Integrating Inductive Definitions in SAT

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Abstract. We investigate techniques for supporting inductive definitions (IDs) in SAT, and report on an implementation, called MIDL, of the resulting solver. This solver was first introduced in [11], as a part of a declarative problem solving framework. We go about our investigation by proposing a new formulation of the semantics of IDs as presented in [2]. This new formulation suggests a way to perform the computational task involved, resulting in an algorithm supporting IDs. We show in detail how to integrate our algorithm with traditional SAT solving techniques. We also point out the similarities with another algorithm that was recently developed for ASP [1]. Indeed, our formulation reveals a very tight relation with stable model semantics. We conclude by an experimental validation of our approach using MIDL.

1 Introduction

This work is motivated by the following observations:

- contemporary SAT solvers exhibit impressive performance;
- SAT provides a poor modelling language;
- the ability to express inductive definitions (or recursion) is present in many important knowledge representation formalisms.

The first two observations provide an impetus to extend SAT with language constructs that yield a better modelling language, without giving up too much on the performance side. Adding to this, the third observation shows what would make a better modelling language. Hence this paper focuses on providing computational support for an extension of SAT with (propositional) inductive definitions.

An extension of classical logic with inductive definitions, FO(ID), is given in [2]. A first model generator for the propositional fragment of this logic, SAT(ID), was presented in [10]. An improved version of this solver was reported on in [11], as a part of a declarative problem solving framework. This paper presents the algorithms of this new solver.

Our work is aimed at closing the computational gap between SAT(ID) and “pure” SAT. Hence we try to show how ID support can be integrated in state-of-the-art SAT solving techniques. First, we give an alternative definition of the semantics of SAT(ID). This alternative definition leads to a natural understanding of the computational task of a SAT(ID) solver. In particular, the new definition contains two properties that could be used as invariants for such a solver. We

show that applying SAT’s *two watched literal* technique [14] on (Clark’s completion of) an ID suffices to satisfy the first property. A more involved algorithm to satisfy the second property is worked out. It turns out to be very similar to an algorithm for finding unfounded sets proposed by [1].¹ As we show a strong relation between SAT(ID) and the stable model semantics [5], which is the semantics used in Answer Set Programming (ASP) [9], this similarity is not unsurprising.

Combining these two algorithms, we end up with an extension of the DPLL algorithm. As such, a lot of techniques that lead to the impressive performance of SAT solvers can be used in a SAT(ID) solver.

Finally, we present results obtained with MIDL, an implementation of the discussed algorithms. A comparison with some SAT and ASP solvers shows that MIDL is competitive with state-of-the art ASP solvers, while there is still an efficiency gap with SAT solvers on problems containing no recursion.

2 Preliminaries

2.1 SAT(ID)

In this section, we introduce SAT(ID), an extension of propositional logic with inductive definitions. We assume familiarity with propositional logic.

A vocabulary Σ is a set of atoms. A literal is an atom P or its negation $\neg P$. An atom P is called a *positive* literal, $\neg P$ a *negative* one. For a literal L , we identify $\neg\neg L$ with L . For a set S of literals, we denote by $\bar{S}$ the set $\{\neg L \mid L \in S\}$, and by S^* the set $S \cup \bar{S}$.

A *definition* over Σ is a finite set of rules of the form $P \leftarrow \varphi$ where $P \in \Sigma$ is an atom and φ is an arbitrary propositional formula over Σ . P is called the *head* of the rule and φ the *body*. For a definition Δ , an atom P occurring as head of a rule in Δ is called a *defined* atom of Δ . All other atoms are called *open* atoms of Δ . The set of all defined, respectively open atoms of Δ is denoted by $Def(\Delta)$, respectively $Open(\Delta)$. We say that an atom occurs positively (negatively) in a propositional formula if it occurs in the scope of an even (odd) number of negations. We call a definition Δ *positive* if each occurrence of an atom in the body of a rule in Δ is positive.

A *SAT(ID) theory* is a set of propositional formulae and definitions.

A three-valued Σ -interpretation I is a function $I : \Sigma \rightarrow \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$. An interpretation is two-valued if it maps no atom to $\mathbf{u}$. The restriction of I to a set $\sigma \subset \Sigma$ is denoted $I|_{\sigma}$. The *truth order* $\leq$ on $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ is induced by $\mathbf{f} \leq \mathbf{u} \leq \mathbf{t}$ and the *precision order* $\leq_p$ by $\mathbf{u} \leq_p \mathbf{f}$ and $\mathbf{u} \leq_p \mathbf{t}$. Both orders pointwise extend to interpretations. Define $\mathbf{f}^{-1} = \mathbf{t}$, $\mathbf{u}^{-1} = \mathbf{u}$ and $\mathbf{t}^{-1} = \mathbf{f}$. An interpretation I on Σ can be extended inductively to propositional formulae over Σ :

$$I(\varphi \wedge \psi) = \min_{\leq}(\{I(\varphi), I(\psi)\}), \quad I(\varphi \vee \psi) = \max_{\leq}(\{I(\varphi), I(\psi)\}), \quad \text{and} \quad I(\neg \varphi) = I(\varphi)^{-1}.$$

¹ We independently developed our algorithm, and believe that the new insights gained by the alternative presentation are a valuable asset.

We say that I satisfies φ , denoted by $I \models \varphi$, if $I(\varphi) = \mathbf{t}$.

We now introduce both the stable and well-founded semantics for definitions. The semantics of definitions in SAT(ID) will be given by the latter. Our presentation is based on [16], where the first three-valued characterisation of the stable model semantics was given: such a characterisation is closer to actual computational processes, where partial interpretations are used.

