quad + \rho_{i,j}^{D}\ \sigma(I_i)\ \sigma(I_j)\ E(LGD_i)\ E(LGD_j). \end{aligned} \quad (12)$$

Also from expression (10) we find that

$$\begin{aligned} Var(L_i) &= Var\left[E\left(L_i \mid I_i\right)\right] + E\left[Var\left(L_i \mid I_i\right)\right] \\ &= E^2(LGD_i)\ q_i\ (1 - q_i) + q_i\ Var\left(LGD_i\right). \end{aligned} \quad (13)$$

From the derivations above, we find that a general expression for $Var(L)$ is given by

$$\begin{aligned} Var(L) &= \sum_{i \neq j}^{n} cov(L_i, L_j) + \sum_{i=1}^{n} Var(L_i) \\ &= \sum_{i \neq j}^{n} \left[\rho_{i,j}^{D}\ \sigma(I_i)\ \sigma(I_j) + q_i\ q_j\right]\ \rho_{i,j}^{LGD}\ \sigma(LGD_i)\ \sigma(LGD_j) \\ &\quad + \sum_{i \neq j}^{n} \rho_{i,j}^{D}\ \sigma(I_i)\ \sigma(I_j)\ E(LGD_i)\ E(LGD_j) \\ &\quad + \sum_{i=1}^{n} q_i \left(E^2(LGD_i)\ (1 - q_i) + Var\left(LGD_i\right)\right). \end{aligned} \quad (14)$$

From expression (12) we obtain that $\rho_{i,j}^{L}$ is given by:

$$\rho^L_{i,j} = \frac{A + B}{\sqrt{Var(L_i)Var(L_j)}} \tag{15}$$

with:

$$\begin{aligned} A &= \left[\rho^D_{i,j}\ \sigma(I_i)\ \sigma(I_j) + q_i\ q_j)\right]\ \rho^{LGD}_{i,j}\ \sigma(LGD_i)\ \sigma(LGD_j) \\ B &= \rho^D_{i,j}\ \sigma(I_i)\ \sigma(I_j)\ E(LGD_i)\ E(LGD_j) \end{aligned} \tag{16}$$

4 An inconsistent model

In case all LGD_i are deterministic, it is straightforward to prove that for any risk pair (i, j) the loss correlation is equal to the default correlation. Therefore, a natural assumption seems that this holds in general:

ASSUMPTION (2a) : $\rho^L_{i,j} = \rho^D_{i,j}$ for all (i, j). (17)

However, LGD is considered to a random variable and the question raises whether assumption (2a) can be made in this case. The inconsistency of assumption (2a) when adding it to the existing set of assumptions can be shown easily by a numerical counterexample:

Taking

$$\begin{aligned} q_i &= 0.001 \\ q_j &= 0.01 \\ E\left(LGD_i\right) &= 0.8 \\ E\left(LGD_j\right) &= 0.2 \\ Var\left(LGD_i\right) &= 0.04 \\ Var\left(LGD_j\right) &= 0.04 \\ \rho^D_{i,j} &= \rho^L_{i,j} = 0.03 \end{aligned}$$

We find from (13) and (15) that:

$$\begin{aligned} Var(L_j) &= 0.00068 \\ Var(L_j) &= 0.00080 \end{aligned}$$

From (15) we find now that

$$\rho^{LGD}_{i,j} = 1.669$$

which is in contradiction with $\rho_{i,j}^{LGD} \leq 1$.

We can conclude that the pair of assumptions (1) and (2a) lead to unacceptable inconsistencies. This makes the assumption (2a) inappropriate for practical use. We remark that the assumption of equality between the loss correlations and default correlations is correct when the LGD_i are assumed to be deterministic. In this case it is even not necessary to assume (2a). It is just a property that follows from the model.

We will now propose two consistent alternatives. Both alternatives still use the concept of a stochastic LGD but replace the inconsistent assumption 2a by another consistent assumption.

5 Consistent methodology I

Consider the individual loss random variables $L_i = I_i \ (LGD_i)$, $i = 1, \cdots, n$, as defined above. As mentioned earlier, throughout this paper we assume that the vector of Default Bernoulli random variables $(I_i, \cdots, I_n)$ and the vector of Loss Given Default random variables $(LGD_1, , \cdots, LGD_n)$ are mutually independent.

