BOOK REVIEW

A Critique of Leyton's Theory of Perception and Cognition

Michael Leyton. **Symmetry, Causality, Mind**. Cambridge, MA: MIT Press, 1992. 630 pp. \$45.00. ISBN 0-262-12163-8.

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Michael Leyton is on the faculty of the Psychology Department at Rutgers University. He received his Ph.D. from the University of California at Berkeley in 1984. He is a recipient of the Presidential Young Investigator Award for outstanding work in cognitive science. *Symmetry, Causality, Mind* represents the synthesis of a theory that goes back to his Ph.D. thesis (1984a) and that was gradually developed in a number of papers, totalling more than 350 pages, published in journals like *Biological Cybernetics* (1984b); *Computer Vision, Graphics and Image Processing* (1985, 1987a, 1987b); *Information Sciences* (1986a); *Journal of Mathematical Psychology* (1986b, 1986c, 1987c); *Artificial Intelligence* (1988); and *Cognitive Science* (1989). The entire first issue of the 1986 volume (Vol. 38) of *Information Sciences*, comprising some 120 pages, was devoted to his theory.

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We acknowledge useful comments on previous drafts by James Cutting, Peter Dodwell, Donald Hoffman, Keith Niall, Richard Chechile, and two anonymous reviewers. Revisions were prepared with support from Grants G.0210.97N and G.0130.98 from the Research Program of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.).

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In *Symmetry, Causality, Mind*, a work of more than 600 pages, Leyton presents a general theory of the human mind and beyond, applicable to such diverse domains as neurophysiology, perception, psychological scales, categorization, linguistics, planning, arts, and politics. We will summarize and discuss this work. We will first clarify how the theory was developed from a persuasive demonstration of the perceived history of a rotated parallelogram into a universally applicable theory of group decomposition and algebraic stability ordering. The focus in this first section will be on Leyton's papers published prior to the 1992 book. These are often highly mathematical. Part of his toolbox are such topics as inner automorphism groups, asymptotically stable equilibria, basins of attraction, fiber bundles on differentiable manifolds, homotopy theory, normed vector spaces, Fourier transforms, universal covering groups, Lie algebras, Peano's axioms, differential forms, central normal subgroups, one-parameter subgroups, etc.

We will then clarify how the theory reached a further level of abstraction in *Symmetry, Causality, Mind*. The central thesis in this book is that all cognitive activity reduces to the recovery of process-history. This thesis is developed into a set of theoretical proposals, principles, hypotheses, heuristics, and strategies that constitute the basis of a radically new theory of the mind with a scope that can be stretched from the perception of simple shapes to the understanding of art work and even the socio-economical, political, and ethical aspects of our technological society. The book is much more philosophical than it is mathematical.

Our major aim is to examine the logical soundness of this body of work when the mathematics and metaphysics are stripped off. Such an in-depth analysis is important because the field of cognitive science is clearly in need of good theoretical work and many experimental psychologists could be attracted to Leyton's work as a source of inspiration. Upon first reading, it seems highly creative, innovative, and impressive. We will argue that there is no ground to expect any direct empirical success based on this theory of cognition as recovering process-history. Its mathematical foundations are shaky and the theoretical principles are formulated in terms that are vague, overly general, and hence unclear. To allow the reader to consult the original sources, we will always indicate where a more complete description of Leyton's theory can be found. Moreover, we will frequently cite from Leyton's work to make our examination as specific as possible.

FROM THE HISTORY OF A ROTATED PARALLELOGRAM TO A NEW THEORY OF PERCEPTION AND COGNITION

In a "convergent series" of experiments (which we could not find in print), Leyton found that when subjects are presented with a rotated parallelogram (illustrated in Fig. 1A), they reference it to a nonrotated one (Fig. 1B), which they reference to a rectangle (Fig. 1C), which they reference to a square (Fig. 1D)" (Leyton, 1985, p. 201). The reduction of the rotated parallelogram to a square is called the *external process-history* of the parallelogram. The square can be reduced to a side that has been rotated over 0° , 90° , 180° , and 270° and the side can be reduced to a point that has been translated over a certain distance. The reduction

FIG. 1. The external history of a rotated parallelogram (adapted from *Information Sciences*, **38**, Michael Leyton, Principles of information structure common to six levels of the human cognitive system, pp. 1–120, Copyright (1986), with permission from Elsevier Science).

of the square to a point is called the *internal process-history* of the square. The external history of the rotated parallelogram and the internal history of the square together constitute the complete process-history of the rotated parallelogram.

The simple example of the process-history of a rotated parallelogram contains the main ingredients of Leyton’s theory of cognition (i.e., process-history, symmetry-asymmetry, group decomposition, stability, and interaction principles). Cognition is the recovery of process-history. Process-history reduces something asymmetrical (rotated parallelogram) to something symmetrical (square). Process-histories can be described as group decompositions. The factors in the group decomposition appear in order of decreasing instability. Some process histories consist of an external history and an internal history, connected by the interaction principle. Leyton has analyzed and elaborated these principles mathematically (e.g., Leyton, 1985; 1986a; 1992, Chaps. 5 and 6) and he has reformulated and expanded them in comprehensive generalizations (e.g., Leyton, 1992, Chaps. 1, 2, and 4). We will first discuss the mathematical aspects of his theory of perception and cognition as a group decomposition with an algebraic stability ordering. We will then discuss the more philosophical generalizations.

PERCEPTION AND COGNITION AS GROUP DECOMPOSITION WITH AN ALGEBRAIC STABILITY ORDERING

Group Decomposition

The transformation of a square into a rotated parallelogram is a linear transformation of the plane and can be described by a 2×2 matrix. The set of all 2×2 invertible matrices with real coefficients and matrix multiplication as its operator, form a group¹ called the *general linear group* GL_2R . In GL_2R we can distinguish the following subgroups:

¹ Let G be a nonempty set. Suppose that for any elements a, b of G there exists a uniquely determined element c of G , which is called the product of a and b , written $c = a \cdot b$. G is called a *group* if (i) the associative law $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ holds, (ii) there exists an element e (called the identity element or unit element of G) such that $a \cdot e = e \cdot a = a$ for any element a of G , and (iii) for any element a of G there exists an element x such that $a \cdot x = x \cdot a = e$. The element x in condition (iii) is called the inverse of a , denoted by a^{-1} . A subset of G is called a *subgroup* if it obeys the same conditions under the same product operator.

(1) A , the group of pure deformations. This is the subgroup consisting of all diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

(2) N , the group of shears. This is the subgroup of all matrices

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

(3) SO_2 , the group of rotations

$$\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}.$$

The matrices A stretch and contract the plane along the X and Y axes (from Fig. 1D to 1C). The matrices N shear the plane along the X axis (from Fig. 1C to 1B). The matrices SO_2 rotate the plane around the origin (from Fig. 1B to 1A).

Every matrix m of GL_2R can be written as the product $m = a \cdot n \cdot r$ where a is a stretch, that is, an element of A , the group of pure deformations, n is an element of N , the group of shears, and r is an element of SO_2 , the group of rotations. This decomposition is unique in the following sense: If $m = a_1 \cdot n_1 \cdot r_1$ is another decomposition then either $a_1 = a$, $n_1 = n$ and $r_1 = r$ or $a_1 = -a$, $n_1 = n$ and $r_1 = -r$. We then have the decomposition

$$GL_2R = A \cdot N \cdot SO_2.$$

This decomposition is a special case of the Iwasawa decomposition theorem for connected semisimple Lie groups of which GL_2R is a special case (Lang, 1975).

The Iwasawa decomposition of GL_2R can be considered as the external process-history of the rotated parallelogram. The rotations SO_2 transform the rotated parallelogram into a nonrotated one. The shears N transform the nonrotated parallelogram into a rectangle and the stretches A transform the rectangle into the unit square.

The internal process-history of a square can be described by a decomposition

$$D_4 \cdot R \cdot \{e\}$$

The trivial group $\{e\}$ represents a point. This point is translated by the group R , the group of translations along a line, to form a side. The sides are then rotated by elements of the group

$$D_4 = \{e, r_{90}, r_{180}, r_{270}, t, tr_{90}, tr_{180}, tr_{270}\},$$

where e is the identity element, r_α a rotation by α° , and t the reflection about the vertical axis bisecting the top and bottom sides of the square.

Two remarks are in place here. First, the decomposition $A \cdot N \cdot SO_2$ is a *reference sequence* that reduces a rotated parallelogram into a square. By reversing the order we obtain the *generation sequence* $SO_2 \cdot N \cdot A$ that generates a rotated parallelogram from a square. The internal process-history $D_4 \cdot R \cdot \{e\}$ is in fact a generation sequence. For external histories Leyton uses reference sequences, whereas for internal histories he uses generation sequences.