Let Σ be a vocabulary, Δ a definition over Σ and I_O an $Open(\Delta)$ -interpretation. Denote by L_Δ the set of all Σ -interpretations extending I_O and define the operator $\Psi_\Delta : L_\Delta \rightarrow L_\Delta$ by $\Psi_\Delta(I)(P) = I(\bigvee_{P \leftarrow \varphi \in \Delta} \varphi)$ if $P \in Def(\Delta)$, and $\Psi_\Delta(I)(P) = I(P)$ otherwise. If Δ is a positive definition, Ψ_Δ is $\leq$ -monotone, and the *least model of Δ extending I_O* is defined as the $\leq$ -least fixpoint of Ψ_Δ .

Let I be a 3-valued Σ -interpretation. The *reduct* of Δ in I , denoted by Δ^I , is the definition obtained by replacing in every rule all open atoms and all negative occurrences of defined atoms P by $I(P)$. The reduct is a positive definition.

Definition 1 (Stable model). *Let Δ be a Σ -definition, I_O an $Open(\Delta)$ -interpretation. Then a Σ -interpretation I is a (three-valued) stable model of Δ extending I_O iff $I|_{Def(\Delta)}$ is the least model of Δ^I extending I_O and $I|_{Open(\Delta)} = I_O$.*

Note that in the standard definition of stable models [5] atoms in $Open(\Delta)$ are considered false. Intuitively, an atom $P \in Open(\Delta)$ here corresponds to an atom P defined by “ $P \leftarrow \text{not } P'$. $P' \leftarrow \text{not } P$.” in the standard definition.

It is shown in [16] that for every definition Δ and $Open(\Delta)$ -interpretation I_O , there exists at least one stable model of Δ extending I_O . Also, the greatest lower bound with respect to $\leq_p$ of the set of all stable models of Δ extending I_O is itself a stable model of Δ extending I_O .

Definition 2 (Well-founded model). *Let Δ be a definition, I_O an $Open(\Delta)$ -interpretation. The well-founded model of Δ extending I_O is the $\leq_p$ -least stable model of Δ extending I_O .*

An interpretation I satisfies a definition Δ , denoted $I \models \Delta$, if I is the well-founded model of Δ extending $I|_{Open(\Delta)}$ and I is two-valued. Finally, I satisfies a SAT(ID) theory T if I satisfies every formula and every definition of T .

Observe that we are only interested in two-valued models of definitions. We call a definition Δ *total* if for every two-valued $Open(\Delta)$ -interpretation I_O , the well-founded model M of Δ extending I_O is two-valued. In this case, M is also the unique stable model of Δ extending I_O . Definitions that are encountered in practice are total, and for these, the well-founded and stable semantics coincide.

Example 1. The definition $\{P \leftarrow \neg P', \quad P' \leftarrow \neg P\}$ is not total (its well-founded model is three-valued) and hence has no model.

Example 2. In the following definition E_{xy} represents the existence of an edge between nodes x and y in a graph, and R_{xy} the reachability of x to y . $\Delta_2 = \{R_{ab} \leftarrow E_{ab} \vee I_{acb}, \quad R_{ac} \leftarrow E_{ac} \vee I_{abc}, \quad I_{acb} \leftarrow R_{ac} \wedge E_{cb}, \quad I_{abc} \leftarrow R_{ab} \wedge E_{bc}\}$. For any interpretation I_O of $\{E_{ab}, E_{bc}, E_{cb}, E_{ac}\}$, Δ_2 has a two-valued well-founded model extending I_O , e.g. for $I_1 = \{E_{ab} \mapsto \mathbf{f}, E_{bc} \mapsto \mathbf{t}, E_{cb} \mapsto \mathbf{t}, E_{ac} \mapsto \mathbf{f}\}$, the model is $I_1 \cup \{R_{ab} \mapsto \mathbf{f}, R_{ac} \mapsto \mathbf{f}, I_{acb} \mapsto \mathbf{f}, I_{abc} \mapsto \mathbf{f}\}$.

2.2 MIDL Normal Form

We extend the CNF format, as used by SAT solvers, to a normal form for SAT(ID) theories. A *clause* is a disjunction $L_1 \vee \dots \vee L_n$ of literals. We denote clauses also by $[L_1, \dots, L_n]$. A CNF theory is a set (conjunction) of clauses. A definition Δ is in MIDL *normal form* (MNF) if each rule $r \in \Delta$ is of the form $P \leftarrow L_1 \vee \dots \vee L_n$ or $P \leftarrow L_1 \wedge \dots \wedge L_n$, with L_i literals and $n \geq 0$, and each atom in $Def(\Delta)$ occurs exactly once as head. A SAT(ID) theory T is in MNF if $T = \Delta \cup \Gamma$, where Δ is a definition in MNF and Γ is a CNF theory.

As with CNF, the advantage of MNF is its simplicity. In particular, MNF makes explicit the data structures that are implemented in ASP systems such as Smodels [18] and clasp [4], where both literals and bodies have a truth value.

There exists a linear transformation from an arbitrary SAT(ID) theory T over Σ to an MNF theory T' over $\Sigma' \supset \Sigma$ such that there is a one-to-one correspondence between models M of T and models M' of T' (with $M'|_{\Sigma} = M$). Hence without loss of generality, we can from now on assume MNF theories.

3 Theory

In the rest of the paper, T denotes a SAT(ID) theory over vocabulary Σ , with definition Δ and CNF part Γ , and I denotes a three-valued Σ -interpretation. A trivial but naive SAT(ID) solving algorithm consists of (1) applying traditional SAT solving techniques to find a model of Γ and (2) subsequently checking whether this model satisfies Δ . Several existing algorithms can be used to compute (in quadratic time) the well-founded model extending a given two-valued interpretation, e.g. [19]. If it turns out that the model does not satisfy Δ , the algorithm tries to find another model of Γ . This algorithm is very inefficient, since in general, most models of Γ are not a model of Δ .² Hence the goal of this work is to interleave (1) and (2), i.e., while constructing the model of Γ , making sure that it can still satisfy Δ .

3.1 Justifications

In this section, we introduce the notion of a *justification*, and use it to provide an alternative characterization of the stable and well-founded model. We then return to model generation for SAT(ID), following an approach suggested by this new characterization.