The simplest consistent methodology derives from additionally assuming that Loss-Given-Defaults are mutually independent:

$$\textbf{ASSUMPTION (2b)}: \qquad \rho_{i,j}^{LGD} = 0 \text{ for all } i \neq j. \tag{18}$$

In this case, we find from the general expression (12) that

$$Cov(L_i, L_j) = \rho_{i,j}^{D} \ \sigma(I_i) \ \sigma(I_j) \ E(LGD_i) \ E(LGD_j)), \qquad i \neq j, \tag{19}$$

or equivalently,

$$\rho_{i,j}^{L} \sigma(L_i) \ \sigma(L_j) \ = \rho_{i,j}^{D} \ \sigma(I_i) \ \sigma(I_j) \ E(LGD_i) \ E(LGD_j)), \qquad i \neq j, \tag{20}$$

From (14) we find now the following expression for the variance of the Portfolio Loss:

$$\begin{aligned} Var(L) \ &= \ \sum_{i \neq j}^{n} \rho_{i,j}^{D} \ \sqrt{q_i(1-q_i)q_j(1-q_j)} \ E(LGD_i) \ E(LGD_j)) \\ &+ \sum_{i=1}^{n} q_i \left(E^2(LGD_i) \ (1-q_i) + Var\left(LGD_i\right) \right). \end{aligned} \tag{21}$$

Finally, using the same theoretical example setting as in Section 4, we find from (20) that

$$\rho_{i,j}^{L} = 0.021$$

and not $\rho_{i,j}^{L} = 0.03$, as it was the case in the inconsistent methodology.

6 Consistent methodology II

Consider the individual loss random variables $L_i = I_i \ (LGD_i)$, $i = 1, \cdots, n$, as defined above. Again we assume that the vector of Default Bernoulli random variables $(I_i, \cdots, I_n)$ and the vector of Loss-Given-Default random variables $(LGD_1, , \cdots, LGD_n)$ are mutually independent. But now we assume that the vector $(LGD_1, , \cdots, LGD_n)$ is a comonotonic vector. This means that $(LGD_1, \cdots, LGD_n)$ has the same distribution as $\left(F_{LGD_1}^{-1}(U), \cdots, F_{LGD_n}^{-1}(U)\right)$, where U is a random variable that is uniformly distributed on the unit interval $(0, 1)$, and $F_{LGD_i}^{-1}$ is the inverse distribution function of the random variable LGD_i. The assumption of comonotonicity implies that the different LGD_i are monotonic functions of a common random variable, which also explains the word comonotonic (common monotonic) and shows that comonotonicity is indeed a strong dependency structure. One intuitively expects that the assumption of comonotonicity for the vector $(LGD_1, , \cdots, LGD_n)$ will give rise to an upper bound for the variance of the Portfolio Loss.

ASSUMPTION (2c) : $cov\,(LGD_i, LGD_j) = cov\left(F_{LGD_i}^{-1}(U), F_{LGD_j}^{-1}(U)\right)$ for all (i, j).

Note that the vectors $(LGD_1, \cdots, LGD_n)$ and $\left(F_{LGD_1}^{-1}(U), \cdots, F_{LGD_n}^{-1}(U)\right)$ have the same marginal distributions, so that the LGD-correlations are given by

$$\rho_{i,j}^{LGD} = \frac{cov\left(F_{LGD_i}^{-1}(U), F_{LGD_j}^{-1}(U)\right)}{\sqrt{Var\,(LGD_i)\ Var\,(LGD_j)}} \tag{22}$$

It is straightforward to show that if $\rho_{i,j}^{LGD} = 1$ for all $i \neq j$ then this implies that $(LGD_1, , \cdots, LGD_n)$ is comonotonic, but the opposite statement is only true if there exists a random variable Y, positive real constants a_i and real constants b_i such that the relation $LGD_i \overset{d}{=} a_i Y + b_i$ holds for $i = 1, 2, \ldots, n$.

However, it can be proven that the comonotonicity of $(LGD_1, , \cdots, LGD_n)$ is equivalent with the maximization of the $\rho_{i,j}^{LGD}$ for all pairs (LGD_i, LGD_j) with $i \neq j$, see Dhaene et al (2000a) for a proof.