Second, the generation process for external histories is different from that for internal histories. A square that is transformed into a rotated parallelogram disappears. However, when a side of a square is rotated to form a square, the original side remains. This difference is formalized by Leyton (1985, p. 208) as follows. The external generation of a set S is the application of a sequence of operations $g_1, g_2, \dots, g_n$ to some set S_0 such that

$$S = g_n \cdot g_{n-1} \cdot \dots \cdot g_1(S_0).$$

The internal generation of a set S , by the operations $g_1, g_2, \dots, g_n$ is defined by

$$S = U_{j=1, n} \{g_j \cdot g_{j-1} \cdot \dots \cdot g_1(S_0)\},$$

that is,

$$S = \{g_1(S_0)\} \cup \{g_2 \cdot g_1(S_0)\} \cup \dots \cup \{g_n \cdot g_{n-1} \cdot \dots \cdot g_1(S_0)\}.$$

This last definition presupposes that $g_1, g_2, \dots, g_n$ are fixed. But that does not work. Let us take the example of the square with generation sequence $D_4 \cdot R \cdot \{e\}$ and let us take $g_1 = e$, $g_2 = t_d$, where t_d is the translation over a certain distance, and $g_3 = r_{90}$. For S_0 we take a single point. Then $S = \{e(S_0)\} \cup \{t_d \cdot e(S_0)\} \cup \{r_{90} \cdot t_d \cdot e(S_0)\}$. This set consists of a point, the point translated over a distance d , and the point translated over a distance d and rotated over 90° . It is clear that in the definition we must take the union for all g_i to obtain all the points of the square. But even then there are problems. The generated sides have infinite length.

The decompositions $A \cdot N \cdot SO_2$ and $D_4 \cdot R \cdot \{e\}$ can be considered as special cases of group decompositions. Group decompositions form the cornerstone of Leyton's theory. Cognition, for example, is defined as a map $D: G \rightarrow S$ from a group G to a stimulus set S , such that G is given a decomposition,

$$G = G_1 \cdot G_2 \cdot \dots \cdot G_n.$$

No precise definition of group decomposition is given, but from Leyton's writings we may infer the following characteristics: (1) each element g of G can be written uniquely as a product $g = g_1 \cdot g_2 \cdot \dots \cdot g_n$ (Leyton, 1986a, p. 10); (2) each G_i is a subgroup of G (Leyton, 1986a, p. 15); and (3) the products $G_{j+1} \cdot G_{j+2} \cdot \dots \cdot G_n$ are subgroups of G for each j (Leyton, 1985, p. 214). From condition (1) it follows that G_i and G_j have no common elements except the identity element (if i does not equal j). There are several problems with this.

First, it should be noted that the Iwasawa decomposition $GL_2R = A \cdot N \cdot SO_2$ is not a group decomposition as defined above. Indeed each element g of GL_2R has in general two decompositions $g = a \cdot n \cdot r$ and $g = (-a) \cdot n \cdot (-r)$. So, the Iwasawa decomposition does not satisfy the first condition of Leyton's definition of group decomposition.

Second, in general, the product $G_i \cdot G_j$ of two subgroups G_i and G_j of a group G is not a subgroup of G . Thus, in general, $G_{j+1} \cdot G_{j+2} \cdot \dots \cdot G_n$ is not a subgroup. In the case of the Iwasawa decomposition, one can easily prove that $N \cdot SO_2$ is not a subgroup of GL_2R .² That means that the Iwasawa decomposition does not satisfy the third condition of the definition of group decomposition. This third condition, however, is essential for the definition of cosets³ and cosets are used to explain grouping phenomena (see below).

Third, the internal history of a square $D_4 \cdot R \cdot \{e\}$ is not a decomposition because $D_4 \cdot R \cdot \{e\}$ is not a group.⁴ But even if it were a group, the decomposition would not satisfy the first condition because $r_{180} \cdot a = e \cdot (-a)$, that is, the decomposition would not be unique. So, although Leyton strongly focuses on the rotated parallelogram when illustrating the application of his theory, we see that his theory does not even hold for this case. In fact, most of the examples given by Leyton are not group decompositions (see below).

Let us come back to the general conception of cognition as a map $D: G \rightarrow S$ from a group G to a stimulus set S . How should we interpret this map? It is remarkable that Leyton's work contains no examples for which this map is explicitly defined. Let us consider the example of the rotated parallelogram again. We then have a map $D: A \cdot N \cdot SO_2 \rightarrow S$. The only reasonable definition for D is that D maps the identity element to the square S_0 , element of S , that is, $D(e) = S_0$ and that $D(a \cdot n \cdot k) = (k \cdot n \cdot a)(S_0)$. From this we can deduce that for all g and h we must have $D(g \cdot h) = h(D(g))$. In particular, $D(g) = D(e \cdot g) = g(D(e))$. Thus, we have $D(a \cdot n \cdot k) = k(D(a \cdot n)) = k(n(D(e))) = k(n(a(D(e)))) = k(n(a(S_0)))$. This seems the only way to establish the link with the foregoing definition of external generation: $S = g_n \cdot g_{n-1} \cdot \dots \cdot g_1(S_0)$.

This interpretation is confirmed by Leyton's theory of categories: "Each member in a category must be characterized as a transformation from that category member which is labeled by the identity element in the describing group" (1986a, p. 59). The stimulus set is the category, and the prototype, to which all elements are referenced, is the element $D(e)$. From the foregoing it also follows that for the external history of the rotated parallelogram, the stimulus set S is the set of all rotated parallelograms. It is not clear whether the map D is one-to-one. In the theory of concepts and categories (see also below), Leyton uses the inverse map D^{-1} , and thus implicitly assumes D is indeed one-to-one (see Leyton, 1986a, p. 68).

To summarize, cognition is modeled as a group decomposition. As a special case, the external and internal process histories of a rotated parallelogram are modeled

² A proof can be found in Hendrickx and Wagemans (1995).

³ Let H be a subgroup of a group G and g be an element of G . The set of elements of the form hg (with h being an element of H) is denoted by Hg and is called a *right coset* of H . A left coset gH is defined similarly. G is the disjoint union of the left (right) cosets of H .

⁴ A proof can be found in Hendrickx and Wagemans (1995).

by the decompositions $A \cdot N \cdot SO_2$ and $D_4 \cdot R \cdot \{e\}$. Both Leyton's theory of group decompositions in general and the application of it to the special case of the process history of a rotated parallelogram are mathematically inconsistent.

Interaction Principles

We have seen that a rotated parallelogram has an external history, described by the external group decomposition $A \cdot N \cdot SO_2$, and an internal history, described by the internal group decomposition $D_4 \cdot R \cdot \{e\}$. The complete history of the rotated parallelogram is then given by the decomposition $A \cdot N \cdot SO_2 \hat{\cdot} D_4 \cdot R \cdot \{e\}$ where $\hat{\cdot}$ is "the product symbol describing their interaction" (Leyton, 1986a, p. 41). The interaction between the external and the internal group is given by the fact that the symmetry axes of the square become eigenspaces (i.e., linear subspaces that are invariant under the transformation) of the most stable external group factors. The X and Y axes become eigenspaces of the stretch which is the most stable transformation. The Y axis becomes the eigenspace of the shear, but not the X axis, because the shear is less stable than the stretch. Neither the X nor Y axis becomes an eigenspace of the rotation, which is the most unstable transformation in the sequence and leaves only the center of rotation invariant.

Leyton generalizes the example of the rotated parallelogram to any stimulus set: "A perceptual organization of a stimulus set, S , is the interaction of two input-space descriptions: (i) A description from an external group, G_E , which describes S within a space of modifications of S , and (ii) a description from an internal group, G_I , which generates S from a subset of itself" (Leyton, 1986a, p. 41). The interaction between these groups is governed by the interaction principles. The FIRST INTERACTION PRINCIPLE specifies that "the symmetry axes of the internal group are eigenspaces of the most stable external group factors"⁵ (Leyton, 1986a, p. 42). The SECOND INTERACTION PRINCIPLE specifies that the more salient a symmetry axis is, the more transformations there are in the external generation sequence of which it becomes an eigenspace (adapted from Leyton, 1986a, p. 42).⁶

There are three problems with these definitions. First, it should be noted that groups do not have symmetry axes, only geometrical figures do, namely, the shapes generated by the internal group. Second, groups do not have eigenspaces only linear transformations do. Thus, the interaction principle should be interpreted as follows: the symmetry axes of the shape generated by the internal group become eigenspaces of the most stable external transformation. Third, the "interaction symbol" $\hat{\cdot}$ is not formally defined. Hence, it makes little sense to say that $G_E \hat{\cdot} G_I$ is a group decomposition as Leyton repeatedly does (e.g., Leyton, 1986a, p. 47).