For a directed graph $G = (V, E)$ and an element $v \in V$, we denote by $Ch_G(v)$ the set $\{w \mid (v, w) \in E\}$. If V is a set of literals, we call a cycle in G *positive*,

² This is a simplified representation of a more intelligent approach, applied by several ASP solvers, e.g. ASSAT [8] and Cmodels [6]. These apply (1) on Clark's completion of the given theory, and in (2) they add a *nogood* loop formula to the original theory for every failed SAT model. Though a viable and sometimes competitive approach, both experimental evaluation and theoretical considerations [7] show that exponentially many iterations may be needed, i.e., the original conclusion still holds. A direct integration of IDs in SAT seems more promising in principle.

negative or *mixed* if it contains respectively only positive, only negative or both kind of literals.

Definition 3 (Justification). A justification J for Δ is a directed graph (Σ^*, E) such that

- for every conjunctive rule $C \leftarrow C_1 \wedge \dots \wedge C_N \in \Delta$, $Ch_J(C) = \{C_1, \dots, C_N\}$ and $Ch_J(\neg C) = \{\neg C_i\}$ for some $i \in [1, N]$, and
- for every disjunctive rule $D \leftarrow D_1 \vee \dots \vee D_N \in \Delta$, $Ch_J(\neg D) = \{\neg D_1, \dots, \neg D_N\}$ and $Ch_J(D) = \{D_i\}$ for some $i \in [1, N]$, and
- for every $L \notin Def(\Delta)^*$, $Ch_J(L)$ is empty.

We denote by J^Δ the greatest common subgraph of all justifications for Δ . This contains at least the subgraphs determined by $Ch_J(C)$ resp. $Ch_J(\neg D)$, for conjunctively resp. disjunctively defined atoms C resp. D . We denote the unique descendant of $\neg C$ resp. D by $\mathcal{D}_J$: $\mathcal{D}_J(\neg C) = \neg C_i$, $\mathcal{D}_J(D) = D_i$.

Example 3. Let $\Delta_3 = \{P \leftarrow Q \vee A, \quad Q \leftarrow P\}$. Then Δ_3 has two justifications, J_1 and J_2 , which differ in $\mathcal{D}_J(P)$: $J_1 = J^{\Delta_3} \cup \{(P, Q)\}$, $J_2 = J^{\Delta_3} \cup \{(P, A)\}$, with $J^{\Delta_3} = \left[\begin{array}{ccc} \neg A & \leftarrow & \neg P \\ & \xrightarrow{\quad} & \neg Q \\ & \xleftarrow{\quad} & Q \end{array} \quad Q \rightarrow P \right]$.

Definition 4 (Stable, well-founded). Let J be a justification for Δ , and I a 3-valued Σ -interpretation. Then J is stable in I iff

- for each $L \in Def(\Delta)^*$ it holds that $I(L) \geq_p I(\bigwedge Ch_J(L))$; and
- any positive cycle in J contains an atom that is false in I .

The first condition is called J supports I , the second J is cycle-safe in I . J is well-founded in I iff J is stable in I and no mixed cycle in J contains literals that are true in I .

Intuitively, the cycle-safeness property expresses that atoms in a positive cycle without external support must be false.³

Example 4. Continuing from Example 3, let $I = \{A \mapsto \mathbf{f}, P \mapsto \mathbf{t}, Q \mapsto \mathbf{t}\}$. Then J_1 supports I and J_2 does not, while J_2 is cycle-safe in I and J_1 is not.

Theorem 1 (Stable, well-founded model). A 2-valued interpretation I is a stable (well-founded) model of Δ extending $I|_{Open(\Delta)}$ iff there exists a justification J that is stable (well-founded) for Δ in I .

We omit the proof for well-founded models due to space restrictions; we include the proof for stable models, however, because it is more instructive.

Proof. Denote by I_1 the Σ -interpretation such that $I_1(P) = I(P)$ for every $P \in Open(\Delta)$ and $I_1(P) = \mathbf{f}$ for all $P \in Def(\Delta)$, let $I_{n+1} = \Psi_{\Delta^I}(I_n)$ for every $n \geq 1$, and finally denote the limit of $(I_n)_{n \geq 1}$ by I' .

³ This intuition can be made explicit by the concept of a *loop formula* [8].

($\Leftarrow$) Assume there exists a stable justification J for Δ in I . From the facts that J supports I , and I is 2-valued, one can easily show by induction that for every $n \geq 1$, $I_n \leq I$. Now, let $V = \{P \in \text{Def}(\Delta) \mid I(P) = \mathbf{t} \neq I'(P)\}$. We show that $V = \emptyset$, hence that $I = I'$.

Assume towards contradiction $V \neq \emptyset$ and let $P \in V$. If P is defined by the rule $P \leftarrow D_1 \vee \dots \vee D_n$, then for all $1 \leq i \leq n$, $I'(D_i) < \mathbf{t}$. Hence $I'(\mathcal{D}_J(P)) < \mathbf{t}$, while $I(\mathcal{D}_J(P)) = \mathbf{t}$. Therefore, $\mathcal{D}_J(P)$ must be a defined atom and $\mathcal{D}_J(P) \in V$. If P is defined by the rule $P \leftarrow C_1 \wedge \dots \wedge C_n$, there exists a C_i such that $I'(C_i) < \mathbf{t}$. As $I(C_i) = \mathbf{t}$, C_i must be defined atom and $C_i \in V$. This proves that every atom of V has at least one child of V in J . Hence, J contains a positive cycle whose atoms are true in I . This contradicts the cycle-safeness of J in I .

($\Rightarrow$) Conversely, assume I is a stable model of Δ , i.e. $I = I'$. We construct a justification J for Δ as follows.

