**A new approach to
the rational interpolation problem:
the vector case**

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ABSTRACT

We generalize our earlier results on rational interpolation which were given in [19] for the scalar case and in [6] for the vector case when all the interpolation points coincide, to the case of vector data given at arbitrary points that may coincide or not. This is the vector valued Newton-Padé problem. We give a recursive algorithm which has the important advantage over other algorithms that we don't need a reordering of the given interpolation data to overcome a singularity in the interpolation table, not even in the non-normal vector case. It also generates all the information needed to give all solutions of the problem.

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Abstract

We generalize our earlier results on rational interpolation which were given in [19] for the scalar case and in [6] for the vector case when all the interpolation points coincide, to the case of vector data given at arbitrary points that may coincide or not. This is the vector valued Newton-Padé problem. We give a recursive algorithm which has the important advantage over other algorithms that we don't need a reordering of the given interpolation data to overcome a singularity in the interpolation table, not even in the non-normal vector case. It also generates all the information needed to give all solutions of the problem.

Keywords: Rational interpolation, Newton-Padé, vector interpolants, simultaneous rational approximation

1 Introduction

In an earlier paper [19], we showed that we could take away several drawbacks of the classical Newton-Padé rational interpolation problem by using a new approach to the rational interpolation problem (**RIP**). We also gave a recursive algorithm which has the important advantage over existing algorithms that we don't need a reordering of the given interpolation data. Our algorithm basically solves the linearized **RIP** and gives a solution that is in some clearly defined sense as simple as possible, but as a side product, it also generates all the information needed to give all solutions, not only for the linearized **RIP** but also for the proper **RIP**. When all the interpolation points coincide, we can reformulate the scalar **RIP** as a scalar minimal partial realization problem. We have generalized the case where all the data are given at one point to vector data [6] and even matrix data [17]. In [18] we allowed data to be given at two points (zero and infinity). In this paper, we generalize this solution method to the case of vector interpolation data with possible non-coinciding interpolation points.

The essential observations made in the above mentioned papers is that the **RIP** can be linearized whereupon it reduces to the recursive solution of nested systems of homogeneous linear equations. The matrices of these systems have a special structure. Thus we have to find an element from their null space. Since these systems have one more column than they have rows, their null space is at least one dimensional. In the case of Padé approximation, these

systems are (block) Toeplitz or (block) Hankel. These matrices appear in many different seemingly unrelated mathematical topics and engineering applications, which might explain their popularity during the last decades. You will find them for example in the theory and applications of orthogonal polynomials, integral equations, digital filtering and signal processing, time series analysis, stochastic processes, inverse scattering, BCH-decoding etc., to name just a few.

The basic algorithms to solve these problems are the Levinson/Schur/Szegő algorithms (for the Toeplitz case) and the Euclidean/Berlekamp-Massey/Kronecker algorithms (for the Hankel case). A basic element in these algorithms, and also in our approach, is the description of all elements in the null space. This is done by constructing a basis. The nesting of the matrices and their special structure make it possible to find such a basis rather efficiently if you know the bases of the previous null spaces. During the last decade much attention has been paid to a multitude of generalizations of these algorithms to all sorts of “special” or “structured” matrices. People have considered matrices close-to-Toeplitz or Hankel, Hilbert matrices and Vandermonde matrices [7] and many others.

Recently, there is a revived interest in the rational interpolation case [14, 15, 16, 2, 3, 1, 19]. The matrices involved here are Löwner matrices or generalized Löwner matrices depending on whether the interpolation points can coincide or not (see [2]). The basic thing to be studied remains, just like in the classical problems, to describe a basis for the null spaces of such matrices and to find an efficient way to generate them. In this paper, you will find a solution to this problem.

Vector rational interpolants have been considered before. The papers by Peter Graves-Morris are the most recent ones. [10, 11, 13]. We refer to his introduction (especially the second one) to find out about the history of the problem and the relation to related problems in the literature like the so called Latin and German polynomial problems. Our algorithm will be able to recover the directional vector interpolants considered in [11] (see section 2).

In this paper we combine the new approach to rational interpolation of [19], and the idea of minimal vector Padé approximation of [6] to the case where vector interpolation data are given. All the results we obtained for the scalar problem are generalized to the vector case. The basic results being the constructive way of defining the basis of the null space, which results in an efficient recursive algorithm (section 3). A “minimal” solution of the original interpolation problem (section 2) can then be readily given (section 4) as well as a parameterization of all solutions, not only for the linearized problem, but also for the proper interpolation problem (section 5).

The absence in our algorithm of a reordering strategy which is needed to overcome a singular situation when the algorithm hits a “block” in the Newton-Padé table in most of the existing scalar methods [8, 12, 9, 20, 21, 22, 23] is a basic advantage. For the vector case, a normal situation was supposed from the start [10, 11, 13]. Our algorithm will also work in singular situations for the vector case.

2 The vector rational interpolation problem (VRIP)

This paper is a continuation of [19] and [6]. Ample motivation for our approach and the definitions and methods introduced can be found there. It might be an advantage to read the simpler cases of scalar interpolants [19], or completely confluent points [6] first. The present paper follows closely the outline of [19].