From (12) we find

$$\begin{aligned} Cov(L_i, L_j) &= \left[\rho_{i,j}^D \ \sigma(I_i) \ \sigma(I_j) + q_i \ q_j)\right] \ cov\left(F_{LGD_i}^{-1}(U), F_{LGD_j}^{-1}(U)\right) \\ &\quad +\rho_{i,j}^D \ \sigma(I_i) \ \sigma(I_j) \ E(LGD_i) \ E(LGD_j). \end{aligned} \tag{23}$$

or equivalently

$$\begin{aligned} \rho_{i,j}^L \ \sigma(L_i) \ \sigma(L_j) &= \left[\rho_{i,j}^D \ \sigma(I_i) \ \sigma(I_j) + q_i \ q_j)\right] \ cov\left(F_{LGD_i}^{-1}(U), F_{LGD_j}^{-1}(U)\right) \\ &\quad +\rho_{i,j}^D \ \sigma(I_i) \ \sigma(I_j) \ E(LGD_i) \ E(LGD_j). \end{aligned} \tag{24}$$

The variance of the Portfolio Loss follows from (14):

$$\begin{aligned} Var(L) &= \sum_{i \neq j}^{n} \left[\rho_{i,j}^D \ \sigma(I_i) \ \sigma(I_j) + q_i \ q_j)\right] \ cov\left(F_{LGD_i}^{-1}(U), F_{LGD_j}^{-1}(U)\right) \\ &\quad + \sum_{i \neq j}^{n} \rho_{i,j}^D \ \sigma(I_i) \ \sigma(I_j) \ E(LGD_i) \ E(LGD_j) \\ &\quad + \sum_{i=1}^{n} q_i \left(E^2(LGD_i) \ (1 - q_i) + Var\left(LGD_i\right)\right). \end{aligned} \tag{25}$$

Assuming that $\rho_{i,j}^D$ and $\rho_{i,j}^D$ are both non-negative for all (i, j), we find by comparing the general expression (12) for $\rho_{i,j}^L$ with the expressions (20) and (24) respectively that:

$$\rho_{i,j}^L[\text{methodI}] \leq \rho_{i,j}^L[\text{consistent method}] \leq \rho_{i,j}^L[\text{method II}] \tag{26}$$

and also that

$$Var(L)[\text{method I}] \leq Var(L)[\text{consistent method}] \leq Var(L)[\text{method II}]. \tag{27}$$

7 Conclusion and final remarks

In this paper, we proved that, when using a stochastic LGD, the method of setting default correlations and loss correlations equal is inconsistent. We proposed two consistent models. These two models are extreme models, in the sense that for any possible consistent method (with all $\rho_{i,j}^{LGD}$ and $\rho_{i,j}^{D}$ being non-negative) the following inequalities will hold:

$$\rho_{i,j}^{L}[\text{methodI}] \leq \rho_{i,j}^{L}[\text{consistent method}] \leq \rho_{i,j}^{L}[\text{method II}] \tag{28}$$

and

$$Var(L)[\text{method I}] \leq Var(L)[\text{consistent method}] \leq Var(L)[\text{method II}]. \tag{29}$$

Note that these inequalities hold as far as the distribution functions of the I_i and of the LGD_i, as well as the correlations $\rho_{i,j}^{D}$ are the same for the different methods. We point out that method I relies on an independency assumption between the different LGD_i's whilst in the most prudent method II, one assumes that the different LGD_i's are comonotonic. The theory of comonotonicity has been extensively studied in a series of papers, starting from Dhaene & Goovaerts (1996) and pursued in Kaas et al (2000) and Dhaene et al (2000a). The theory has been applied to a number of important financial and actuarial problems such as the pricing of Asian and Basket options in a Black & Scholes model, the setting of provisions and required capitals in an insurance context and the determination of optimal portfolio strategies. We refer to Albrecher et al (2005), Dhaene et al. (2002b), Dhaene et al. (2004), Vanduffel et al. (2002) and Vanduffel et al. (2005) where these applications have been investigated.

Finally, we remark that all the results in this paper continue to hold if we generalize the model to the case that the defaults $(I_1, \cdots, I_n)$ depend on some conditioning random vector $(Q_1, \cdots, Q_n)$ such that

$$Q_i = \Pr\left[I_i = 1 \mid Q_i\right]. \tag{30}$$

This leads to

$$\Pr\left[I_i = 1\right] = E\left(Q_i\right) = q_i. \tag{31}$$

Hence, the probability of default of risk i can be interpreted as the expectation of the conditioning random variable Q_i in this case.