Stability

A rotated parallelogram is reduced to a square through a sequence of transformations: a rotation followed by a shear which is followed by a stretch. The order

⁵ By *group factor* Leyton means one of the groups G_i in a group decomposition $G = G_1 \cdot G_2 \cdot \dots \cdot G_n$.

⁶ The formulation of the SECOND INTERACTION LAW in the same paper is incomprehensible to us. So, we have taken the definition from Leyton (1985, p. 221). As Leyton's terminology changes from paper to paper, we have been obliged to reformulate it a little.

of the transformations is one of increasing stability. Rotation is more destabilizing than stretch or shear, so removing rotation introduces more stability than removing stretch or shear. According to Leyton, the stability associated with a transformation can be characterized by the size of the eigenspace of the transformation, that is, “the amount of subspace that is invariant” (Leyton, 1985, p. 219). A stretch leaves the X and Y axes invariant and is thus more stable than a shear that leaves only the X axis invariant. A shear in its turn is more stable than a rotation that leaves only the origin invariant.

A precise definition of stability was given in Leyton (1986a, p. 15) in terms of an *algebraic stability ordering* on a Lie group G . The definition is highly technical and is discussed in detail elsewhere (Hendrickx & Wagemans, 1995). It is shown there, first, that Leyton’s definition is inconsistent, second, that the Iwasawa decomposition does not allow an algebraic stability ordering, and, finally, that Leyton’s work contains no correct examples of an algebraic stability ordering.

Cognitive Maximization Principle

Leyton defines cognition of a stimulus set S as a map $D: G \rightarrow S$ such that G is given a decomposition, $G = G_1 \cdot G_2 \cdot \dots \cdot G_n$, defined by an algebraic stability ordering. For a given stimulus set, however, there will be several possible maps. The *cognitive maximization principle* gives us a criterion to make the right choice. This principle states that “given a stimulus set S , the cognitive system chooses a description of S that is algebraically maximized and stability maximized” (Leyton, 1986a, p. 20). A *stability maximized description* $D: G \rightarrow S$ is one in which the members of $D(G)$ are more stable than for other descriptions or, in other words, one where as many of the trajectories as possible occur as nearly as possible to the stable end of the associated stability ordering. An *algebraically maximized description* is one with a minimal number of generators that are maximally used in accounting for the stimuli.

The point is that stability maximization is only defined for groups that have an associated stability ordering, that is, Lie groups. But Lie groups cannot be described by a finite number of generators.⁷ Thus, it is not possible to select an algebraically maximized description by taking one with a minimal number of generators. Hence, Leyton’s cognitive maximization principle is invalid.

COGNITION AS GROUP DECOMPOSITION: APPLICATIONS TO DIVERSE AREAS IN COGNITIVE SCIENCE

Perceptual Organization (Leyton, 1985; 1986a; 1992, pp. 349–370)

The foregoing theory is used by Leyton to explain a number of perceptual organization phenomena. Here we will limit the discussion to two of them: the orientation and form problem and the Johansson motion phenomenon.

⁷ A proof can be found in Hendrickx and Wagemans (1995).

Orientation and form problem. A typical example of the orientation and form problem is the square-diamond phenomenon of Mach (1886/1959); when a square is presented in a 45° orientation, it is no longer perceived as a square but as a diamond instead. The perceptual changes resulting from orientation changes have later been studied more systematically by Goldmeier (1936/1972), Rock (1973), and many others. Leyton’s explanation of the orientation and form problem is based on Cartesian reference frames. The internal history of an orthonormal Cartesian reference frame is given by the group $(Z_2 \times Z_2)(x_{sd})R^2$. R^2 represents the two-dimensional plane and the semidirect product with $Z_2 \times Z_2$ represents the fact that reflection about the axes transforms the reference frame into itself. The external history of an arbitrary Cartesian reference frame is $A \cdot N \cdot SO_2$, that is, the same history as for the rotated parallelogram. The complete history is then given by $(A \cdot N \cdot SO_2) \hat{=} ((Z_2 \times Z_2)(x_{sd})R^2)$.

“The basis for the impression that a Cartesian reference frame has been imposed on a perceptual organization, is this: The group decomposition sequence $(A \cdot N \cdot SO_2) \hat{=} ((Z_2 \times Z_2)(x_{sd})R^2)$ defining the frame, occurs as a subsequence of the full group decomposition sequence $G_{E1} \cdot G_{E2} \cdot \dots \cdot G_{En} \cdot G_{I1} \cdot G_{I2} \cdot \dots \cdot G_{In}$ defining the organization” (Leyton, 1986a, p. 47). No definition of “subsequence” is given. The intuitive interpretation that a subsequence of $G_1 \cdot G_2 \cdot \dots \cdot G_n$ is a sequence $G_p \cdot G_{p+1} \cdot \dots \cdot G_q$ with $1 \leq p \leq q \leq n$ is not applicable here. For the square-diamond phenomenon, for example, the group decomposition sequence for the square or diamond is $A \cdot N \cdot SO_2 \hat{=} D_4 \cdot R \cdot \{e\}$. This sequence cannot contain the group decomposition sequence of the Cartesian reference frame as a subsequence. However, without a proper definition of “subsequence,” Leyton’s theory of Cartesian reference frames is empty.

Unfortunately, Leyton’s informal application of the principle to the orientation and form problem is incorrect. Consider the perceptually ambiguous shape in Fig. 2. It can either be seen as a sheared square or as a stretched diamond. Leyton’s explanation for this is that the Cartesian reference frame can be imposed on this figure in two ways. The possible external coordinates are (e_A, n, r_1) and (a, e_N, r_2) . The corresponding internal histories are those of a square and of a diamond, respectively (Leyton, 1985, p. 230).

This explanation is flawed. The external history of the stretched diamond is given by (a, e_N, r_2) . All these transformations leave the sides of the diamond equal. The

FIG. 2. Ambiguous figure (adapted from *Information Sciences*, 38, Michael Leyton, Principles of information structure common to six levels of the human cognitive system, pp. 1–120, Copyright (1986), with permission from Elsevier Science): Figure (A) can be seen as (B) a sheared square or (C) a stretched diamond.

external history of the sheared square is given by (e_A, n, r_1) . The stretch n applied to the initial square returns a parallelogram with unequal sides. These sides are not changed by the rotation. However, the sheared square of Fig. 2B must have all sides equal. That means that (e_A, n, r_1) cannot be the external history of the sheared square as Leyton states. A nontrivial stretch is required. Moreover, the stretch has to be calculated such that after the shear the sides become equal, as can be seen in Fig. 3. As a consequence, the two interpretations, sheared square and stretched diamond, are not equally plausible.

Johansson motion phenomenon. Johansson (1950/1994) showed that two points moving perpendicularly and in phase (Fig. 4A) are not perceived as such. They are perceived as moving on a diagonal line that itself is moving as indicated in Fig. 4B. Johansson explained this phenomenon by decomposing the velocity vectors of the points into a component along the axis joining the two points and a component perpendicular to it (Fig. 4C). In other words, absolute motion M = common motion C^* relative motion R . Similar decomposition schemes have later been used in explanations of several other motion phenomena (e.g., Cutting, 1981; Cutting & Proffitt, 1982; Johansson, 1973; Restle, 1979).

Leyton's argument runs as follows. The internal history of the system is given by $(Z_2 \times Z_2) \cdot \{e\}$. The points P_i are encoded by $\{e\}$. The $Z_2 \times Z_2$ component expresses the fact that the points are reflectionally symmetric about the C - and R -diagonals. The external group that acts on the internal system is $R \cdot A$. The group of stretches A stretches the points P_i along the R -diagonal. The translation group R translates the R -diagonal along the C -diagonal. The complete system is then given by $R \cdot A \hat{=}$ $(Z_2 \times Z_2) \cdot \{e\}$. The symmetry axes of the internal system, the R - and C -diagonals, become eigenspaces of the R and A transformations.

Several remarks must be made. First, it should be noted that Leyton's theory, that was developed for static shapes, is now generalized to moving shapes. Second, the operations A and R should be applied in sequence whereas the movement of the points along the common axis and the movement of the axis itself occur in parallel. Leyton (p. 368) introduces the notation $[R \cdot A]_P$ to indicate that the processes act in parallel. This, however, is a contradiction in terms because the product $R \cdot A$ by definition means sequential application of transformations. Third, the notion of eigenspace is only defined for linear transformations like A , not for translations R . It thus makes no sense to say, as Leyton (1985, p. 233) does, that a symmetry axis

FIG. 3. Ambiguous figure revisited.