The field, finite or infinite, we are working with, is denoted by $\mathbf{K}$. The set of polynomials with coefficients of the field $\mathbf{K}$, is denoted by $\mathbf{K}[z]$. The set of polynomial m -vectors, i.e., the

set of all vectors whose m elements belong to $\mathbf{K}[z]$, is denoted by $\mathbf{K}^m[z]$. Suppose we want to interpolate the given data $f_i \in \mathbf{K}^m$, in the interpolation points $z_i \in \mathbf{K}, i = 1, 2, \dots, p$, by a vector rational function $n(z)/d(z)$ with $n(z) \in \mathbf{K}^m[z]$ and $d(z) \in \mathbf{K}[z]$. The division by $d(z)$ is taken componentwise, i.e.

$$n(z)/d(z) = \begin{bmatrix} n_1(z)/d(z) \\ n_2(z)/d(z) \\ \vdots \\ n_m(z)/d(z) \end{bmatrix}.$$

Because we want to keep the exposition here rather informal, we don't allow a component of f_i to have the value ∞ . Hence, $d \neq 0$. In future publications, we shall show that the same approach we use here is also valid when the data can have a component value equal to infinity or when the interpolation point is equal to infinity (compare with [14]). For the same reason, we assume that all the interpolation points $z_i, i = 1, 2, \dots, p$ are different. The confluent case is handled in section 6. Note that the previous problem is equivalent to the problem of finding a polynomial $(m+1)$ -tuple $(n_1(z), n_2(z), \dots, n_m(z), d(z))$, denoted as $(n(z), d(z))$, not divisible by $z - z_i, i = 1, 2, \dots, p$, i.e. at least one of the polynomials $n_1(z), n_2(z), \dots, n_m(z), d(z)$ is not divisible by $z - z_i$, satisfying $f_i d(z_i) = n(z_i), i = 1, 2, \dots, p$. Similar to the polynomial interpolation problem, we want to restrict the complexity of the vector rational function $n(z)/d(z)$. The complexity of a polynomial can be indicated by its degree. We shall base our definition of the complexity of a polynomial $(m+1)$ -tuple $(n(z), d(z))$ on the degrees of the polynomials $n_1(z), n_2(z), \dots, n_m(z)$ and $d(z)$. To indicate the relative importance of the $(m+1)$ degrees, the shift parameters $s = (s_1, s_2, \dots, s_m) \in \mathbf{Z}^m$ are given. We define the s -degree α of $(n(z), d(z))$ as $\alpha = \max(\deg n_1(z) - s_1, \deg n_2(z) - s_2, \dots, \deg n_m(z) - s_m, \deg d(z))$ (We define $\deg 0 = -\infty$). Given the shift parameters $s \in \mathbf{Z}^m$, we take the s -degree as the complexity of the polynomial $(m+1)$ -tuple $(n(z), d(z))$. We can now define the following two vector rational interpolation problems (**VRIPs**).

Definition 2.1 (Linearized Vector Rational Interpolation Problem (LVRIP)) *The interpolation points $z_i \in \mathbf{K}$ and data $f_i \in \mathbf{K}^m, i = 1, 2, \dots, p$ and the shift parameters $s \in \mathbf{Z}^m$ are given. We look for a polynomial $(m+1)$ -tuple $(n(z), d(z))$ of minimal s -degree, satisfying the linearized rational interpolation conditions*

$$f_i d(z_i) = n(z_i), i = 1, 2, \dots, p. \quad (2.1)$$

Definition 2.2 (Proper Vector Rational Interpolation Problem (PVRIP)) *This is the linearized rational interpolation problem with the additional condition that $(n(z), d(z))$ is not divisible by $z - z_i, i = 1, 2, \dots, p$.*

We give now the relation between our **LVRIP** and the directional vector-valued interpolants (**DIRIs**) of [11]. For the **DIRI** problem, we also have to consider $N+1$ interpolation points $z_i, i = 0, 1, \dots, N$ with corresponding function values $f(z_i) = v_i \in \mathbf{C}^m$. Besides these, there are given k_l independent directions $w_k \in \mathbf{C}^m, k = 1, \dots, k_l$ and associated multiplicities m_k , such that $\sum_{k=1}^{k_l} m_k = M \leq N$. Along the direction w_k , the numerator degree should be reduced to $N - m_k$. This means the following. Let $P(z)$ be the polynomial m -tuple, representing the numerator of the **DIRI** and let $Q(z)$ be the corresponding denominator. Then $P(z) \cdot w_k$ should have a degree at most $N - m_k$, where $a \cdot b$ denotes the usual Euclidean inner product of the two m -vectors a and b . This condition we can rewrite as follows.

Let $W \in \mathbb{C}^{m \times m}$ be a matrix whose first k_l columns are the directional vectors w_k and the remaining columns are added to make W a regular square matrix. Then

$$\text{degree}[P_1(z), \dots, P_m(z)]W \leq [N - m_1, \dots, N - m_{k_l}, N, \dots, N].$$

The trailing $m - k_l$ bounds impose no extra conditions so that we can consider them as redundant. We only need them as a trick to define the invertible matrix W . Now the interpolation conditions $P(z_i)/Q(z_i) = v_i$ can be rewritten as the equivalent conditions (consider $P(z)$ and v_i as row-vectors) $P(z_i)W/Q(z_i) = v_iW$, $i = 0, 1, \dots, N$. Rename $N + 1$ as p , $P(z)W$ as $n(z)$, $Q(z)$ as $d(z)$ and v_iW as f_{i+1} and you will recognize our **LVRIP** with shift parameters $s_j = N - M - m_j$, $M = \sum_1^{k_l} m_k$. Thus it turns out that one of the solutions that our algorithm will produce with appropriate shift parameters, will be a **DIRI**.