- J contains J^Δ .
- For every $D \in \text{Def}(\Delta)$ with rule $D \leftarrow D_1 \vee \dots \vee D_m$ in Δ , let $\mathcal{D}_J(D) = D_1$ if $I(D) = \mathbf{f}$. If $I(D) = \mathbf{t}$, we distinguish between two cases. If there exists a $D_i \in \text{Open}(\Delta) \cup \text{Def}(\Delta)$ with $I(D_i) = \mathbf{t}$, let $\mathcal{D}_J(D) = D_i$ for such a D_i . Otherwise, for some $n \geq 3$ we have $I_n(D) \neq \mathbf{t}$, but $I_{n+1}(D) = \mathbf{t}$. Then let $\mathcal{D}_J(D) = D_i$ for a D_i such that $I_n(D_i) = \mathbf{t}$, but $I_{n-1}(D_i) \neq \mathbf{t}$.
- For every $C \in \text{Def}(\Delta)$ with rule $C \leftarrow C_1 \wedge \dots \wedge C_m$, let $\mathcal{D}_J(\neg C) = \neg C_1$ if $I(C) = \mathbf{t}$. If $I(C) = \mathbf{f}$, let $\mathcal{D}_J(\neg C) = \neg C_i$ for some C_i such that $I(C_i) = \mathbf{f}$.

It can easily be verified that J supports I . We define the *derivation level* of a literal L , denoted $dl(L)$, as follows: if $L \in \text{Def}(\Delta)$ with $I(L) = \mathbf{t}$, then $dl(L) = n + 1$, with n such that $I_n(L) \neq \mathbf{t}$ and $I_{n+1}(L) = \mathbf{t}$; in all other cases, $dl(L) = 0$. Hence, our construction of J is such that for each child Q in J of any atom $P \in \text{Def}(\Delta)$ with $I(P) = \mathbf{t}$, $dl(Q)$ is strictly lower than $dl(P)$. Therefore J cannot contain a positive cycle with true atoms, i.e., J is also cycle-safe in I .

Theorem 1 suggests an approach to compute models of a definition: namely, maintain a justification that is stable for the partial interpretation at each moment in the computation. We show in Section 3.2 how to maintain support, and in Section 3.3 how to maintain cycle-safeness.

3.2 Two Watched Literals

Contemporary SAT solvers have *unit propagation* as their propagation mechanism: whenever in an input clause $[L_1, \dots, L_n]$ all L_i are false, except one, this so-called *unit literal* is made true. In all state-of-the art solvers, unit propagation is executed by means of the *two watched literals scheme* (2WL). In this scheme, in each clause $[L_1, \dots, L_n]$ two literals $W_1 = L_i$ and $W_2 = L_j$ for some $i \neq j$ are “watched”, and the statement $I(W_1 \vee W_2) = \mathbf{t} \vee I(W_1 \wedge W_2) = \mathbf{u}$, called the *2WL invariant*, must be satisfied at all times. Hence, when either W_1 or W_2 becomes false, a replacement watch has to be found (the watch has to be “moved”). When no suitable replacement is found, i.e., all other literals are false, the remaining watch is made true. In contrast, when any non-watched literal becomes false,

nothing needs to be done. We denote the function mapping clauses $[L_1, \dots, L_n]$ to the set of their watched literals $\{W_1, W_2\}$ by F_{2WL} .

To apply 2WL on definitional rules, we first introduce some terminology.

Definition 5. The completion of a rule⁴ $P \leftarrow \varphi$ is given by the clausal form of $P \equiv \varphi$. The completion of Δ , denoted by $\text{comp}(\Delta)$, is the union of the completions of all $r \in \Delta$.

Example 5. $\text{comp}(\Delta_3) = \{\neg P \vee Q \vee A, \quad P \vee \neg Q, \quad P \vee \neg A, \quad \neg P \vee Q\}$.

Observe that for a clause A in the completion or a rule r , either the head of r or its negation occurs in A : we denote this occurrence by Head_A . I.e., $\text{Head}_{[D, \neg D_i]} = D$, $\text{Head}_{[\neg D, D_1, \dots, D_n]} = \neg D$, etc. Now, when applying 2WL on $\text{comp}(\Delta)$, F_{2WL} naturally induces justifications: let for every rule $r \in \Delta$ and every clause A in the completion of r , W_A be a literal in $F_{2WL}(A)$ such that $W_A \neq \text{Head}_A$. Then the set of all edges $(\neg \text{Head}_A, W_A)$ forms a justification for Δ . Remark that the binary clauses in the completion induce the graph J^Δ . Multiple justifications are induced when for some clauses A in the completion $\text{Head}_A \notin F_{2WL}(A)$. We call the watches in $[\neg D, D_1, \dots, D_n]$ respectively $[C, \neg C_1, \dots, \neg C_n]$ of the completion of a rule r the *watches of r* and denote them by $W_1(r)$ and $W_2(r)$, or simply W_1, W_2 if r is clear from the context. The following property is easy to verify.

Proposition 1. If I and F_{2WL} satisfy the 2WL invariant on $\text{comp}(\Delta)$, then F_{2WL} induces at least one justification J for Δ such that J supports I , called a supported justification.

Example 6. Let $\Delta_6 = \{P \leftarrow Q \wedge R, \quad Q \leftarrow P \vee \neg S\}$, and let $F_{2WL}([P, \neg Q, \neg R]) = \{\neg Q, \neg R\}$ and $F_{2WL}([\neg Q, P, S]) = \{\neg Q, P\}$. Then F_{2WL} induces $J_1 = J^{\Delta_6} \cup \{(Q, P), (\neg P, \neg Q)\}$ and $J_2 = J^{\Delta_6} \cup \{(Q, P), (\neg P, \neg R)\}$. The interpretation $I = \{P \mapsto \mathbf{f}, Q \mapsto \mathbf{f}, R \mapsto \mathbf{t}, S \mapsto \mathbf{t}\}$ satisfies the 2WL invariants on $\text{comp}(\Delta_6)$. J_1 supports I , but J_2 does not: $I(\neg P) = \mathbf{t}$, whereas $I(\bigwedge \text{Ch}_{J_2}(\neg P)) = I(\neg R) = \mathbf{f}$.

