CONSTRUCTING EMBEDDED LATTICE RULES FOR MULTIVARIATE INTEGRATION

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Abstract. Lattice rules are a family of equal-weight cubature formulas for approximating high-dimensional integrals. By now it is well established that good generating vectors for lattice rules having n points can be constructed component-by-component for integrands belonging to certain weighted function spaces, and that they can achieve the optimal rate of convergence. Although the lattice rules constructed this way are extensible in dimension, they are not extensible in n , thus when n is changed the generating vector needs to be constructed anew. In this paper we introduce a new algorithm for constructing good generating vectors for embedded lattice rules which can be used for a range of n while still being extensible in dimension. By using an adaptation of the fast component-by-component construction algorithm (which makes use of fast Fourier transforms), we are able to obtain good generating vectors for thousands of dimensions and millions of points, under both product weight and order-dependent weight settings, at the cost of $\mathcal{O}(dn(\log(n))^2)$ operations. With a sufficiently large number of points and good overall quality, these embedded lattice rules can be used for practical purposes in the same way as a low-discrepancy sequence. We show for a range of weight settings in an unanchored Sobolev space that our embedded lattice rules achieve the same (optimal) rate of convergence $\mathcal{O}(n^{-1+\delta})$, $\delta > 0$, as those constructed for a fixed number of points, and that the implied constant only gains a factor of 1.30 to 1.55.

Key words. Numerical integration, Quasi-Monte Carlo methods, Rank-1 lattice rules, Fast component-by-component construction, Embedded lattice rules, Extensible lattice sequences

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1. Introduction. Integrals with hundreds or even thousands of dimensions occur in many practical applications in areas such as statistics and mathematical finance. After some appropriate transformations, these integrals can be formulated over the d -dimensional unit cube $[0, 1]^d$:

$$I_d(f) = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_d) dx_1 \cdots dx_d = \int_{[0,1]^d} f(\mathbf{x}) d\mathbf{x},$$

where f is a function of d variables and $\mathbf{x} = (x_1, \dots, x_d)$. In this paper we develop an algorithm for constructing “embedded lattice rules” for evaluating such integrals, in the mathematical setting of “weighted reproducing kernel Hilbert spaces”. The details about our methods and our function space settings will be discussed in the next section. Here in the Introduction we shall give a general overview of the historical background relevant to the new construction.

When d is large, it is too costly to approximate these integrals using product rules based on low-dimensional integration formulae. The *Monte Carlo method* is an equal-weight cubature rule of the form

$$Q_{n,d}(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(\mathbf{t}_k),$$

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where the n cubature points $\mathbf{t}_0, \dots, \mathbf{t}_{n-1}$ are independent and identically distributed uniform random samples from $[0, 1]^d$. The expected error for the Monte Carlo method is $\sigma(f)/\sqrt{n}$, where $\sigma^2(f)$ is the variance of f . This leads to a rate of convergence of $\mathcal{O}(n^{-1/2})$ and the convergence is independent of the dimension d provided that $f \in L_2([0, 1]^d)$. *Quasi-Monte Carlo (QMC) methods* have the same form as the Monte Carlo method, but the cubature points $\mathbf{t}_0, \dots, \mathbf{t}_{n-1}$, instead of being generated randomly, are chosen deterministically, in the hope of obtaining a better rate of convergence than the Monte Carlo order $\mathcal{O}(n^{-1/2})$. QMC error bounds are typically in the form of a Koksma-Hlawka-type inequality

$$|I_d(f) - Q_{n,d}(f)| \leq D(\mathbf{t}_0, \dots, \mathbf{t}_{n-1}) V(f), \quad (1.1)$$

where $D(\mathbf{t}_0, \dots, \mathbf{t}_{n-1})$ is the *discrepancy* which measures the non-uniformity of the point set, and $V(f)$ measures the variability of f . Error bounds of this form separate the dependence on the cubature points from the dependence on the integrand. In general we do not have control over the integrand and so we focus on choosing cubature points so that the discrepancy is as small as possible. QMC point sets with discrepancy of order $\mathcal{O}(n^{-1}(\log n)^d)$ or better are collectively known as *low-discrepancy point sets* (see e.g. [12]). This includes, for example, the well-known Sobol' sequences and Niederreiter sequences. Note that the word “sequences” used in this context implies a special feature: the construction of the point set is independent of the number of points n and the dimension d . Thus these low-discrepancy sequences are “extensible” in the sense that if we need more cubature points or if we need to increase the dimension, we only need to compute the additional points or the additional components of the points.

Lattice rules are a family of QMC rules which were traditionally used to approximate integrals of periodic functions. The oldest and simplest form of lattice rules, now known as *rank-1 lattice rules*, have the point set

$$\left\{ \left\{ \frac{k\mathbf{z}}{n} \right\} : k = 0, 1, \dots, n-1 \right\}.$$

Here $\mathbf{z} = (z_1, \dots, z_d)$, known as the *generating vector*, is a d -dimensional integer vector having no factor in common with n , and the braces around a vector indicate that each component of the vector is to be replaced by its fractional part. Lattice point sets are low-discrepancy point sets although their constructions are typically not extensible. For a long time the most popular way to construct lattice rules in higher dimensions was to take the generating vector of the *Korobov form*:

$$\mathbf{z} = (1, a, a^2, \dots, a^{d-1}) \bmod n, \quad 1 \leq a \leq n-1, \gcd(a, n) = 1. \quad (1.2)$$

For a given value of n and a given value of d , an optimal value of a can be chosen by minimizing some discrepancy measure. However, this value of a is only good for the specific pair of n and d , and so if either n or d is changed, we need to obtain a new value of a . (One could use a chosen value of a for higher dimensions simply by taking higher powers of a modulo n , in which case the projections of the points over consecutive dimensions will remain stationary. Unfortunately, this does not guarantee that the discrepancy measure which we are interested in will remain small.)

In 2002 Sloan and his collaborators (see e.g. [16]) developed the “component-by-component (CBC)” construction: the components of $\mathbf{z}$ are constructed one at a time to minimize the *worst-case error* in some weighted function space. The details about

worst-case analysis and weighted function spaces will be given in the next section. For now we can think of the worst-case error as some form of discrepancy which measures the quality of our lattice rule. Let $e_{n,d}(\mathbf{z}) = e_{n,d}(z_1, \dots, z_d)$ denote the worst-case error for a rank-1 lattice rule with generating vector $\mathbf{z}$. The CBC algorithm goes like this:

ALGORITHM 1 (CBC Algorithm). *Given n ,*

1. *Set $z_1 = 1$.*
2. *For each $d = 2, 3, \dots$,*
with $z_1, \dots, z_{d-1}$ fixed,
choose z_d from the set $\{1 \leq z \leq n-1 : \gcd(z, n) = 1\}$
to minimize the worst-case error $e_{n,d}(z_1, \dots, z_{d-1}, z_d)$.

This construction is extensible in d in the sense that if further dimensions are needed, the algorithm can be restarted in an obvious way. It is, however, not extensible in n and so a different value of n means that a completely new $\mathbf{z}$ must be constructed.

It has been proved by Hickernell and Niederreiter in 2003 (see [8]) that good generating vectors for extensible lattice sequences do exist, that is, there exists some vector $\mathbf{z}$ which is good for all n and all d . However their proof is non-constructive and it leaves no clues as to how to find such a vector $\mathbf{z}$. Since n can be arbitrarily large, the components of the generating vector $\mathbf{z}$ can also be arbitrarily large, making it rather difficult to find one generating vector that is good simultaneously for all values of n . Back in 2000, Hickernell and his collaborators (see [7]) had already experimented with the construction of extensible lattice sequences in an empirical setting. By assuming that

$$n = b^m, \quad b \geq 2, \quad m_1 \leq m \leq m_2, \quad \text{and} \quad 1 \leq d \leq d_{\max},$$

they designed an algorithm which finds a generating vector of the Korobov form (1.2), with the parameter a chosen by minimizing a loss function

$$G_{m_1, m_2, d_{\max}}(a) := \max_{\substack{m_1 \leq m \leq m_2 \\ 1 \leq d \leq d_{\max}}} \frac{D_{b^m, d}(a)}{D_{b^m, d}^{(\text{asy})}}.$$

Here $D_{b^m, d}(a)$ is some discrepancy measure for a d -dimensional b^m -point lattice rule with Korobov parameter a , and $D_{b^m, d}^{(\text{asy})}$ is some asymptotically derived normalization chosen to ensure that for various values of m and d the discrepancy $D_{b^m, d}(a)$ is scaled appropriately for comparison.

Lattice rules constructed in [7] are not so much “extensible” as “embedded” in their nature: the distinction being that extensible rules would leave n unbounded whilst embedded rules have n bounded. Therefore, with each d -dimensional generating vector $\mathbf{z}$, we have a sequence of point sets

$$\{L_m\}_{m_1 \leq m \leq m_2},$$

where each L_m is the point set for a lattice rule with b^m points, and all the points in L_m are also points of L_{m+1} . For example, when $b = 2$, the number of points doubles as we move along the sequence, and each rule in the sequence contains all the points from the previous rule together with an equal number of new points. This embedding property is very attractive in practice because not only can we easily increase the number of points without wasting any function evaluations, but also we can estimate the error based on the results from successive rules in the sequence.

Finally we come to the point where we can discuss our new construction. Our algorithm is similar to that of [7], but rather than making use of the Korobov form, our algorithm is based on CBC constructions and so there is no need to assume that d is bounded. For each m from m_1 to m_2 , we use Algorithm 1 to find a generating vector $\mathbf{z}^{(m)}$ for a b^m -point rule and we store the resulting worst-case error $e_{b^m,d}(\mathbf{z}^{(m)})$ for each $d = 1, 2, \dots$. We define a new error measure for our embedded lattice rule by

$$X_{m_1, m_2, d}(\mathbf{z}) := \max_{m_1 \leq m \leq m_2} \frac{e_{b^m, d}(\mathbf{z})}{e_{b^m, d}(\mathbf{z}^{(m)})}, \quad (1.3)$$

where we use $e_{b^m, d}(\mathbf{z}^{(m)})$ as our reference error to provide a good normalization. Our new algorithm goes like this:

ALGORITHM 2 (CBC Algorithm for Embedded Lattice Rules). *Given b , m_1 , m_2 , and the pre-computed values of $e_{b^m, d}(\mathbf{z}^{(m)})$ for each $m_1 \leq m \leq m_2$ and each $d = 1, 2, \dots$,*

1. Set $z_1 = 1$.
2. For each $d = 2, 3, \dots$,
 with $z_1, \dots, z_{d-1}$ fixed,
 choose z_d from the set $\{1 \leq z \leq b^{m_2} - 1 : \gcd(z, b) = 1\}$
 to minimize the error measure $X_{m_1, m_2, d}(z_1, \dots, z_{d-1}, z_d)$.

Just as is the case for the algorithm in [7], our new algorithm does not come with an *a priori* error bound, nor does it give us a fully extensible lattice sequence that can be used for arbitrarily large n . Having said that, in addition to the bonus that d need not be bounded, our algorithm does have two advantages over the algorithm of [7]. Firstly, it is proved in [10] and in [4] that the CBC algorithm can achieve a rate of convergence of $\mathcal{O}(n^{-1+\delta})$, $\delta > 0$, if we “randomly shift” the lattice points (see below). Thus the worst-case error $e_{b^m, d}(\mathbf{z}^{(m)})$ can be said to be almost optimal, which means that we are using a fairly reliable normalization. The quantity $X_{m_1, m_2, d}(\mathbf{z})$ represents how much worse our chosen vector $\mathbf{z}$ is compared with the vectors $\mathbf{z}^{(m)}$ which are the near-optimal choices for different $n = b^m$. With all the benefits of having embedded rules, we would be happy with a value of $X_{m_1, m_2, d}(\mathbf{z})$ that is no more than 2, and this indeed turns out to be the case in all our computations. Thus in practice our construction comes with an *a posteriori* worst-case error bound that is of optimal order, and within a factor of 2 of that achieved by the original CBC algorithm.