FIG. 4. The Johansson motion phenomenon (adapted from *Information Sciences*, **38**, Michael Leyton, Principles of information structure common to six levels of the human cognitive system, pp. 1–120, Copyright (1986), with permission from Elsevier Science).

“must be wired as an eigenspace of the analyzer system R.” Moreover, no explanation is given as to which symmetry axis is wired as an eigenspace of which analyzer.

Shape Perception

In the area of shape perception, Leyton’s work has quite gradually evolved from a rather simple extension of a specific analysis of smoothed local symmetry (Brady, 1983; Brady & Asada, 1984) to a more general theory of process-grammar applicable to all shapes bounded by a closed smoothly curved contour (i.e., shapes that can be represented as sequences of primitive contour fragments, called *codons*; see D. D. Hoffman & Richards, 1984; Richards & D. D. Hoffman, 1985).

Smoothed local symmetry (Leyton, 1987c; 1992, pp. 391–413). In Hendrickx and Wagemans (1995) it is shown that Leyton’s theory of smoothed local symmetry is fundamentally flawed. For more details we refer the reader to that report.

Process-grammar (Leyton, 1988; 1989; 1992, pp. 16–34, 55–69, 79–84, 398–413). Suppose we have a smooth, closed, and nonintersecting curve. The curvature variation

FIG. 5. The four possible kinds of extrema (adapted from *Artificial Intelligence*, **34**, Michael Leyton, A process-grammar for shape, pp. 213–247, Copyright (1988), with permission from Elsevier Science).

along the contour can be represented by a function of the type shown in Fig. 5. M and m denote local maximum and local minimum, respectively, and $+$ and $-$ denote positive and negative curvature. The curvature graph can be used to describe transformations of the original shape. First, we can pull the M^+ extremum up or the m^- extremum down. The curvature graph remains qualitatively the same and so does the corresponding shape. Leyton calls these transformations *continuation at M^+ and m^-* . We can do the same at m^+ and M^- . However, if we pull m^+ down beyond the horizontal axis, we get the transformation represented in Fig. 6A. This transformation is called *continuation at m^+* and can be represented by the rewrite rule $Cm^+ : m^+ \rightarrow 0m^-0$. The corresponding shape is transformed as in Fig. 6B. Similarly, the *continuation at M^-* gives the rewrite rule $CM^- : M^- \rightarrow 0M^+0$. Another transformation results from indenting one of the extrema as indicated in Fig. 7A. This is called *bifurcation at M^+* . The corresponding rewrite rule is $BM^+ : M^+ \rightarrow M^+m^+M^+$. The corresponding shape is transformed as in Fig. 7B. For each of the other extrema, there is a corresponding rewrite rule. The set of all these rewrite rules is called the *process-grammar*:

$$Cm^+ : m^+ \rightarrow 0m^-0$$

$$CM^- : M^- \rightarrow 0M^+0$$

$$BM^+ : M^+ \rightarrow M^+m^+M^+$$

$$Bm^- : m^- \rightarrow m^-M^-m^-$$

$$Bm^+ : m^+ \rightarrow m^+M^+m^+$$

$$BM^- : M^- \rightarrow M^-m^-M^-.$$

All of this is elementary geometry. What has it to do with perception? Leyton's argument runs as follows. From curvature extrema we can infer symmetry axes using Leyton's Symmetry-Curvature Duality Theorem. This theorem states that "any segment of a smooth planar curve bounded by two consecutive curvature extrema of the same type (either both maxima or both minima) has a unique differential symmetry axis and this axis terminates at the curvature extremum of the opposite type" (Leyton, 1988, p. 217). From the symmetry axes we can infer the process-history of the shape. This is expressed in the Interaction Principle: "The symmetry axes of a perceptual organization are interpreted as the principle directions along which processes are most likely to act or have acted" (ibid., p. 218). The sequence of rewrite rules that reduces a shape to an initial shape then constitutes a process-history for the shape.

The four types of extrema correspond to semantic terms that people use to classify processes: M^+ = protrusion, m^- = indentation, m^+ = squashing, M^- = internal resistance. The process-grammar rules have a corresponding interpretation:

Cm^+ squashing continues till it indents.

CM^- internal resistance continues till it protrudes.

BM^+ a protrusion bifurcates, e.g., a node becomes a lobe.

Bm^- an indentation bifurcates, e.g., an inlet becomes a bay.

FIG. 6. Continuation at m^+ (adapted from *Artificial Intelligence*, 34, Michael Leyton, A process-grammar for shape, pp. 213–247, Copyright (1988), with permission from Elsevier Science): (A) the deformation of the curvature curve and (B) the corresponding deformation of the shape.

FIG. 7. Bifurcation at M^+ (adapted from *Artificial Intelligence*, **34**, Michael Leyton, A process-grammar for shape, pp. 213–247, Copyright (1988), with permission from Elsevier Science); (A) the deformation of the curvature curve and (B) the corresponding deformation of the shape.

Bm^+ a protrusion is introduced.

BM^- an indentation is introduced.

There are several problems with this process-grammar. First, it should be noted that all shapes are reduced to a sequence of curvature extrema (which is also true for the coding theory of D. D. Hoffman & Richards, 1984). The exact form and length of the segments between the extrema plays no role. There are good reasons to doubt the psychological plausibility of this reduction of two-dimensional shape to a one-dimensional sequence of curvature extrema (e.g., Kurbat, 1994; Waeytens, Hanouille, Wagemans, & d'Ydewalle, 1994; Wagemans, Hanouille, Waeytens, & d'Ydewalle, 1994).

Second, all curves that contain segments of constant curvature are excluded because these segments contain an infinite number of extrema whereas shapes generated by the process-grammar rules have only a finite number of extrema. This means, for example, that the familiar bottle shape with its two parallel lines is excluded from Leyton's "shape space" (for a more thorough discussion of this type of limitations, see Kurbat, 1994).

Third, different shapes can have identical process-histories. A shape can have different extrema of the same type and the grammar rules do not specify which extremum is to be expanded. For example, the shapes Q6 and Q8 from Leyton's shape space (Leyton, 1988, pp. 220–230) are both generated from Q4 by applying the rule Cm^+ . This contradicts the Well-Definedness Principle that "the histories defined in cognitive representations are well-defined, that is, they are effective procedures" (Leyton, 1992, p. 158). The application of a process-history to an initially given shape should give one well-defined shape.

Soon after Leyton's process-grammar was published, others noticed that the process-grammar does not allow one to quantitatively derive a later shape from an earlier one and that the continuous family of intermediate shapes is therefore unspecified (Lin, Liang, & C.-T. Chen, 1989). A computational model based on the idea of elastic interpolation was then proposed to solve this problem. However, the specific examples used to demonstrate the computational model reveal that it is a typical engineering solution tailored to a specific problem, not a plausible account of human perception of shape history in general. For example, one of the test cases for interpolating between a start shape and a goal shape had to be split in two steps. Regardless of the practical usefulness of this interpolation tool, it is clear that it is not derived from Leyton's general idea of group decomposition.

Fourth, the process-grammar rules are purely formal rules that introduce new curvature extrema into a curve. The interpretation of these rules in terms like "squashing" and "resistance" is purely arbitrary. There is no evidence that people actually use these terms to interpret curves.

Fifth, much is left to the imagination of the reader. The "psychological" aspect of the theory is severely underdeveloped. For example, it is not clear what is meant by "processes." Leyton (1988, p. 213) writes, "We also develop a formal grammar by which someone, who has two views of an entity⁸ at two developmental stages,

⁸ Some of Leyton's favorite examples of shape entities are tumor, cloud, island, embryo.

can infer the processes that produced the second stage from the first.” Does this mean that the process-grammar rules allow one to deduce the physical production processes that produced the object? He writes, “The purpose of the grammar is to bridge any two shapes with a plausible developmental history” (ibid., p. 231). What is a developmental history? Is it the actual history of the object?

Symmetry axes are interpreted as traces of processes. But who does the interpretation? What is the nature of the interpretation process? For example, it is not clear whether the interpretation process is automatic, as one might expect from a perceptual system, or a conscious activity. The square-parallelogram example suggests that process-recovery occurs spontaneously and independent of will (as with illusions). However, the process-histories of Raphael's *Alba Madonna* (Leyton, 1992, Sect. 8.7) that are deduced with the help of the process-grammar rules are clearly the result of reflection. Also, it is not clear whether process-recovery changes with growing familiarity with the stimulus set. The illusions discussed by Leyton suggest that the recovery process remains the same. On the other hand, one may suspect that after having analyzed Raphael's *Alba Madonna* once, the recovery process will become a little easier later on. *Symmetry, Causality, Mind* does not give an answer to these questions. On the contrary, the notion of process-history is stretched even further in Leyton's book (see below).