Our algorithm applies the 2WL scheme on the completion of Δ , hence maintains at least one supported justification. Next section shows to maintain amongst the supported justifications at least one that is also cycle-safe, hence stable.

3.3 Cycle-Safeness

Assume that for a given I and F_{2WL} , at least one supported justification for Δ is cycle-safe. Now, moving a watch in $\text{comp}(\Delta)$ changes the induced justifications. Hence we have to evaluate which type of watch moves may lead to the introduction of a positive cycle in any of those, i.e., to the invalidation of cycle-safeness.⁵ If r is a conjunctive rule with head C , $\text{Ch}_J(C)$ is fixed, hence moving $W_1(r)$

⁴ Recall that in MNF each defined atom has exactly one defining rule.

⁵ The same reasoning was used to introduce the concept of a *source pointer* in Smodels [18], which corresponds to the unique descendant ($\mathcal{D}_J(\cdot)$) concept here.

or $W_2(r)$ cannot introduce a positive cycle in any induced justification. If r is a disjunctive rule with head D , and $W_1(r)$ or $W_2(r)$ is moved, a positive cycle may be introduced if the move is to a defined atom in r 's body.

It is highly inefficient—mainly for technical reasons, related to cache behaviour—to interrupt unit propagations to verify whether there is indeed a positive cycle, and to fix that problem if so. The alternative is to delay this cycle testing/repairing. To do so, atoms that could be in a positive cycle are marked as “cycle sources”. We formalize this by means of the following concept.

Definition 6 (Cycle-safe up to a set of atoms). *Let J be a directed graph of literals, S a set of atoms. J is cycle-safe up to S if any positive cycle in J contains an atom in S . Elements $s \in S$ are called cycle sources.*

Observe that when all atoms in S are false in I , cycle-safeness up to S implies cycle-safeness in I . Hence we consider an algorithm that maintains a partial interpretation I , a watch function F_{2WL} , and a set of cycle sources S , and has as invariant: at least one justification induced by F_{2WL} and supported by I is cycle-safe up to S . To satisfy this invariant without interrupting unit propagations, the head of a disjunctive rule r must be added to S whenever one of r 's watches is moved to a defined atom. To also obtain a *stable* justification, the algorithm must try to remove all non-false atoms from S , but only *after* unit propagations reached a fixpoint. Atoms in S should be made false only when they are false in all three-valued stable models of Δ that are refinements of I . The following algorithm makes S smaller while preserving the invariant.

1. Select some $P \in S$ with $I(P) \neq f$.
2. If F_{2WL} induces some J with J cycle-safe up to $S \setminus \{P\}$, remove P from S and stop.
3. Try to change F_{2WL} such that it satisfies the 2WL invariant and induces J with J cycle-safe up to $S \setminus \{P\}$. If this succeeds: remove P from S and stop.
4. Set $I(P) := f$.

This algorithm can be repeated until all remaining atoms in S are false in I . The main challenge is how to perform Step 3 in an efficient way. Also, we must be able to prove that this sub-algorithm is complete, so that step 4 is justified, i.e., any justification supported in I necessarily exhibits a positive cycle. This is the subject of Section 4.1.

3.4 Identifying a Unique Supported Justification

Let $P_1, P_2 \in S$, and let J_1, J_2 be different supported justifications, and suppose J_1 is cycle-safe up to $S \setminus \{P_1\}$, while J_2 is cycle-safe up to $S \setminus \{P_2\}$. We cannot conclude from this that there is a supported justification that is cycle-safe up to $S \setminus \{P_1, P_2\}$. The task of making S smaller would be greatly simplified if a unique justification J could be maintained during the whole algorithm.

One way to do so is to change F_{2WL} , such that the functionality of the two watches is distinguished, i.e., the function now maps to a pair (W_1, W_2) instead

of a set $\{W_1, W_2\}$. We assign W_1 as the single watch that induces a justification. As such, for each clause A in $\text{comp}(\Delta)$ with $F_{2WL}(A) = (W_{1,A}, W_{2,A})$, $W_{1,A}$ should not be equal to Head_A . Then the set of all edges $(\neg \text{Head}_A, W_{1,A})$ forms a *unique* justification. This change requires some adaptations of 2WL: whenever $W_{1,A}$ is moved to Head_A , and whenever $W_{1,A}$ becomes false and all other literals in the clause, except for $W_{2,A}$, are false, $W_{1,A}$ and $W_{2,A}$ have to be swapped. Proposition 1 can be adapted accordingly: by applying the new 2WL scheme, the unique induced justification supports the partial interpretation.⁶

Observe that this new strategy also means that moving $W_2(r)$ in a disjunctive rule r does not change the induced justification, hence does not generate a cycle source: thus only about half as many cycle sources are produced.

4 Algorithm

For the whole of Section 4, let J be a justification and S a set of atoms, such that J supports I and is cycle-safe up to S , and let $P \in S$.

4.1 Justifying Cycle Sources

We want to remove P from S . This can be done when J is cycle-safe up to $S \setminus \{P\}$. If this is not the case for the current J , a search for an alternative justification has to begin, i.e., for some atoms A , $\mathcal{D}_J(A)$ has to be changed. We now investigate which atoms need to be considered.

Example 7. Consider following sub-graph of a justification J , where P is a cycle source: $\left[U \rightarrow P \begin{array}{c} \rightarrow \\ \leftarrow \end{array} Q \rightarrow R \quad V \right]$. At least one of the outgoing edges from P or Q has to change its child node. Changing it to U , however, would introduce a new cycle, while changing it to R or V would not. It follows that only $\mathcal{D}_J(P)$, $\mathcal{D}_J(Q)$ and/or $\mathcal{D}_J(U)$ need to be changed. Generalizing: the ancestor atoms of a cycle may need to change their outgoing edges, the descendants do not.

We express the observation from this example using following concepts.