The second advantage of our algorithm is that, surprisingly, it has a relatively low computational cost. Nuyens and Cools [13, 14] developed a fast implementation of the CBC algorithm by making use of fast Fourier transforms, which allows the construction with n in the millions and d in the thousands within a feasible time frame. Thus the pre-computation of $\mathbf{z}^{(m)}$ for each m can be done in a fast way. In fact if we take b to be prime, our new algorithm based on minimizing $X_{m_1, m_2, d}(\mathbf{z})$ can also be done in a fast way, and the cost depends only on m_2 rather than the sum of all m . The total computational cost of Algorithm 2, including the pre-computations of Algorithm 1, is $\mathcal{O}(dn(\log(n))^2)$ operations.

The description above was a bit simplified: it is actually “randomly-shifted” rank-1 lattice rules that we construct generating vectors for. *Shifted rank-1 lattice rules* are QMC rules with the point set

$$\left\{ \left\{ \frac{k\mathbf{z}}{n} + \mathbf{\Delta} \right\} : k = 0, 1, \dots, n-1 \right\},$$

where $\mathbf{z}$ is the generating vector as in a rank-1 lattice rule, and $\mathbf{\Delta} \in [0, 1]^d$ is the *shift*. In particular, we are interested in using *random* shifts. There are several advantages

in doing this: we only need to determine the generating vector; the error expression is simpler (see §2); and most importantly, we can obtain a probabilistic error estimate (see §5).

The paper is organized as follows. In §2 we discuss weighted reproducing kernel Hilbert spaces. We consider weighted Korobov spaces and weighted Sobolev spaces, as it is already known that the CBC algorithm leads to lattice rules which achieve the optimal rate of convergence in these spaces, under suitable conditions on the weights. In §3 we outline some important properties of embedded lattice rules. In §4 we discuss the implementation of our new algorithm under two different weight settings: the product weights and the order-dependent weights. We consider lattice rules with $n = b^m$ points, and for simplicity we consider only prime base b . In §5 we provide some notes on how to use these embedded lattice rules as a sequence and how to obtain practical error estimation by using random shifts. In §6 we carry out some numerical experiments to see how well our new lattice rules perform for some finance problems. Finally a brief summary is given in §7. In the Appendix we list one generating vector up to 360 dimensions (obtained with $b = 2$, $m_1 = 10$ and $m_2 = 20$) which is good for n ranging from 1024 to 1048576.

2. Function space setting. In this section we briefly review the mechanics behind worst-case analysis in a reproducing kernel Hilbert space and the concept of weighted function spaces. For more details, see e.g. [6, 17, 5].

2.1. Worst-case analysis in reproducing kernel Hilbert spaces. Suppose that our integrands belong to some reproducing kernel Hilbert space H . A reproducing kernel $K(\mathbf{x}, \mathbf{y})$ is a symmetric and positive definite function satisfying $K(\mathbf{x}, \cdot) \in H$ for all $\mathbf{x} \in [0, 1]^d$, and $\langle K(\mathbf{x}, \cdot), f \rangle_H = f(\mathbf{x})$ for all $f \in H$ and all $\mathbf{x} \in [0, 1]^d$. The existence of a reproducing kernel follows from the Riesz representation theorem, under the assumption that point evaluations are bounded linear functionals in H . We assume that the function $h(\mathbf{x}) := \int_{[0, 1]^d} K(\mathbf{x}, \mathbf{y}) d\mathbf{y}$ belongs to the space H . Then multivariate integration I_d is a well-defined continuous linear functional, with h as its representer in H , and the *initial error* for multivariate integration is

$$\|I_d\| = \|h\|_H = \left(\int_{[0, 1]^{2d}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} \right)^{1/2}.$$

To ensure that H is embedded in $L_2([0, 1]^d)$, we assume that $\int_{[0, 1]^d} K(\mathbf{x}, \mathbf{x}) d\mathbf{x} < \infty$ which also guarantees that the initial error is finite.

The *worst-case error* for a QMC rule $Q_{n,d}$ with point set $P_n = \{\mathbf{t}_0, \dots, \mathbf{t}_{n-1}\}$ in H is defined by

$$e_{n,d}(P_n, K) := \sup_{f \in H, \|f\|_H \leq 1} |I_d(f) - Q_{n,d}(f)|.$$

Clearly $|I_d(f) - Q_{n,d}(f)| \leq e_{n,d}(P_n, K) \|f\|_H$. Thus the worst-case error can be thought of as a type of discrepancy measure (see (1.1)). Using the reproducing property of K one can derive an expression for the squared worst-case error

$$\begin{aligned} e_{n,d}^2(P_n, K) &= \int_{[0, 1]^{2d}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} - \frac{2}{n} \sum_{k=0}^{n-1} \int_{[0, 1]^d} K(\mathbf{t}_k, \mathbf{x}) d\mathbf{x} + \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} K(\mathbf{t}_k, \mathbf{t}_\ell). \quad (2.1) \end{aligned}$$

This explicit expression is especially useful when the reproducing kernel has a simple closed form.

The error expression is simpler for *shift-invariant* kernels. A shift-invariant kernel is one which satisfies

$$K(\{\mathbf{x} + \Delta\}, \{\mathbf{y} + \Delta\}) = K(\mathbf{x}, \mathbf{y}) = K(\{\mathbf{x} - \mathbf{y}\}, \mathbf{0}) \quad \forall \mathbf{x}, \mathbf{y}, \Delta \in [0, 1]^d.$$

For any arbitrary kernel K , there is an associated shift-invariant kernel K^{sh} defined by

$$K^{\text{sh}}(\mathbf{x}, \mathbf{y}) := \int_{[0,1]^d} K(\{\mathbf{x} + \Delta\}, \{\mathbf{y} + \Delta\}) d\Delta.$$

Let $P_n + \Delta := \{\{\mathbf{t} + \Delta\} : \mathbf{t} \in P_n\}$ denote the point set obtained by adding a shift Δ to every point in P_n and then taking the fractional part of every component. One can easily derive the important property that

$$\begin{aligned} e_{n,d}^2(P_n, K^{\text{sh}}) &= \int_{[0,1]^d} e_{n,d}^2(P_n + \Delta, K) d\Delta \\ &= - \int_{[0,1]^d} K^{\text{sh}}(\mathbf{x}, \mathbf{0}) d\mathbf{x} + \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} K^{\text{sh}}(\{\mathbf{t}_k - \mathbf{t}_\ell\}, \mathbf{0}). \end{aligned} \quad (2.2)$$

We want to use randomly-shifted rank-1 lattice rules in the function space with kernel K . It makes sense to study this “shift-averaged” error given by (2.2). In addition, the double sum reduces to a single sum for lattice rules, explicitly we have

$$e_{n,d}^2(\mathbf{z}) = - \int_{[0,1]^d} K^{\text{sh}}(\mathbf{x}, \mathbf{0}) d\mathbf{x} + \frac{1}{n} \sum_{k=0}^{n-1} K^{\text{sh}}(\{\frac{k\mathbf{z}}{n}\}, \mathbf{0}). \quad (2.3)$$

We shall refer to this quantity as *the worst-case error for randomly-shifted rank-1 lattice rules*, and it is this worst-case error which we shall consider throughout the rest of the paper. Note that $K^{\text{sh}} = K$ if K is already shift-invariant.

2.2. Weighted function spaces. In this paper we consider reproducing kernel Hilbert spaces H with kernels of the form

$$K(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{u} \subseteq \{1, \dots, d\}} \gamma_{\mathbf{u}} \prod_{j \in \mathbf{u}} \eta(x_j, y_j),$$

with $\eta(x, y) = \eta(y, x)$. This general form of kernels has been considered in [17, 5]. The space H is a direct sum of 2^d reproducing kernel Hilbert spaces $H_{\mathbf{u}}$, where each $H_{\mathbf{u}}$ is a $|\mathbf{u}|$ -dimensional tensor-product space with kernel $K_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \mathbf{y}_{\mathbf{u}}) = \gamma_{\mathbf{u}} \prod_{j \in \mathbf{u}} \eta(x_j, y_j)$. Functions in $H_{\mathbf{u}}$ depend only on the set of variables $\mathbf{x}_{\mathbf{u}} := \{x_j : j \in \mathbf{u}\}$. Functions $f \in H$ have the natural decomposition

$$f(\mathbf{x}) = \sum_{\mathbf{u} \subseteq \{1, \dots, d\}} f_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}), \quad \text{with} \quad f_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) = \langle K_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \cdot), f \rangle_H \quad \text{and} \quad f_{\mathbf{u}} \in H_{\mathbf{u}}.$$

We associate a *weight* parameter $\gamma_{\mathbf{u}}$ with each set of variables $\mathbf{x}_{\mathbf{u}}$, which describes the level of *interaction* between the variables in $\mathbf{x}_{\mathbf{u}}$. The 2^d weights $\gamma_{\mathbf{u}}$ together model the relative importance between various sets of variables. Roughly speaking, a small

γ_u means that the contribution of the term f_u is small compared with the other terms. To fix the scaling we set $\gamma_\emptyset = 1$. The limiting case of $\gamma_u = 0$ implies that there is no f_u term in the decomposition, or in other words, there is no interaction between the variables in $\mathbf{x}_u$.

The worst-case error for randomly-shifted rank-1 lattice rules (see (2.3)) is expressed in terms of the associated shift-invariant kernel

$$K^{\text{sh}}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{u} \subseteq \{1, \dots, d\}} \gamma_u \prod_{j \in \mathbf{u}} \omega(\{x_j - y_j\}), \quad \text{with} \quad \omega(x) = \int_0^1 \eta(\{x + \Delta\}, \Delta) d\Delta.$$

Note that ω is symmetric, i.e. $\omega(x) = \omega(1-x)$. For simplicity we impose the additional condition $\int_0^1 \omega(x) dx = 0$. The initial error is then 1 and the worst-case error (2.3) then satisfies

$$e_{n,d}^2(\mathbf{z}) = -1 + \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\mathbf{u} \subseteq \{1, \dots, d\}} \gamma_u \prod_{j \in \mathbf{u}} \omega\left(\left\{\frac{kz_j}{n}\right\}\right). \quad (2.4)$$

(If the initial error were not 1, we could consider instead the relative error, that is, we divide $e_{n,d}^2(\mathbf{z})$ by the squared initial error. The expression for the relative error would be of the same form as (2.4), but with rescaled weights γ_u .)

To have a function space that best describes our integrand f , we must choose the weights γ_u according to the dimension structure of f . (See [3] for two notions of *effective dimension*, and see [19] for numerical schemes to estimate them.) When the nominal dimension is d , there are 2^d weights γ_u to decide upon. Here are a few options to help limit the choices:

(i) *Product weights*: Here $\gamma_u := \prod_{j \in \mathbf{u}} \gamma_{\{j\}}$, that is, the weight associated with the set of variables $\mathbf{x}_u$ is automatically assigned the product of the weights for each individual variable x_j in this set. The product-weight setting leads to a tensor product space H , and the worst-case error for randomly-shifted rank-1 lattice rules satisfies

$$e_{n,d}^2(\mathbf{z}) = -1 + \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^d \left(1 + \gamma_j \omega\left(\left\{\frac{kz_j}{n}\right\}\right)\right).$$

(ii) *Order-dependent weights*: Here $\gamma_u := \Gamma_{|\mathbf{u}|}$, that is, the weight associated with the set of variables $\mathbf{x}_u$ depends only on the cardinality of $\mathbf{u}$. In this case the worst-case error for randomly-shifted rank-1 lattice rules can be written as

$$e_{n,d}^2(\mathbf{z}) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^d \Gamma_\ell \sum_{\substack{\mathbf{u} \subseteq \{1, \dots, d\} \\ |\mathbf{u}|=\ell}} \prod_{j \in \mathbf{u}} \omega\left(\left\{\frac{kz_j}{n}\right\}\right).$$

(iii) *Finite-order weights*: Here $\gamma_u = 0$ for all $|\mathbf{u}| > q$, with some fixed number q . The finite-order-weight setting can be very useful because many high-dimensional integrals in practice do appear to have a finite order of just 2 or 3. We often combine the finite-order-weight setting with the order-dependent-weight setting.

We end this section with two important examples of such spaces.