Psychological Scales (Leyton, 1986a)

Leyton's theory of psychological scales contains many errors. In Hendrickx and Wagemans (1995) a detailed discussion is given of his argument (Leyton 1986a, p. 33) that the decimal expansion of a number can be described as an “asymmetric, sequential, explicit reference frame.” It is shown there that this argument is fundamentally flawed.

Concepts and Categories (Leyton, 1986a)

A category is defined by Leyton (1986a, p. 60) as follows: “A category is the basin of attraction of the prototype under the deviational dynamic [*sic*] defined by the algebraic stability ordering σ_M .” What does this mean? Let us take an example of a category whose elements can be characterized by two real numbers. These elements can be represented as points in the plane. Following Rosch (1975), one of the elements is a prototype for the category, to which the other elements are referenced asymmetrically. The reference structure can be represented by a vector field; the direction of the vector gives the direction of the reference and the length of the vector gives the reference strength.

We can now understand Leyton's definition. We have a dynamic process because category elements are referenced to and thus attracted towards the prototype. The dynamic process is deviational because elements are deviations of the prototype. The prototype is an attractor because it attracts the elements. The category itself is the basin of attraction of the prototype, in the sense of the set of elements that are attracted by the attractor.

Once more, we must discuss several problems with this formulation. First, the definition presupposes that the elements of the category can be characterized by real numbers. This follows from the fact that algebraic stability orderings are defined for Lie groups, and Lie groups are locally like R^n . As a consequence, all categories must have an infinite number of elements. More precisely, a category must have the same cardinality as R^n for some $n > 0$. The category of integer numbers, an example given by Leyton himself, does not satisfy this condition and is thus not a category according to his own definition.

Second, whereas there certainly exist categories whose elements may be characterized by real numbers (e.g., the category of real numbers) this seems to be the exception rather than the rule. How are the elements of categories like humans, birds, cars, etc., to be characterized by real numbers?

Third, the category must be given the structure of a group. Certainly, if the elements of a category can be characterized by real numbers then the group structure of R^n can be used. In that case the elements are just vectors. But whereas adding vectors is a meaningful operation, what does it mean to add humans, birds, cars?

Fourth, we must define an algebraic stability ordering. This implies the definition of a group decomposition (the trivial decomposition $G = G$) and of a vector field such that the prototype becomes an attractor. This can, of course, be done in an infinite number of ways. But can it be done in a meaningful way? What does it mean when the reference strength of a human towards the prototype human is 100?

Fifth, the definition of algebraic stability ordering implies that there can be only one prototype in a category. This contradicts Leyton's own statement that "the properties cognitively characterizing a category are not common to its members, but are held by only a few particular members called the prototypes," for example, "line orientations are referenced to the vertical and horizontal" (1986a, p. 58).

Leyton's work does not contain one single, completely worked-out example of a category (i.e., including a definition of the Lie group and of the algebraic stability ordering). He discusses only one example namely the category of cups (bowls), characterized by the ratio of their width to their height. But he does not specify either the corresponding Lie group (R^+ with multiplication and trivial decomposition) or the algebraic stability ordering.

Conclusion

The theory of cognition as group decomposition has been developed into new accounts of theories such as Fourier's description of simple cells, Chomsky's theory of grammatical structure and function, and robot trajectory planning (all in Leyton, 1986a). All of these applications suffer from similar problems as those mentioned above; the elements that appear in Leyton's theoretical approach of these phenomena as instantiations of group decomposition and algebraic stability ordering (e.g., transformations, generation sequences, interaction) do not obey the mathematical requirements to make this theory work.

Given that the mathematical tools do not allow the theory to capture all of the phenomena under its umbrella of group decomposition, perhaps we should turn to

the more conceptual foundations underlying the theory. They are formulated in a less technical, more philosophical way in *Symmetry, Causality, Mind* as recovery of causal process-history.

PERCEPTION AND COGNITION AS RECOVERY OF CAUSAL PROCESS-HISTORY

For Leyton the example of how a rotated parallelogram is successively referred to more symmetric shapes as precursors in its causal history (see Fig. 1) is prototypical for all perception and cognition. Perceiving is a matter of recovering or reconstructing the causal history of the perceived object. This recovery of causal history pervades all mental activity. It obeys certain general principles that are also found in other areas in which the human mind plays a role, like linguistics, arts, and politics. Hence, the successive reference phenomenon contains the grains of Leyton's central thesis in *Symmetry, Causality, Mind* and its elaboration into some forty proposals, principles, hypotheses, heuristics, and strategies.⁹

Recovering Process-History

The process-history idea is embodied in the Representation-is-Explanation Principle: "Any cognitive representation is the description of a stimulus as a state (or set of states) in a history that causally explains the stimulus to the organism" (p. 157). This principle is supplemented by the Cognitive Characterization Principle: "A cognitive system is one that creates and manipulates causal explanations" (p. 157). According to the Well-Definedness Principle, "the histories defined in cognitive representations are well-defined, i.e., they are effective procedures" (p. 158). "There is no concept that more clearly embodies the notion of well-defined causal interaction than *machine*. A machine is an object that behaves completely mechanically, i.e., without choice or creativity" (p. 158).

Any machine is defined by five components: (a) *input* actions that are applied to the machine; (b) *output* actions that are produced by the machine in response to the input actions; (c) the *states* of the machine, that is, the properties of the machine that determine which output actions the machine will produce in response to its input actions; (d) the *First Causal Relationship* **Input** $\times$ **State** $\rightarrow$ **Output** that specifies which output is produced for a given input when the machine is in a given state; and (e) the *Second Causal Relationship* **Input** $\times$ **State** $\rightarrow$ **Next State** that specifies what is the resulting new state of the machine. A sequence of input actions will be called a *program*. We can now reformulate the Representation-is-Explanation Principle as the Representational Machine Analogy: "Any cognitive representation of a stimulus is the description of the stimulus as a state (or set of states) of a programmed machine. That is, cognition attempts to define any environment as a collection of machines" (p. 159).

As an example (p. 173), a cup can be defined in terms of the five components specifying a machine as follows: (a) inputs, e.g., lifting; (b) outputs, e.g., conveying liquid; (c) states, e.g., the shape; (d) First Causal Relationship, e.g., lifting (input)

⁹ From now on, we will refer to Leyton (1992) whenever we cite his work.

acts on shape (state) → conveying liquid (output); (e) Second Causal Relationship, e.g., lifting (input) acts on shape (state) → same shape (next state). The following is an example of a process-history: cup was lifted and conveyed liquid leaving shape unchanged, cup was put back producing sound when touching the table and leaving shape unchanged, cup was filled again producing sound of falling liquid and leaving shape unchanged, cup was lifted again conveying new liquid leaving shape unchanged, cup slipped out of hand producing sound of breaking porcelain and changing the state from one piece to many pieces.

The Representational Machine Analogy states that cognition represents a stimulus as a programmed-machine. Computer science has established a number of programming principles that are also relevant to process histories. One of these principles is that the basic problem of computing is the mastering of complexity. Minimizing program complexity is exactly the requirement made in the History Minimization Principle according to which “the inferred process-history contains minimal distinguishability” (p. 70). The branch of computer science that deals with the minimization of control complexity is called *structured programming*. This brings us to the Structured Programming Hypothesis: “Perception is a structured programmer; i.e., it applies techniques of structured programming to construct the histories it assigns to explain the environment” (p. 183).

Structured programming has three basic principles: a restricted set of standardized control structures, decomposition, and hierarchical organization. First, the *standardized control structures* are sequential execution of statements, selection between two statements, and repeated execution of a statement. Second, *decomposition* means decomposing a program into independent components, called *modules*. Perception decomposes process-histories into modules in order to reduce complexity. These modules are, in fact, what are psychologically understood as *processes*. The advantages of decomposing a history into processes are the same as the advantages in computer science of decomposing a program into modules, namely: “(1) The modules or processes are constructing blocks that can be used in building other programs or histories. (2) The programs or histories can be more easily modified or revised (e.g., if new information is acquired) by modifying only the relevant module or process. That is, the effects of revision do not multiply through the whole program or history. In particular, errors in the history can be more easily isolated and changed” (p. 187). For example, the above process-history of the cup must be changed when perception realizes that the cup is not of porcelain but of iron. Modular programming will make such a change easier. (3) “The programs or histories are more comprehensible because they can be understood in pieces, i.e., modules or processes” (p. 187). Unlike real programmers, perception will not get lost in its own programs. Third, *hierarchical organization* results from nested control, that is, the execution of a program loop within another program loop.