Definition 7. Let A be a defined atom, J a justification. Define $\text{Top}_J(A)$ as the set of atoms $\{B \mid \text{there is a non-empty path of atoms in } J \text{ from } B \text{ to } A\}$.

Definition 8. Let A be a defined atom, J a justification, S a set of atoms. A is justified in J up to S iff for any positive cycle in J , either the cycle contains an element of S , or $A \notin \text{Top}_J(B)$ for any atom B in the cycle.

Clearly, if $P \notin \text{Top}_J(P)$, there is no positive cycle through P . The intuition behind Definition 8 is that J might contain positive cycles through P : if J contains no path of atoms from A to any atom B in such a cycle, then A is

⁶ Actually, the 2WL invariant must be refined to the following: if one of the watches is false, then all other literals of the clause are false, except for the other watch. Current SAT solvers already maintain this invariant for efficiency reasons.

justified. Hence the following formalization of the intuition from Example 7: any atom outside $Top_J(P)$ is justified in J up to $S \setminus \{P\}$. This means that we can restrict the search for an alternative justification to $Top_J(P)$.

Remark 1. It is easy to adjust the definition of Top_J to take into account the *strongly connected components* (SCCs) of the positive atom dependency graph. Then the search can be further restricted to those atoms in $Top_J(P)$ that are in the same SCC as P . For simplicity, we stick to definition 7 in this presentation.

During the search, a set $\mathcal{N}$ of atoms is maintained, of which it is not known yet whether they are justified up to $S \setminus \{P\}$ in the current justification J . In other words, these atoms may be in, or may positively depend on, a positive cycle in J . Initially, $\mathcal{N}$ is set to $Top_J(P)$; after that, the goal of the algorithm is to decrease $\mathcal{N}$ until $P \notin \mathcal{N}$. Informally, we call removing an atom A from $\mathcal{N}$, “justifying” A . Any non-false literal from outside $\mathcal{N}$ can be used for justifying atoms in $\mathcal{N}$. E.g., let $D \leftarrow D_1 \vee \dots \vee D_N$ be a rule with $\mathcal{D}_J(D) = D_i$, $D, D_i \in \mathcal{N}$ and for some D_j , $D_j \notin \mathcal{N}$ and $I(D_j) \neq \mathbf{f}$, then we can safely remove D from $\mathcal{N}$ by changing $\mathcal{D}_J(D)$ to D_j . For a conjunction, the head can only be justified by showing that all body literals are outside $\mathcal{N}$. The algorithm employs a function $f_\wedge$ for this, which maps conjunctively defined atoms to one of their body literals. Note that other cycle sources than P could be in $Top_J(P)$; justifying them during the process effectively means that they can be safely removed from S .

This search for an alternative justification J' proceeds downward from P , from head to body atoms. As such it tries to justify atoms as close to P as possible, the intuition being that in most cases, a solution can be found close to P . To avoid double work, a set $\mathcal{T}$ of “touched” atoms is also maintained. When an atom is touched, it is added to $\mathcal{Q}$: this set contains the atoms for which search can still be expanded, i.e., it is the “working queue”. When an atom is justified, it is not only removed from $\mathcal{N}$, but also from $\mathcal{T}$. Hence when $\mathcal{Q}$ is empty (the whole search space has been visited), but $\mathcal{T} \neq \emptyset$, the search proved unable to produce a good justification for the atoms in $\mathcal{T}$ —i.e., in the current interpretation, they have to be false.⁷

Algorithm 1 is the result of the above reasoning. Here we provide a correctness and completeness result with respect to this algorithm.

Theorem 2. *Let I be an interpretation, S a set of atoms, J a justification such that J supports I and cycle-safe up to S , and $P \in S$. Then Algorithm 1 applied on these inputs terminates. Let U be the output, and S' and J' respectively the set of cycle sources and the justification after applying the algorithm. Then J' supports I and is cycle-safe up to S' , and also one of the following holds:*

- $U = \emptyset$, $J' = J$ and $S' = S \setminus \{P\}$;
- $U = \emptyset$, $P \in Top_J(P)$ and $S \setminus Top_J(P) \subseteq S' \subseteq S \setminus \{P\}$;
- $U \neq \emptyset$, $S \setminus Top_J(P) \subsetneq S' \subseteq S$, and for each $Q \in U$, $\max_{\leq} \{I'(Q) \mid I' \text{ is a three-valued stable model extending } I|_{Open(\Delta)}\} = \mathbf{f}$.

⁷ They are in an *unfounded set* [20].

In the third of the possible outcomes, U is an unfounded set. It then also holds that each $Q \in U$ is in a positive cycle in J' through P . Observe that in the second and third possible outcome the set S' is not precisely defined: some atoms from $Top_J(P)$ may be removed from S in Step 22 of the algorithm.

4.2 Overall Algorithm

The difference between the overall MIDL algorithm and the DPLL algorithm as used in SAT solvers lies entirely in an adapted initialization phase, and an augmentation of DPLL's unit propagation. The former must make sure that the invariants are satisfied: a straightforward (though slightly naive) initialization phase may set $S := Def(\Delta)$. The latter is replaced by Algorithm 2. Here, the function `UnitProp()` applies unit propagation by means of the (new) 2WL scheme on Γ and on $comp(\Delta)$, as described in Section 3. At the end of this step, the invariants “ J supports I ” and “ J is cycle-safe up to S ” are satisfied, and a while-loop begins, trying to decrease S . The loop applies `Justify( $P$ )` on every non-false cycle source P ,⁸ continuing unit propagations as soon as an unfounded set is found. At the end of Algorithm 2, S is made as small as possible. We then have “ J is cycle-safe up to S ”, and “ $\forall A \in S : I(A) = \mathbf{f}$ ”, hence also “ J is cycle-safe in I ”. We can conclude that I is a three-valued stable model.

By repeated application of Algorithm 2 on new choice literals, we obtain a two-valued stable model. A standard well-founded model checking algorithm can subsequently verify whether that model is also well-founded. This check may fail when Δ is not total, causing backtracking. However, all definitions encountered in practice are total. We demonstrate the algorithm in next example.