Example 1 (Weighted Korobov spaces). The weighted Korobov space consists of one-periodic functions with absolutely convergent Fourier series, see e.g. [5]. The

reproducing kernel is already shift-invariant, with

$$\eta(x, y) = \omega(\{x - y\}) \quad \text{and} \quad \omega(x) = \sum_{h \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i h x}}{|h|^\alpha},$$

where $\alpha > 1$ is a smoothness parameter which corresponds to the rate of decay of the Fourier coefficients. It is known that good (unshifted or randomly-shifted) rank-1 lattice rules can be constructed using the CBC algorithm to achieve the optimal rate of convergence $\mathcal{O}(n^{-\alpha/2+\delta})$, $\delta > 0$ (see [10, 4]). Note that when α is an even integer, ω can be written in terms of the Bernoulli polynomial of degree α :

$$\omega(x) = \frac{(2\pi)^\alpha}{(-1)^{\frac{\alpha}{2}-1}\alpha!} B_\alpha(x).$$

Example 2 (Weighted Sobolev spaces). The weighted Sobolev space (with smoothness one) consists of functions with square-integrable mixed first derivatives, see e.g. [17]. These functions are in general non-periodic. There are a few variants of these spaces depending on how the inner product is defined. One popular variant, now commonly referred to as the *unanchored* Sobolev space, corresponds to

$$\eta(x, y) = \frac{1}{2} B_2(\{x - y\}) + \left(x - \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \quad \text{and} \quad \omega(x) = B_2(x) = x^2 - x + \frac{1}{6},$$

which satisfies the condition $\int_0^1 \omega(x) dx = 0$. This space has a direct association with the ANOVA decomposition of functions (see e.g. [3, 19]). It is known that good randomly-shifted rank-1 lattice rules can be constructed using the CBC algorithm to achieve the optimal rate of convergence $\mathcal{O}(n^{-1+\delta})$, $\delta > 0$, in a probabilistic sense (see [10, 4]).

3. Properties of embedded lattice rules. Before we get into the ideas behind the fast implementations of CBC algorithms, we need to review some fundamental properties of embedded lattice rules.

Given an integer base $b \geq 2$, a fixed dimension d , and any integer vector $\mathbf{z} \in \mathbb{Z}^d$ whose components are all co-prime with b , define

$$L_m = L_m(\mathbf{z}) := \left\{ \left\{ \frac{k\mathbf{z}}{b^m} \right\} : k = 0, 1, \dots, b^m - 1 \right\}, \quad m = 0, 1, \dots \quad (3.1)$$

The following properties hold for all $m \geq 0$.

PROPERTY 1. Clearly L_m is a point set for a rank-1 lattice rule with b^m distinct points.

PROPERTY 2. The generating vector for the lattice point set L_m can be reduced modulo b^m to give the same point set, i.e., $L_m(\mathbf{z}) = L_m(\mathbf{z} \bmod b^m)$.

PROPERTY 3. The lattice point set L_m is embedded in L_{m+1} , i.e., $L_m \subset L_{m+1}$.

PROPERTY 4. The points in L_m are exactly those points in L_{m+1} whose indices k in (3.1) are multiples of b .

For a given m , we can construct a generating vector $\mathbf{z}$ using Algorithm 1 with $n = b^m$ so that L_m is a “good” lattice point set with b^m points. Due to Property 2, we can restrict our search for the components of $\mathbf{z}$ to a finite set as indicated in Algorithm 1, with the largest possible choice being $b^m - 1$. Note that the goodness of the point set L_m in no way implies that all smaller lattice point sets embedded in L_m are good. On the contrary, it is quite possible that the same generating vector $\mathbf{z}$ leads to a very poor point set L_ℓ for $\ell < m$ (as will be demonstrated in §6.1).

This brings us to Algorithm 2, where a generating vector $\mathbf{z}$ is constructed with the hope that the lattice point sets L_m , for all values of m in a finite range $m_1 \leq m \leq m_2$, would be good simultaneously. The assumption that m is bounded from above by m_2 means that the maximum number of points is b^{m_2} ; hence we can restrict the components of $\mathbf{z}$ to be at most $b^{m_2} - 1$, ensuring a finite search space. The quality of this generating vector $\mathbf{z}$ is assessed by the error measure $X_{m_1, m_2, d}(\mathbf{z})$ defined in (1.3), which compares the worst-case error of $\mathbf{z}$ with the worst-case error of the near-optimal generating vectors constructed by Algorithm 1 for each m in our range. Put differently, $X_{m_1, m_2, d}(\mathbf{z})$ measures how much worse than optimal our “one for all” vector $\mathbf{z}$ is, and Algorithm 2 selects a vector $\mathbf{z}$ so that $X_{m_1, m_2, d}(\mathbf{z})$ is as small as possible for m in the prescribed range. This can be described as the “mini-max” optimization of the worst-case error ratios. The quality of $\mathbf{z}$ as a “one for all” vector is assured as long as $X_{m_1, m_2, d}(\mathbf{z})$ is uniformly bounded by a small constant, say 2.

4. Fast construction of embedded lattice rules. Before we can use Algorithm 2, we need to use Algorithm 1 to find the near-optimal generating vectors for each m from m_1 to m_2 and store the worst-case errors in each dimension. The apparent cost of Algorithm 1 for n points up to dimension d is $\mathcal{O}(d^2 n^2)$, and so the cost of the pre-computations for Algorithm 2 up to $n = b^{m_2}$ points can be $\mathcal{O}(d^2 n^2 \log(n))$. Once this is done, the cost of Algorithm 2 appears to be another $\mathcal{O}(d^2 n^2 \log(n))$. Thus unless we can speed up the computations in some way, Algorithm 2 would be a mere theoretical exercise.

Three ideas are used to obtain a fast alternative for Algorithm 2 (the first two were already known for Algorithm 1):

(i) The nature of the CBC algorithm is iterative with respect to the dimension. Therefore we will store previous calculations, sacrificing memory for speed. This reduces the construction cost of Algorithm 1 to $\mathcal{O}(dn^2)$.

(ii) The calculation of the worst-case errors resembles a structured matrix-vector product which can be computed in a fast way. Thus the construction cost of Algorithm 1 can be dramatically reduced, to $\mathcal{O}(dn \log(n))$, implying that the cost of the pre-computations for Algorithm 2 is $\mathcal{O}(dn(\log(n))^2)$.

(iii) The calculation of the worst-case error $e_{n, d}(\mathbf{z})$ for the maximum number of points $n = b^{m_2}$ contains within it all worst-case errors $e_{b^m, d}(\mathbf{z})$ for $m \leq m_2$. This means that, with a slight adaptation of the algorithm, the actual cost of Algorithm 2 is only $\mathcal{O}(dn \log(n))$ while the cost for those pre-computations based on Algorithm 1 is $\mathcal{O}(dn(\log(n))^2)$. In other words, we get the embedded rule almost for free.

In the next three subsections we will explain these ideas as independently of each other as possible.

4.1. Matrix-vector formulation. Suppose we are at the point in the CBC algorithm where we want to choose the d th component z_d . It makes sense to consider the squared worst-case error $e_{n, d}^2(\mathbf{z}) = e_{n, d}^2(z_1, \dots, z_d)$ as a function of z_d , and we denote it by $E_d^2(z_d)$. First we will rewrite the expression for $E_d^2(z_d)$ in a form that is more suitable for recursion with respect to d . For computational efficiency we only consider two types of weights.

(i) *Product weights:*

$$E_d^2(z_d) = -1 + \frac{1}{n} \sum_{k=0}^{n-1} \left(1 + \gamma_d \omega \left(\left\{ \frac{kz_d}{n} \right\} \right) \right) \underbrace{\prod_{j=1}^{d-1} \left(1 + \gamma_j \omega \left(\left\{ \frac{kz_j}{n} \right\} \right) \right)}_{p_{d-1}(k)}$$

$$= e_{n,d-1}^2(z_1, \dots, z_{d-1}) + \frac{\gamma_d}{n} \sum_{k=0}^{n-1} \omega \left(\left\{ \frac{kz_d}{n} \right\} \right) p_{d-1}(k), \quad (4.1)$$

where $e_{n,0}^2 := 0$ and the products $p_d(k)$ are defined recursively by

$$\begin{aligned} p_0(k) &:= 1, \\ p_d(k) &:= \left(1 + \gamma_d \omega \left(\left\{ \frac{kz_d}{n} \right\} \right) \right) p_{d-1}(k). \end{aligned} \quad (4.2)$$

(ii) *Order-dependent weights (which may be of finite order q):*

$$\begin{aligned} E_d^2(z_d) &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^{\min(q,d)} \Gamma_\ell \left(\underbrace{\sum_{\substack{\mathbf{u} \subseteq \{1, \dots, d-1\} \\ |\mathbf{u}|=\ell}} \prod_{j \in \mathbf{u}} \omega \left(\left\{ \frac{kz_j}{n} \right\} \right)}_{p_{d-1,\ell}(k)} \right. \\ &\quad \left. + \omega \left(\left\{ \frac{kz_d}{n} \right\} \right) \underbrace{\sum_{\substack{\mathbf{u} \subseteq \{1, \dots, d-1\} \\ |\mathbf{u}|=\ell-1}} \prod_{j \in \mathbf{u}} \omega \left(\left\{ \frac{kz_j}{n} \right\} \right)}_{p_{d-1,\ell-1}(k)} \right) \\ &= e_{n,d-1}^2(z_1, \dots, z_{d-1}) + \frac{1}{n} \sum_{k=0}^{n-1} \omega \left(\left\{ \frac{kz_d}{n} \right\} \right) \left(\sum_{\ell=1}^{\min(q,d)} \Gamma_\ell p_{d-1,\ell-1}(k) \right), \end{aligned} \quad (4.3)$$

where $e_{n,0}^2 := 0$ and the products $p_{d,\ell}(k)$ are defined recursively by

$$\begin{aligned} p_{d,0}(k) &:= 1, \\ p_{d,\ell}(k) &:= p_{d-1,\ell}(k) + \omega \left(\left\{ \frac{kz_d}{n} \right\} \right) p_{d-1,\ell-1}(k). \end{aligned} \quad (4.4)$$

Let $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$ denote the set of the integers modulo n , and U_n the multiplicative group of integers modulo n , i.e.,

$$U_n := \{u \in \mathbb{Z}_n : \gcd(u, n) = 1\},$$

with the cardinality of U_n given by the Euler totient function $\phi(n) = |U_n|$. We need to evaluate $E_d^2(z_d)$ for every choice of $z_d \in U_n$, which suggests the definition of the vectors

$$\mathbf{E}_d^2 := [E_d^2(z)]_{z \in U_n}, \quad \mathbf{p}_d := [p_d(k)]_{k \in \mathbb{Z}_n}, \quad \mathbf{p}_{d,\ell} := [p_{d,\ell}(k)]_{k \in \mathbb{Z}_n}, \quad (4.5)$$

and the matrix

$$\mathbf{\Omega}_n := \left[\omega \left(\left\{ \frac{kz}{n} \right\} \right) \right]_{\substack{z \in U_n \\ k \in \mathbb{Z}_n}} = \left[\omega \left(\frac{kz \bmod n}{n} \right) \right]_{\substack{z \in U_n \\ k \in \mathbb{Z}_n}}. \quad (4.6)$$

We now observe from (4.1) and (4.3) that the squared worst-case error vector $\mathbf{E}_d^2$ can be expressed in terms of a matrix-vector product with the matrix $\mathbf{\Omega}_n$ as

$$\mathbf{E}_d^2 := \mathbf{1}_{\phi(n)} e_{n,d-1}^2(z_1, \dots, z_{d-1}) + \frac{\gamma_d}{n} \mathbf{\Omega}_n \mathbf{p}_{d-1} \quad (\text{product})$$

$$\mathbf{E}_d^2 := \mathbf{1}_{\phi(n)} e_{n,d-1}^2(z_1, \dots, z_{d-1}) + \frac{1}{n} \mathbf{\Omega}_n \left(\sum_{\ell=1}^{\min(q,d)} \Gamma_\ell \mathbf{p}_{d-1,\ell-1} \right) \quad (\text{order-dependent})$$

where $\mathbf{1}_t$ denotes a vector of ones of length t . Algorithm 1 picks the value of $z_d \in U_n$ which corresponds to the smallest entry in $\mathbf{E}_d^2$. Then it is clear from (4.2) and (4.4) that the vectors $\mathbf{p}_d$ or $\mathbf{p}_{d,\ell}$ for the next iteration can be obtained recursively via

$$\mathbf{p}_d := (\mathbf{1}_n + \gamma_d \mathbf{\Omega}_n(z_d)) .* \mathbf{p}_{d-1} \quad (\text{product}) \quad (4.7)$$

$$\mathbf{p}_{d,\ell} := \mathbf{p}_{d-1,\ell} + \mathbf{\Omega}_n(z_d) .* \mathbf{p}_{d-1,\ell-1} \quad (\text{order-dependent}) \quad (4.8)$$

where $\mathbf{\Omega}_n(z_d)$ denotes the row of $\mathbf{\Omega}_n$ corresponding to the chosen z_d , and the operator $.*$ denotes the element-wise vector multiplication. Since the vectors $\mathbf{p}_{d-1}$ or $\mathbf{p}_{d-1,\ell}$ are no longer needed in the next iteration, we can simply overwrite $\mathbf{p}_{d-1}$ with $\mathbf{p}_d$ or overwrite $\mathbf{p}_{d-1,\ell}$ with $\mathbf{p}_{d,\ell}$. Hence, starting with the vectors $\mathbf{p}_0 := \mathbf{1}_n$ or $\mathbf{p}_{0,\ell} := \mathbf{1}_n$, we require only $\mathcal{O}(n)$ storage for product weights and $\mathcal{O}(qn)$ storage for order-dependent weights of finite order q .