In [11], the link between the German polynomial problem and the construction of **DIRIs** is explained. Thus our method is also able to find solutions of the German polynomial problem.

3 The solution of the LVRIP

In this and the subsequent sections we shall work constantly with polynomial $(m + 1)$ -tuples. These will be split into an m -tuple of numerator polynomials and one scalar denominator polynomial. If the $(m + 1)$ -tuple is denoted by e.g., $v(z)$, then the corresponding numerator m -tuple is denoted by $n_v(z)$ and the associated denominator polynomial as $d_v(z)$. Thus $v(z) = (n_v(z), d_v(z))$. It will happen that we have a sequence of polynomial $(m + 1)$ -tuples : $v_j(z)$. In order not to complicate the notation, we shall indicate the numerator and denominator also with that index : $v_j(z) = (n_j(z), d_j(z))$.

We can write (2.1) as a set of linear homogeneous equations:

$$R_i v(z) = 0, i = 1, 2, \dots, p \quad (3.2)$$

where $R_i v(z)$ is the i -th residual of the polynomial $(m + 1)$ -tuple $v(z) = (n_v(z), d_v(z))$ defined as $R_i v(z) = f_i d_v(z_i) - n_v(z_i)$. We denote by $S_{p,\alpha}$ the vector space of all polynomial $(m + 1)$ -tuples $(n(z), d(z))$ having s -degree $\leq \alpha$ and satisfying (2.1) or, what is the same, solving (3.2). It is easy to see that a basis $BS_{p,\alpha}$ for the vector space $S_{p,\alpha}$, $-\infty < \alpha < +\infty$, can be based on $(m + 1)$ polynomial $(m + 1)$ -tuples $v_p^j(z) = (n_p^j(z), d_p^j(z))$, $j = 1, 2, \dots, m + 1$, as will be shown in the sequel. Note that we use a subscript p to indicate that the polynomial $(m + 1)$ -tuple satisfies (3.2). We say that it is an $(m + 1)$ -tuple of level p . The $m + 1$ polynomial $(m + 1)$ -tuples of level p are numbered with a superscript j . Thus $v_p^j(z)$ is the j th $(m + 1)$ -tuple of level p . Without loss of generality, we can assume that the $v_p^j(z)$ are ordered with respect to increasing s -degree. We define the residual vector space $R_{i,p}^j$, $i > 0, 1 \leq j \leq m + 1$ as follows

$$R_{i,p}^j = \text{span}_{k=1}^j R_i v_p^k(z).$$

In other words, $R_{i,p}^j$ is the vector space spanned by the i -th residuals of the j polynomial $(m + 1)$ -tuples of level p having the smallest s -degree. The basis $BS_{p,\alpha}$ can now be described using the following notation. Given an integer number α and a polynomial $(m + 1)$ -tuple $x(z)$ having s -degree α_x , we define the set of polynomial $(m + 1)$ -tuples $\{x(z)\}^\alpha$ as follows:

$$\begin{aligned} \{x(z)\}^\alpha &= \emptyset, & \text{if } \alpha < \alpha_x \\ \{x(z)\}^\alpha &= \{x(z), zx(z), \dots, z^{\alpha - \alpha_x} x(z)\}, & \text{if } \alpha \geq \alpha_x. \end{aligned}$$

We shall prove the following theorem:

Theorem 3.1 For each $p \geq 0$, there exist $(m+1)$ polynomial $(m+1)$ -tuples $v_p^j(z)$, $j = 1, 2, \dots, m+1$ (from level p) such that for each α , $|\alpha| \leq \infty$, a basis $BS_{p,\alpha}$ for $S_{p,\alpha}$ is given by

$$BS_{p,\alpha} = \cup_{j=1}^{m+1} V_{p,\alpha}^j \text{ with } V_{p,\alpha}^j = \{v_p^j(z)\}^\alpha.$$

The residual space $R_{i,p}^{m+1}$ is equal to $\mathbf{K}^m$ for $i > p$.

Proof. We shall prove the theorem by induction. It is clear that the theorem is true for $p = 0$ if we choose e.g., $v_0^j(z) = e_j$, $j = 1, 2, \dots, m+1$, with e_j the $(m+1)$ -tuple with all the components equal to zero except the j -th one which is equal to one. Note that $R_{i,0}^{m+1} = \mathbf{K}^m$ for $i > 0$. We assume that the theorem is true for a certain value for p . Without loss of generality we can assume that the polynomial $(m+1)$ -tuples $v_p^j(z)$ are ordered such that the s -degrees α_p^j of $v_p^j(z)$ satisfy $\alpha_p^j \leq \alpha_p^{j+1}$ for $j = 1, 2, \dots, m$.