Comments

The main problem with *Symmetry, Causality, Mind* is the use of a large number of terms such as history, causality, stimulus, machine, process, program, action, and memory, whose meaning changes from one page to another and in the end becomes

so broad as to become useless. This is especially so for the central concept of process-history. The following points concerning process-histories need some clarification: (a) What is a process-history? (b) What is the nature of the recovery process? (c) The process-history of what is recovered? (d) Who recovers the process-history?

First, process-history can mean the real history of an object or the perceived history of a shape. An example of the latter is the rotated parallelogram which is seen as originating from a nonrotated parallelogram, then a rectangle, and finally, a square (see Fig. 1). Whenever a physical object has a shape that is as simple as one of the steps in this so-called successive reference phenomenon (e.g., the rectangular cup, see p. 174 of Leyton's book), the stimulus has two process-histories: the perceived process-history and the real history (e.g., the way it is produced in a pottery or a factory). Even if one would consider the transformation from a square to a rotated parallelogram a history, to call it a causal history seems to require much imagination.

Process-history can mean past history or future history. It can mean the production history of an object or the history of its use. On page 173 of the book, where the process-history of the cup is discussed, we read that "the past history is the history that created the shape of the cup" whereas "the future history concerns the *use* of the cup." Now the future of today is the past of tomorrow. Nevertheless, the use of the cup is always future history. The terms past and future as they are used in this book have a meaning that is different from their generally accepted meaning.

A process-history is a program, that is, a discrete sequence of actions. The history of a side of a square, for example, is described (on pp. 189–190) as a succession of printing points at a distance of $1/1000$ of an inch. That means that if perception perceives a side of 1 inch length, it constructs a process-history and goes 1000 times through the loop of printing one point; that is an unlikely picture. The notion of *action* itself is again very broad. It can mean "print a point," "rotate a parallelogram," "lift a cup," "scratch a surface," and "kick a bin." Is "bring a cup to the lips" one action or is it a succession of "lift vertically," "move horizontally," and "turn the cup"? With such a broad definition, it should come as no surprise that everything can be explained as a process-history.

Second, the nature of the recovery process is variously described as "is understood as having originated from" (p. 9), "assigns a trace interpretation" (p. 45), "one will try to explain" (p. 57), "the observer will assume" (p. 70), "is taken to be" (p. 78), "processes are conjectured" (p. 80), "cannot help but see" (p. 138), "causally explains" (p. 157), "perception is a programmer" (p. 183). The recovery process can be a perceptual process but also a process of inference. It can be unconscious and observer independent as in the case of the rotated parallelogram or conscious and observer dependent as the reconstruction of the process-history of Raphael's *Alba Madonna* in Chapter 8.

Third, what is being recovered is variously described as "the present moment" (p. 4), "shape" (p. 4), "some aspect of the percept" (p. 45), and "stimulus" (p. 157). Mostly it is the process-history of an object or shape that is being recovered. But it can also be the process-history of the flux of light rays that make the object visible, as in the discussion on shape-from-shading (p. 99). It can be the history of the observer of the object, as in Chapter 9 on politics: "In arriving in another country,

the refugee is presented with objects (...) from which the refugee cannot infer his or her own past" (p. 588). It can be the history of "a person, a culture, the animate and inanimate environment, oneself" (p. 601).

Fourth, the recoverer is variously described as "an individual" (p. 4), "the perceptual system" (p. 46), "the observer" (p. 70), "the reader" (p. 84), "interpolation theorists" (p. 138), "the organism" (p. 157), "the cognizer" (p. 166), and "cognition" (p. 273). The term *observer* is also very broad (e.g., it can mean a "piece of neuronal material in the brain," p. 154).

In sum, the concept of process-history is so broad and so vague as to be useless. The same is true of many of the other concepts used in this book. For example, memory is "a physical object (1) that is in the present of some observer and (2) that has features that the observer causally explains" (p. 154). The second condition is redundant because of the Representation-is-Explanation Principle. Thus, any object that is observed is memory. But when the observer turns his head, the object ceases to be memory. It becomes memory again when the observer takes another look at it. Memory is not only a physical object, it is also energy according to the Energy-is-Memory Principle (p. 76): "The energy of a system can be regarded as memory of the causal interactions that transferred the energy to the system."

This point is made clear by the Galileo experiment (pp. 74–75): "A ball is held at position *P* on one of the planes. The ball is then released, and it rolls down the plane to the bottom and then begins to roll up the second plane. Galileo discovered that, at whatever angle the second plane is inclined, the ball ends its movement up the second plane at the same height as the height at which it started on the first plane; i.e., the same level as position *P*". The classical explanation is that the potential energy of the ball is converted into kinetic energy when the ball rolls down and is converted back into potential energy when the ball goes up. The book reinterprets this in terms of memory (p. 76): "Thus [the ball] *remembers* the height at which it started because its energy has to equal the energy it had when it started. Therefore the conservation of energy is equivalent to the existence of memory." This reinterpretation opens up new lines of thought (which may strike some readers as ridiculous). Due to friction, the ball will not attain the same height: "the ball is forgetting." When the experimenter positions the ball at *P*, "the ball is learning."

Further Principles

In addition to the principles discussed to date, there are a number of other principles. The Symmetry-Asymmetry principles occupy a special place in the book. According to the Asymmetry Principle, "an asymmetry in the present is understood as having originated from a past symmetry" (p. 9). The Symmetry Principle states that "a symmetry in the present is understood as having always existed" (p. 10). Symmetry is equated with *indistinguishability* and asymmetry with *distinguishability*. The concepts of distinguishability and indistinguishability are so general that almost any theory or phenomenon becomes a special case of the Asymmetry Principle (see below).

The Stability Principle states that "the more symmetric a configuration is, the more stable it is understood to be" (p. 13), where stability is defined as "temporal

symmetry; i.e., the absence of change through time" (p. 12). Counterexamples are easy to find. If we pile ten cubes one upon another the resulting tower will be understood to be relatively stable. If we replace the cubes by balls the resulting configuration will appear highly unstable, although it is equally symmetric as a configuration, or even more symmetric if one also considers the higher degree of symmetry of a circle compared to a square.

The Second System Principle states that "increased asymmetry over time can occur in a system only if the system has a causal interaction with a second system" (p. 13). In that case, however, the two systems together constitute a system with increased asymmetry and no interaction with another system. This principle seems inconsistent.

The Prototype Hypothesis states that "shape prototypes are nested repetitive Euclidean hierarchies" (p. 53). But the book does not tell what *nested repetitive Euclidean hierarchies* are. Hence, this principle remains empty. Perhaps the idea of sequential group decomposition as in the successive reference phenomenon is implicated, but we have already indicated some fundamental problems with its mathematical aspects.

The Shape-is-Time Hypothesis states that "shape equals the history that created it" (p. 73). The equality should be understood literally, not metaphorically. "In fact, this would mean that shape is literally the history that created it" (p. 73). It is clear that shape and history get a meaning here that is different from their generally accepted meaning.

COGNITION AS RECOVERY OF PROCESS-HISTORY: APPLICATIONS TO DIVERSE AREAS IN COGNITIVE SCIENCE AND BEYOND

The general theory summarized and discussed above is applied to several different domains. Chapter 3 deals with the application of the theory to computational vision. Chapter 6, with its 180 pages, involves an extensive discussion of the application of the theory to the problems of perception. The first part, some 130 pages, is devoted to the perception of simple shapes and more detail about the process-history of a rotated parallelogram. The second part of Chapter 6 is devoted to the perception of complex shapes, which is also discussed in Chapters 1 and 2. Further applications are in the areas of linguistics, art, and politics. Because we have already highlighted some problems with Leyton's theory of perceptual organization and shape perception, we will now focus on the other applications.

Radical Computational Vision (Chapter 3)

Whereas Standard Computational Vision tries to reconstruct a 3D object from a 2D projection, Radical Computational Vision goes one step further, that is, it takes the 3D object to its process-history. Thus, Radical Computational Vision = Standard Computational Vision + process-history. The chapter discusses a number of approaches from computer vision: shape-from-shading, shape-from-texture, shape-from-contour, shape-from-stereo, shape-from-motion. All these approaches fall under the Asymmetry Principle because they all use some Uniformity Assumption

to explain something that is not (or less) uniform. Consider, for example, shape-from-stereo (p. 130):

The two eyes each have a view of some object. We will consider, for the moment, the simplest possible object, a dot. Thus the image in the left eye is a single dot and the image in the right eye is a single dot. The assumption made by the perceptual system is that these *two* dots, one in one eye and one in the other, arose from a *single* dot in the environment.... This means that a distinguishability, two dots, is converted into an indistinguishability, one dot.