The trick now is to order the indices $z \in U_n$ and $k \in \mathbb{Z}_n$ in (4.5) and (4.6) in a clever way to allow fast matrix-vector multiplications.

4.2. Fast matrix-vector product. To get an idea of the general concept, we first consider the matrix $\mathbf{\Omega}_n$ for n prime. Using the Rader factorization [15, 18], this matrix can be reordered such that it has a circulant submatrix of size $(n-1) \times (n-1)$. This is achieved by taking the indices as $z = g^i$ and $k = g^{-i'}$ for $0 \leq i, i' \leq n-2$, where g is a primitive root of n (i.e. g is a generator for the cyclic group U_n). We denote this particular ordering of $\mathbf{\Omega}_n$ by $\mathbf{\Omega}_n^{(g)}$.

The effect of such reordering is explained most easily by an example. Take $n = 11$, then the possible components for the generating vector which we want to consider are

$$U_{11} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Since n is prime, we can always find a generator for this group. For our example we choose $g = 2$. The ordering for the z indices we want to use is then

$$[g^i : 0 \leq i \leq 9] = [1, 2, 4, 8, 5, 10, 9, 7, 3, 6].$$

For the indices $k \neq 0$ we take the negative powers of g

$$[g^{-i'} : 0 \leq i' \leq 9] = [1, 6, 3, 7, 9, 10, 5, 8, 4, 2],$$

which can easily be seen as reversing the order of the z indices, but keeping the first index fixed (the zero-th power of g). This reordering, together with a column of zeros for $k = 0$, gives a matrix with the following structure

$$\mathbf{\Omega}_{11}^{(2)} = \omega\left(\frac{1}{11}\right) \underbrace{\begin{bmatrix} 1 & 6 & 3 & 7 & 9 & 10 & 5 & 8 & 4 & 2 & 0 \\ 2 & 1 & 6 & 3 & 7 & 9 & 10 & 5 & 8 & 4 & 0 \\ 4 & 2 & 1 & 6 & 3 & 7 & 9 & 10 & 5 & 8 & 0 \\ 8 & 4 & 2 & 1 & 6 & 3 & 7 & 9 & 10 & 5 & 0 \\ 5 & 8 & 4 & 2 & 1 & 6 & 3 & 7 & 9 & 10 & 0 \\ 10 & 5 & 8 & 4 & 2 & 1 & 6 & 3 & 7 & 9 & 0 \\ 9 & 10 & 5 & 8 & 4 & 2 & 1 & 6 & 3 & 7 & 0 \\ 7 & 9 & 10 & 5 & 8 & 4 & 2 & 1 & 6 & 3 & 0 \\ 3 & 7 & 9 & 10 & 5 & 8 & 4 & 2 & 1 & 6 & 0 \\ 6 & 3 & 7 & 9 & 10 & 5 & 8 & 4 & 2 & 1 & 0 \end{bmatrix}}_{C_{11}}.$$

As was proved in [13], we can halve the number of rows and the number of columns of the submatrix $\mathbf{C}_{11}$ due to the symmetry $\omega(x) = \omega(1-x)$. More specifically, we can write $\mathbf{C}_{11}$ in a reduced form (denoted by $\sim$ instead of $=$)

$$\mathbf{\Omega}_{11}^{(2)} \sim \omega\left(\frac{1}{11} \underbrace{\begin{bmatrix} 1 & 6 & 3 & 7 & 9 & 0 \\ 2 & 1 & 6 & 3 & 7 & 0 \\ 4 & 2 & 1 & 6 & 3 & 0 \\ 8 & 4 & 2 & 1 & 6 & 0 \\ 5 & 8 & 4 & 2 & 1 & 0 \end{bmatrix}}_{\tilde{\mathbf{C}}_{11}}\right),$$

where we remember that the full matrix $\mathbf{\Omega}_{11}^{(2)}$ can be obtained by doubling each partition vertically and also horizontally up to the double bar.

A matrix with a structure like the submatrix $\mathbf{C}_{11}$ (as well as $\tilde{\mathbf{C}}_{11}$) is called a *circulant matrix* and is completely defined by its first column $\mathbf{c}$. Moreover, matrix-vector multiplication with a circulant matrix of order n can be done in $\mathcal{O}(n \log(n))$ using fast Fourier transforms.

LEMMA 4.1 (Fast matrix-vector product with a circulant matrix). *A matrix-vector product with a circulant matrix $\mathbf{C}_n$ of order n , with first column $\mathbf{c}$, takes $\mathcal{O}(n \log(n))$ operations instead of the usual $\mathcal{O}(n^2)$, and uses $\mathcal{O}(n)$ memory, when using its similarity transform combined with a fast Fourier algorithm for calculating the discrete Fourier transform $\mathbf{F}_n$ and its inverse $\mathbf{F}_n^{-1}$:*

$$\begin{aligned} \mathbf{C}_n \mathbf{x} &= \mathbf{F}_n^{-1} \text{diag}(\mathbf{F}_n \mathbf{c}) \mathbf{F}_n \mathbf{x} \\ &= \text{IFFT}(\text{FFT}(\mathbf{c}) .* \text{FFT}(\mathbf{x})). \end{aligned}$$

A fast CBC algorithm based on this fast matrix-vector product was first introduced for n prime in [13] and later extended for composite n in [14]. The general idea for composite n is that the complete matrix $\mathbf{\Omega}_n$ can be partitioned in blocks which have a circulant or block-circulant structure. Basically, we consider a vertical partition $\mathbf{A}_{d_i}^{(g)}$ for each divisor d_i of n

$$\mathbf{\Omega}_n^{(g)} := \left[\mathbf{A}_{d_1}^{(g)} \mid \mathbf{A}_{d_2}^{(g)} \mid \dots \mid \mathbf{A}_{d_\nu}^{(g)} \right],$$

where g is a generator (or technically, a set of generators) for the group U_n , and each $\mathbf{A}_{d_i}^{(g)}$ is constructed out of circulant (or technically, nested block circulant) matrices $\mathbf{B}_{n/d_i}^{(g)}$. The full details can be found in [14] where it is described how the ordering is done and why a matrix-vector multiplication with $\mathbf{\Omega}_n^{(g)}$ takes $\mathcal{O}(n \log(n))$ time using $\mathcal{O}(n)$ memory.

For embedded lattice rules, we shall restrict ourselves to n being a power of a prime p , because in this case the matrix $\mathbf{\Omega}_{p^m}^{(g)}$ is of a particularly simple form. For this case we have $|U_{p^m}| = p^{m-1}(p-1)$. We will state the results for $p \neq 2$ and $p = 2$ in two separate theorems below; both results follow from [14, Theorem 4 and Corollary 4]. These two theorems differ in subtle yet crucial ways: the form of the matrices $\mathbf{B}_{p^m}^{(g)}$ differs, and there are two degenerate cases $\mathbf{B}_{2^1}^{(5)}$ and $\mathbf{B}_{2^0}^{(5)}$ for $p = 2$, while there is only one degenerate case $\mathbf{B}_{p^0}^{(g)}$ for $p \neq 2$. In the theorems $\mathbf{1}_t \otimes \mathbf{B}$ denotes the stacking of t replications of a matrix $\mathbf{B}$.

THEOREM 4.2 (Block partitioning $\mathbf{\Omega}_{p^m}$, p prime, $p \neq 2$). *The matrix*

$$\mathbf{\Omega}_{p^m} := \left[\omega\left(\frac{kz \bmod p^m}{p^m}\right) \right]_{\substack{z \in U_{p^m} \\ k \in \mathbb{Z}_{p^m}}},$$

with $\omega(x) = \omega(1-x)$, can be ordered with respect to the divisors of p^m as

$$\Omega_{p^m}^{(g)} := \left[\mathbf{1}_{p^0} \otimes \mathbf{B}_{p^m}^{(g)} \mid \mathbf{1}_{p^1} \otimes \mathbf{B}_{p^{m-1}}^{(g)} \mid \cdots \mid \mathbf{1}_{p^{m-1}} \otimes \mathbf{B}_{p^1}^{(g)} \mid \mathbf{1}_{p^{m-1}(p-1)} \otimes \mathbf{B}_{p^0}^{(g)} \right],$$

where g is a generator for U_{p^m} and $\mathbf{B}_{p^0}^{(g)} := [\omega(0)]$. For $\ell \geq 1$, each matrix $\mathbf{B}_{p^\ell}^{(g)}$ (associated with a divisor $p^{m-\ell}$) is circulant of the form

$$\mathbf{B}_{p^\ell}^{(g)} := \left[\begin{array}{c|c} \mathbf{M}_{p^\ell}^{(g)} & \mathbf{M}_{p^\ell}^{(g)} \\ \hline \mathbf{M}_{p^\ell}^{(g)} & \mathbf{M}_{p^\ell}^{(g)} \end{array} \right],$$

with circulant blocks

$$\mathbf{M}_{p^\ell}^{(g)} := \left[\omega \left(\frac{kz \bmod p^\ell}{p^\ell} \right) \right]_{\substack{z \in \langle\langle g \rangle\rangle_{p^\ell} \\ k \in \langle\langle g^{-1} \rangle\rangle_{p^\ell}}}.$$

Here $\langle\langle g \rangle\rangle_{p^\ell}$ denotes the ordered set $[g^i \bmod p^\ell : 0 \leq i \leq \phi(p^\ell)/2 - 1]$, i.e., half of the group U_{p^ℓ} in generator ordering.

THEOREM 4.3 (Block partitioning Ω_{2^m}). The matrix

$$\Omega_{2^m} := \left[\omega \left(\frac{kz \bmod 2^m}{2^m} \right) \right]_{\substack{z \in U_{2^m} \\ k \in \mathbb{Z}_{2^m}}},$$

with $\omega(x) = \omega(1-x)$, can be ordered with respect to the divisors of 2^m as

$$\Omega_{2^m}^{(5)} := \left[\mathbf{1}_{2^0} \otimes \mathbf{B}_{2^m}^{(5)} \mid \mathbf{1}_{2^1} \otimes \mathbf{B}_{2^{m-1}}^{(5)} \mid \cdots \mid \mathbf{1}_{2^{m-1}} \otimes \mathbf{B}_{2^1}^{(5)} \mid \mathbf{1}_{2^{m-1}(2-1)} \otimes \mathbf{B}_{2^0}^{(5)} \right],$$

where $\mathbf{B}_{2^0}^{(5)} := [\omega(0)]$ and $\mathbf{B}_{2^1}^{(5)} := [\omega(1/2)]$. For $\ell \geq 2$, each matrix $\mathbf{B}_{2^\ell}^{(5)}$ (associated with a divisor $2^{m-\ell}$) is circulant of the form

$$\mathbf{B}_{2^\ell}^{(5)} := \left[\begin{array}{c|c} \mathbf{M}_{2^\ell}^{(5)} & \mathbf{M}_{2^\ell}^{(5)} \\ \hline \mathbf{M}_{2^\ell}^{(5)} & \mathbf{M}_{2^\ell}^{(5)} \end{array} \right], \quad (4.9)$$

with circulant blocks

$$\mathbf{M}_{2^\ell}^{(5)} := \left[\omega \left(\frac{kz \bmod 2^\ell}{2^\ell} \right) \right]_{\substack{z \in \langle 5 \rangle_{2^\ell} \\ k \in \langle 5^{-1} \rangle_{2^\ell}}}.$$

Here $\langle 5 \rangle_{2^\ell}$ denotes the ordered set $[5^i \bmod 2^\ell : 0 \leq i \leq 2^{\ell-2} - 1]$, i.e., half of the group U_{2^ℓ} in generator ordering.