The proof is constructive because an algorithm is given to compute a possible choice for the set of all the polynomial $(m+1)$ -tuples $v_{p+1}^j(z)$ for level $p+1$, given all the $v_p^j(z)$ of the previous level. First of all we shall describe how to compute a nonzero polynomial $(m+1)$ -tuple $a_{p+1}(z) \in S_{p+1,\alpha}$, not divisible by $z - z_{p+1}$, which always exists, having minimal s -degree α . This polynomial $(m+1)$ -tuple $a_{p+1}(z)$ together with m other polynomial $(m+1)$ -tuples (up to reordering), will give us $v_{p+1}^j(z)$, $j = 1, 2, \dots, m+1$. Because $a_{p+1}(z)$ has to be in $S_{p+1,\alpha} \subset S_{p,\alpha}$, we can write $a_{p+1}(z)$ as a linear polynomial combination of the polynomial $(m+1)$ -tuples $v_p^j(z)$. Because $S_{p,\alpha} = \{(0, 0, \dots, 0)\}$ with $\alpha < \alpha_p^1$, α has to be greater than or equal to α_p^1 . By induction, we know that $R_{p+1,p}^{m+1} = \mathbf{K}^m$. Consequently, m of the $(m+1)$ residuals $R_{p+1}v_p^j(z)$, $j = 1, 2, \dots, m+1$, are linearly independent. For the moment, we forget the minimality condition for $a_{p+1}(z)$. We show that $a_{p+1}(z)$ can not be a linear polynomial combination of only those $v_p^j(z)$ having linearly independent residuals. Suppose $a_{p+1}(z)$ is such a linear polynomial combination:

$$a_{p+1}(z) = \sum_{j \in J} p_j(z) v_p^j(z)$$

with $J \neq \emptyset$ and all the residuals $R_{p+1}v_p^j$, $j \in J$ linearly independent. Because $a_{p+1}(z)$ has to satisfy the $(p+1)$ -th interpolation condition, the residual $R_{p+1}a_{p+1}(z)$ has to be equal to the zero m -vector, i.e.,

$$R_{p+1}a_{p+1}(z) = \sum_{j \in J} p_j(z_{p+1}) R_{p+1}v_p^j(z)$$

has to be equal to zero. Because the residuals into play are linearly independent, this means that $p_j(z_{p+1}) = 0$ for $j \in J$. Hence, $z - z_{p+1}$ divides $a_{p+1}(z)$. Therefore, to compute a possible choice for $a_{p+1}(z)$, we have to consider a linear polynomial combination of $v_p^j(z)$ where the corresponding residuals are linearly dependent. To keep the s -degree α as small as possible, we compute the smallest j such that $\dim R_{p+1,p}^j < j$. Then we know that the residual corresponding to $v_p^j(z)$ can be written as

$$R_{p+1}v_p^j(z) = \sum_{k=1}^{j-1} c_k R_{p+1}v_p^k(z).$$

Therefore, $v_p^j - \sum_{k=1}^{j-1} c_k v_p^k$ is a possible choice for $a_{p+1}(z)$ if we can prove that $a_{p+1}(z)$ is then not divisible by $z - z_{p+1}$. If $a_{p+1}(z)$ would be divisible by $z - z_{p+1}$, then $a'(z) = a_{p+1}(z)/(z - z_{p+1}) \in S_{p,\alpha-1}$ or $a'(z)$ can be written as a unique linear polynomial combination based on the polynomial $(m+1)$ -tuples $v_p^k(z)$ having s -degree smaller than α . Hence, $v_p^j(z)$ is not involved. Because $a_{p+1}(z) = (z - z_{p+1})a'(z)$, $a_{p+1}(z)$ can be written as a unique linear

polynomial combination not involving $v_p^j(z)$. However, this is in contradiction to the fact that $v_p^j(z)$ is a component of $a_{p+1}(z)$.

We shall prove now that $a_{p+1}(z)$ together with the m polynomial $(m+1)$ -tuples $(z - z_{p+1})v_p^k(z), k \neq j$ are a possible choice for $v_{p+1}^j(z), j = 1, 2, \dots, m+1$ by proving that for each α a basis for $S_{p+1,\alpha}$ is given by $BS_{p+1,\alpha} = \cup_{k \neq j} \{(z - z_{p+1})v_p^k(z)\}^\alpha \cup \{a_{p+1}(z)\}^\alpha$. Take $a(z) \in S_{p+1,\alpha}$, then $a(z)$ can be written in a unique way as $a(z) = ca_{p+1}(z) + a'(z)$ with $a'(z)$ divisible by $z - z_{p+1}$. If $a(z)$ has s -degree smaller than the s -degree of $a_{p+1}(z)$, then $a(z)$ is divisible by $z - z_{p+1}$. Hence, in this case $c = 0$. Otherwise, $a_{p+1}(z)$ would not have minimal s -degree. When the s -degree of $a(z)$ is greater than or equal to the s -degree of $a_{p+1}(z)$, we can always choose c such that $a(z) - ca_{p+1}(z) = (n_a(z), d_a(z)) - c(n_{p+1}(z), d_{p+1}(z))$ is divisible by $z - z_{p+1}$, i.e. such that

$$d_a(z_{p+1}) - cd_{p+1}(z_{p+1}) = 0 \quad (3.3)$$

and

$$n_a(z_{p+1}) - cn_{p+1}(z_{p+1}) = 0. \quad (3.4)$$

Because $a_{p+1}(z)$ is not divisible by $z - z_{p+1}$, either the scalar $d_{p+1}(z_{p+1}) \neq 0$ or the m -vector $n_{p+1}(z_{p+1}) \neq 0$. Suppose $d_{p+1}(z_{p+1}) \neq 0$, the other case is similar. We can determine c as $c = d_a(z_{p+1})/d_{p+1}(z_{p+1})$. Because $R_{p+1}a(z) = 0$ and $R_{p+1}a_{p+1}(z) = 0$, we obtain:

$$f_{p+1}(d_a(z_{p+1}) - cd_{p+1}(z_{p+1})) = n_a(z_{p+1}) - cn_{p+1}(z_{p+1}). \quad (3.5)$$