What the theory does not explain is which dot in the left eye corresponds to which dot in the right eye. What the theory does not explain is why the whole image does not collapse into one single point as the Asymmetry Principle requires.

Linguistics (Chapter 7)

Languages are described in modern linguistics by Phrase-Structure Grammars that specify how the sentences of the language can be generated from a finite set of Phrase-Structure Rules. For example, the Phrase-Structure Rule

$$NP \rightarrow DET, A, N$$

specifies that a noun phrase *NP* can be generated by generating successively a determiner, an adjective, and a noun. If we take *DET* = the, *A* = angry, and *N* = student we obtain

$$NP \rightarrow \text{the angry student.}$$

Leyton argues that the Phrase-Structure Rule $NP \rightarrow DET, A, N$ should be replaced by the Accumulation Rule

$$N \rightarrow A(N) \rightarrow DET(A(N)) = NP$$

because the relation between *DET*, *A*, and *N* is not symmetric, as suggested by the Phrase-Structure Rule, but asymmetric. The word *angry* describes a property of *student*, not vice versa. Similarly, the word *the* describes a property of *angry student*. The asymmetry should be made explicit in the rule.

Leyton relates this new approach to the theory of *topic-comment structure* as elaborated by American functionalists (Li, 1976). In the example, *student* is the topic, *angry* is a comment on *student*, and *the* is a comment on *angry student*. The topic-comment structure of a sentence also describes the *information structure* of the sentence. In fact, the terms *topic* and *comment* are alternative terms for *old information* and *new information*, respectively. A comment is the addition of new information to old information. In the Accumulation Rule new information is successively added (accumulated) to old information.

The problem with this theory is that it is explained using a simple example and that it is hard to see how it generalizes to more complicated cases. For a fuller

discussion of Leyton's theory of linguistics we refer to Hendrickx and Wagemans (1995).

Arts (Chapter 8)

This chapter opens with the First Art-Work Principle: "An art-work is the process-history inferred from it" (p. 478). This principle is illustrated by an in-depth analysis of three paintings by Picasso and one by Raphael using the rules of the process-grammar.

The analysis of Picasso's *Woman Ironing* illustrates the full power of the theory by showing that the deeper meaning of a painting may be extracted from a small and even hardly visible part of a picture (pp. 480–492):

The meaning of this painting is obvious. It shows a woman, bruised and pummelled by the outside world, bent over by exhaustion, and using what little strength she has left to hold herself upright and to carry out a simple task that has become too burdensome.... The meaning of the painting is conveyed in its process-structure, and this structure is established by our rules for the recovery of history from curvature extrema.... The most dominant process is the upward one terminating at the tip of the shoulder. This describes the push of the woman upwards against immense exhaustion.... Now, when one ... examines the shoulder in greater detail, one finds that the shoulder has a flattening at the top.... [The tip of the shoulder] represents an outward force of the woman. [With the flattening of the tip], an inward counter-force of the environment has been introduced; i.e., in flattening the tip of the shoulder, the environment is literally *squashing* her intent.... Let us consider the shoulder still further.... What we find is that the shoulder has a dip in the top.... The force squashing the woman's intent has continued further until it has actually invaded her.... Because the woman's shoulder, which was once merely squashed or bruised is now actually penetrated by the environment.

So much for the dip in the tip of the shoulder.

Whereas the dip in the tip of the shoulder of Picasso's *Woman Ironing* can be made visible with a magnifying-glass, the lines that form the basis for the analysis of Raphael's *Alba Madonna* are completely invisible. That the analysis of a work of art should be based on invisible lines is no cause for worry; it might make the analysis all the more profound. A twenty-five page in-depth analysis of the straight and curved lines in the picture reveals that the meaning of the picture is *convergence*. Then, with a masterful sleight of hand the author switches from convergence to *divergence* and concludes (p. 563):

Clearly, the meaning of divergence is *outgrowth*. The central role of the Madonna, in the Christian mythology, is the birth that she gives. And the birth brings not only physical life but spiritual life. This process of spiritual emergence and flowering is captured in the divergences of the painting.

According to the First Art-Work Principle, an art-work is the process-history inferred from it. However, all perception is process-recovery. Thus the First Art-Work Principle does not distinguish an art-work from an arbitrary object. What distinguishes an art-work from an arbitrary object is explained in the Second Art-Work Principle: "An art-work allows the *maximal amount of history* that a person

is capable of inferring from any percept; i.e., the *maximal amount of causal explanation* that a person is capable of assigning to any percept” (p. 565). The maximal amount of history is equal to the maximal amount of asymmetry and the amount of asymmetry is quantified by complexity. This brings us to the Complexity-Level Hypothesis.

This hypothesis states that “(1) Each individual has a preferred level of environmental complexity, a level that the individual seeks, and finds extremely appetitive. (2) As environmental complexity moves away from this level, i.e., as complexity is raised or lowered from this level, the environment becomes more and more aversive” (p. 567). Talking and walking, for example, are attempts to increase complexity (p. 575). Everybody likes to talk and walk now and then. But nobody likes to talk or walk all day and night. It is clear then that there must be a preferred level of talking and walking complexity and that an increase or decrease in complexity is experienced as being unpleasant.

Politics (Chapter 9)

In this chapter, the principles of the book are used to analyze political subjugation. Political systems achieve assent by two methods: (a) the destruction of *internal* history and (b) the destruction of *external* history.

Internal history is destroyed through death. Death is an example of symmetry. This can be seen as follows (p. 585):

The body ... represents a massive degree of *asymmetry*.... The functions of living, e.g., functions such as eating and self-protection, are to maintain these asymmetries. At death, these functions can no longer be carried out and the body begins to decay. The important thing to understand is that the decay is a process of *symmetrization*. The highly asymmetric living body degrades into material that is successively more and more homogeneous. Quite simply, death is an example of symmetry.

External history is destroyed by removal of the objects from which the past can be reconstructed. This can be achieved in three ways. The external history of the *refugee* is destroyed in that “in arriving in another country, the refugee is presented with objects that contain memory only of other people; i.e., from which the refugee cannot infer his or her own past” (p. 588). The external history of the *political prisoner* is destroyed in that the walls of the prison “physically separate the prisoner from his or her external memory and therefore cause a substantial reduction in the computational capacity of the prisoner” (p. 590). The external history of the *slave* is destroyed in that “the slave is forced to produce only the external memory of others” (p. 591).

The destruction of internal and external history is achieved through the symmetrizing effects of *technology*, as follows (pp. 595–597):

The structure of production, marketing, and consumption, in the technological society, requires the mass symmetrization of the human population, and the mass symmetrization of the physical environment. [Standardization of production leads to] mass duplication of a product, an example of symmetry.... The goal of advertising is to create greater and greater symmetry in the behavior, and therefore, history of a population.... It is at this

point that our Symmetry Principle becomes crucial. The principle states that history cannot be extracted from a symmetry. The consequence, therefore, of the mass symmetrization created by the technological society, is the mass destruction of history.... Therefore, the supposedly beneficial consequences of a technological society—a thriving and expanding economy—are formally indistinguishable from the detrimental effects—genocide and mass political subjugation in the form of refugee populations, concentration camps, labor camps, slave populations, etc. Both the beneficial and the detrimental effects lead to one end: the mass destruction of history.

The chapter ends with a more positive note (pp. 601–602):

In a world in which the full might of technological and economic power is being used to destroy history, it is not surprising that a new ethical principle should be proposed. It is such a principle that we wish now to offer: THE HISTORY ETHIC. The history ethic is the commitment to revealing, to the fullest, the history of an entity—a person, a culture, the animate and inanimate environment, oneself—and to interact with the entity by virtue of its fullest history. It is possible that all the great ethical principles of the past come from the History Ethic; e.g., the commandment ‘Thou shall not kill’ is an injunction not to cause the physical symmetrization of a person and therefore an injunction to maintain our capacity to extract history from the person.... The great ethical principles of the past would therefore be injunctions against different modes of destroying history or equivalently against different modes of destroying our capacity to think.

In other words, the ten commandments of Moses should be replaced by the one commandment of Michael Leyton:

Thou shall process histories.