We will illustrate this theorem with an example shortly. First we discuss the effect of the symmetry of ω . Note that Theorem 4.3 rests upon the partition

$$U_{2^\ell} = \langle 5 \rangle_{2^\ell} \cup (-1) \langle 5 \rangle_{2^\ell}, \quad \text{for } \ell \geq 2. \quad (4.10)$$

If we do not make use of the symmetry of ω , formula (4.9) would actually look like

$$\widehat{\mathbf{B}}_{2^\ell}^{(5)} := \left[\begin{array}{c|c} \mathbf{M}_{2^\ell}^{(5)} & \mathbf{M}_{2^\ell}^{(5)\star} \\ \hline \mathbf{M}_{2^\ell}^{(5)\star} & \mathbf{M}_{2^\ell}^{(5)} \end{array} \right], \text{ with } \begin{cases} \mathbf{M}_{2^\ell}^{(5)} = \left[\omega \left(\frac{kz \bmod 2^\ell}{2^\ell} \right) \right]_{\substack{z \in \langle 5 \rangle_{2^\ell} \\ k \in \langle 5^{-1} \rangle_{2^\ell}}}, \\ \mathbf{M}_{2^\ell}^{(5)\star} = \left[\omega \left(\frac{-kz \bmod 2^\ell}{2^\ell} \right) \right]_{\substack{z \in \langle 5 \rangle_{2^\ell} \\ k \in \langle 5^{-1} \rangle_{2^\ell}}}, \end{cases}$$

and the stacking of the matrices $\widehat{\mathbf{B}}_{2^\ell}^{(5)}$ would be

$$\widehat{\mathbf{\Omega}}_{2^m}^{(5)} := \left[\mathbf{R}_{2^0} \widehat{\mathbf{B}}_{2^m}^{(5)} \mid \cdots \mid \mathbf{R}_{2^{m-2}} \widehat{\mathbf{B}}_{2^2}^{(5)} \mid \mathbf{1}_{2^{m-1}} \otimes \mathbf{B}_{2^1}^{(5)} \mid \mathbf{1}_{2^{m-1}} \otimes \mathbf{B}_{2^0}^{(5)} \right], \quad (4.11)$$

where $\mathbf{R}_{2^\ell} := \mathbf{I}_2 \otimes \mathbf{1}_{2^\ell} \otimes \mathbf{I}_{2^{m-\ell-2}}$, which stacks the top and bottom half of $\widehat{\mathbf{B}}_{2^{m-\ell}}^{(5)}$ separately. The symmetry of ω implies $\mathbf{M}_{2^\ell}^{(5)} = \mathbf{M}_{2^\ell}^{(5)*}$ from which the theorem follows.

We consider now the example $n = 2^4 = 16$, initially without using the symmetry of ω . Following (4.10), the first half of U_{16} is generated by 5

$$[5^i \bmod 16 : 0 \leq i \leq 3] = [1, 5, 9, 13],$$

where i stays less than $\phi(n)/2$ since this is only half of the group U_{16} . The second half is a ‘copy’ of this subgroup, but multiplied with $(-1) \equiv 15 \pmod{16}$

$$[(-1) 5^i \bmod 16 : 0 \leq i \leq 3] = [15, 11, 7, 3].$$

With this row ordering we can associate a column ordering to get the circulant structure. This can be achieved by reversing the row ordering except for the first element. We obtain the first submatrix for the divisor $2^0 = 1$

$$\widehat{\mathbf{B}}_{16}^{(5)} = \omega\left(\frac{1}{16} \begin{bmatrix} 1 & 13 & 9 & 5 & 15 & 3 & 7 & 11 \\ 5 & 1 & 13 & 9 & 11 & 15 & 3 & 7 \\ 9 & 5 & 1 & 13 & 7 & 11 & 15 & 3 \\ 13 & 9 & 5 & 1 & 3 & 7 & 11 & 15 \\ 15 & 3 & 7 & 11 & 1 & 13 & 9 & 5 \\ 11 & 15 & 3 & 7 & 5 & 1 & 13 & 9 \\ 7 & 11 & 15 & 3 & 9 & 5 & 1 & 13 \\ 3 & 7 & 11 & 15 & 13 & 9 & 5 & 1 \end{bmatrix}\right).$$

The submatrices for the other divisors can be constructed similarly, and we get

$$\widehat{\mathbf{B}}_8^{(5)} = \omega\left(\frac{1}{8} \begin{bmatrix} 1 & 5 & 7 & 3 \\ 5 & 1 & 3 & 7 \\ 7 & 3 & 1 & 5 \\ 3 & 7 & 5 & 1 \end{bmatrix}\right), \quad \widehat{\mathbf{B}}_4^{(5)} = \omega\left(\frac{1}{4} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\right), \quad \mathbf{B}_2^{(5)} = \omega\left(\frac{1}{2} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\right), \quad \mathbf{B}_1^{(5)} = \omega\left(\begin{bmatrix} 0 & \\ & 0 \end{bmatrix}\right).$$

Note that all these matrices (except for the last two trivial ones) have a block-circulant-with-circulant-blocks structure, which is consistent with the two partitions in their multiplicative groups (see (4.10)). Using (4.11), the full matrix looks like this (where italic font means redundancy)

$$\widehat{\mathbf{\Omega}}_{16}^{(5)} = \omega\left(\frac{1}{16} \begin{bmatrix} 1 & 13 & 9 & 5 & 15 & 3 & 7 & 11 & 2 & 10 & 14 & 6 & 4 & 12 & 8 & 0 \\ 5 & 1 & 13 & 9 & 11 & 15 & 3 & 7 & 10 & 2 & 6 & 14 & 4 & 12 & 8 & 0 \\ 9 & 5 & 1 & 13 & 7 & 11 & 15 & 3 & 2 & 10 & 14 & 6 & 4 & 12 & 8 & 0 \\ 13 & 9 & 5 & 1 & 3 & 7 & 11 & 15 & 10 & 2 & 6 & 14 & 4 & 12 & 8 & 0 \\ 15 & 3 & 7 & 11 & 1 & 13 & 9 & 5 & 14 & 6 & 2 & 10 & 12 & 4 & 8 & 0 \\ 11 & 15 & 3 & 7 & 5 & 1 & 13 & 9 & 6 & 14 & 10 & 2 & 12 & 4 & 8 & 0 \\ 7 & 11 & 15 & 3 & 9 & 5 & 1 & 13 & 14 & 6 & 2 & 10 & 12 & 4 & 8 & 0 \\ 3 & 7 & 11 & 15 & 13 & 9 & 5 & 1 & 6 & 14 & 10 & 2 & 12 & 4 & 8 & 0 \end{bmatrix}\right).$$

We now apply the symmetry of ω by mapping each $k \in \mathbb{Z}_n$ to $\min(n-k, k)$ and we get the matrix

$$\mathbf{\Omega}_{16}^{(5)} = \omega\left(\frac{1}{16} \begin{bmatrix} 1 & 3 & 7 & 5 & 1 & 3 & 7 & 5 & 2 & 6 & 2 & 6 & 4 & 4 & 8 & 0 \\ 5 & 1 & 3 & 7 & 5 & 1 & 3 & 7 & 6 & 2 & 6 & 2 & 4 & 4 & 8 & 0 \\ 7 & 5 & 1 & 3 & 7 & 5 & 1 & 3 & 2 & 6 & 2 & 6 & 4 & 4 & 8 & 0 \\ 3 & 7 & 5 & 1 & 3 & 7 & 5 & 1 & 6 & 2 & 6 & 2 & 4 & 4 & 8 & 0 \\ 1 & 3 & 7 & 5 & 1 & 3 & 7 & 5 & 2 & 6 & 2 & 6 & 4 & 4 & 8 & 0 \\ 5 & 1 & 3 & 7 & 5 & 1 & 3 & 7 & 6 & 2 & 6 & 2 & 4 & 4 & 8 & 0 \\ 7 & 5 & 1 & 3 & 7 & 5 & 1 & 3 & 2 & 6 & 2 & 6 & 4 & 4 & 8 & 0 \\ 3 & 7 & 5 & 1 & 3 & 7 & 5 & 1 & 6 & 2 & 6 & 2 & 4 & 4 & 8 & 0 \end{bmatrix}\right),$$

as promised in Theorem 4.3. We can then represent this matrix in a reduced form

$$\Omega_{16}^{(5)} \sim \omega\left(\frac{1}{16} \left[\begin{array}{cccc|cc|c||c|c} 1 & 3 & 7 & 5 & 2 & 6 & 4 & 8 & 0 \\ 5 & 1 & 3 & 7 & 6 & 2 & 4 & 8 & 0 \\ 7 & 5 & 1 & 3 & 2 & 6 & 4 & 8 & 0 \\ 3 & 7 & 5 & 1 & 6 & 2 & 4 & 8 & 0 \end{array} \right] \right). \quad (4.12)$$

If we write out the reordering from Theorems 4.2 and 4.3 explicitly by means of permutations $\Pi_g^\top$ and $\Pi_{g^{-1}}$ on the rows and columns induced by g , we obtain

$$\mathbf{y} = \Omega_{p^m} \mathbf{x} \quad \Leftrightarrow \quad \Pi_g^\top \mathbf{y} = \underbrace{\Pi_g^\top \Omega_{p^m} \Pi_{g^{-1}}}_{\Omega_{p^m}^{(g)}} \Pi_{g^{-1}}^\top \mathbf{x} = \Omega_{p^m}^{(g)} \Pi_{g^{-1}}^\top \mathbf{x}.$$

Note that the permutations do not complicate matters, since the starting vectors are all ones and we know that the vectors

$$\begin{aligned} \mathbf{p}_d^{(g^{-1})} &:= \Pi_{g^{-1}}^\top \mathbf{p}_d = (\mathbf{1}_n + \gamma_d \Omega_{p^m}^{(g)}(z_d)) .* \mathbf{p}_{d-1}^{(g^{-1})} && \text{(product)} \\ \mathbf{p}_{d,\ell}^{(g^{-1})} &:= \Pi_{g^{-1}}^\top \mathbf{p}_{d,\ell} = \mathbf{p}_{d-1,\ell}^{(g^{-1})} + \Omega_{p^m}^{(g)}(z_d) .* \mathbf{p}_{d-1,\ell-1}^{(g^{-1})} && \text{(order-dependent)} \end{aligned}$$

have the same symmetry structure as the rows of the matrix $\Omega_{p^m}^{(g)}$, which means that we can keep a reduced version in memory and never have to refer to the unpermuted version. Also, from the resulting vector $\mathbf{E}_d^2$ we only need the index of the minimal entry, from which we obtain z_d via the sequence $\langle g \rangle_{p^m}$.

Combining this with Lemma 4.1, we observe that we only need the fast Fourier transforms of the first column of the smaller circulant matrices in the reduced matrix. This requires $\mathcal{O}(n)$ memory. A matrix-vector product with a vector of length $n \times 1$ involves $m+1$ (the number of divisors) smaller matrix-vector products with circulant matrices, with a total cost of

$$\sum_{\ell=0}^m \phi(p^{m-\ell}) \log(\phi(p^{m-\ell})) = \mathcal{O}(n \log(n)).$$

We can calculate all these circulant matrix-vector products in turn, starting from the smallest matrix up to the largest matrix, whilst accumulating the results. In each step the previous result vector is replicated $p-1$ times before the next result vector is added. The summing alone has a memory complexity of $\mathcal{O}(n)$ and a time complexity of $\mathcal{O}(n)$, which is negligible against the $\mathcal{O}(n \log(n))$ cost of the matrix-vector products. Again we specify this as a specialized version of the general case in [14, Theorem 5]. (There is no need for a special summing order as is the case for general n in [14].)