Acknowledgement 1 *The authors would like to thank Emil Valdez for fruitful discussions on this topic. They also thank two anonymous referees and an editor for their helpful comments which improved the paper. Jan Dhaene, Marc Goovaerts and Steven Vanduffel acknowledge the financial support by the Onderzoeksfonds K.U.Leuven (GOA/02: Actuariële, financiële en statistische aspecten van afhankelijkheden in verzekerings- en financiële portefeuilles).*

References

[1] Albrecher, H., Dhaene, J., Goovaerts, M.J. & Schoutens, W. (2003). "Static hedging of Asian options under Lévy models: the comonotonic approach", Research Report OR 0365, Department of Applied Economics, K.U.Leuven.

[2] Basel Committee for Banking Supervision (2004) "International convergence of capital measurement and capital standards, a revised framework".

[3] Credit Suisse Financial Products (1997) "CreditRisk+, A Credit Risk Management Framework", Technical document, available online at *www.csfb.com/creditrisk.*

[4] Deloitte & Touch. (2004). "Deloitte's Gloabl Risk Management Survey", available online at *www.deloitte.com.*

[5] Dhaene, J.; Denuit, M.; Goovaerts, M.J.; Kaas, R.; Vyncke, D. (2002a). "The concept of comonotonicity in actuarial science and finance: theory", *Insurance: Mathematics & Economics 31, 3-33.*

[6] Dhaene, J.; Denuit, M.; Goovaerts, M.J.; Kaas, R.; Vyncke, D. (2002b). "The concept of comonotonicity in actuarial science and finance: applications", *Insurance: Mathematics & Economics 31, 133-161.*

[7] Dhaene, J. & Goovaerts, M.J. (1996). "Dependency of risks and stop-loss order", *ASTIN Bulletin* 26, 201-212.

[8] Dhaene, J., Vanduffel, S., Goovaerts, M.J., Kaas, R. & Vyncke, D. (2004). "Comonotonic approximations for optimal portfolio selection problems", *Journal of Risk and Insurance*, to appear.

[9] Gollinger, T.L. & Morgan, J.B.(1993). "Calculation of an Efficient Frontier for a Commercial Loan Portfolio", *Journal of Portfolio Management, 39-46.*

[10] JP Morgan & Co. Inc. (1997). "Credit Metrics", *Technical Document,* availabale online at www.creditmetrics.com.

[11] Hull, J. & White, A. (2001). "Valuing Credit Default Swaps II: Modelling Default Correlations", *Journal of Derivatives* **8(3)**, 12-21.

[12] Kaas, R.; Goovaerts, M.J., Dhaene, J.; Denuit, M. (2001). "Modern Actuarial Risk Theory", Kluwer Academic Publishers, pp. 328.

[13] Kaas, R., Dhaene, J. & Goovaerts, M. (2000). "Upper and lower bounds for sums of random variables", *Insurance: Mathematics and Economics*, 27, 151-168.

[14] Kealhofer, S. (1995) "Managing default risk in derivative portfolios". Derivative Credit Risk: Advances in Measurement and Management. Risk Publications, London.

[15] Stevenson, B.G. & Fadil, M. (1995). "Modern Portfolio Theory: Can it Work for Commercial Loans?", *Commercial Lending Review* **10(2).**

[16] Vanduffel, S., Dhaene, J., Goovaerts, M. & Kaas, R. (2002). "The hurdle-race problem", *Insurance: Mathematics and Economics,* **33(2)**, 405-413.

[17] Vanduffel, S., Dhaene, J., Goovaerts, M.(2005) "On the evaluation of Saving-consumption plans", *Journal of Pension Economics and Finance,* to be published.

[18] Wilson, T. (1997a) "Porfolio credit risk: Part I", *Risk* (September), 111-117.

[19] Wilson, T. (1997a) "Porfolio credit risk: Part II", *Risk* (October), 56-61.

[20] Zhou, C. (2001). "An Analysis of Default Correlations and Multiple Defaults", *Review of Financial Studies* **14(2)**, 555-576.