Assuming (3.3) is true, (3.4) follows easily from (3.5). From $a(z) = ca_{p+1}(z) + a'(z)$ with $a'(z)$ divisible by $z - z_{p+1}$, it follows that $a(z)$ can be written in a unique way as $a(z) = ca_{p+1}(z) + (z - z_{p+1})a''(z)$ with $a''(z) \in S_{p,\alpha-1}$. Using the basis for $S_{p,\alpha-1}$ based on the $(m+1)$ polynomial $(m+1)$ -tuples $v_p^k(z), k = 1, 2, \dots, m+1$, it is easy to see that $\cup_{k \neq j} \{(z - z_{p+1})v_p^k(z)\}^\alpha \cup \{a_{p+1}(z)\}^\alpha$ forms a possible basis for $S_{p+1,\alpha}$.

It remains to be shown that $R_{i,p+1}^{m+1} = \mathbf{K}^m, i > p+1$. By definition, we have that

$$\begin{aligned} R_{i,p+1}^{m+1} &= \text{span}_{k \neq j} \{R_i(z - z_{p+1})v_p^k(z), R_i(v_p^j - \sum_{l=1}^{j-1} c_l v_p^l)\} \\ &= \text{span}_{k \neq j} \{(z_i - z_{p+1})R_i v_p^k(z), R_i v_p^j - \sum_{l=1}^{j-1} c_l R_i v_p^l\}. \end{aligned}$$

Because $z_i \neq z_{p+1}$ and $R_{i,p}^{m+1} = \mathbf{K}^m$ for $i > p+1$, it is clear that $R_{i,p+1}^{m+1} = \mathbf{K}^m, i > p+1$ and this proves the theorem completely. $\square$

The proof of the theorem contains an algorithm to compute the $(m+1)$ polynomial $(m+1)$ -tuples $v_p^j(z), j = 1, 2, \dots, m+1$ at each level p knowing the $v_{p-1}^j(z)$ of the previous level. The first one $v_p^1(z)$, will have a minimal s -degree and it will be a solution for the **LVRIP**. To be able to give a compact formulation of the algorithm, we introduce the following notations:

- We group the $(m+1)$ polynomial $(m+1)$ -tuples $v_p^j(z)$ together in an $(m+1) \times (m+1)$ polynomial matrix as follows

$$\begin{aligned} G_p(z) &= \begin{bmatrix} v_p^1(z) & v_p^2(z) & \dots & v_p^{m+1}(z) \end{bmatrix} \\ &= \begin{bmatrix} n_p^1(z) & n_p^2(z) & \dots & n_p^{m+1}(z) \\ d_p^1(z) & d_p^2(z) & \dots & d_p^{m+1}(z) \end{bmatrix}. \end{aligned}$$

- The elementary polynomial operations, which we shall apply on $(m + 1) \times (m + 1)$ polynomial matrices, are denoted as follows:
 - $E_{i,j}$ denotes the polynomial matrix describing a switching of the columns i and j ,
 - $E_i(p(z))$ denotes the polynomial matrix describing the multiplication of column i by a polynomial $p(z) \in \mathbf{K}[z]$,
 - $E_{i,j}(c)$ denotes the polynomial matrix describing the addition to column i of column j multiplied by the scalar $c \in \mathbf{K}$.
- U denotes the $(m + 1) \times (m + 1)$ identity matrix.

We can summarize the algorithm as in algorithm VRIP below.

Algorithm 1: VRIP{minimal vector rational interpolation}

{ Initialization }
 G_0 is initialized as $G_0(z) = UP_0 = V_0(z)$,
 where P_0 is a sequence of elementary polynomial operations
 of the form $E_{i,j}$, i.e. describing a permutation matrix,
 taking care of the ordering of the v_0^j with respect
 to increasing s -degree.
 for $p = 0, 1, 2, \dots$
 Look for the smallest j such that $R_{p+1}v_p^j(z)$
 can be written as a linear combination of the previous residuals:
 $R_{p+1}v_p^j(z) = \sum_{k=1}^{j-1} c_k R_{p+1}v_p^k(z)$.
 If we set $V'_{p+1}(z) = \prod_{k=1}^{j-1} E_{j,k}(c_k) \prod_{k \neq j} E_k(z - z_{p+1})$,
 then we get $G_{p+1}(z)$ by permutation of $G_p(z)V'_{p+1}(z)$
 $G_{p+1}(z) = G_p(z)V_{p+1}(z)$ with
 $V_{p+1}(z) = V'_{p+1}(z)P_{p+1}$
 where P_{p+1} is a permutation matrix reordering the polynomial
 vectors with respect to their s -degree.
 endfor

It is clear that we can write:

$$G_p(z) = V_0(z)V_1(z) \cdots V_p(z) \text{ with}$$

$$\begin{aligned} V_0(z) &= \begin{bmatrix} x_0(z) & t_0(z) \\ y_0(z) & u_0(z) \end{bmatrix} \\ &= P_0 \end{aligned}$$

and for $i \geq 1$

$$\begin{aligned} V_i(z) &= \begin{bmatrix} x_i(z) & t_i(z) \\ y_i(z) & u_i(z) \end{bmatrix} \\ &= \prod_{k=1}^{j_i-1} E_{j_i,k}(c_k^i) \prod_{k \neq j_i} E_k(z - z_i), \end{aligned}$$

where j_i and c_k^i are the j and c_k values needed in the algorithm to go from level $i - 1$ to level i . The partitioning of the $V_i(z)$ matrices into $x_i(z), y_i(z), t_i(z)$ and $u_i(z)$ is such that $y_i(z)$ is a scalar polynomial.