THEOREM 4.4 (Fast matrix-vector product $\Omega_{p^m} \mathbf{p}$). *The matrix-vector product $\Omega_{p^m} \mathbf{p}$ for the vectors $\mathbf{p}$ defined in (4.7) and (4.8) can be done in the permuted space in $\mathcal{O}(n \log(n))$ time with $\mathcal{O}(n)$ memory by doing each of the matrix-vector products with the matrices $\mathbf{B}_{p^\ell}^{(g)}$ in a fast way.*

4.3. Exploiting the embedding structure. In Theorems 4.2 and 4.3 we used implicitly the property that a generator g for the group U_{p^m} also generates U_{p^ℓ} for all $\ell \leq m$, with the obvious adjustment for $p=2$ where 5 generates U_{2^ℓ} in two partitions for all $\ell \leq m$. This actually follows from the fact (see e.g. [1, §24.3.4]) that if we have a generator r for U_p , then a generator g for U_{p^ℓ} , $\ell \geq 2$, can be found by taking

$$g = \begin{cases} r + p, & \text{if } r^{p-1} \equiv 1 \pmod{p^2}, \\ r, & \text{otherwise.} \end{cases} \quad (4.13)$$

It can be checked that if one searches for generators in increasing numerical order starting from 2, then the first p for which the smallest generator of U_p does not generate U_{p^2} is as high as 40487 (and the first p for which one of its generators does not generate U_{p^2} is $p = 29$). In other words, as long as p is smaller than 40487, the smallest generator of U_p also generates U_{p^ℓ} for all $\ell \geq 1$. Furthermore, the most logical choice in practice is $p = 2$ for which we can always pick 5 as a pseudo-generator. This gives us a way to extend a matrix for p^m points to a matrix for p^{m+1} points by simply adding one block in front.

We now show the extension to $\Omega_{32}^{(5)}$ starting from $\Omega_{16}^{(5)}$:

$$\Omega_{32}^{(5)} = \left[B_{32}^{(5)} \left| \frac{\Omega_{16}^{(5)}}{\Omega_{16}^{(5)}} \right. \right],$$

where we have recursively

$$\Omega_{16}^{(5)} = \left[B_{16}^{(5)} \left| \frac{\Omega_8^{(5)}}{\Omega_8^{(5)}} \right. \right], \Omega_8^{(5)} = \left[B_8^{(5)} \left| \frac{\Omega_4^{(5)}}{\Omega_4^{(5)}} \right. \right], \Omega_4^{(5)} = \left[B_4^{(5)} \left| \frac{\Omega_2^{(5)}}{\Omega_2^{(5)}} \right. \right],$$

with $\Omega_2^{(5)} = [B_2^{(5)} | B_1^{(5)}]$. In reduced form (and a common scaling of $1/32$) we have

$$\Omega_{32}^{(5)} \sim \omega\left(\frac{1}{32} \begin{bmatrix} 1 & 13 & 9 & 11 & 15 & 3 & 7 & 5 & \left| \begin{array}{c} 2 & 6 & 14 & 10 \\ 10 & 2 & 6 & 14 \\ 14 & 10 & 2 & 6 \\ 6 & 14 & 10 & 2 \end{array} \right| \begin{array}{c} 4 & 12 \\ 12 & 4 \\ 4 & 12 \\ 12 & 4 \end{array} \left| \begin{array}{c} 8 \\ 8 \\ 8 \\ 8 \end{array} \right| \left\| \begin{array}{c} 16 \\ 16 \\ 16 \\ 16 \end{array} \right\| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \end{bmatrix} \right).$$

In §4.2 we explained how to sum up the result vectors of a matrix-vector multiplication with the matrix Ω_{p^m} in $\mathcal{O}(n \log(n))$ time. In fact, since n is of the form p^m , any method used to sum up the intermediate result vectors would have been sufficient to get at the $\mathcal{O}(n \log(n))$ time complexity (see [14]). By accumulating the result vectors from the smallest matrix upwards, we obtain intermediate result vectors which are precisely the matrix-vector products with the matrices Ω_{p^ℓ} for ℓ going from 0 up to $m - 1$. Thus in an attempt to compute $e_{p^m,d}$, we get along the way the values of $e_{p^\ell,d}$ for ℓ from 0 to $m - 1$ at no extra cost.

The availability of all these intermediate results is a nice side effect since then the cost for calculating $X_{m_1,m_2,d}$, which involves $e_{p^\ell,d}$ for all ℓ from m_1 to m_2 , is no worse than the cost for calculating just $e_{p^{m_2},d}$. This brings us to the main result of this paper.

THEOREM 4.5 (Fast construction of embedded lattice rules). *Construction of embedded lattice rules up to $n = p^{m_2}$ points, with p prime, using Algorithm 2 can be done in $\mathcal{O}(dn \log(n))$ time, with $\mathcal{O}(n)$ memory for product weights and $\mathcal{O}(qn)$ memory for order-dependent weights of order q . The total construction cost including the pre-calculations of Algorithm 1 is therefore $\mathcal{O}(dn(\log(n))^2)$.*

5. Using embedded lattice rules.

5.1. Ordering of lattice points. The main advantage of having a sequence of embedded lattice rules $L_0 \subset L_1 \subset L_2 \subset \dots$ is that we can increase the number of points without having to throw away already computed function values. When the

base is b (not necessary prime anymore), we increase the number of points by a factor of b each step along the sequence. For $\ell \geq 1$, let $\Delta L_\ell := L_\ell \setminus L_{\ell-1}$ denote the set of additional points from the point set $L_{\ell-1}$ to L_ℓ . We recall from Property 4 that the points in $L_{\ell-1}$ are generated by the indices k which are multiples of b in $\mathbb{Z}_{b^\ell}$. Thus we can write

$$L_m = L_0 \cup \left(\bigcup_{\ell=1}^m \Delta L_\ell \right), \quad (5.1)$$

with $L_0 = \{\mathbf{0}\}$ and

$$\Delta L_\ell = \left\{ \left\{ \frac{k\mathbf{z}}{b^\ell} \right\} : k \in (\mathbb{Z}_{b^\ell} \setminus b\mathbb{Z}_{b^{\ell-1}}) \right\}. \quad (5.2)$$

Letting $m \rightarrow \infty$ in (5.1) leads formally to an extensible lattice sequence L_∞ with an infinite number of points. In this paper we consider lattice point sets where the number of points is finite, that is, the union in (5.1) only goes to a predefined upper limit m_2 . But, due to the fast construction, this limit can be taken rather large.

Note that the set of additional points ΔL_ℓ from the point set $L_{\ell-1}$ to L_ℓ actually comprises of $b-1$ shifted versions of $L_{\ell-1}$. We put this forward as another property of embedded lattice rules (see also [7, Theorem 2.3]).

PROPERTY 5. The lattice point set L_m can be considered as the union of b shifted versions (dependent on $\mathbf{z}$) of the lattice point set L_{m-1} , as well as the union of b^2 shifted versions of L_{m-2} , and so on. Trivially, L_m is the union of b^m shifted versions of the initial point $\mathbf{0}$.

Explicitly, we have

$$L_m = \bigcup_{k_0=0}^{b-1} \cdots \bigcup_{k_\ell=0}^{b-1} \left(L_{m-\ell-1} + \frac{k_\ell \mathbf{z}}{b^{m-\ell}} + \cdots + \frac{k_0 \mathbf{z}}{b^m} \right) \quad \text{for all } \ell = 0, \dots, m-1,$$

where the addition is interpreted as in the definition $P_n + \Delta := \{\{\mathbf{t} + \Delta\} : \mathbf{t} \in P_n\}$ (the inner braces indicate taking the fractional part).

With the generating vector constructed by Algorithm 2 for n ranging from b^{m_1} to b^{m_2} , we can start with an approximation of the integral using the point set L_{m_1} , and then gradually add blocks of additional points $\Delta L_{m_1+1}, \Delta L_{m_1+2}, \dots, \Delta L_{m_2}$ until we are satisfied with the error (provided that m_2 is large enough). This usage of the sequence of embedded lattice rules is justified by the knowledge that every lattice point set L_m is good for $m_1 \leq m \leq m_2$.

A more interesting usage would be to apply the lattice rule one point at a time, i.e., as a low-discrepancy sequence. Since we may end up with an incomplete lattice point set, there is no guarantee that these additional points will be well distributed over the unit cube, and we could even end up with these points all falling in one half of the cube. In fact, we are sure to get a bad ordering of the points if we generate the points from each block ΔL_ℓ according to the natural order of the k indices in (5.2). The solution to this inappropriate ordering of the points is given by replacing k/b^ℓ in (5.2) with the radical inverse function, possibly in combination with Gray code ordering. These schemes provide a better ordering of the numbers k/b^ℓ in the unit interval, and this property is preserved under multiplication by $\mathbf{z}$ (and then taking the fractional part) so that the ordering of every one-dimensional projection of the lattice points is good. Both schemes are well-known tools borrowed from the vast literature on digital sequences, see for example [12].

The *radical inverse* function takes an integer in base b and mirrors it around the radix point. For an integer $k = \sum_{\ell=0}^{m-1} k_\ell b^\ell = (k_{m-1} \dots k_0)_b$ the radical inverse is given as $\phi_b(k) = \sum_{\ell=0}^{m-1} k_\ell b^{-(\ell+1)} = (0.k_0 k_1 \dots k_{m-1})_b$. The *Gray code* variant uses this same transformation, but starts with a Gray code ordering in base b of the k indices (i.e., successive indices only differ in one b -ary digit). The latter is computationally more efficient since then the next index can be obtained from the previous one by a simple digit change and the actual radical inversion can be avoided.

We summarize the usage of these schemes as a property of embedded lattice rules. Note that they preserve the embedding property $L_0 \subset L_1 \subset \dots$ of the lattice point sets, while providing a better ordering of the points within each block of additional points ΔL_ℓ . More discussions can be found in [7].

PROPERTY 6. The lattice point set L_m can be used as a low-discrepancy sequence if the points are generated in the order

$$\mathbf{t}_k = \{\varphi_b(k) \mathbf{z}\} \quad \text{for } k = 0, 1, \dots, b^m - 1,$$

where $\varphi_b(k)$ denotes either the radical inverse $\phi_b(k)$ of k in base b , or the corresponding Gray code variant.

We end this subsection with a simple illustration for the most practical case of $b = 2$:

$$\begin{array}{c} \underbrace{t_0 = 0}_{L_0}, \underbrace{t_1 = \frac{1}{2}}_{\Delta L_1}, \underbrace{t_2, t_3}_{\Delta L_2}, \underbrace{t_4, t_5, t_6, t_7}_{\Delta L_3}, \underbrace{t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}}_{\Delta L_4}, \dots \\ \underbrace{\hspace{1.5cm}}_{L_1} \\ \underbrace{\hspace{2.5cm}}_{L_2} \\ \underbrace{\hspace{3.5cm}}_{L_3} \\ \underbrace{\hspace{4.5cm}}_{L_4} \end{array}$$

The number of points doubles as we move along the sequence, and the set of additional points ΔL_ℓ is generated by $\{k\mathbf{z}/2^\ell\}$ with k running through all the odd numbers up to $2^\ell - 1$. In Table 5.1 we show how the points are ordered under radical inverse and the Gray code variant.

5.2. Error estimation. There is a comprehensive discussion on error estimation in [7] including error estimation based on the embedding structure of the lattice rules. Here we only focus on the simplest form of error estimation using random shifts.

In practice, once we have found a good generating vector $\mathbf{z}$, we generate a number of independent random shifts $\Delta_1, \Delta_2, \dots, \Delta_w$ and compute the approximations $Q_1, Q_2, \dots, Q_w$, where Q_i is the shifted lattice rule approximation to the integral $I_d(f)$ based on the generating vector $\mathbf{z}$ and the shift Δ_i . Then we take the average

$$\bar{Q} := \frac{1}{w}(Q_1 + Q_2 + \dots + Q_w)$$

as our final approximation to $I_d(f)$. An unbiased estimate for the standard error of our approximation is given by

$$\hat{\sigma} := \sqrt{\frac{1}{w(w-1)} \sum_{i=1}^w (Q_i - \bar{Q})^2}.$$

Using the Chebyshev inequality we can obtain a $(100 - 100/v^2)\%$ confidence interval $\bar{Q} \pm v\hat{\sigma}$ (with $v > 1$).