GENERAL DISCUSSION

Summary and Conclusions

We have presented a two-pronged critique of Leyton’s theory of perception and cognition. First, we have analyzed the mathematical aspects of Leyton’s theory of perception and cognition as a map $D: G \rightarrow S$ from a group G to a stimulus set S , such that G is given a decomposition, $G = G_1 \cdot G_2 \cdot \dots \cdot G_n$, and an associated algebraic stability ordering. In many cases, no precise definitions were given, so essential clarifications were lacking. In other instances, the underlying mathematical assumptions were found to be self-contradictory. All in all, we must conclude that Leyton’s mathematics, although quite impressive to nonmathematicians, is deficient. Second, we have discussed the more conceptual principles of Leyton’s theory of perception and cognition as recovery of causal process- history. Again, we had to point out many ambiguities and unsolved questions even with respect to the basic notion of process-history (What exactly is recovered? Who recovers? How is it recovered?). Some of the derivative theoretical principles were found to be contradictory, and other principles are so general that they are devoid of meaning.

We have illustrated how far Leyton has pushed the idea of process-history and its many associated theoretical principles in areas like computational vision, linguistics, arts, and politics. In all of these domains, the metaphysical principles of

Symmetry, Causality, Mind seemed to work only because the meaning of underlying notions like process, history, memory, and symmetry, was stretched beyond limits. We have demonstrated how the applications inherit the problems from their underlying foundations. The idea of stability-ordered group decomposition was shown to be mathematically ill-founded in all of the phenomena where Leyton claims to have uncovered asymmetric, sequential referencing.

Despite the large differences in style between Leyton's mathematical papers and his more philosophical book, the problems are remarkably similar. For example, we have indicated some mathematical problems in the theoretical account of the perceived history of the rotated parallelogram, the steppingstone to the whole framework, and we have indicated a confusion between the reference sequence and the generation sequence. In the same spirit, we have noticed a fundamental ambiguity in the more conceptual elaborations of the notion of process-history (e.g., from cup to square, or to the real process having produced the cup in a pottery or factory). As another example, many of the mathematical problems were caused by the fact that the group decomposition was not unique, just as many of the conceptual problems had to do with the fact that arbitrary choices had to be made when the general principles were applied to concrete cases (e.g., linguistic utterances, specific parts or configurations in a painting). Not only is it possible to translate the mathematical theory of stability-ordered group decomposition into a metaphysical theory of causal process-history recovery (as Leyton did), it is also possible to find counterparts of the theoretical problems in the two versions.

In sum, the mathematical and conceptual foundations of Leyton's theory of perception and cognition are shaky. In the final section of this paper we will discuss what is left when the mathematical and metaphysical bombast are stripped from Leyton's work.

On What Remains of an Ill-Founded Theory

Can Leyton's theory be salvaged if we forget about group decomposition and algebraic stability ordering, that is, if we discard the mathematics? We are tempted to conclude that this is not the case. Leyton's way of bringing diverse topics like perceptual organization, shape perception, categorization, language, and more, together in one single approach is intimately based on the mathematics of group decomposition which was supposed to be applicable to all of them. If all of these fail to fulfill the essential conditions for this group decomposition to work, as we have suggested above, what remains is nothing more than the original theory formulated to explain the phenomena (for example, Johansson's vector decomposition to explain the visual perception of some motion configurations, Brady's smoothed-local-symmetry analysis of shape, and Rosch's idea of prototypes).

In our opinion, the more general, nonmathematical idea that the human mind is tuned to the processes that give rise to perceived objects might survive if the specific mathematical conceptualization in terms of transformation groups is proven untenable. However, this is not a very original idea. Indeed, Corballis (1994) noted that similar ideas can be traced back to the ecological approach to perception and action (including cognition) of Gibson (1979). We also find it related to Shepard's notions of

psychophysical complementarity (1981), resonant kinematics (1984), and perceptual-cognitive universals (1994). Regardless of the solidity of the mathematical basis, these approaches have one major advantage compared to Leyton's: They have inspired a great deal of empirical work yielding well-established phenomena that will survive the theoretical fashions which prevail for a relatively short time.

The empirical work presented in published articles in which Leyton's work is cited does not follow Leyton in his general approach to perception and cognition as a map $D: G \rightarrow S$ with a group decomposition, $G = G_1 \cdot G_2 \cdot \dots \cdot G_n$, and an associated algebraic stability ordering. Nor do they follow-up on one of Leyton's specific proposals. Indeed, we claim that this would be virtually impossible because the general principles, with the incorrect mathematics being stripped off, cannot be reformulated in specific, falsifiable predictions. For example, sometimes Leyton's paper about perceptual organization (Leyton, 1986c) is mentioned for its focus on symmetry and invariance under transformations (e.g., K. C. Chen & M.-T. Chen, 1993; Leeuwenberg & van der Helm, 1991; Pani, 1994; Pani, Zhou, & Friend, 1997; van der Helm & Leeuwenberg, 1991, 1996; van der Helm, van Lier, & Leeuwenberg, 1992; Wagemans, 1993). Many others refer to Leyton's paper about process-history (Leyton, 1989) for its focus on curvature singularities (e.g., Burbeck & Pizer, 1995; Elder & Zucker, 1993; Forbus, Nielsen, & Faltings, 1991; Kurbat, 1994; Landau & Jackendoff, 1993; Rom & Medioni, 1993; Siddiqi & Kimia, 1995).

Another question in relation to Leyton's deficient mathematics is whether it could be reworked into a coherent whole. This we do not exclude. Impossibility theorems are notoriously hard to prove and human ingenuity can be amazing. But such a rewriting is the author's task, not the reader's. In some of the areas which Leyton's theory covers, group theory has been used with some degree of success. For example, in visual perception, group theory has played an important role (e.g., Cassirer, 1938/1944), especially in relation to an invariants-based approach to object recognition (e.g., Van Gool, Moons, Pauwels, & Wagemans, 1994; Wagemans, Lamote, & Van Gool, 1997; Wagemans, Van Gool, & Lamote, 1996) and in the ecological approach of Gibson (1979).

More specifically, W. C. Hoffman (1966) has proposed the Lie algebra of a transformation group as a mathematical model of form perception. This so-called Lie transformation group theory of neuropsychology (LTG/NP) has been made more accessible to psychologists by Dodwell (1983) and has been applied successfully to Gestalt phenomena (e.g., W. C. Hoffman, 1994; W. C. Hoffman & Dodwell, 1985). Direct empirical support has also been provided (e.g., Caelli & Dodwell, 1982, 1984; Dodwell & G. K. Humphrey, 1990; G. K. Humphrey, Muir, Dodwell, & D. E. Humphrey, 1988; Wilkinson & Dodwell, 1980).

Another valid application of group theory to perception is contained in a set of papers by M.-T. Chen and K. C. Chen. They started to study the isometry group and some of its subgroups in two-dimensional figures (M.-T. Chen & K. C. Chen, 1982). The isometry group was then extended to the larger similarity group and its subgroups (M.-T. Chen & K. C. Chen, 1987). In their third paper, they applied group-theoretic principles to the perception of objects in three-dimensional space, using the isometry group and some of its subgroups (K. C. Chen & M.-T. Chen, 1993). This isometry group was then extended to the larger space similarity group

and its subgroups (K. C. Chen & M.-T. Chen, 1999). In all of these cases, the mathematics is sound and the experimental results confirm the group model (see also Wagemans, Vanden Bossche, Segers, & d'Ydewalle, 1994).

One final instance of an improvement of Leyton's mathematics is the use of Lie group structure by Carlton and Shepard (1990a, 1990b) to induce a connection on a manifold and then to study the geodesics on this manifold to explain psychological phenomena (e.g., mental rotation). Carlton and Shepard's starting point is that the human mind has (partly) internalized the external world. The external world can be described by the formalism of differentiable manifolds as has been made clear by modern physics (relativity theory, elementary particle physics). Carlton and Shepard suggest that the internalized world can be described by differentiable manifolds in the same way even though the manifolds may be of a more abstract nature. It remains to be seen whether this approach will lead to further empirical success but at least it is mathematically sound and therefore worthy of elaboration and testing.

Final Conclusion

The basic problem with Leyton's work is that it is vague and ill-specified in its nonmathematical version and mathematically incorrect when the idea of successive referencing is made more specific. More than 600 pages in *Symmetry, Causality, Mind* did not suffice to make the theory of cognition as the recovery of causal process-history specific enough to be testable and therefore scientifically promising. There is some hope that the mathematics that Leyton has used in his earlier attempts to substantiate the theory, namely, group theory and differential geometry, will turn out to be useful in some areas of perception. Mathematically correct applications can be found already. However, with Cutting (1986, 1987), we believe that the problem with group theory is that it may model but not explain perception (which may be the case for all mathematical theories or models of psychological mechanisms; see Uttal, 1990). All in all, group theory and differential geometry are in themselves important disciplines. But what remains to be elucidated is the link with perception and cognition.

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