Example 8. Let $\Delta_8 = \{P \leftarrow Q \vee R. \quad Q \leftarrow P. \quad R \leftarrow A.\}$. Initially, everything is unknown. Suppose $\mathcal{D}_J(P) = R$. A sample run of the algorithm might make P true (by choice), which makes Q true by unit propagation. S is still empty, so a new choice is made: say, $\neg A$. This makes R false by unit propagation, which forces $\mathcal{D}_J(P)$ to change to Q , so that P must be added to S . Next, `Justify( $P$ )` finds the unfounded set $\{P, Q\}$, so both atoms are made false, which yields a conflict. By backtracking, A is made true, propagating to R and P . P is still in S , so `Justify( $P$ )` is run again, changing $\mathcal{D}_J(P)$ back to R , thus creating a cycle safe justification. The resulting interpretation is 2-valued, hence a model of Δ_8 .

Theorem 3 (Correctness & Completeness). *Let T be a $SAT(ID)$ theory in MNF. Given input T , MIDL returns a model of T when such a model exists and concludes that T is unsatisfiable otherwise.*

This can easily be proven by completeness of standard backtracking, and correctness and completeness of both `UnitProp()` and `Justify()`.

⁸ Observe that not removing false atoms from S renders it unnecessary to change S during backtracking.

Result: A set of atoms U

```

1  $\mathcal{N} := \text{Top}_J(P)$ ;
2 if  $P \notin \mathcal{N}$  then  $S := S \setminus \{P\}$ ; return  $\emptyset$  ;
3  $\mathcal{T} := \mathcal{Q} := \{P\}$ ;
4 while  $\mathcal{Q} \neq \emptyset$  do
5   Remove an atom  $Q$  from  $\mathcal{Q}$  ;
6   if  $Q$  is disjunctively defined then
7     % Let the rule for  $Q$  be  $Q \leftarrow D_1 \vee \dots \vee D_n$ .
8      $\mathcal{D} := \{D_i \mid 1 \leq i \leq n, I(D_i) \neq f\}$ . % Note that  $\mathcal{D} \neq \emptyset$ .
9     if  $\exists D_i \in \mathcal{D} \setminus \mathcal{N}$  then
10      |  $\mathcal{D}_J(Q) := D_i$  ;  $\text{BUProp}(Q)$  ;
11    else
12      | Add  $\mathcal{D} \setminus \mathcal{T}$  to  $\mathcal{Q}$  ; Add  $\mathcal{D}$  to  $\mathcal{T}$  ;
13  else
14    % Let the rule for  $Q$  be  $Q \leftarrow C_1 \wedge \dots \wedge C_n$ .
15     $\mathcal{C} := \{C_i, 1 \leq i \leq n \mid C_i \in \mathcal{N}\}$ ;
16    if  $\mathcal{C} = \emptyset$  then
17      |  $\text{BUProp}(Q)$  ;
18    else
19      | if  $\exists C_i \in \mathcal{T}$  then  $f_\wedge(Q) := C_i$  ;
20      | else Choose a  $C_i \in \mathcal{C}$ ;  $f_\wedge(Q) := C_i$ ; Add  $C_i$  to  $\mathcal{Q}$  and to  $\mathcal{T}$  ;
21 return  $\mathcal{T}$  ;
22 function  $\text{BUProp } Q$ 
23   Remove  $Q$  from  $\mathcal{N}$ ,  $\mathcal{T}$  and  $\mathcal{Q}$  ;
24   if  $Q \in S$  then  $S := S \setminus \{Q\}$  ;
25   if  $Q = P$  then return  $\emptyset$  ;
26   for each  $C \in \mathcal{T}$  with  $f_\wedge(C) = Q$  do
27     | Push  $C$  on  $\mathcal{Q}$  ;
28   for each  $D \in \mathcal{T}$  with rule  $r_D = D \leftarrow \dots \vee Q \vee \dots$  do
29     |  $\mathcal{D}_J(D) := Q$  ;  $\text{BUProp}(D)$  ;
30 end function

```

Algorithm 1: *Justify*(P): justifying a cycle source P

```

1 repeat
2    $\text{UnitProp}()$ ;
3    $U := \emptyset$ ;
4   while  $U = \emptyset$  and  $\exists P \in S$  such that  $I(P) \neq f$  do
5     | Let  $P \in S$  such that  $I(P) \neq f$ ;
6     |  $U := \text{Justify}(P)$ ;
7     |  $I(Q) := f$  for every  $Q \in U$ ;
8 until  $U = \emptyset$  ;

```

Algorithm 2: SAT(ID) replacement of SAT's unit propagation

5 Evaluation

On MIDL. We have implemented the above algorithms in a system called MIDL [12, 13]. The following features that make contemporary SAT solvers efficient and robust have been implemented in MIDL: good choice heuristics (VSIDS and VMTF), clause learning and backjumping, restarts, compact encoding of binary clauses. Some other important features could make it more mature still, such as preprocessing, clause deletions, and compact encoding of ternary clauses.

On SAT(ID) experimenting. SAT(ID) is a relatively new logic, hence few solvers exist. The main purpose of this paper, however, is to demonstrate how to integrate ID support in a SAT solver, and to illustrate that this integration is viable. Hence we perform experiments on two different fronts: one concerning ID support, where we compare with ASP solvers on problems containing inductive definitions, and one concerning SAT solving, where we compare with both ASP and SAT solvers on problems not containing any inductive definitions.⁹

Are IDs needed? Observe that the need for using IDs can sometimes be avoided by using elaborate encodings, e.g. for the Hamiltonian circuit problem.¹⁰ The propositional instances obtained this way are over 50 times the size of those using an ID encoding, for graphs with 150 nodes, and were simply too big for graphs with 200 nodes. In Table 1, the column for MiniSAT refers to these instances. Also a reduction from SAT(ID) to SAT is a possibility [15]. The IDSAT solver implements this, but yielded only time-outs on the instances of Table 1 (using MiniSAT). Clearly, an integrated approach is superior.