TABLE 5.1
Ordering of lattice points

	k	natural order	radical inverse	Gray code variant
L_0	0	0	0	0
ΔL_1	1	$1/2 = 0.5$	$(0.1)_2 = 0.5$	$(0.1)_2 = 0.5$
ΔL_2	2	$1/4 = 0.25$	$(0.01)_2 = 0.25$	$(0.11)_2 = 0.75$
	3	$3/4 = 0.75$	$(0.11)_2 = 0.75$	$(0.01)_2 = 0.25$
ΔL_3	4	$1/8 = 0.125$	$(0.001)_2 = 0.125$	$(0.011)_2 = 0.375$
	5	$3/8 = 0.375$	$(0.101)_2 = 0.625$	$(0.111)_2 = 0.875$
	6	$5/8 = 0.625$	$(0.011)_2 = 0.375$	$(0.101)_2 = 0.625$
	7	$7/8 = 0.875$	$(0.111)_2 = 0.875$	$(0.001)_2 = 0.125$
ΔL_4	8	$1/16 = 0.0625$	$(0.0001)_2 = 0.0625$	$(0.0011)_2 = 0.1875$
	9	$3/16 = 0.1875$	$(0.1001)_2 = 0.5625$	$(0.1011)_2 = 0.6875$
	10	$5/16 = 0.3125$	$(0.0101)_2 = 0.3125$	$(0.1111)_2 = 0.9375$
	11	$7/16 = 0.4375$	$(0.1101)_2 = 0.8125$	$(0.0111)_2 = 0.4375$
	12	$9/16 = 0.5625$	$(0.0011)_2 = 0.1875$	$(0.0101)_2 = 0.3125$
	13	$11/16 = 0.6875$	$(0.1011)_2 = 0.6875$	$(0.1101)_2 = 0.8125$
	14	$13/16 = 0.8125$	$(0.0111)_2 = 0.4375$	$(0.1001)_2 = 0.5625$
	15	$15/16 = 0.9375$	$(0.1111)_2 = 0.9375$	$(0.0001)_2 = 0.0625$
$\vdots$				

6. Numerical experiments.

6.1. Results from CBC construction. We construct good generating vectors of embedded lattice rule up to 360 dimensions using Algorithm 2 for n ranging from 2^{10} to $2^{20} \approx 1$ million (i.e. $b = 2$, $m_1 = 10$, and $m_2 = 20$). Our search criterion is the quantity $X_{m_1, m_2, d}(\mathbf{z})$ defined in (1.3), i.e.,

$$X_{10, 20, d}(\mathbf{z}) = \max_{10 \leq m \leq 20} \frac{e_{2^m, d}(\mathbf{z})}{e_{2^m, d}(\mathbf{z}^{(m)})},$$

where $\mathbf{z}^{(m)}$ is the generating vector constructed using Algorithm 1 for a 2^m -point rule (which we refer to as a good “fixed” rule). Let $\mathbf{z}^*$ denote the vector resulting from Algorithm 2. Then the quantity $X_{10, 20, d}(\mathbf{z}^*)$ tells us how far off the vector $\mathbf{z}^*$ is compared with the vectors $\mathbf{z}^{(m)}$ in terms of the maximum (over m) ratio of the worst-case errors.

We consider weighted Sobolev spaces (see Example 2.2) with both product weights and order-dependent weights. In particular, we consider equal product weights, decaying product weights, order-2 weights and order-3 weights. Our choices of weights together with the maxima of $X_{10, 20, d}$ over all $1 \leq d \leq 360$ are given in Table 6.1.

TABLE 6.1
Maximum of $X_{10, 20, d}(\mathbf{z}^*)$ for various weights

Weights	$\max_d X_{10, 20, d}(\mathbf{z}^*)$	Weights	$\max_d X_{10, 20, d}(\mathbf{z}^*)$
product 1	1.53	product $1/j$	1.41
product 0.5	1.48	product $1/j^2$	1.31
product 0.3	1.47	order-2	1.43
product 0.1	1.41	order-3 1	1.42
product 0.05	1.37	order-3 0.5	1.55
product 0.01	1.30	order-3 0.3	1.45
product 0.9^j	1.50	order-3 0.1	1.47
product 0.75^j	1.42	order-3 0.05	1.45
product 0.5^j	1.30	order-3 0.01	1.43

The product-type weights in the table are self-explanatory: “product 0.5” means $\gamma_j = 0.5$ and “product $1/j$ ” means $\gamma_j = 1/j$. We shall explain briefly our choices of order-2 and order-3 weights. Firstly, since all one-dimensional projections of our lattice rules are left-rectangle rules with all points distinct, the weight Γ_1 in an order-dependent weight setting has no essential effect on the choice of the rule. We thus set $\Gamma_1 = 1$. Secondly, if we decide to construct a rule based on order-2 weights (i.e. if $\Gamma_\ell = 0$ for $\ell > 2$), then the rule is unique regardless of how we choose Γ_2 , since in this case Γ_2 only rescales the worst-case error. Therefore without loss of generality we set $\Gamma_2 = 1$. In other words, such a rule is universally applicable for all order-2 problems. For order-3 weights, the rule depends only on the ratio Γ_3/Γ_2 . Hence we use Γ_3 to specify the importance of order-3 interactions compared with order-2 interactions. In our table “order-3 0.1” corresponds to the choice $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = 0.1$. Note that having equal product weights $\gamma_j = r$ is equivalent to having order-dependent weights with $\Gamma_\ell = r^\ell$. The generating vector for the order-2 rule is given in the Appendix. From Table 6.1, the worst-case errors for the order-2 rule in the Appendix are within a factor of 1.43 of the worst-case errors obtained with the results from the original CBC construction.

Starting from $X_{10,20,1}(\mathbf{z}^*) = 1$, the empirical observation is that in most cases the numbers $X_{10,20,d}(\mathbf{z}^*)$ increase initially as d increases, then they decrease from some dimension onward. We see from Table 6.1 that $X_{10,20,d}(\mathbf{z}^*)$ is substantially less than 1.6 for all our choices of weights. Thus we conclude that, in all our experiments, the embedded lattice rules are at most 1.6 times worse than the near-optimal rules constructed from Algorithm 1 for specific values of n . Put differently, we have rules with the optimal rate of convergence, but with a constant which is a factor of at most 1.6 times larger.

Enlarging the range of m reduces the overall quality. Fig. 6.1 shows the profile of $X_{10,m_2,d}(\mathbf{z}^*)$ with increasing d for order-2 rules with $m_2 = 15, 20$, and 25. It is understandable that by allowing the range of n to grow, the quality of the point set is sacrificed. But we observe that the quality remains very good for all feasible values of m_2 .

Using initial segments of a good embedded lattice rule works as expected. The criterion $X_{10,20,d}(\mathbf{z}^*)$ ensures that the resulting vector $\mathbf{z}^*$ is good for a 2^m -point rule for all values of m between 10 and 20. In other words, every point set in the embedded sequence $L_{10} \subset L_{11} \subset \dots \subset L_{20}$ is good. (We refer to the smaller lattice point sets L_ℓ with $\ell < 20$ as the “initial segments” of the point set L_{20} .) In Fig. 6.2 we plot the worst-case error ratios

$$\frac{e_{2^m,d}(\mathbf{z}^*)}{e_{2^m,d}(\mathbf{z}^{(m)})}$$

for the case of the order-2 rule in the Appendix and every $m = 10, \dots, 20$.

As expected, the maximum of all these graphs corresponds to the graph for $X_{10,20,d}(\mathbf{z}^*)$ in Fig. 6.1. For most values of m we do better than $X_{10,20,d}(\mathbf{z}^*)$ promises, but we note that all the individual curves actually stay close to $X_{10,20,d}(\mathbf{z}^*)$. We also see that these curves cross each other in an erratic manner, which is important, otherwise we could redefine the error measure to make use of some assumed ordering. That said, in general it can be observed that the curves for higher m tend to be closer to the top.

For high dimensions we see that the curves for $m = 10$ and $m = 12$ depart from the general trend. This might be explained by observing that the differences

FIG. 6.1. Profile of $X_{10,m_2,d}(\mathbf{z}^*)$ for order-2 rules with $m_2 = 15, 20$, and 25 as d increases

FIG. 6.2. Initial segments of a good embedded lattice rule

between the worst-case errors of different choices for a new component decrease as the dimension increases. In other words, the different choices of generating vectors are equally bad when there are very few points in very high dimensions.

Using initial segments of a good fixed lattice rule does not work. We could wonder if it would be satisfactory to just use the initial segments from a good

FIG. 6.3. *Initial segments of a good fixed lattice rule*

2^{20} -point lattice rule with generating vector $\mathbf{z}^{(20)}$ constructed by Algorithm 1. (As before, by initial segments we mean the smaller lattice point sets L_ℓ embedded in L_{20} with $\ell < 20$.) If these initial segments would be any good, then there would be little point to use Algorithm 2. In Fig. 6.3 we plot the worst-case error ratios

$$\frac{e_{2^m,d}(\mathbf{z}^{(20)})}{e_{2^m,d}(\mathbf{z}^{(m)})}$$

for every $m = 10, \dots, 20$.

It is obvious from the figure that using a large good fixed lattice rule for a maximum number of points $n = 2^{m_2}$ tells you nothing about the goodness of the initial segments of this lattice rule. While $X_{10,20,d}(\mathbf{z}^*)$ is bounded by 1.45 for the good embedded lattice rule, for the fixed lattice rule, the corresponding ratio is 35 times worse than the optimal fixed lattice rule for $m = 12$. Once again there is no direct ordering in these curves, but it is obvious that the curves corresponding to higher m are lower on the graph. For $m = 20$ we have a horizontal line at 1, and the line just above this constant line is the one for $m = 19$. It is not surprising that using half of the points of a good fixed lattice rule would also give a reasonable rule. The maximum of all these curves is clearly very bad. We can draw a very clear conclusion from this graph: the embedded lattice rule constructed by Algorithm 2 works very well in comparison with a more simple-minded strategy.

6.2. Pricing an Asian option. We test the performance of the embedded lattice rules constructed in this paper for calculating the pricing of an Asian call option. This problem is extensively documented, both in the financial literature and in the QMC literature [9, 11, 2]. For an Asian option (also called average option) the payoff is determined by the average value of the underlier (i.e. a stock price) during its life time. We will consider a (European style) Asian option with arithmetic average of

d equally spaced stock prices S_j at times $t_j = jT/d$, with expiry time T . The payoff for such a call option can be written as

$$C(T) = \max \left(\frac{1}{d} \sum_{j=1}^d S_j - K, 0 \right) = \left(\frac{1}{d} \sum_{j=1}^d S_j - K \right)^+,$$

where K is the strike price at the expiry time $T = t_d$. We assume the risk-neutral measure of the Black-Scholes model for the path of the stock prices

$$S_j = S_0 \exp \left((r - \sigma^2/2) t_j + \sigma w_j \right),$$

where S_0 is the initial stock price, r is the risk-free interest rate, σ the volatility of the stock price and the w_j are samples of a Brownian motion at times t_j , i.e. w_j is normally distributed with variance t_j , or $w_j \sim \mathcal{N}(0, t_j)$. This sampled Brownian motion has mean 0 and covariance matrix $\Sigma = [\min(t_i, t_j)]_{i,j=1,\dots,d}$.