It follows that $n_p^1(z)/d_p^1(z)$, the solution of the **LVRIP**, is the p -th convergent of the generalized continued fraction:

$$\frac{x_0(z) + t_0(z) \frac{x_1(z) + t_1(z) \frac{\dots}{y_1(z) + u_1(z) \frac{\dots}{\dots}}}{y_0(z) + u_0(z) \frac{x_1(z) + t_1(z) \frac{\dots}{y_1(z) + u_1(z) \frac{\dots}{\dots}}}{y_1(z) + u_1(z) \frac{\dots}{\dots}}}{\dots} \quad (3.6)$$

with $A/B = AB^{-1}$. Note that we can also write a similar continued fraction whose p -th convergent is $n_p^j(z)/d_p^j(z)$, $j > 1$. Remember that the polynomial $(m + 1)$ -tuple $a_p(z)$ which featured in the proof of the theorem is one of the $v_p^j(z)$. More precisely, it is the solution of the **LVRIP** with the additional condition that this polynomial tuple is not divisible by $z - z_p$. The above continued fraction reduces to an ordinary continued fraction, at least formally, if all the $t_i(z)$ vanish. It still would have vector valued convergents. Since the multiplications with the $V_i(z)$ matrices represent more general transformations, we get the more general form of the fractions. It is however possible to get ordinary continued fractions as well. See e.g., [5, Theorem 2.9].

The algorithm, described above, can be implemented in several ways. The linearized rational interpolation conditions (2.1) can be written in terms of the classical basis $1, z, z^2, z^3, \dots$, involving classical block-Vandermonde matrices. In this case, the multiplication of a polynomial by $z - z_{p+1}$, needed in the algorithm, is straightforward. Another possibility is to write the interpolation conditions (2.1) in terms of a basis of orthogonal polynomials $\phi_0(z), \phi_1(z), \phi_2(z), \dots$, involving generalized block-Vandermonde matrices. The multiplication by $z - z_{p+1}$ of a polynomial written in terms of this basis of orthogonal polynomials can be carried out using the recurrence relation for orthogonal polynomials: $\phi_k(z) = \lambda_k(z - \alpha_k)\phi_{k-1}(z) - \beta_k\phi_{k-2}(z)$. More generally, we can use any basis as long as the shift operation, i.e., the multiplication by z of a polynomial written in terms of this basis, can be carried out in a simple and efficient way.

If we consider $G_p(z) = [v_p^1(z) \ v_p^2(z) \ \dots \ v_p^{m+1}(z)] = V_0(z)V_1(z)\dots V_p(z)$ as an abstract layered medium, with layers described by $V_i(z)$, each step of the algorithm, given above, computes the residuals $R_{p+1}v_p^j(z)$, needed to compute the next layer $V_{p+1}(z)$, and connects this new layer to the layered medium. So we can call this a layer adjoining algorithm. However, it is not necessary to obtain the layered medium $G_p(z)$ explicitly. If we know the separate layers $V_i(z)$, we can represent the layered medium by the continued fraction (3.6). To compute the layer $V_p(z)$, we need to know the residuals $R_p v_{p-1}^j(z)$. Suppose we know all the residuals $R_i v_{p-1}^j(z)$ of level $p - 1$, with $i \geq p$. It is easy to design an algorithm computing the next layer $V_p(z)$ and transforming all the residuals of the previous step into the residuals of level p , using the formula

$$[R_i v_p^1(z) \ R_i v_p^2(z) \ \dots \ R_i v_p^{m+1}(z)] = [R_i v_{p-1}^1(z) \ R_i v_{p-1}^2(z) \ \dots \ R_i v_{p-1}^{m+1}(z)] V_p(z).$$

We call this algorithm a layer peeling one, because at each step p we peel off the layer $V_p(z)$. Compared to the layer adjoining algorithm, the layer peeling one has the drawback that we have to know all the residuals $R_i v_0^j(z)$, with $i \geq 1$ at the beginning of the algorithm. As

an advantage, we mention that the layer peeling algorithm is easier to parallelize than the other one. From the previous explanation, it should be clear that also a mixture of a layer adjoining and layer peeling algorithm is possible. For more information, we refer to [4].

4 The solution of the PVRIP

The algorithm **VRIP** solved the **LVRIP**, but from the $m + 1$ polynomial $(m + 1)$ -tuples v_p^j , $j = 1, \dots, m + 1$, it constructs, we can now easily find a solution of the proper interpolation problem **PVRIP** as described in the following theorem.