Experiments and discussion. We show the experimental results in Table 1.¹¹ The **first set** of results concerns Hamiltonian cycle problems, where we compare MIDL with the ASP solvers clasp [4], Smodels [18], Cmodels [6], and Smodels_{cc} [21], and with MiniSAT [3] on an alternative encoding. The instances have respectively (100, 800), (150, 1200) and (200, 1800) nodes and edges (4 instances of each kind). Too big ground files are denoted ‘#’. We can conclude that MIDL and clasp are comparable, and outperform all other solvers. The **second set** concerns Hitori puzzles of size 50×50 , compared against the same ASP solvers. These puzzles contain inductive definitions, but with very few cycles: on most instances, Cmodels needed to create no or only one loop formula. It outperformed both MIDL and the other ASP solvers, despite its use of a somewhat outdated SAT solver. The **last set** concerns Blocked 28-queens problems, where we also compare with the SAT solvers siege [17] and MiniSAT. We observe an efficiency gap between pure SAT solvers, and ID supporting solvers. Interestingly,

⁹ Naturally, problem encodings differ according to the formalism used. We have tried to ensure fair comparisons, e.g., we haven’t used aggregate expressions.

¹⁰ See <http://www.cs.sfu.ca/research/groups/mxp/examples/view.php?f=hc>.

¹¹ Version numbers of the programs used are: Minisat 1.14, siege 4, Clasp 1.0.3, Smodels 2.32, Cmodels 3.67, SmodelsCC 1.08, IDSAT 0.9.5, and MIDL 2.1.1. All experiments run on a P4 2.8GHz with 1GB memory. Problem instances of Hamiltonian cycle and of Blocked N -queens are taken from the benchmark website Asparagus, <http://asparagus.cs.uni-potsdam.de/>. Results of all benchmark instances are available in attachment, but left out here due to space restrictions.

clasp	MiDL	Cmodels	MiniSAT	Smodels _{cc}	Smodels	Cmodels	clasp	MiDL	Smodels	Smodels _{cc}
1.05	2.28	19.35	163.8	194.43	210.6	1.32	6.04	8.15	17.63	51.55
0.47	0.48	3.92	7.22	285.86	> 300	1.00	6.50	7.19	16.26	45.87
0.48	1.36	2.37	5.77	> 300	> 300	0.94	6.53	7.24	16.40	45.98
0.48	0.67	2.35	6.18	24.33	8.63	1.18	7.62	8.66	20.75	68.21
0.48	0.69	1.96	15.95	77.70	> 300	0.90	5.24	5.70	22.36	55.80
0.79	1.51	5.05	55.15	181.35	> 300	1.25	7.97	6.86	18.72	52.78
1.07	1.37	7.77	> 300	> 300	> 300	0.93	0.38	0.61	1.37	1.37
0.91	3.75	5.59	179.18	185.29	> 300	1.56	6.68	12.82	18.43	58.31
0.87	1.22	8.57	196.12	59.54	19.69	0.67	0.37	0.63	1.64	1.25
1.78	4.65	146.07	#	151.90	47.51	3.94	9.00	25.14	27.72	78.76
2.27	4.26	20.73	#	167.15	51.32	1.20	5.86	7.08	23.03	62.33
1.51	4.02	13.37	#	> 300	> 300	1.15	9.64	8.32	22.36	72.89
1.45	3.41	14.40	#	> 300	> 300	1.17	6.69	7.55	22.54	75.05
			MiniSat	Siege	clasp	MiDL	Cmodels	Smodels	Smodels _{cc}	
			0.54	2.53	3.26	4.68	5.66	24.16	58.29	
			0.62	2.14	2.18	4.40	8.38	17.44	44.61	
			0.10	1.65	4.20	2.33	2.90	1.79	2.28	
			0.23	2.27	1.41	3.61	0.43	8.20	12.74	
			0.34	0.60	1.02	1.43	1.22	5.16	12.17	
			0.37	0.85	1.29	2.26	1.79	18.81	70.48	
			2.26	11.87	14.37	18.54	28.64	124.30	> 300	
			0.19	0.28	0.58	1.23	0.59	5.36	15.60	
			0.20	0.54	1.02	3.67	1.30	12.21	8.41	

Table 1. Running times (sec.) of Hamiltonian Cycle (left), Hitori (right) and Blocked N -queens (below) problem instances. First lines show averages.

Cmodels’ performance is not as good as that of siege and MiniSAT. By analogy, a possible explanation for the worse performance of the ID solvers is that their implementation details are less advanced than those of siege and MiniSAT.

6 Conclusions, Related and Future Work

In this paper, we have illustrated an approach to integrate ID support in contemporary SAT solving algorithms. In particular, we have shown how SAT’s 2WL scheme can be reused to also determine justifications, and how such justifications can be used to find models of IDs. Finally, we reported on an implementation of the algorithms discussed.

A related approach to SAT(ID) solving works by reduction to SAT [15].¹² An algorithm for finding unfounded sets, presented in [1], is closely related to our Algorithm 1. In particular, the following sets have similar functionalities: $\text{Set} \sim \mathcal{T}$, $\text{Ext} \sim \mathcal{Q}$, $\text{Sink} \sim (\Sigma \setminus \mathcal{N})$. The precise relation between **Source** and S is unclear.

The most obvious item of future work is to convert theory into practice, by actually extending an existing contemporary SAT solver such as MiniSat with IDs. On the other hand, extending MiDL with additional features such as preprocessing may make it more mature and robust. There is also room for experimenting with variants of Algorithm 1. Finally, it is a goal of our project to make SAT more expressive by adding relevant modelling constructs: for instance, we intend to also support aggregate expressions in MiDL.

¹² The solver described in this paper had mostly time-outs in our experiments.

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