The value of the Asian option at time 0 can then be written as the actualized expected payoff at time T ,

$$\begin{aligned} C(0) &= \mathbb{E}(\exp(-rT) C(T)) \\ &= \exp(-rT) \int_{\mathbb{R}^d} \left(\frac{1}{d} \sum_{j=1}^d S_0 \exp \left((r - \sigma^2/2) t_j + \sigma w_j \right) - K \right)^+ \mathcal{N}(\mathbf{w}; \mathbf{0}, \Sigma) d\mathbf{w}. \end{aligned}$$

We transform this integral to $[0, 1]^d$ by using the inverse of the cumulative normal distribution function, Φ^{-1} , and deal with the correlation in our sample points by using a factorization of the covariance matrix, $\Sigma = \mathbf{A}^\top \mathbf{A}$. Then our lattice rule approximation to $C(0)$ is given by

$$\tilde{C}(0) = \frac{\exp(-rT)}{n} \sum_{k=0}^{n-1} \left(\frac{1}{d} \sum_{j=1}^d S_0 \exp \left((r - \sigma^2/2) t_j + \sigma w_{k,j} \right) - K \right)^+,$$

where $\mathbf{w}_k = \mathbf{A} \Phi^{-1}(\mathbf{t}_k)$ with lattice point $\mathbf{t}_k$ for $k = 0, 1, \dots, n-1$.

For our particular example we choose the following parameters:

$$S_0 = 100, \quad r = 0.1, \quad \sigma = 0.2, \quad T = 1, \quad K = 100, \quad d = 100.$$

We choose the principal component analysis method to factorize the matrix Σ , that is, $\mathbf{A} = [\sqrt{\lambda_1} \mathbf{v}_1, \dots, \sqrt{\lambda_d} \mathbf{v}_d]$ where $\lambda_1 \geq \dots \geq \lambda_d$ are the eigenvalues of Σ and $\mathbf{v}_1, \dots, \mathbf{v}_d$ are the corresponding unit-length eigenvectors.

In Table 6.2 we present the results obtained with order-2 weights using 10 random shifts. As a comparison we include also results obtained by the Monte Carlo method (where we also used 10 random shifts). In Fig. 6.4 we plot the graphs of the standard errors obtained by adding one point at a time. (The values of n in both Table 6.2 and Fig. 6.4 correspond to the number of points used from the sequence. Due to the use of 10 random shifts, the actual number of function evaluations in each case is $10n$.) The results for the order-2 rule clearly show a rate of convergence close to $\mathcal{O}(n^{-1})$ while the Monte Carlo error decays like $\mathcal{O}(n^{-1/2})$. Furthermore, the graph assures us that it is quite all right to use the embedded lattice rules as a sequence, since the standard error goes down nicely (with some minor zigzag behavior, as expected) as the number of points is increased one at a time.

TABLE 6.2
Results of the Asian option for Monte Carlo and the order-2 lattice rule

n	order-2 lattice rule		Monte Carlo	
	integral	standard error	integral	standard error
1024	7.11310	5.07e-03	7.13816	8.97e-02
2048	7.10685	1.77e-03	7.10988	5.98e-02
4096	7.10252	1.56e-03	7.12438	4.84e-02
8192	7.10261	6.47e-04	7.11989	2.54e-02
16384	7.10304	4.17e-04	7.12116	2.10e-02
32768	7.10324	2.34e-04	7.12404	1.49e-02
65536	7.10280	1.18e-04	7.11410	1.14e-02
131072	7.10293	4.70e-05	7.11427	9.58e-03
262144	7.10287	1.91e-05	7.10980	5.05e-03
524288	7.10285	1.69e-05	7.10636	5.00e-03
1048576	7.10285	8.68e-06	7.10709	1.69e-03

FIG. 6.4. Standard errors of the Asian option for Monte Carlo and the order-2 lattice rule

7. Conclusion. We presented an algorithm for constructing good embedded lattice rules that are extensible in the dimension d and can be used for any number of points n in a pre-defined range. The algorithm utilized both the component-by-component construction principle and the fast implementation idea. For product weights and order-dependent weights, the total construction cost is $\mathcal{O}(dn(\log(n))^2)$ operations, with a linear memory dependence on n . Numerical experiments with the unanchored Sobolev space demonstrated that these rules achieve the same optimal rate of convergence $\mathcal{O}(n^{-1+\delta})$, $\delta > 0$, as those constructed for a fixed number of points, and that the implied constant only gains a factor of 1.30 to 1.55.

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REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN, eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, vol. 55 of National Bureau of Standards Applied Mathematics Series, U.S. Government Printing Office, Washington, D.C., 1964.
- [2] P. P. BOYLE, Y. LAI, AND K. S. TAN, *Pricing options using lattice rules*, North Amer. Act. J., (2005), pp. 50–76.
- [3] R. E. CAFLISCH, W. MOROKOFF, AND A. OWEN, *Valuation of mortgage backed securities using Brownian bridges to reduce effective dimension*, J. Comput. Finance, (1997), pp. 27–46.
- [4] J. DICK, *On the convergence rate of the component-by-component construction of good lattice rules*, J. Complexity, 20 (2004), pp. 493–522.
- [5] J. DICK, I. H. SLOAN, X. WANG, AND H. WOŹNIAKOWSKI, *Good lattice rules in weighted Korobov spaces with general weights*, Numer. Math. To appear.
http://www.maths.unsw.edu.au/applied/files/2003/amr03_24.pdf.
- [6] F. J. HICKERNELL, *Lattice rules: How well do they measure up?*, in Random and Quasi-Random Point Sets, P. Hellekalek and G. Larcher, eds., Springer-Verlag, 1998, pp. 109–166.
- [7] F. J. HICKERNELL, H. S. HONG, P. L'ÉCUYER, AND C. LEMIEUX, *Extensible lattice sequences for quasi-Monte Carlo quadrature*, SIAM J. Sci. Comput., 22 (2000), pp. 1117–1138.
- [8] F. J. HICKERNELL AND H. NIEDERREITER, *The existence of good extensible rank-1 lattices*, J. Complexity, 19 (2003), pp. 286–300.
- [9] C. JOY, P. P. BOYLE, AND K. S. TAN, *Quasi-Monte Carlo methods in numerical finance*, Management Science, (1996), pp. 926–938.
- [10] F. Y. KUO, *Component-by-component constructions achieve the optimal rate of convergence for multivariate integration in weighted Korobov and Sobolev spaces*, J. Complexity, 19 (2003), pp. 301–320.
- [11] C. LEMIEUX AND P. L'ÉCUYER, *Efficiency improvement by lattice rules for pricing Asian options*, in Proceedings of the 30th conference on Winter simulation, D. J. Medeiros, E. F. Watson, J. Carson, and M. Manivannan, eds., IEEE, 1998, pp. 579–586.
- [12] H. NIEDERREITER, *Random Number Generation and Quasi-Monte Carlo Methods*, no. 63 in Regional Conference Series in Applied Mathematics, SIAM, 1992.
- [13] D. NUYENS AND R. COOLS, *Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces*, Math. Comp., 75 (2006), pp. 903–920.
- [14] ———, *Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points*, J. Complexity, 22 (2006), pp. 4–28.
- [15] C. M. RADER, *Discrete Fourier transforms when the number of data samples is prime*, Proc. IEEE, 5 (1968), pp. 1107–1108.
- [16] I. H. SLOAN, F. Y. KUO, AND S. JOE, *Constructing randomly shifted lattice rules in weighted Sobolev spaces*, SIAM J. Numer. Anal., 40 (2002), pp. 1650–1665.
- [17] I. H. SLOAN, X. WANG, AND H. WOŹNIAKOWSKI, *Finite-order weights imply tractability of multivariate integration*, J. Complexity, 20 (2004), pp. 46–74.
- [18] C. F. VAN LOAN, *Computational Frameworks for the Fast Fourier Transform*, vol. 10 of Frontiers in Applied Mathematics, SIAM, 1992.
- [19] X. WANG AND K. T. FANG, *The effective dimension and quasi-Monte Carlo integration*, J. Complexity, 19 (2003), pp. 101–124.

Appendix. Order-2 embedded lattice rule for $2^{10} \leq n \leq 2^{20}$.

d	z_d	d	z_d	d	z_d	d	z_d	d	z_d	d	z_d
1	1	61	390955	121	151423	181	455605	241	8601	301	83785
2	182667	62	202915	122	492123	182	227317	242	118219	302	11217
3	302247	63	230815	123	166279	183	251285	243	503827	303	163315
4	433461	64	304301	124	335461	184	295275	244	368993	304	293397
5	160317	65	452315	125	295093	185	476155	245	439439	305	386597
6	94461	66	383101	126	98821	186	291155	246	153211	306	338761
7	481331	67	70519	127	118331	187	294443	247	282303	307	88653
8	252345	68	346393	128	371123	188	31913	248	252999	308	337581
9	358305	69	291605	129	166909	189	518445	249	28617	309	230703
10	221771	70	397773	130	369553	190	481917	250	514327	310	140519
11	48157	71	92713	131	310959	191	326981	251	422201	311	421913
12	489023	72	391775	132	130595	192	488711	252	465011	312	416151
13	438503	73	131613	133	417025	193	447541	253	160907	313	197025
14	399693	74	74351	134	264103	194	39629	254	191687	314	350607
15	200585	75	382127	135	453203	195	394681	255	226205	315	262579
16	169833	76	219343	136	497985	196	379411	256	254941	316	510879
17	308325	77	297125	137	200509	197	335309	257	480375	317	451713
18	247437	78	88545	138	269743	198	93541	258	322857	318	8619
19	281713	79	89837	139	461919	199	188491	259	456621	319	50451
20	424209	80	191863	140	45927	200	152371	260	446175	320	193793
21	244841	81	506647	141	394663	201	408829	261	109049	321	155739
22	205461	82	441649	142	299155	202	217957	262	285305	322	23611
23	336811	83	240063	143	81299	203	65145	263	436731	323	437689
24	359375	84	239067	144	112403	204	462013	264	290763	324	100267
25	86263	85	310875	145	447473	205	220627	265	290439	325	439671
26	370621	86	211625	146	480325	206	368989	266	25363	326	211341
27	422443	87	147791	147	105053	207	273585	267	480371	327	22951
28	284811	88	370849	148	328455	208	373297	268	153337	328	73405
29	231547	89	149445	149	513239	209	285793	269	406899	329	220037
30	360239	90	340329	150	322199	210	405807	270	90863	330	169733
31	505287	91	493031	151	47537	211	63693	271	78537	331	408633
32	355195	92	336897	152	485183	212	382573	272	358757	332	64171
33	52937	93	202595	153	490687	213	84291	273	69087	333	44141
34	344561	94	518247	154	214311	214	444801	274	431749	334	122149
35	286935	95	369599	155	61871	215	226471	275	384083	335	258491
36	312429	96	50453	156	359761	216	166195	276	472419	336	215021
37	513879	97	133655	157	509981	217	170689	277	298375	337	11507
38	171905	98	395941	158	192829	218	368423	278	291499	338	295341
39	50603	99	13871	159	17075	219	509169	279	187231	339	41981
40	441451	100	248123	160	486463	220	243221	280	6967	340	378867
41	164379	101	380725	161	463461	221	191447	281	338357	341	261545
42	139609	102	314879	162	438237	222	200867	282	411965	342	56423
43	371213	103	106903	163	444353	223	194633	283	447817	343	445605
44	152351	104	94505	164	381279	224	226469	284	135463	344	362017
45	138607	105	499197	165	425405	225	413665	285	156061	345	504897
46	441127	106	479671	166	179723	226	148007	286	356943	346	335763
47	157037	107	76553	167	203897	227	284505	287	224431	347	417407
48	510073	108	351609	168	300411	228	459795	288	452203	348	58033
49	281681	109	13815	169	35087	229	324557	289	284195	349	80239
50	380297	110	342069	170	501519	230	422149	290	489071	350	372353
51	208143	111	446599	171	172275	231	368087	291	299537	351	116163
52	497641	112	374429	172	278507	232	166133	292	313173	352	287617
53	482925	113	342011	173	473739	233	401441	293	436265	353	483795
54	233389	114	8365	174	141429	234	44077	294	316447	354	310473
55	238553	115	424079	175	117025	235	457535	295	8353	355	75779
56	121499	116	184381	176	516465	236	122611	296	512723	356	141943
57	137783	117	203853	177	204743	237	27489	297	522699	357	418435
58	463115	118	7937	178	505099	238	392389	298	453557	358	305609
59	168681	119	302301	179	359937	239	327231	299	512829	359	152901
60	70699	120	444747	180	359503	240	87841	300	214315	360	129869