Theorem 4.1 *Let $v_p^j(z)$, $j = 1, 2, \dots, m + 1$ be defined by theorem 3.1. If $v_p^1(z)$ is not a solution of the **PVRIP**, then $\sum_{k=1}^l a_k v_p^k(z)$ is, where l is the smallest value such that the first l columns of $G_p(z)$ evaluated for $z = z_i$, $i = 1, 2, \dots, p$, have for each value i rank 1. The l -tuple $(a_1, a_2, \dots, a_l)$ is not allowed to be an element of a specific $(l - 1)$ dimensional subspace of $\mathbf{K}^l$ for each interpolation point z_i . The latter condition should prevent that the proposed solution would be divisible by $z - z_i$.*

Proof. When $v_p^1(z)$ is divisible by $z - z_i$ for a certain i , $1 \leq i \leq p$, then $v_p^1(z)$ is a solution of the **LVRIP** but not of the **PVRIP**. It is clear that $V_i(z)$ has full rank $m + 1$ if $z \neq z_i$ and has rank 1 if $z = z_i$. Hence, $G_p(z) = V_0(z)V_1(z) \cdots V_p(z)$ has full rank $m + 1$ if $z \neq z_i$ and has rank 1 if $z = z_i$. Therefore, to get a solution of smallest s -degree, we can take the linear polynomial combination $\sum_{k=1}^l a_k v_p^k(z)$ as a candidate where l is chosen as indicated above. For a smaller value of l , there is at least one z_i for which the first l columns of $G_p(z_i)$ would form a matrix of rank 0. Hence, we could not find a linear polynomial combination of the $v_p^k(z)$, $k = 1, 2, \dots, l$, not divisible by $z - z_i$. However, with the value of l as chosen above, $\sum_{k=1}^l a_k v_p^k(z)$ is not divisible by $z - z_i$, $i = 1, 2, \dots$ as long as the l -tuple $(a_1, a_2, \dots, a_l)$ doesn't belong to the solution space of $G_p^l(z_i)[a_1 \dots a_l]^T = 0$ with $G_p^l(z)$ the first l columns of $G_p(z)$. We have chosen l such that $G_p^l(z_i)$ has rank 1. Hence, the solution space has dimension $(l - 1)$ and is the specific subspace of $\mathbf{K}^l$, we are referring to above. $\square$

5 A parameterization of all solutions of the LVRIP and PVRIP

To describe all possible solutions of the linearized and proper interpolation problems, we can formulate the following theorems.

Theorem 5.1 *If the s -degree α_p^1 of $v_p^1(z)$ is smaller than the s -degree α_p^2 of $v_p^2(z)$, then there is only one rational function $n_p^1(z)/d_p^1(z)$ which solves the **LVRIP**. Otherwise a parameterization of all solutions is given by:*

$$\begin{bmatrix} n_p^a(z) \\ d_p^a(z) \end{bmatrix} = \begin{bmatrix} n_p^1 & n_p^2 & \cdots & n_p^l \\ d_p^1 & d_p^2 & \cdots & d_p^l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_l \end{bmatrix}$$

with $a = (a_1, a_2, \dots, a_l) \neq (0, 0, \dots, 0) \in \mathbf{K}^l$ and l is such that $\alpha_p^1 = \alpha_p^2 = \cdots = \alpha_p^l < \alpha_p^{l+1}$ ($\alpha_p^{m+2} = +\infty$). If $a \neq ka'$ with $k \in K$, then $n_p^a(z)/d_p^a(z)$ and $n_p^{a'}(z)/d_p^{a'}(z)$ are different vector rational functions.

Proof. That the rational approximant constructed is a solution of the **LVRIP** is trivial. We shall only prove the assertion of the last sentence. For the parameter l -tuple $a \neq ka'$, we set

$$\begin{bmatrix} n_p^a(z) & n_p^{a'}(z) \\ d_p^a(z) & d_p^{a'}(z) \end{bmatrix} = \begin{bmatrix} n_p^1(z) & n_p^2(z) & \cdots & n_p^l(z) \\ d_p^1(z) & d_p^2(z) & \cdots & d_p^l(z) \end{bmatrix} \begin{bmatrix} a_1 & a_1' \\ a_2 & a_2' \\ \vdots & \vdots \\ a_l & a_l' \end{bmatrix}.$$

Because the right-hand side of the previous equation has rank 2 for $z \neq z_i, i = 1, 2, \dots, p$, we derive that the left-hand side has rank 2 for an infinite number of values for z . Hence, the two vector rational functions $n_p^a(z)/d_p^a(z)$ and $n_p^{a'}(z)/d_p^{a'}(z)$ are different. $\square$

Concerning a parameterization of all solutions of the **PVRIP**, we have the following.

Theorem 5.2 *If $v_p^1(z)$ which solves the problem **LVRIP** is also a solution of the problem **PVRIP**, the two cases of the previous theorem also apply here, with the only exception that $(a_1, a_2, \dots, a_l)$ is not allowed to be an element of a specific $(l-1)$ -dimensional subspace of $\mathbf{K}^l$ for each interpolation point $z_i, i = 1, 2, \dots, p$. If $v_p^1(z)$ is not a solution of the **PVRIP**, a parameterization of all solutions of the problem is given by:*

$$\begin{bmatrix} n_p^a(z) \\ d_p^a(z) \end{bmatrix} = \begin{bmatrix} n_p^1 & n_p^2 & \cdots & n_p^{l'} \\ d_p^1 & d_p^2 & \cdots & d_p^{l'} \end{bmatrix} \begin{bmatrix} a_1(z) \\ a_2(z) \\ \vdots \\ a_{l'}(z) \end{bmatrix}$$

with l' related to the l of theorem 4.1 by $\alpha_p^l = \alpha_p^{l+1} = \cdots = \alpha_p^{l'} < \alpha_p^{l'+1}$ and $\deg a_j(z) \leq \alpha_p^l - \alpha_p^j, j = 1, 2, \dots, l'$. Moreover $a = (a_1(z_i), a_2(z_i), \dots, a_{l'}(z_i))$ is not allowed to be an element of a specific $(l'-1)$ -dimensional subspace of $\mathbf{K}^{l'}$ for each interpolation point $z_i, i = 1, 2, \dots, p$. As before we can show that, except for a scaling factor, for different polynomial l' -tuples $(a_1(z), a_2(z), \dots, a_{l'}(z))$, we get different vector rational functions.

The proof goes along the same lines as the previous one, making additional use of the result of the previous section. We leave the details to the reader.

6 The confluent case

When for certain interpolation points z_i , not only the function value f_i is given but also the values of one or several consecutive derivatives, we get the confluent **LVRIP** and **PVRIP**. If we consider the interpolation data for the “confluent” points in the natural ordering, i.e., function value, first derivative value, second derivative value, ..., then we can prove in a similar way that the same algorithm can be used to compute the polynomial $(m+1)$ -tuples $v_p^j(z)$ of the confluent problem. The parameterization of all solutions of the **LVRIP** and **PVRIP** is also valid in this case. The completely confluent problem was discussed as a minimal vector Padé problem in [6], but for the general confluent problem, as well as the case where we consider interpolation data around the interpolation point ∞ and the case where the interpolation point has to be a pole of the rational function requires a much more careful setting of the problem. This shall be worked out in a future publication on a more formal basis.

7 An example

We take the following interpolation data: $z_1 = 0, f_1 = [1 \ 0]^T, z_2 = 1, f_2 = [2 \ 1]^T, z_3 = 2, f_3 = [2 \ 1]^T$. We use the Chebyshev polynomials $T_k(z)$ as a basis for the set of polynomials: $T_0(z) = 1, T_1(z) = z$ and $T_{k+1}(z) = 2zT_k(z) - T_{k-1}(z)$ for $k \geq 1$. Hence, the multiplication of $T_k(z)$ by $z - z_{p+1}$ can be written as $(z - z_{p+1})T_0(z) = T_1(z) - z_{p+1}T_0(z)$ and $(z - z_{p+1})T_k(z) = T_{k+1}(z)/2 - z_{p+1}T_k(z) + T_{k-1}(z)/2$ for $k \geq 1$. The algorithm of section 3 for $s = (s_1, s_2) = (0, 1)$ generates the following $V_i(z)$ -matrices:

$$V_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V_1 = \begin{bmatrix} z & 0 & 0 \\ 0 & 1 & z \\ 0 & 1 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} z-1 & 0 & 1 \\ 0 & z-1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, V_3 = \begin{bmatrix} 0 & z-2 & 0 \\ 1 & 0 & z-2 \\ 1 & 0 & 0 \end{bmatrix}.$$

The vector rational function solving the **LVRIP** is unique and equal to

$$n_3^1(z)/d_3^1(z) = \begin{bmatrix} 2T_1(z) \\ T_1(z) \\ T_1(z) \end{bmatrix}.$$

This solution has minimal s -degree 1. However, it is not a solution of the **PVRIP**, because $T_1(z) = z$ is a common factor of the numerator and denominator polynomials. Hence, 0 is an "unattainable point". Theorem 5.2 gives all solutions of the **PVRIP** having a minimal s -degree 2, with $\deg a_1(z) \leq 1, a_1(1) \neq 0, a_1(2) \neq 0$ and $a_3 \neq 0$ as

$$\begin{bmatrix} v_3^1(z) & v_3^2(z) & v_3^3(z) \end{bmatrix} \begin{bmatrix} a_1(z) \\ a_2 \\ a_3 \end{bmatrix}$$

with

$$\begin{aligned} G_3(z) &= \begin{bmatrix} v_3^1(z) & v_3^2(z) & v_3^3(z) \end{bmatrix} \\ &= \begin{bmatrix} 2T_1(z) & 0 & \frac{1}{2}T_2(z) - 3T_1(z) + \frac{5}{2}T_0(z) \\ T_1(z) & \frac{1}{4}T_3(z) - \frac{3}{2}T_2(z) + \frac{11}{4}T_1(z) - \frac{3}{2}T_0(z) & 0 \\ T_1(z) & 0 & \frac{1}{2}T_2(z) - 3T_1(z) + \frac{5}{2}T_0(z) \end{bmatrix}. \end{aligned}$$

8 Conclusion

In this short paper we have described how the ideas of rational interpolation without re-ordering the interpolation points can be generalized to the case where vector data are given. The approximant is minimal with respect to the s -degree complexity measure. An efficient algorithm, allowing several possible implementations is given, yet it describes precisely what operations have to be performed. The algorithm finds a solution which solves a linearized interpolation problem. However, it is shown how the results of the algorithm can be used to generate not only all the possible solutions of this problem but also how all the solutions for the proper rational interpolation problem can be constructed.

The treatment is kept informal to make the ideas clear without being troubled by exceptional cases and notational burden. For this reason we supposed in the main development that all the interpolation points are different. However, as we mentioned in section 6, the results for the confluent case are basically the same. In the case where all the interpolation points coincide, we recover the minimal vector Padé approximants. For the same reason of simplicity, we did not allow the interpolation points to be at infinity or prescribed poles at the interpolation points. All these generalizations and more elaborate proofs will be given in a later publication.

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